ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 25, nº 3-4 (1997), p. 829-839

<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_829_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXV (1997), pp. 829-839

Periodicity and Almost Periodicity in Markov Lattice Semigroups

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Let K be a compact metric space. A continuous semiflow $\phi : \mathbb{R}_+ \times K \to K$ on K defines a strongly continuous Markov lattice semigroup $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{C}(K))$ acting on the Banach space $C(K)$ of all complex-valued continuous functions on K (endowed with the uniform norm) and expressed by

$$T(t)\,f = f \circ \phi_t \tag{1}$$

for all $f \in C(K)$ and all $t \in \mathbb{R}_+$.

If $x \in K$ is a periodic point of ϕ, the functions $t \mapsto f(\phi_t(x))$ are continuous periodic functions on $\mathbb{R}_+$ for all $f \in C(K)$: a fact which imposes constraints on the spectral structure of the infinitesimal generator X of T. Milder restrictions on the spectrum of X are implied by the existence of almost periodic orbits, of asymptotically stable points and of non-wandering points for ϕ. Some of these constraints, together with their consequences on the behaviour of T and of ϕ, are discussed in this article.

In its final section, the paper corrects an error in [5], that was kindly pointed out to the author by C. J. K. Batty.

1. The following results have been established in [5]. Let $\mathcal{E}$ be a complex Banach space and let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ be a uniformly bounded, strongly continuous semigroup of continuous linear operators acting on $\mathcal{E}$. Let $X : \mathcal{D}(X) \subset \mathcal{E} \to \mathcal{E}$ be the infinitesimal generator of T. Let $M \geq 1$ be such that $\|T(t)\| \leq M$ for all $t \geq 0$.

Consider now the dual semigroup $T^+ : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E}^+)$ of T, and let $X^+ : \mathcal{D}(X^+) \subset \mathcal{E}^+ \to \mathcal{E}^+$ be its infinitesimal generator (see, e.g., [3] for the definition). For $g \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, the topological dual of $\mathcal{E}$, $\langle g, \lambda \rangle$ will denote the value of λ on g. For $\theta \in \mathbb{R}$, the set $\mathcal{H}'_{i\theta}$ of all λ in the topological dual $\mathcal{E}'$ of $\mathcal{E}$ for which the limit

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a \mathrm{e}^{-i\theta t} \langle T(t) f, \lambda \rangle dt \tag{2}$$

exists for all $f \in \mathcal{E}$, is a linear subspace of $\mathcal{E}'$ which contains $\ker(X^+ - i\theta I) \oplus \overline{\mathcal{R}(X^+ - i\theta I)}$ (where $\overline{\mathcal{R}(X^+ - i\theta I)}$ is the closure of the range $\mathcal{R}(X^+ - i\theta I)$ of

$X^+ - i\theta I)$. Since $\mathcal{E}'$ is sequentially weak-star complete, the equation

$$\langle f, R_{i\theta}\lambda\rangle = \lim_{a\to+\infty} \frac{1}{a}\int_0^a e^{-i\theta t}\langle T(t)f, \lambda\rangle dt \ \ \forall f \in \mathcal{E} \tag{3}$$

defines a continuous linear operator $R_{i\theta} : \mathcal{H}'_{i\theta} \to \mathcal{E}'$ which is a projector with norm $\leq M$, whose range is $\ker(X^+ - i\theta I)$ and whose restriction to $\ker(X^+ - i\theta I) \oplus \overline{\mathcal{R}(X^+ - i\theta I)}$ coincides with the spectral projector defined on $\ker(X^+ - i\theta I) \oplus \overline{\mathcal{R}(X^+ - i\theta I)}$ by the ergodic theorem applied to $X^+ - i\theta I$.

The hypothesis on the existence of the limit (2) for all $f \in \mathcal{E}$ and all $\theta \in \mathbb{R}$, is satisfied if $\lambda \in \mathcal{E}'$ is such that the functions $t \mapsto \langle T(t)f, \lambda\rangle$ are asymptotically almost periodic for all $f \in \mathcal{E}$.

2. Let K be a compact metric space. To avoid trivialities, suppose that K contains more than one point. Let $\phi : \mathbb{R}_+ \times K \to K$ be a continuous semiflow on K, (see, e.g., [1]), and let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{C}(K))$ be the strongly continuous Markov lattice semigroup, acting on the Banach space $\mathcal{E} = C(K)$ of all complex-valued continuous functions on K, endowed with the uniform norm, defined by (1) for all $f \in C(K)$ and all $t \in \mathbb{R}_+$. The infinitesimal generator X of T is a derivation.

For any $t \in \mathbb{R}_+$, $T(t)$ is a linear contraction; it is an isometry if, and only if, ϕ_t is surjective, and is a surjective isometry if, and only if, ϕ_t is a homeomorphism of K onto K.

If

$$\|T(t_o)f\| < \|f\| \ \text{ for some } t_o > 0 \text{ and some } f \in \mathcal{E}, \tag{4}$$

and if $t \geq 0$, then

$$\|T(t_o + t)f\| = \|T(t)(T(t_o)f)\| \leq \|T(t_o)f\| < \|f\|.$$

Thus, if (4) holds, then $\|T(s)f\| < \|f\|$ for all $s \geq t_o$. Equivalently, if there exists $\epsilon > 0$ such that $\|T(\epsilon)g\| = \|g\|$ for all $g \in \mathcal{E}$, then $\|T(s)g\| = \|g\|$ for all $g \in \mathcal{E}$ and all $s \in [0, \epsilon]$. Thus, if (4) holds, then $t_o > \epsilon$. Suppose now that, furthermore, $\|T(t_o - s)g\| = \|g\|$ for all $g \in \mathcal{E}$ and some $s \in (0, \epsilon)$. Then

$$\|T(t_o)g\| = \|T(s)T(t_o - s)g\| = \|T(t_o - s)g\| = \|g\|,$$

contradicting (4). The set

$$S = \{t \in \mathbb{R}_+^* : T(t) \text{ is not an isometry}\}$$

is either empty or an open half line.

Proposition 1. *If $T(t)$ is a contraction of $\mathcal{E}$ for any $t \geq 0$, the set of all t for which $T(t)$ is an isometry is either $\mathbb{R}_+$ or the empty set.*

In other words, either $S = \emptyset$ or $S = \mathbb{R}_+^*$ [(1)].

PROOF. If $S \neq \mathbb{R}_+^*$, there is $\epsilon > 0$ such that $(0, \epsilon] \cap S = \emptyset$, and therefore $T(t)$ is an isometry for all $t \in (0, \epsilon]$. Let ($S \neq \emptyset$ and let) $t_1 = \inf S$. Hence $t_1 > 0$. Choose σ in such a way that

$$0 < 2\sigma < \epsilon, \quad \sigma < t_1.$$

Then $t_1 + \sigma \in S$ and $0 < t_1 - \sigma \notin S$. Since

$$t_1 - \sigma = t_1 + \sigma - 2\sigma > t_1 + \sigma - \epsilon,$$

$T(t_1 - \sigma)$ is not an isometry, contradicting the definition of t_1. □

As a consequence of this result and of Theorems 1 and 2 of [4], the following theorem holds.

THEOREM 1. *If ϕ_t is surjective for some $t > 0$, the derivation X is a conservative and m-dissipative operator whose spectrum is non-empty.*

Let $x \in K$ be such that the functions

$$t \mapsto \langle T(t)f, \delta_x \rangle = f(\phi_t(x))$$

are asymptotically almost periodic on $\mathbb{R}_+$ for all $f \in C(K)$. Then, for any $\theta \in \mathbb{R}$, the limit

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a \mathrm{e}^{-i\theta t} \langle T(t)f, \delta_x \rangle dt = \lim_{a \to +\infty} \frac{1}{a} \int_0^a \mathrm{e}^{-i\theta t} f(\phi_t(x)) dt$$

exists for all $f \in C(K)$, showing that $\delta_x \in \mathcal{H}'_{i\theta}$. As before, let $R_{i\theta}\,\delta_x \in C(K)'$ be defined by

$$\langle f, R_{i\theta}\,\delta_x \rangle = \lim_{a \to +\infty} \frac{1}{a} \int_0^a \mathrm{e}^{-i\theta t} f(\phi_t(x)) dt \quad \forall f \in C(K);$$

$R_{i\theta}\,\delta_x$ is (represented by) a Borel measure on K, i.e.,

$$\begin{aligned} \int f\, dR_{i\theta}\,\delta_x &= \langle f, R_{i\theta}\,\delta_x \rangle \\ &= \lim_{a \to +\infty} \frac{1}{a} \int_0^a \mathrm{e}^{-i\theta t} f(\phi_t(x)) dt \quad \forall f \in C(K). \end{aligned}$$

(1) The proof holds for any normed vector space $\mathcal{E}$ and for every map $T : \mathbb{R}_+^* \to \mathcal{L}(\mathcal{E})$ such that $T(t)$ is a contraction and $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t, t_1, t_2 \in \mathbb{R}_+^*$.

For all $f \in C(K)$, $\lambda \in \mathcal{H}'_{i\theta}$ and $s \geq 0$,

$$\begin{aligned}\langle f \circ \phi_s, R_{i\theta}\lambda\rangle &= \lim_{a\to+\infty}\frac{1}{a}\int_0^a \mathrm{e}^{-i\theta t}\langle f\circ\phi_s\circ\phi_t, \lambda\rangle dt \\ &= \lim_{a\to+\infty}\frac{1}{a}\int_0^a \mathrm{e}^{-i\theta t}\langle f\circ\phi_{s+t}, \lambda\rangle dt \\ &= \mathrm{e}^{i\theta s}\lim_{a\to+\infty}\frac{1}{a}\int_s^{s+a} \mathrm{e}^{-i\theta t}\langle f\circ\phi_t, \lambda\rangle dt \\ &= \mathrm{e}^{i\theta s}\left\{\lim_{a\to+\infty}\frac{a+s}{a}\frac{1}{a+s}\int_0^{a+s} \mathrm{e}^{-i\theta t}\langle f\circ\phi_t, \lambda\rangle \right. \\ &\quad \left. - \lim_{a\to+\infty}\frac{1}{a}\int_0^s \mathrm{e}^{-i\theta t}\langle f\circ\phi_t, \lambda\rangle dt\right\} \\ &= \mathrm{e}^{i\theta s}\lim_{a\to+\infty}\frac{1}{a}\int_0^a \mathrm{e}^{-i\theta t}\langle f\circ\phi_t, \lambda\rangle dt \\ &= \mathrm{e}^{i\theta s}\langle f, R_{i\theta}\lambda\rangle.\end{aligned}$$

In particular,

$$\langle f\circ\phi_s, R_{i\theta}\,\delta_x\rangle = \mathrm{e}^{i\theta s}\langle f, R_{i\theta}\,\delta_x\rangle$$

for all $f \in C(K)$ and $s \geq 0$.

As a consequence of the ergodic theorem for asymptotically almost periodic functions, $R_{i\theta}\,\delta_x \neq 0$ if, and only if, θ is a frequency of the asymptotically almost periodic function $t \mapsto f(\phi_t(x))$ for some $f \in C(K)\backslash\{0\}$, i.e., if, and only if, [5], $i\theta \in p\sigma(X)\cup p\sigma(X^+)$, where $p\sigma$ denotes the point spectrum. For $\theta = 0$, $R_0\,\delta_x$ is a Borel probability measure which is ϕ_s-invariant for all $s \geq 0$ and whose support is $\overline{O^+(x)}$[(2)].

If x is a periodic point of the continuous flow ϕ, the functions $t \mapsto f(\phi_t(x))$, are periodic for all $f \in C(K)$, and the support of $R_0\,\delta_x$ is the forward orbit $O^+(x)$. Let ϕ_s be uniquely ergodic for some $s > 0$, i.e., [6], suppose that there is only one ϕ_s-invariant Borel probability measure μ on K. If x_o and x_1 are two periodic points of ϕ, then $\mu = R_0\,\delta_{x_o} = R_0\,\delta_{x_1}$. That proves

THEOREM 2. *If ϕ is a continuous flow on the compact metric space K, and if ϕ_s is uniquely ergodic for some $s > 0$, then ϕ has a periodic orbit at most.*

3. Let ϕ be topologically transitive, i.e., $\overline{O^+(x_o)} = K$ for some $x_o \in K$. If $\kappa \in \mathbf{C}$ is an eigenvalue of X, and g_1, g_2 are two eigenfunctions of X corresponding to κ, then $g_1(x_o)g_2(x_o) \neq 0$, and therefore

$$\frac{g_2(\phi_t(x_o))}{g_1(\phi_t(x_o))} = \frac{\mathrm{e}^{\kappa t}g_2(x_o)}{\mathrm{e}^{\kappa t}g_1(x_o)} = \frac{g_2(x_o)}{g_1(x_o)}$$

(2) This fact can be viewed as a generalization of Theorem 6.16 of [6] to continuous semiflows.

for all $t \in \mathbb{R}_+$, showing that $\dim_{\mathbf{C}} \ker(X - \kappa I) = 1$. Furthermore, the eigenfunctions corresponding to $\kappa = 0$, i.e. the ϕ-invariant continuous functions, are constant on K.

If $f \in \ker(X - \kappa I)\setminus\{0\}$ and x is a periodic point of ϕ, with period τ, then

$$f(x) = f(\phi_\tau(x)) = e^{\kappa\tau} f(x).$$

Therefore, either $\kappa\tau = 2n\pi i$ for some $n \in \mathbb{Z}$, or $f(x) = 0$. By the same argument, if ω is another eigenvalue of X and $h \in \ker(X - \omega I)\setminus\{0\}$, then either $\omega\tau = 2m\pi i$ for some $m \in \mathbb{Z}$, or $h(x) = 0$. Hence, if $f(x)\,h(x) \neq 0$, κ and ω are linearly dependent over $\mathbb{Z}$.

Suppose again that ϕ is topologically transitive. Then the sets $\{y \in K : f(y) \neq 0\}$ and $\{y \in K : h(y) \neq 0\}$ are dense open sets of K, and the following theorem holds.

THEOREM 3. *If the continuous semiflow ϕ is topologically transitive and the set of its periodic points is dense, either the point spectrum of X is empty, or all the eigenvalues of X are rational multiples of some point of $i\mathbb{R}$.*

Let $\zeta \in p\sigma(X)$ and $g \in \ker(X - \zeta I)\setminus\{0\}$, so that

$$g \circ \phi_t = e^{\zeta t} g \quad \forall t \in \mathbb{R}_+.$$

If there is $x \in K$ such that the functions $t \mapsto f(\phi_t(x))$ are asymptotically almost periodic for all $f \in C(K)$, then

$$e^{\zeta t}\langle g, R_{i\theta}\,\delta_x\rangle = \langle g \circ \phi_t, R_{i\theta}\,\delta_x\rangle = e^{i\theta t}\langle g, R_{i\theta}\,\delta_x\rangle,$$

i.e.

$$(e^{(i\theta - \zeta)t} - 1)\langle g, R_{i\theta}\,\delta_x\rangle = 0$$

for all $t \geq 0$ and all $\theta \in \mathbb{R}$. Hence, either $\theta = -i\zeta$ or

$$\langle g, R_{i\theta}\,\delta_x\rangle = 0. \tag{5}$$

If $\theta = -i\zeta$, then

$$\begin{aligned}\langle g, R_{i\theta}\,\delta_x\rangle &= \lim_{a\to+\infty} \frac{1}{a}\int_0^a e^{-i\theta t} e^{\zeta t}\,dt\ \langle g, \delta_x\rangle\\ &= \langle g, \delta_x\rangle = g(x).\end{aligned}$$

If $\omega \in \mathbf{C}$ is an eigenvalue of $T(s)$ for some $s > 0$, there is some $n \in \mathbb{Z}$ such that

$$\zeta_n := \log\omega + \frac{2n\pi i}{s} \in p\sigma(X),$$

and the eigenspace $\ker(T(s) - \omega I)$ is the closure of the linear subspace of $C(K)$ spanned by all $\ker(X - \zeta_n I)$ for which $\zeta_n \in p\sigma(X)$, [2].

If there is a frequency θ of the asymptotically almost periodic function $t \mapsto f(\phi_t(x))$, for some $f \in C(K)$, such that $e^{i\theta s}$ is *not* an eigenvalue of $T(s)$, then (5) holds for all eigenfunctions g of X. Suppose now that, for some $s > 0$, ϕ_s has a topological discrete spectrum, i.e., all eigenfunctions of $T(s)$ span a dense linear suspace of $C(K)$. Thus, (5) - holding on a dense subspace of $C(K)$ - implies that $R_{i\theta}\,\delta_x = 0$: which is absurd. Hence

$$e^{i\theta s} \in p\sigma(T(s)),$$

and therefore

$$i\theta + \frac{2n\pi i}{s} \in p\sigma(X)$$

for some $n \in \mathbb{Z}$. Since $p\sigma(X) \in p\sigma(X')$, where X' is the dual operator of X, the following proposition holds.

PROPOSITION 2. *If $x \in K$ is such that the functions $t \mapsto f(\phi_t(x))$ are asymptotically almost periodic for all $f \in C(K)$ and if ϕ_s has a topological discrete spectrum for some $s > 0$, then $p\sigma(X') \cap i\mathbb{R} \neq \emptyset$.*

4. Let d be a distance defining the metric topology of K. A point $x \in K$ will be said to be an *asymptotically almost periodic point* of ϕ if, for all $\delta > 0$, there exist $\alpha \geq 0$ and $l > 0$ such that every interval $[s, s+l]$, with $s \geq 0$, contains some τ such that

$$d(\phi_{t+\tau}(x), \phi_t(x)) < \delta \tag{6}$$

for all $t \geq \alpha$. Since

$$|d(\phi_{t+\tau}(x), x) - d(\phi_t(x), x)| \leq d(\phi_{t+\tau}(x), \phi_t(x)),$$

if $x \in K$ is an asymptotically almost periodic point of ϕ, the function $\mathbb{R}_+ \ni t \mapsto d(\phi_t(x), x)$ is asymptotically almost periodic.

If (6) is only required to hold when $t = 0$, the point x is said to be *almost periodic*.

Since K is compact, for any $f \in C(K)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that, if $d(x_1, x_2) < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. If x is asymptotically almost periodic for ϕ, choosing α and l as above, then

$$|f(\phi_{t+\tau}(x)) - f(\phi_t(x))| < \epsilon \quad \forall t \geq \alpha.$$

That proves the following lemma.

LEMMA 1. *If $x \in K$ is an asymptotically almost periodic point of the continuous semiflow ϕ, for every $f \in C(K)$ the function $\mathbb{R}_+ \ni t \mapsto f(\phi_t(x))$ is asymptotically almost periodic.*

The point will be said to be *asymptotically stable* for the semiflow ϕ if, for every $\epsilon > 0$ and every $\alpha > 0$, there is some $t \geq \alpha$ such that

$$d(\phi_t(x), x) \leq \epsilon. \tag{7}$$

All almost periodic points are asymptotically stable.

Let $\phi : \mathbb{R} \times K \to K$ be a continuous flow, and let $T : \mathbb{R} \to \mathcal{L}(C(K))$ be the strongly continuous group defined by (1) for all $t \in \mathbb{R}$ and all $f \in C(K)$.

THEOREM 4. *Let $x \in K$. If the functions $\mathbb{R} \ni t \mapsto f(\phi_t(x))$ are almost periodic for all $f \in C(K)$, the point $x \in K$ is asymptotically stable for the restriction of ϕ to $\mathbb{R}_+$.*

PROOF. If $x \in K$ is not asymptotically stable, there are some $\epsilon > 0$ and some $\alpha > 0$ such that

$$t > \alpha \implies d(\phi_t(x), x) > \epsilon \tag{8}$$

Let $B(x, \epsilon)$ be the open ball, with center x and radius ϵ for the distance d. Let $f \in C(K)$ be such that

$$\operatorname{Supp} f \subset B(x, \epsilon) \text{ and } f(x) \neq 0. \tag{9}$$

Then

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a e^{-i\theta t} f(\phi_t(x))\, dt = 0$$

for all $\theta \in \mathbb{R}$. Hence, all the frequencies of the almost periodic function $t \mapsto f(\phi_t(x))$ vanish. Thus the function is constant, contradicting (9).

COROLLARY 1. *If the group T is weakly almost periodic, every point of K is asymptotically stable.*

Suppose there is some $c > 0$ such that

$$d(\phi_t(u), \phi_t(v)) \leq c\, d(u, v) \quad \forall u, v \in K, \ \forall t \geq 0; \tag{10}$$

ϕ will then be called a c-contractive semiflow (a contractive semiflow when $c = 1$).

If (10) is satisfied and if $x \in K$ is an almost ~~periodic~~ point of ϕ, (6) holds for all $t \geq 0$. As a consequence, the function $t \mapsto d(\phi_t(x), x)$ is asymptotically almost periodic.

PROPOSITION 3. *All asymptotically stable points of the continuous semiflow ϕ are non-wandering.*

If ϕ is c-contractive for some $c > 0$, all non-wandering points are asymptotically stable.

PROOF. If x is asymptotically stable, for all $\epsilon > 0$ and all $\alpha > 0$ there is some $t \geq \alpha$ satisfying (7). Since $\phi_t(x) \in B(x, 2\epsilon)$, then

$$x \in B(x, 2\epsilon) \cap \phi_t{}^{-1}(B(x, 2\epsilon)),$$

showing that x is a non-wandering point.

Conversely, let x be a non-wandering point, and suppose there are $\epsilon_o > 0$ and $\alpha_o > 0$ such that

$$d(\phi_\tau(x), x) \geq \epsilon_o \quad \forall \tau \geq \alpha_o. \tag{11}$$

Choose $\tau_o > \alpha_o$, and let $\sigma \in (0, \frac{\epsilon_o}{2c})$. There exists $\delta > 0$ - which can be assumed $< \frac{\epsilon_o}{2}$ - such that, if $d(x, y) < \delta$, then $d(\phi_{\tau_o}(x), \phi_{\tau_o}(y)) < \sigma$, i.e.,

$$\phi_{\tau_o}(B(x, \delta)) \subset B(\phi_{\tau_o}(x), \sigma).$$

Since x is non-wandering, there is some $\tau \geq \tau_o$ such that

$$\phi_\tau{}^{-1}(B(x, \delta)) \cap B(x, \delta) \neq \emptyset,$$

and therefore, being

$$\begin{aligned}\phi_\tau{}^{-1}(B(x, \delta) \cap \phi_\tau(B(x, \delta))) &= \phi_\tau{}^{-1}(B(x, \delta)) \cap \phi_\tau{}^{-1} \circ \phi_\tau(B(x, \delta)) \\ &\supset \phi_\tau{}^{-1}(B(x, \delta)) \cap B(x, \delta),\end{aligned}$$

also

$$B(x, \delta) \cap \phi_\tau(B(x, \delta)) \neq \emptyset.$$

Since, by (10),

$$\begin{aligned}d(\phi_\tau(x), \phi_\tau(y)) &= d(\phi_{\tau-\tau_o} \circ \phi_{\tau_o}(x), \phi_{\tau-\tau_o} \circ \phi_{\tau_o}(y)) \\ &\leq c\, d(\phi_{\tau_o}(x), \phi_{\tau_o}(y)) < c\sigma < \frac{\epsilon_o}{2}\end{aligned}$$

whenever $d(x, y) < \delta$, then

$$\phi_\tau(B(x, \delta)) \subset B\left(\phi_\tau(x), \frac{\epsilon_o}{2}\right).$$

Choose any

$$z \in B(x, \delta) \cap \phi_\tau(B(x, \delta)).$$

Thus, $z \in B(\phi_\tau(x), \frac{\epsilon_o}{2})$, i.e., $d(\phi_\tau(x), z) < \frac{\epsilon_o}{2}$. Since $d(x, z) < \delta < \frac{\epsilon_o}{2}$, then

$$d(\phi_\tau(x), x) \leq d(\phi_\tau(x), z) + d(x, z) < \frac{\epsilon_o}{2} + \frac{\epsilon_o}{2} = \epsilon_o,$$

contradicting (11). □

5. If the forward orbit of $x \in K$ is not dense, there are $u \in K$ and $r > 0$ such that

$$B(u, r) \cap O^+(x) = \emptyset.$$

If (10) holds, and if $y \in K$ is such that $d(x, y) < \frac{r}{2c}$, then

$$d(\phi_t(x), \phi_t(y)) \leq c\, d(x, y) < \frac{r}{2},$$

and therefore

$$\begin{aligned} d(u, \phi_t(y)) &\geq |d(u, \phi_t(x)) - d(\phi_t(x), \phi_t(y))| \\ &> r - \frac{r}{2} = \frac{r}{2} \quad \forall\ t \in \mathbb{R}_+. \end{aligned}$$

Thus,

$$y \in B\left(x, \frac{r}{2c}\right) \Rightarrow B\left(u, \frac{r}{2}\right) \cap O^+(y) = \emptyset.$$

That proves

Lemma 2. *If* (10) *holds, the set of points of K whose forward orbits are dense, is closed.*

Let $\phi_s(K) = K$ for some $s > 0$. Then, the set of points of K whose forward orbits are dense, is either empty or a dense G_δ, [6]. Hence, the following proposition holds.

Proposition 4. *If* (10) *holds, and if ϕ_s is surjective and topologically transitive for some $s > 0$, then every point of K has a dense orbit.*

As a consequence, ϕ has no fixed point and a periodic orbit at most. If x is a periodic point with period $\tau > 0$, then

$$K = O^+(x) = \{\phi_t(x) : 0 \leq t \leq \tau\}.$$

Thus, K is homeomorphic to the circle $\mathbb{R}\backslash\tau\mathbb{Z}$ and the map $t \mapsto \phi_t(x)$ is topologically conjugate to the the restriction to $\mathbb{R}_+$ of the covering map $\mathbb{R} \to \mathbb{R}\backslash\tau\mathbb{Z}$.

If $y \neq x$, then $y = \phi_r(x)$ for some $r \in (0, \tau)$, and therefore

$$\begin{aligned} \phi_\tau(y) &= \phi_\tau(\phi_r(x)) = \phi_{\tau+r}(x) \\ &= \phi_r(\phi_\tau(x)) = \phi_r(x) = y. \end{aligned}$$

Hence, the period σ of y is $\sigma \leq \tau$, and $x = \phi_t(y)$ for some $t \in (0, \sigma)$. Being

$$\begin{aligned} \phi_\sigma(x) &= \phi_\sigma(\phi_t(y)) = \phi_{\sigma+t}(y) \\ &= \phi_t(\phi_\sigma(x)) = \phi_t(y) = x, \end{aligned}$$

then $\tau \leq \sigma$, and, in conclusion, $\sigma = \tau$, proving thereby the following theorem.

THEOREM 5. *If the c-contractive continuous semiflow* $\phi : \mathbb{R}_+ \times K \to K$ *has a periodic orbit and is such that* ϕ_s *is surjective and topologically transitive for some* $s > 0$, *then* K *is homeomorphic to a circle, and* ϕ *is topologically conjugate to the restriction to* $\mathbb{R}_+$ *of the group of rotations of* $\mathbb{R}^2$.

THEOREM 6. *If* (10) *holds and if the set of all periodic points of the c-contractive semiflow* ϕ *is dense in* K, *then* ϕ *is asymptotically almost periodic at all points of* K.

PROOF. Let $x \in K$ and let $\{x_\nu\}$ be a sequence of periodic points $x_\nu \in K$ converging to x. If $t > 0$,

$$d(\phi_t(x), x) \leq d(\phi_t(x), \phi_t(x_\nu)) + d(\phi_t(x_\nu), x_\nu) + d(x_\nu, x).$$

For any $\epsilon > 0$ there is an index ν_o such that, whenever $\nu \geq \nu_o$, $d(x_\nu, x) < \epsilon$. Let $\tau > 0$ be the period of x_{ν_o}. Then, for any integer $p \geq 1$,

$$\begin{aligned} d(\phi_{p\tau}(x), x) &\leq d(\phi_{p\tau}(x), \phi_{p\tau}(x_{\nu_o})) + d(\phi_{p\tau}(x_{\nu_o}), x_{\nu_o}) + d(x_{\nu_o}, x) \\ &= d(\phi_{p\tau}(x), \phi_{p\tau}(x_{\nu_o})) + d(x_{\nu_o}, x) \\ &< (c+1)\epsilon. \end{aligned}$$

Since every interval $[s, s+2\tau]$ contains some $p\tau$, the point x is almost periodic and therefore asymptotically almost periodic. □

6. C. J. K. Batty has kindly pointed out to me that Theorem 6 of [5] is not correct. In fact, the inclusion lenght $l > 0$ appearing in the inequality (16) of [5] depends on x and λ, and - as x and λ vary - may increase to ∞. To make (16) a uniform estimate - i.e., an estimate holding for all x and λ chosen as in i) and ii) of [5] - assume that T fulfills, besides i) and ii), the following condition:

iii) there exists $\epsilon_o \in (0, \sqrt{2})$ such that, for every choice of x and λ satisfying i) and such that $\langle x, \lambda \rangle = 1$, the set of lenghts $l > 0$ for which (12) holds is bounded.

A correct version of Theorem 6 of [5] can be phrased as follows.

THEOREM 7. *If the function* $\langle T(\bullet)x, \lambda \rangle$ *is asymptotically almost periodic for all* $x \in \mathcal{D}(X)$ *and all* $\lambda \in \mathcal{D}(X^+)$ *and if* i) *and* iii) *hold, then the set* $(p\sigma(X) \cup p\sigma(X^+) \cap i\mathbb{R}$ *is discrete.*

EXAMPLE. Let T be the unitary group in the Hilbert space l^2 generated by the self-adjoint linear operator X defined on the standard basis $\{e_n : n \in \mathbb{Z}\}$ of l^2 by

$$X\, e_n = \operatorname{sign}(n)\, i \left(\sum_0^{|n|} \frac{1}{p} \right) e_n$$

if $n \neq 0$, and by $X\, e_0 = 0$. The group T is almost periodic and satisfies iii), but is not uniformly almost periodic.

Condition iii) shall be added to the hypotheses of Theorems 9 of [5]. Theorem 10 can be correctly stated, with the same notations as in [5], as follows.

THEOREM 8. *Let the semigroup defined in* B *of* [5] *be strongly asymptotically almost periodic. If* $p\sigma(X) = \emptyset$, *the function* $T(\bullet)x$ *vanishes at* $+\infty$ *for all* $x \in C(K)$. *If* $p\sigma(X) \neq \emptyset$, *and if* iii) *holds, there is* $\omega > 0$ *such that*

$$p\sigma(X) \cap i\mathbb{R} = \{in\omega : n \in \mathbb{Z}\}$$

and, for every $x \in \mathcal{E}$, $T(\bullet)x$ *is the sum of a continuous function vanishing at* $+\infty$ *and of a periodic function with period* ω.

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