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Periodicity and Almost Periodicity in Markov Lattice Semigroups

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Let $K$ be a compact metric space. A continuous semiflow $\phi : \mathbb{R}_+ \times K \to K$ on $K$ defines a strongly continuous Markov lattice semigroup $T : \mathbb{R}_+ \to \mathcal{L}(C(K))$ acting on the Banach space $C(K)$ of all complex-valued continuous functions on $K$ (endowed with the uniform norm) and expressed by

$$T(t) f = f \circ \phi_t$$

for all $f \in C(K)$ and all $t \in \mathbb{R}_+$.

If $x \in K$ is a periodic point of $\phi$, the functions $t \mapsto f(\phi_t(x))$ are continuous periodic functions on $\mathbb{R}_+$ for all $f \in C(K)$: a fact which imposes constraints on the spectral structure of the infinitesimal generator $X$ of $T$. Milder restrictions on the spectrum of $X$ are implied by the existence of almost periodic orbits, of asymptotically stable points and of non-wandering points for $\phi$. Some of these constraints, together with their consequences on the behaviour of $T$ and of $\phi$, are discussed in this article.

In its final section, the paper corrects an error in [5], that was kindly pointed out to the author by C. J. K. Batty.

1. The following results have been established in [5]. Let $E$ be a complex Banach space and let $T : \mathbb{R}_+ \to \mathcal{L}(E)$ be a uniformly bounded, strongly continuous semigroup of continuous linear operators acting on $E$. Let $X : D(X) \subset E \to E$ be the infinitesimal generator of $T$. Let $M > 1$ be such that $\|T(t)\| \leq M$ for all $t \geq 0$.

Consider now the dual semigroup $T^+ : \mathbb{R}_+ \to \mathcal{L}(E^+)$ of $T$, and let $X^+ : D(X^+) \subset E^+ \to E^+$ be its infinitesimal generator (see, e.g., [3] for the definition). For $g \in E$ and $\lambda \in E'$, the topological dual of $E$, the topological dual $\mathcal{H}'_{i\theta}$ of all $\lambda$ in the topological dual $E'$ of $E$ for which the limit

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} \langle T(t) f, \lambda \rangle dt$$

exists for all $f \in E$, is a linear subspace of $E'$ which contains $\ker(X^+ - i\theta I) \oplus \overline{\mathcal{R}(X^+ - i\theta I)}$ (where $\overline{\mathcal{R}(X^+ - i\theta I)}$ is the closure of the range $\mathcal{R}(X^+ - i\theta I)$ of
$X^+ - i\theta I)$. Since $\mathcal{E}'$ is sequentially weak-star complete, the equation

\begin{equation}
\langle f, R_{i\theta} \lambda \rangle = \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} \langle T(t)f, \lambda \rangle dt \quad \forall f \in \mathcal{E}
\end{equation}

defines a continuous linear operator $R_{i\theta} : \mathcal{H}'_{i\theta} \to \mathcal{E}'$ which is a projector with norm $\leq M$, whose range is $\ker(X^+ - i\theta I)$ and whose restriction to $\ker(X^+ - i\theta I)$ coincides with the spectral projector defined on $\ker(X^+ - i\theta I)$ by the ergodic theorem applied to $X^+ - i\theta I$.

The hypothesis on the existence of the limit (2) for all $f \in \mathcal{E}$ and all $\theta \in \mathbb{R}$, is satisfied if $\lambda \in \mathcal{E}'$ is such that the functions $t \mapsto \langle T(t)f, \lambda \rangle$ are asymptotically almost periodic for all $f \in \mathcal{E}$.

2. Let $K$ be a compact metric space. To avoid trivialities, suppose that $K$ contains more than one point. Let $\phi : \mathbb{R}_+ \times K \to K$ be a continuous semiflow on $K$, (see, e.g., [1]), and let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{C}(K))$ be the strongly continuous Markov lattice semigroup, acting on the Banach space $\mathcal{E} = \mathcal{C}(K)$ of all complex-valued continuous functions on $K$, endowed with the uniform norm, defined by (1) for all $f \in \mathcal{C}(K)$ and all $t \in \mathbb{R}_+$. The infinitesimal generator $X$ of $T$ is a derivation.

For any $t \in \mathbb{R}_+$, $T(t)$ is a linear contraction; it is an isometry if, and only if, $\phi_t$ is surjective, and is a surjective isometry if, and only if, $\phi_t$ is a homeomorphism of $K$ onto $K$.

If

\begin{equation}
\|T(t_o)f\| < \|f\| \quad \text{for some } t_o > 0 \text{ and some } f \in \mathcal{E},
\end{equation}

and if $t \geq 0$, then

$$
\|T(t_o + t)f\| = \|T(t(T(t_o)f))\| \leq \|T(t_o)f\| < \|f\|.
$$

Thus, if (4) holds, then $\|T(s)f\| < \|f\|$ for all $s \geq t_o$. Equivalently, if there exists $\epsilon > 0$ such that $\|T(\epsilon)g\| = \|g\|$ for all $g \in \mathcal{E}$, then $\|T(s)g\| = \|g\|$ for all $g \in \mathcal{E}$ and all $s \in [0, \epsilon]$. Thus, if (4) holds, then $t_o > \epsilon$. Suppose now that, furthermore, $\|T(t_o - s)g\| = \|g\|$ for all $g \in \mathcal{E}$ and some $s \in (0, \epsilon)$. Then

$$
\|T(t_o)g\| = \|T(s)T(t_o - s)g\| = \|T(t_o - s)g\| = \|g\|,
$$

contradicting (4). The set

$$
S = \{t \in \mathbb{R}_+^* : T(t) \text{ is not an isometry}\}
$$

is either empty or an open half line.

**Proposition 1.** If $T(t)$ is a contraction of $\mathcal{E}$ for any $t \geq 0$, the set of all $t$ for which $T(t)$ is an isometry is either $\mathbb{R}_+$ or the empty set.
In other words, either $S = \emptyset$ or $S = \mathbb{R}^*_+(1)$.

**Proof.** If $S \neq \mathbb{R}^*_+$, there is $\epsilon > 0$ such that $(0, \epsilon] \cap S = \emptyset$, and therefore $T(t)$ is an isometry for all $t \in (0, \epsilon]$. Let $(S \neq \emptyset$ and let $t_1 = \inf S$. Hence $t_1 > 0$. Choose $\sigma$ in such a way that

$$0 < 2\sigma < \epsilon, \quad \sigma < t_1.$$ 

Then $t_1 + \sigma \in S$ and $0 < t_1 - \sigma \not\in S$. Since

$$t_1 - \sigma = t_1 + \sigma - 2\sigma > t_1 + \sigma - \epsilon,$$

$T(t_1 - \sigma)$ is not an isometry, contradicting the definition of $t_1$. $\square$

As a consequence of this result and of Theorems 1 and 2 of [4], the following theorem holds.

**Theorem 1.** If $\phi_t$ is surjective for some $t > 0$, the derivation $X$ is a conservative and $m$-dissipative operator whose spectrum is non-empty.

Let $x \in K$ be such that the functions

$$t \mapsto \langle T(t)f, \delta_x \rangle = f(\phi_t(x))$$

are asymptotically almost periodic on $\mathbb{R}_+$ for all $f \in C(K)$. Then, for any $\theta \in \mathbb{R}$, the limit

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} \langle T(t)f, \delta_x \rangle dt = \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} f(\phi_t(x)) dt$$

exists for all $f \in C(K)$, showing that $\delta_x \in \mathcal{H}_{i\theta}$. As before, let $R_{i\theta} \delta_x \in C(K)'$ be defined by

$$\langle f, R_{i\theta} \delta_x \rangle = \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} f(\phi_t(x)) dt \quad \forall f \in C(K);$$

$R_{i\theta} \delta_x$ is (represented by) a Borel measure on $K$, i.e.,

$$\int f \, dR_{i\theta} \delta_x = \langle f, R_{i\theta} \delta_x \rangle = \frac{1}{a} \int_0^a e^{-i\theta t} f(\phi_t(x)) dt \quad \forall f \in C(K).$$

(1) The proof holds for any normed vector space $\mathcal{E}$ and for every map $T : \mathbb{R}^*_+ \to \mathcal{L}(\mathcal{E})$ such that $T(t)$ is a contraction and $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t, t_1, t_2 \in \mathbb{R}^*_+$. 
For all \( f \in C(K) \), \( \lambda \in \mathcal{H}_i \) and \( s \geq 0 \),

\[
(f \circ \phi_s, R_{i \theta} \lambda) = \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} (f \circ \phi_s \circ \phi_t, \lambda) dt \\
= \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} (f \circ \phi_{s+t}, \lambda) dt \\
= e^{i\theta s} \lim_{a \to +\infty} \frac{1}{a} \int_s^{s+a} e^{-i\theta t} (f \circ \phi_t, \lambda) dt \\
= e^{i\theta s} \left\{ \lim_{a \to +\infty} \frac{a+s}{a} \int_0^{a+s} e^{-i\theta t} (f \circ \phi_t, \lambda) dt \right\} \\
= e^{i\theta s} \frac{1}{a} \int_0^a e^{-i\theta t} (f \circ \phi_t, \lambda) dt \\
= e^{i\theta s} (f, R_{i \theta} \lambda).
\]

In particular,

\[
(f \circ \phi_s, R_{i \theta} \delta_x) = e^{i\theta s} (f, R_{i \theta} \delta_x)
\]

for all \( f \in C(K) \) and \( s \geq 0 \).

As a consequence of the ergodic theorem for asymptotically almost periodic functions, \( R_{i \theta} \delta_x \neq 0 \) if, and only if, \( \theta \) is a frequency of the asymptotically almost periodic function \( t \mapsto f(\phi_t(x)) \) for some \( f \in C(K) \setminus \{0\} \), i.e., if, and only if, \( [5] \), \( i \theta \in p\sigma(X) \cup p\sigma(X^+) \), where \( p\sigma \) denotes the point spectrum. For \( \theta = 0 \), \( R_0 \delta_x \) is a Borel probability measure which is \( \phi_s \)-invariant for all \( s \geq 0 \) and whose support is \( O^+(x) \).

If \( x \) is a periodic point of the continuous flow \( \phi \), the functions \( t \mapsto f(\phi_t(x)) \), are periodic for all \( f \in C(K) \), and the support of \( R_0 \delta_x \) is the forward orbit \( O^+(x) \). Let \( \phi_s \) be uniquely ergodic for some \( s > 0 \), i.e., [6], suppose that there is only one \( \phi_s \)-invariant Borel probability measure \( \mu \) on \( K \).

If \( x_0 \) and \( x_1 \) are two periodic points of \( \phi \), then \( \mu = R_0 \delta_{x_0} = R_0 \delta_{x_1} \). That proves

**Theorem 2.** If \( \phi \) is a continuous flow on the compact metric space \( K \), and if \( \phi_s \) is uniquely ergodic for some \( s > 0 \), then \( \phi \) has a periodic orbit at most.

3. Let \( \phi \) be topologically transitive, i.e., \( O^+(x_0) = K \) for some \( x_0 \in K \).

If \( \kappa \in \mathbb{C} \) is an eigenvalue of \( X \), and \( g_1, g_2 \) are two eigenfunctions of \( X \) corresponding to \( \kappa \), then \( g_1(x_0)g_2(x_0) \neq 0 \), and therefore

\[
g_2(\phi_t(x_0)) = e^{\kappa t} g_2(x_0) = \frac{g_2(x_0)}{g_1(x_0)} g_1(\phi_t(x_0))
\]

(2) This fact can be viewed as a generalization of Theorem 6.16 of [6] to continuous semiflows.
for all \( t \in \mathbb{R}_+ \), showing that \( \dim_{\mathbb{C}} \ker(X - \kappa I) = 1 \). Furthermore, the eigenfunctions corresponding to \( \kappa = 0 \), i.e. the \( \phi \)-invariant continuous functions, are constant on \( K \).

If \( f \in \ker(X - \kappa I) \setminus \{0\} \) and \( x \) is a periodic point of \( \phi \), with period \( \tau \), then

\[
f(x) = f(\phi_\tau(x)) = e^{\kappa \tau} f(x).
\]

Therefore, either \( \kappa \tau = 2n\pi i \) for some \( n \in \mathbb{Z} \), or \( f(x) = 0 \). By the same argument, if \( \omega \) is another eigenvalue of \( X \) and \( h \in \ker(X - \omega I) \setminus \{0\} \), then either \( \omega \tau = 2m\pi i \) for some \( m \in \mathbb{Z} \), or \( h(x) = 0 \). Hence, if \( f(x) h(x) \neq 0 \), \( \kappa \) and \( \omega \) are linearly dependent over \( \mathbb{Z} \).

Suppose again that \( \phi \) is topologically transitive. Then the sets \( \{ y \in K : f(y) \neq 0 \} \) and \( \{ y \in K : h(y) \neq 0 \} \) are dense open sets of \( K \), and the following theorem holds.

**Theorem 3.** If the continuous semiflow \( \phi \) is topologically transitive and the set of its periodic points is dense, either the point spectrum of \( X \) is empty, or all the eigenvalues of \( X \) are rational multiples of some point of \( \mathbb{R} \).

Let \( \zeta \in p\sigma(X) \) and \( g \in \ker(X - \zeta I) \setminus \{0\} \), so that

\[
g \circ \phi_t = e^{\zeta t} g \quad \forall t \in \mathbb{R}_+.
\]

If there is \( x \in K \) such that the functions \( t \mapsto f(\phi_t(x)) \) are asymptotically almost periodic for all \( f \in C(K) \), then

\[
e^{\zeta t} g(R_{t\theta} \delta_x) = \langle g \circ \phi_t, R_{t\theta} \delta_x \rangle = e^{i\theta t} \langle g, R_{t\theta} \delta_x \rangle,
\]

i.e.

\[
(e^{i(\theta - \zeta)t} - 1) \langle g, R_{t\theta} \delta_x \rangle = 0
\]

for all \( t \geq 0 \) and all \( \theta \in \mathbb{R} \). Hence, either \( \theta = -i\zeta \) or

(5) \[
\langle g, R_{t\theta} \delta_x \rangle = 0.
\]

If \( \theta = -i\zeta \), then

\[
\langle g, R_{t\theta} \delta_x \rangle = \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} e^{\zeta t} dt \langle g, \delta_x \rangle
= \langle g, \delta_x \rangle = g(x).
\]

If \( \omega \in \mathbb{C} \) is an eigenvalue of \( T(s) \) for some \( s > 0 \), there is some \( n \in \mathbb{Z} \) such that

\[
\zeta_n := \log \omega + \frac{2n\pi i}{s} \in p\sigma(X),
\]

and the eigenspace \( \ker(T(s) - \omega I) \) is the closure of the linear subspace of \( C(K) \) spanned by all \( \ker(X - \zeta_n I) \) for which \( \zeta_n \in p\sigma(X) \), [2].
If there is a frequency \( \theta \) of the asymptotically almost periodic function 
\( t \mapsto f(\phi_t(x)) \), for some \( f \in C(K) \), such that \( e^{it\theta} \) is not an eigenvalue of 
\( T(s) \), then (5) holds for all eigenfunctions \( g \) of \( X \). Suppose now that, for some 
\( s > 0 \), \( \phi_s \) has a topological discrete spectrum, i.e., all eigenfunctions of \( T(s) \) 
span a dense linear suspace of \( C(K) \). Thus, (5) - holding on a dense subspace 
of \( C(K) \) - implies that \( R_t \delta_x = 0 \): which is absurd. Hence

\[
e^{it\theta} \in p\sigma(T(s)),
\]

and therefore

\[
i\theta + \frac{2n\pi i}{s} \in p\sigma(X)
\]

for some \( n \in \mathbb{Z} \). Since \( p\sigma(X) \in p\sigma(X') \), where \( X' \) is the dual operator of \( X \), 
the following proposition holds.

**PROPOSITION 2.** If \( x \in K \) is such that the functions \( t \mapsto f(\phi_t(x)) \) are asymp-
totically almost periodic for all \( f \in C(K) \) and if \( \phi_s \) has a topological discrete 
spectrum for some \( s > 0 \), then \( p\sigma(X') \cap i\mathbb{R} \neq \emptyset. \)

4. Let \( d \) be a distance defining the metric topology of \( K \). A point \( x \in K \) 
will be said to be an asymptotically almost periodic point of \( \phi \) if, for all \( \delta > 0 \), 
there exist \( \alpha \geq 0 \) and \( l > 0 \) such that every interval \([s, s + l] \), with \( s \geq 0 \), 
contains some \( \tau \) such that

\[
d(\phi_{t+\tau}(x), \phi_t(x)) < \delta
\]

for all \( t \geq \alpha \). Since

\[
|d(\phi_{t+\tau}(x), x) - d(\phi_t(x), x)| \leq d(\phi_{t+\tau}(x), \phi_t(x)),
\]

if \( x \in K \) is an asymptotically almost periodic point of \( \phi \), the function \( \mathbb{R}_+ \ni t \mapsto d(\phi_t(x), x) \) is asymptotically almost periodic.

If (6) is only required to hold when \( t = 0 \), the point \( x \) is said to be almost 
periodic.

Since \( K \) is compact, for any \( f \in C(K) \) and any \( \epsilon > 0 \), there exists \( \delta > 0 \) 
such that, if \( d(x_1, x_2) < \delta \), then \( |f(x_1) - f(x_2)| < \epsilon \). If \( x \) is asymptotically 
almost periodic for \( \phi \), choosing \( \alpha \) and \( l \) as above, then

\[
|f(\phi_{t+\tau}(x)) - f(\phi_t(x))| < \epsilon \quad \forall \ t \geq \alpha.
\]

That proves the following lemma.

**LEMMA 1.** If \( x \in K \) is an asymptotically almost periodic point of the continuous 
semiflow \( \phi \), for every \( f \in C(K) \) the function \( \mathbb{R}_+ \ni t \mapsto f(\phi_t(x)) \) is asymptotically 
almost periodic.
The point will be said to be \textit{asymptotically stable} for the semiflow $\phi$ if, for every $\epsilon > 0$ and every $\alpha > 0$, there is some $t \geq \alpha$ such that

\begin{equation}
    d(\phi_t(x), x) \leq \epsilon.
\end{equation}

All almost periodic points are asymptotically stable.

Let $\phi : \mathbb{R} \times K \to K$ be a continuous flow, and let $T : \mathbb{R} \to \mathcal{L}(C(K))$ be the strongly continuous group defined by (1) for all $t \in \mathbb{R}$ and all $f \in C(K)$.

\textbf{Theorem 4.} Let $x \in K$. If the functions $\mathbb{R} \ni t \mapsto f(\phi_t(x))$ are almost periodic for all $f \in C(K)$, the point $x \in K$ is asymptotically stable for the restriction of $\phi$ to $\mathbb{R}_+$.

\textbf{Proof.} If $x \in K$ is not asymptotically stable, there are some $\epsilon > 0$ and some $\alpha > 0$ such that

\begin{equation}
    t > \alpha \implies d(\phi_t(x), x) > \epsilon
\end{equation}

Let $B(x, \epsilon)$ be the open ball, with center $x$ and radius $\epsilon$ for the distance $d$. Let $f \in C(K)$ be such that

\begin{equation}
    \text{Supp } f \subset B(x, \epsilon) \quad \text{and} \quad f(\epsilon) \neq 0.
\end{equation}

Then

\begin{equation}
    \lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} f(\phi_t(x)) \, dt = 0
\end{equation}

for all $\theta \in \mathbb{R}$. Hence, all the frequencies of the almost periodic function $t \mapsto f(\phi_t(x))$ vanish. Thus the function is constant, contradicting (9).

\textbf{Corollary 1.} If the group $T$ is weakly almost periodic, every point of $K$ is asymptotically stable.

Suppose there is some $c > 0$ such that

\begin{equation}
    d(\phi_t(u), \phi_t(v)) \leq c \, d(u, v) \quad \forall u, v \in K, \forall t \geq 0;
\end{equation}

$\phi$ will then be called a $c$-contractive semiflow (a contractive semiflow when $c = 1$).

If (10) is satisfied and if $x \in K$ is an almost periodic point of $\phi$, (6) holds for all $t \geq 0$. As a consequence, the function $t \mapsto d(\phi_t(x), x)$ is asymptotically almost periodic.

\textbf{Proposition 3.} All asymptotically stable points of the continuous semiflow $\phi$ are non-wandering.

If $\phi$ is $c$-contractive for some $c > 0$, all non-wandering points are asymptotically stable.
PROOF. If $x$ is asymptotically stable, for all $\epsilon > 0$ and all $\alpha > 0$ there is some $t \geq \alpha$ satisfying (7). Since $\phi_t(x) \in B(x, 2\epsilon)$, then

$$ x \in B(x, 2\epsilon) \cap \phi_t^{-1}(B(x, 2\epsilon)),$$

showing that $x$ is a non-wandering point.

Conversely, let $x$ be a non-wandering point, and suppose there are $\epsilon_0 > 0$ and $\alpha_o > 0$ such that

$$d(\phi_t(x), x) \geq \epsilon_0 \ \forall \ \tau \geq \alpha_o. \ (11)$$

Choose $\tau_o > \alpha_o$, and let $\sigma \in (0, \frac{\epsilon_0}{2\epsilon_0})$. There exists $\delta > 0$ - which can be assumed $< \frac{\epsilon_0}{2}$ - such that, if $d(x, y) < \delta$, then $d(\phi_{\tau_o}(x), \phi_{\tau_o}(y)) < \sigma$, i.e.,

$$\phi_{\tau_o}(B(x, \delta)) \subset B(\phi_{\tau_o}(x), \sigma).$$

Since $x$ is non-wandering, there is some $\tau \geq \tau_o$ such that

$$\phi^{-1}_t(B(x, \delta)) \cap B(x, \delta) \neq \emptyset,$$

and therefore, being

$$\phi^{-1}_t(B(x, \delta) \cap \phi_t(B(x, \delta))) = \phi^{-1}_t(B(x, \delta)) \cap \phi^{-1}_t \circ \phi_t(B(x, \delta)) \supset \phi^{-1}_t(B(x, \delta)) \cap B(x, \delta),$$

also

$$B(x, \delta) \cap \phi_t(B(x, \delta)) \neq \emptyset.$$  

Since, by (10),

$$d(\phi_t(x), \phi_t(y)) = d(\phi_{t-\tau_o} \circ \phi_{\tau_o}(x), \phi_{t-\tau_o} \circ \phi_{\tau_o}(y)) \leq c \cdot d(\phi_{\tau_o}(x), \phi_{\tau_o}(y)) < c \sigma < \frac{\epsilon_o}{2}$$

whenever $d(x, y) < \delta$, then

$$\phi_t(B(x, \delta)) \subset B \left( \phi_t(x), \frac{\epsilon_o}{2} \right).$$

Choose any

$$z \in B(x, \delta) \cap \phi_t(B(x, \delta)).$$

Thus, $z \in B(\phi_t(x), \frac{\epsilon_o}{2})$, i.e., $d(\phi_t(x), z) < \frac{\epsilon_o}{2}$. Since $d(x, z) < \delta < \frac{\epsilon_o}{2}$, then

$$d(\phi_t(x), x) \leq d(\phi_t(x), z) + d(x, z) < \frac{\epsilon_o}{2} + \frac{\epsilon_o}{2} = \epsilon_o,$$

contradicting (11). \qed
5. If the forward orbit of $x \in K$ is not dense, there are $u \in K$ and $r > 0$ such that

$$B(u, r) \cap O^+(x) = \emptyset.$$  

If (10) holds, and if $y \in K$ is such that $d(x, y) < \frac{r}{2c}$, then

$$d(\phi_t(x), \phi_t(y)) \leq c d(x, y) < \frac{r}{2},$$  
and therefore

$$d(u, \phi_t(y)) \geq |d(u, \phi_t(x)) - d(\phi_t(x), \phi_t(y))|$$

$$> r - \frac{r}{2} = \frac{r}{2} \quad \forall \quad t \in \mathbb{R}_+. $$

Thus,

$$y \in B\left(x, \frac{r}{2c}\right) \Rightarrow B\left(u, \frac{r}{2}\right) \cap O^+(y) = \emptyset.$$  

That proves

**Lemma 2.** If (10) holds, the set of points of $K$ whose forward orbits are dense, is closed.

Let $\phi_s(K) = K$ for some $s > 0$. Then, the set of points of $K$ whose forward orbits are dense, is either empty or a dense $G_\delta$, [6]. Hence, the following proposition holds.

**Proposition 4.** If (10) holds, and if $\phi_s$ is surjective and topologically transitive for some $s > 0$, then every point of $K$ has a dense orbit.

As a consequence, $\phi$ has no fixed point and a periodic orbit at most. If $x$ is a periodic point with period $\tau > 0$, then

$$K = O^+(x) = \{\phi_t(x) : 0 \leq t \leq \tau\}.$$  

Thus, $K$ is homeomorphic to the circle $\mathbb{R} \setminus \tau \mathbb{Z}$ and the map $t \mapsto \phi_t(x)$ is topologically conjugate to the the restriction to $\mathbb{R}_+$ of the covering map $\mathbb{R} \rightarrow \mathbb{R} \setminus \tau \mathbb{Z}$.

If $y \neq x$, then $y = \phi_r(x)$ for some $r \in (0, \tau)$, and therefore

$$\phi_t(y) = \phi_t(\phi_r(x)) = \phi_{t+r}(x)$$

$$= \phi_r(\phi_t(x)) = \phi_r(x) = y.$$  

Hence, the period $\sigma$ of $y$ is $\sigma \leq \tau$, and $x = \phi_t(y)$ for some $t \in (0, \sigma)$. Being

$$\phi_\sigma(x) = \phi_\sigma(\phi_t(y)) = \phi_{\sigma+t}(y)$$

$$= \phi_t(\phi_\sigma(x)) = \phi_t(y) = x,$$  

then $\tau \leq \sigma$, and, in conclusion, $\sigma = \tau$, proving thereby the following theorem.
THEOREM 5. If the $c$-contractive continuous semiflow $\phi : \mathbb{R}_+ \times K \to K$ has a periodic orbit and is such that $\phi_s$ is surjective and topologically transitive for some $s > 0$, then $K$ is homeomorphic to a circle, and $\phi$ is topologically conjugate to the restriction to $\mathbb{R}_+$ of the group of rotations of $\mathbb{R}^2$.

THEOREM 6. If (10) holds and if the set of all periodic points of the $c$-contractive semiflow $\phi$ is dense in $K$, then $\phi$ is asymptotically almost periodic at all points of $K$.

PROOF. Let $x \in K$ and let $\{x_v\}$ be a sequence of periodic points $x_v \in K$ converging to $x$. If $t > 0$,

$$d(\phi_t(x), x) \leq d(\phi_t(x), \phi_t(x_v)) + d(\phi_t(x_v), x_v) + d(x_v, x).$$

For any $\epsilon > 0$ there is an index $v_0$ such that, whenever $v > v_0$, $d(x_v, x) < \epsilon$. Let $\tau > 0$ be the period of $x_{v_0}$. Then, for any integer $p \geq 1$,

$$d(\phi_{p\tau}(x), x) \leq d(\phi_{p\tau}(x), \phi_{p\tau}(x_{v_0})) + d(\phi_{p\tau}(x_{v_0}), x_{v_0}) + d(x_{v_0}, x)$$

$$= d(\phi_{p\tau}(x), \phi_{p\tau}(x_{v_0})) + d(x_{v_0}, x)$$

$$< (c + 1)\epsilon.$$  

Since every interval $[s, s + 2\pi]$ contains some $p\tau$, the point $x$ is almost periodic and therefore asymptotically almost periodic. $\square$

6. C. J. K. Batty has kindly pointed out to me that Theorem 6 of [5] is not correct. In fact, the inclusion length $l > 0$ appearing in the inequality (16) of [5] depends on $x$ and $\lambda$, and - as $x$ and $\lambda$ vary - may increase to $\infty$. To make (16) a uniform estimate - i.e., an estimate holding for all $x$ and $\lambda$ chosen as in i) and ii) of [5] - assume that $T$ fulfills, besides i) and ii), the following condition:

iii) there exists $\epsilon_o \in (0, \sqrt{2})$ such that, for every choice of $x$ and $\lambda$ satisfying i) and such that $\langle x, \lambda \rangle = 1$, the set of lengths $l > 0$ for which (12) holds is bounded.

A correct version of Theorem 6 of [5] can be phrased as follows.

THEOREM 7. If the function $\langle T(x), \lambda \rangle$ is asymptotically almost periodic for all $x \in D(X)$ and all $\lambda \in D(X^1)$ and if i) and iii) hold, then the set $(p\sigma(X) \cup p\sigma(X^1) \cap i\mathbb{R})$ is discrete.

EXAMPLE. Let $T$ be the unitary group in the Hilbert space $l^2$ generated by the self-adjoint linear operator $X$ defined on the standard basis $\{e_n : n \in \mathbb{Z}\}$ of $l^2$ by

$$X e_n = \text{sign}(n) i \left( \sum_{0}^{\lfloor |n| \rfloor} \frac{1}{p} \right) e_n$$

if $n \neq 0$, and by $X e_0 = 0$. The group $T$ is almost periodic and satisfies iii), but is not uniformly almost periodic.

Condition iii) shall be added to the hypotheses of Theorems 9 of [5]. Theorem 10 can be correctly stated, with the same notations as in [5], as follows.
Theorem 8. Let the semigroup defined in B of [5] be strongly asymptotically almost periodic. If \( p\sigma(X) = \emptyset \), the function \( T(\bullet)x \) vanishes at \(+\infty\) for all \( x \in C(K) \). If \( p\sigma(X) \neq \emptyset \), and if iii) holds, there is \( \omega > 0 \) such that

\[
p\sigma(X) \cap i\mathbb{R} = \{in\omega : n \in \mathbb{Z}\}
\]

and, for every \( x \in \mathcal{E} \), \( T(\bullet)x \) is the sum of a continuous function vanishing at \(+\infty\) and of a periodic function with period \( \omega \).

REFERENCES


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