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## On Homogenization of Solutions of Boundary Value Problems in Domains, Perforated Along Manifolds

M. LOBO – O. A. OLEINIK – M. E. PEREZ – T. A. SHAPOSHNIKOVA

This paper is dedicated to the memory of E. De Giorgi,  
 a great mathematician of the XX century

1. In this paper the problem of homogenization of solutions to the Poisson equation in domains, perforated along a manifold, is considered with the Neumann boundary condition, with the Dirichlet boundary condition or the mixed condition on cavities. Some particular problems of this kind were considered in [1], [2]. The same method can be applied also to boundary value problems in domains, perforated in some subdomains. The short note on these results is published in [3]. We study here also the corresponding spectral problems. E. De Giorgi was one of the first mathematicians, who considered homogenization problems [4].

Let  $\Omega$  be a bounded domain in  $R^n$  with a smooth boundary  $\partial\Omega$  and  $\gamma$  be a manifold in  $\overline{\Omega}$ . Let  $P_j$  ( $j = 1, \dots, N(\varepsilon)$ ) with  $N(\varepsilon) \leq d_0\varepsilon^{1-n}$ ,  $d_0 = \text{const}$  be a point such that  $P_j \in \gamma$ ;  $\varepsilon$  is a small parameter. We denote by  $G^j(a_\varepsilon^j)$  a domain which belongs to  $\Omega$ , has a smooth boundary  $\partial G^j(a_\varepsilon^j)$ ,  $P_j \in \overline{G^j(a_\varepsilon^j)}$ , the diameter of  $G^j(a_\varepsilon^j)$  is  $a_\varepsilon^j$  and  $a_\varepsilon^j \leq C_0\varepsilon$ ,  $C_0 = \text{const} > 0$ ,  $G^j(a_\varepsilon^j) \cap G^i(a_\varepsilon^i) = \emptyset$  for  $i \neq j$ . We consider all possible behavior of  $a_\varepsilon^j$  as  $\varepsilon \rightarrow 0$ .

We set

$$G_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} G^j(a_\varepsilon^j), \quad \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S'_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j),$$

$$S_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j) \cap \Omega, \quad \Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus S_\varepsilon.$$

We assume that  $G^j(a_\varepsilon^j)$  are such that any function  $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  can be extended on  $\Omega$  as a function  $\tilde{u} \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  in such a way that

$$(1) \quad \|\tilde{u}\|_{H_1(\Omega, \Gamma_\varepsilon)} \leq K_1 \|u\|_{H_1(\Omega_\varepsilon, \Gamma_\varepsilon)},$$

$$(2) \quad \|\nabla \tilde{u}\|_{L_2(\Omega)} \leq K_2 \|\nabla u\|_{L_2(\Omega_\varepsilon)},$$

where  $K_j$  here and in what follows are constants which do not depend on  $\varepsilon$ .

The space  $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  is defined as a closure of  $C^\infty(\overline{\Omega_\varepsilon})$ -functions, which are equal to zero in a neighbourhood of  $\Gamma_\varepsilon$ , in the norm

$$\|u\|_{H_1(\Omega_\varepsilon, \Gamma_\varepsilon)} \equiv \left( \int_{\Omega_\varepsilon} (u^2 + |\nabla u|^2) dx \right)^{1/2}.$$

The cases, when it is possible, are considered in [5], [6].

In this domain  $\Omega_\varepsilon$  we consider the boundary value problems for the equation

$$(3) \quad -\Delta u_\varepsilon = f$$

with the boundary conditions

$$(4) \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad S_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,$$

or

$$(5) \quad \frac{\partial u_\varepsilon}{\partial \nu} + \beta u_\varepsilon = 0 \quad \text{on} \quad S_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \beta(x) \geq \beta_0 = \text{const} > 0,$$

or

$$(6) \quad u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon,$$

where  $\nu$  is an outward unit normal vector to  $S_\varepsilon$ .

2. Let us consider the problem (3), (4) (the Neumann condition on  $S_\varepsilon$ ).

We define a weak solution of the problem (3), (4) as a function  $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  which satisfies the integral identity

$$(7) \quad A_\varepsilon(u_\varepsilon, \varphi) \equiv \int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla \varphi) dx = \int_{\Omega_\varepsilon} f \varphi dx$$

for any function  $\varphi \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ .

We also assume that for functions  $u$  from the space  $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  the Friedrichs inequality is valid:

$$(8) \quad \|u\|_{L_2(\Omega_\varepsilon)} \leq K_0 \|\nabla u\|_{L_2(\Omega_\varepsilon)}$$

with the constant  $K_0$ , independent on  $\varepsilon$ . This inequality is satisfied if, for example,  $\Gamma \subset \Gamma_\varepsilon$  and  $\Gamma$  is a smooth piece of  $\partial\Omega$  with a positive measure on  $\partial\Omega$ ,  $\Gamma \cap \overline{G_\varepsilon} = \emptyset$  (see [5]).

From (7) and the Friedrichs inequality it follows that  $\|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leq K_3$ .

Using the Riesz theorem, it is easy to prove that the problem (3), (4) has a unique weak solution in  $\Omega_\varepsilon$ .

Let the function  $v_0$  be a solution of the problem

$$(9) \quad -\Delta v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

and  $f$  is a smooth function in  $\bar{\Omega}$ .

First we consider the case when  $S'_\varepsilon \cap \partial\Omega = \emptyset$ . Using the integral identity (7) for problem (3),(4) and the integral identity for the problem (9), we get

$$(10) \quad \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx = \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v_0) dx + \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx,$$

where  $\tilde{u}_\varepsilon$  is an extension of  $u_\varepsilon$  in  $\Omega$ , such that  $\tilde{u}_\varepsilon - v_0$  satisfies (1),(2). From (10) we have

$$(11) \quad \begin{aligned} \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 &\leq \max_{\Omega} |f| |G_\varepsilon|^{1/2} \|\tilde{u}_\varepsilon - v_0\|_{L_2(G_\varepsilon)} \\ &+ \max_{\Omega} |\nabla v_0| |G_\varepsilon|^{1/2} \|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(G_\varepsilon)}. \end{aligned}$$

Using inequalities (1), (2) for  $\tilde{u}_\varepsilon - v_0$ , and the Friedrichs inequality, we obtain from (11) that

$$\begin{aligned} K_4 \|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega)} &\leq \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} \leq K_5 \left( \max_{\Omega} |f| + \max_{\Omega} |\nabla v_0| \right) |G_\varepsilon|^{1/2} \\ &\leq K_6 |G_\varepsilon|^{1/2} \leq K_7 (\max_j a_\varepsilon^j)^{n/2} \varepsilon^{(1-n)/2}, \end{aligned}$$

where  $|G|$  is the measure of the set  $G$ .

Therefore,

$$(12) \quad \|u_\varepsilon - v_0\|_{H^1(\Omega_\varepsilon)}^2 \leq K_8 (\max_j a_\varepsilon^j)^n \varepsilon^{1-n}.$$

Let us consider the case, when  $S'_\varepsilon \cap \partial\Omega \neq \emptyset$ , and let  $G^j(a_\varepsilon^j)$ ,  $j = 1, \dots, M(\varepsilon)$ , be such that  $\overline{G^j(a_\varepsilon^j)} \cap \partial\Omega \neq \emptyset$ ,  $M(\varepsilon) \leq d_1 \varepsilon^{-n+2}$ ,  $d_1 = \text{const} > 0$ , and  $|\partial G^j(a_\varepsilon^j)| \leq d_2 (a_\varepsilon^j)^{n-1}$ . We set

$$I_\varepsilon = \partial\Omega \cap \bigcup_{j=1}^{M(\varepsilon)} \overline{G^j(a_\varepsilon^j)}.$$

Since  $v_0$  is a smooth function, we derive from the Green formula that

$$(13) \quad \int_{\Omega} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx = \int_{\Omega} f(\tilde{u}_\varepsilon - v_0) dx + \int_{I_\varepsilon} \frac{\partial v_0}{\partial \nu} (\tilde{u}_\varepsilon - v_0) ds.$$

From integral identity (7) with  $\varphi = (u_\varepsilon - v_0)$  and (13), we obtain

$$(14) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx &= \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx - \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v_0) dx \\ &- \int_{I_\varepsilon} \frac{\partial v_0}{\partial \nu} (\tilde{u}_\varepsilon - v_0) ds \equiv J^\varepsilon. \end{aligned}$$

Let us estimate  $J^\varepsilon$ . Using (1), (2), the Friedrichs inequality and the imbedding theorem, we get

$$\begin{aligned}
 \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 &\leq K_9\{|G_\varepsilon|^{1/2}(\|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} \\
 &\quad + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\Omega_\varepsilon)}) + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\partial\Omega)}|I_\varepsilon|^{1/2}\} \\
 (15) \qquad \qquad \qquad &\leq K_{10}\{|G_\varepsilon|^{1/2} + |I_\varepsilon|^{1/2}\}\|\tilde{u}_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \\
 &\leq K_{11}(|G_\varepsilon|^{1/2} + |I_\varepsilon|^{1/2})\|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}.
 \end{aligned}$$

Due to assumptions on  $M(\varepsilon)$  and  $\partial G^j(a_\varepsilon^j)$  we have

$$(16) \qquad \qquad \qquad |I_\varepsilon| \leq K_{12}(\max_j a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

From estimates (15), (16) it follows that

$$(17) \qquad \qquad \qquad \|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{13} \max_j (a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

Hence, we proved the following theorem.

**THEOREM 1.** *Let  $u_\varepsilon$  be a weak solution of the problem (3), (4),  $v_0$  be a solution of the problem (9),  $S'_\varepsilon \cap \partial\Omega = \emptyset$ . Then*

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{14}(\max_j a_\varepsilon^j)^n \varepsilon^{1-n}.$$

*If  $S'_\varepsilon \cap \partial\Omega \neq \emptyset$  and  $M(\varepsilon) \leq d_1 \varepsilon^{2-n}$ ,  $|\partial G^j(a_\varepsilon^j)| \leq d_2 (a_\varepsilon^j)^{n-1}$ , then*

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{15} \max_j (a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

We note that in the proof of Theorem 1 the assumption that  $P_j \in \gamma$  is not used.

**3.** Let us consider the problem (3), (5) (the mixed boundary condition). Let  $\beta(x) \geq \beta_0 = \text{const} > 0$  and  $v_0$  be a solution of problem (9). Then the function  $w_\varepsilon = u_\varepsilon - v_0$  is a weak solution of the problem

$$\begin{aligned}
 -\Delta w_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, \quad w_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \\
 (18) \qquad \frac{\partial w_\varepsilon}{\partial \nu} + \beta(x)w_\varepsilon &= -\left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0\right) \quad \text{on } S_\varepsilon.
 \end{aligned}$$

From the integral identity for the problem (18) we have

$$(19) \qquad \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx + \int_{S_\varepsilon} \beta(x)w_\varepsilon^2 ds = - \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0\right)w_\varepsilon ds.$$

For the right-hand side of (19) we get

$$\begin{aligned}
 & \left| \int_{S_\varepsilon} \left( \frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right) w_\varepsilon ds \right| \\
 (20) \quad & \leq \frac{1}{2} \beta_0 \int_{S_\varepsilon} w_\varepsilon^2 ds + K_{16} \int_{S_\varepsilon} \left( \frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right)^2 ds \\
 & \leq \frac{1}{2} \int_{S_\varepsilon} \beta(x)w_\varepsilon^2 ds + K_{16} \int_{S_\varepsilon} \left( \frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right)^2 ds.
 \end{aligned}$$

From inequalities (20) and (19) it follows that

$$(21) \quad \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 \leq K_{17} \left\| \frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right\|_{L_2(S_\varepsilon)}^2.$$

Taking into account that  $v_0(x)$  is a smooth function, we obtain

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{18}|S_\varepsilon| \leq K_{19}(\max_j a_\varepsilon^j)^{n-1} \varepsilon^{1-n}.$$

From this estimate we derive the following theorem.

**THEOREM 2.** *Let  $\beta(x) \geq \beta_0 = \text{const} > 0$ ,  $u_\varepsilon$  be a solution of the problem (3), (5),  $v_0$  be a solution of the problem (9). Assume that  $|\partial G^j(a_\varepsilon^j)| \leq K_{20}(a_\varepsilon^j)^{n-1}$ ,  $\eta_\varepsilon \equiv (\max_j a_\varepsilon^j)\varepsilon^{-1} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then  $\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and*

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{21}\eta_\varepsilon^{n-1}.$$

In the proof of Theorem 2 we do not use that  $P_j \in \gamma$ .

**4.** Assume that  $\gamma$  is a domain in the hyperplane  $\{x : x_1 = 0\}$  and  $\gamma = \Omega \cap \{x : x_1 = 0\}$ ,  $Q = \{x : -1/2 < x_j < 1/2, j = 1, \dots, n\}$ ,  $G'_\varepsilon = \bigcup_{z \in Z} (a_\varepsilon G_0 + \varepsilon z)$ , where  $G_0$  is a smooth domain and  $a_\varepsilon \leq C\varepsilon$ ,  $C = \text{const}$ ,  $\overline{G_0} \subset Q$ ,  $Z$  is the set of vectors with integer components. Let  $a_\varepsilon \varepsilon^{-1} \rightarrow C_0 = \text{const} > 0$  as  $\varepsilon \rightarrow 0$  and  $C_0 \overline{G_0} \subset Q$ .

We set

$$\begin{aligned}
 & \Pi_\varepsilon = \Omega \cap \{x : |x_1| < \varepsilon/2\}, \Pi_\varepsilon^* = \Pi_\varepsilon \setminus \overline{G'_\varepsilon}, \gamma_\varepsilon^\pm = \Omega \cap \{x : x_1 = \pm \varepsilon/2\}, \\
 & \Omega_\varepsilon^+ = \Omega \cap \{x : x_1 > \varepsilon/2\}, \Omega_\varepsilon^- = \Omega \cap \{x : x_1 < -\varepsilon/2\}, G_\varepsilon = G'_\varepsilon \cap \Omega, \\
 (22) \quad & \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \gamma_\varepsilon^+ \cup \Pi_\varepsilon^* \cup \gamma_\varepsilon^- \cup \Omega_\varepsilon^-, \\
 & S_\varepsilon = \partial G_\varepsilon \cap \Omega, \Gamma_\varepsilon = \partial \Omega_\varepsilon \setminus S_\varepsilon, l_\varepsilon = \overline{G_\varepsilon} \cap \partial \Omega.
 \end{aligned}$$

We assume that  $\beta(x) = \beta_0 = \text{const} > 0$ . Let  $v$  be a weak solution of the problem

$$(23) \quad \begin{aligned} -\Delta v &= f \quad \text{in } \Omega \setminus \gamma, \quad v = 0 \quad \text{on } \partial\Omega, \\ [v]|_\gamma &= 0, \quad \left[ \frac{\partial v}{\partial x_1} \right] |_\gamma = \mu v|_\gamma, \end{aligned}$$

where  $\mu = \beta_0 C_0^{n-1} |\partial G_0|$ ,  $[\varphi]|_\gamma = \varphi(x_1 + 0, x') - \varphi(x_1 - 0, x')$ ,  $(x_1, x') \in \gamma$ ,  $x' = (x_2, \dots, x_n)$ .

We assume that  $|v(x)| < K_{22}$  for  $x \in \overline{\Omega}$ ,  $|\nabla v(x)| \leq K_{23}$  in  $\Omega^+ = \Omega \cap \{x : x_1 > 0\}$  and  $|\nabla v(x)| \leq K_{24}$  in  $\Omega^- = \Omega \cap \{x : x_1 < 0\}$ . From the integral identity for the problem (3), (5) we obtain

$$(24) \quad \begin{aligned} \int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla(u_\varepsilon - v)) dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds + \beta_0 \int_{S_\varepsilon} v(u_\varepsilon - v) ds \\ = \int_{\Omega_\varepsilon} f(u_\varepsilon - v) dx. \end{aligned}$$

Using the assumptions on  $v(x)$  and the Green formula we get

$$(25) \quad \begin{aligned} \int_{\Omega} (\nabla v, \nabla(\tilde{u}_\varepsilon - v)) dx + \beta_0 C_0^{n-1} |\partial G_0| \int_{\gamma} v(\tilde{u}_\varepsilon - v) dx' \\ = \int_{\Omega} f(\tilde{u}_\varepsilon - v) dx + \int_{I_\varepsilon} \frac{\partial v}{\partial \nu} (\tilde{u}_\varepsilon - v) ds, \end{aligned}$$

where  $I_\varepsilon = \partial\Omega \cap \overline{G}_\varepsilon$ ,  $\tilde{u}_\varepsilon - v$  is an extension of  $u_\varepsilon - v$  in  $\Omega$  such that (1), (2) are satisfied.

From (24), (25) we derive

$$(26) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v)|^2 dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds \\ = \beta_0 \{ C_0^{n-1} |\partial G_0| \int_{\gamma} v(\tilde{u}_\varepsilon - v) dx' - \int_{S_\varepsilon} v(u_\varepsilon - v) ds \} + P_\varepsilon, \end{aligned}$$

where

$$(27) \quad P_\varepsilon \equiv - \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v) dx - \int_{I_\varepsilon} \frac{\partial v}{\partial \nu} (\tilde{u}_\varepsilon - v) ds.$$

LEMMA 1. Assume that  $v \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$  and  $(a_\varepsilon \varepsilon^{-1} - C_0) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $\Omega_\varepsilon$  is defined by (22). Then

$$(28) \quad \left| \int_{S_\varepsilon} v ds - C_0^{n-1} |\partial G_0| \int_{\gamma} v dx' \right| \leq K_{25} \{ \sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0| \} \|v\|_{H_1(\Omega)}.$$

PROOF. Consider the function  $\theta_\varepsilon(y)$ ,  $y = \varepsilon^{-1}x$ , as a solution of the problem

$$(29) \quad \begin{cases} \Delta_y \theta_\varepsilon = 0 & y \in Q \setminus \overline{a_\varepsilon \varepsilon^{-1} G_0} = Y_\varepsilon, \\ \frac{\partial \theta_\varepsilon}{\partial \nu} = 1 & \text{on } a_\varepsilon \varepsilon^{-1} \partial G_0, \\ \frac{\partial \theta_\varepsilon}{\partial y_1} = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| & \text{on } \Sigma_0 = \partial Q \cap \{y : y_1 = -1/2\}, \\ \frac{\partial \theta_\varepsilon}{\partial y_1} = 0 & \text{on } \Sigma_1 = \partial Q \cap \{y : y_1 = 1/2\}. \\ \theta_\varepsilon(y) & \text{is 1-periodic in } y' = (y_2, \dots, y_n). \end{cases}$$

Since  $|a_\varepsilon \varepsilon^{-1} \partial G_0| = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0|$ , the problem (29) has a unique weak solution to within a constant. We define the constant in such a way that

$$\int_{Y_\varepsilon} \theta_\varepsilon(y) dy = 0.$$

From the integral identity for problem (29) it follows that

$$(30) \quad \|\nabla \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leq K_{26}.$$

Indeed,

$$(31) \quad \int_{Y_\varepsilon} |\nabla_y \theta_\varepsilon|^2 dy = -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \int_{\Sigma_0} \theta_\varepsilon dy' + \int_{a_\varepsilon \varepsilon^{-1} \partial G_0} \theta_\varepsilon ds_y.$$

From the imbedding theorem and the Poincaré inequality it follows that

$$(32) \quad \|\theta_\varepsilon\|_{L_2(\Sigma_0)} \leq K_{27} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}, \|\theta_\varepsilon\|_{L_2(a_\varepsilon \varepsilon^{-1} \partial G_0)} \leq K_{28} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}.$$

The inequalities (31) and (32) imply (30). We set

$$P_j^\varepsilon = \frac{\partial \theta_\varepsilon}{\partial y_j}, \quad j = 1, \dots, n.$$

For the vector-function  $P^\varepsilon(y) = (P_1^\varepsilon(y), \dots, P_n^\varepsilon(y))$  we have

$$\begin{aligned} \operatorname{div}_y P^\varepsilon &= 0, \quad \text{if } y \in Y_\varepsilon, \quad (P^\varepsilon, \nu) = 1 \quad \text{on } a_\varepsilon \varepsilon^{-1} \partial G_0, \\ (P^\varepsilon, \nu) &= -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \quad \text{on } \Sigma_0 \quad \text{and} \quad (P^\varepsilon, \nu) = 0 \quad \text{on } \Sigma_1, \end{aligned}$$

where  $\nu = (\nu_1, \dots, \nu_n)$  is a unit outward normal vector to  $\partial Y_\varepsilon$ . We denote by  $T_\varepsilon$  the set of cells of the form  $(\varepsilon Q + \varepsilon z) \setminus (a_\varepsilon G_0 + \varepsilon z)$  which have a nonempty



intersection with  $\Pi_\varepsilon^*$  and  $\partial\Omega$ . Let  $\Pi_\varepsilon^1 = \Pi_\varepsilon^* \cup T_\varepsilon$ . We extend the function  $v$  on  $T_\varepsilon$  by setting  $v = 0$  on  $T_\varepsilon \setminus \Omega$ . It is easy to see that

$$(33) \quad \begin{aligned} \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx &= \int_{S_\varepsilon} (P^\varepsilon, v)vds + \int_{\gamma_\varepsilon^-} (P^\varepsilon, v)vdx' \\ &= \int_{S_\varepsilon} vds - (a_\varepsilon\varepsilon^{-1})^{(n-1)}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx'. \end{aligned}$$

From (33) it follows that

$$(34) \quad \begin{aligned} \left| \int_{S_\varepsilon} vds - C_0^{n-1}|\partial G_0| \int_\gamma vdx' \right| &\leq \left| \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx \right| \\ &+ \left| (a_\varepsilon\varepsilon^{-1})^{n-1}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx' - C_0^{n-1}|\partial G_0| \int_\gamma vdx' \right|, \end{aligned}$$

Let us estimate the right-hand side of (34). We have

$$(35) \quad I_1^\varepsilon \equiv \left| \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx \right| \leq \int_{\Pi_\varepsilon^1} |\nabla v||P^\varepsilon(y)|dx.$$

Therefore,

$$(36) \quad I_1^\varepsilon \leq \left( \int_{\Pi_\varepsilon^1} |\nabla_y\theta|^2dx \right)^{1/2} \|v\|_{H_1(\Omega)}.$$

It is easy to see that

$$(37) \quad \|\nabla_y\theta\|_{L_2(\varepsilon Y_\varepsilon)}^2 \leq K_{29}\varepsilon^n.$$

Since  $\Pi_\varepsilon^1$  can contain sets of the form  $\varepsilon Y_\varepsilon + \varepsilon z$  no more than  $a_1\varepsilon^{1-n}$ ,  $a_1 = \text{const} > 0$ , from (36) and (37) we derive that

$$(38) \quad I_1^\varepsilon \leq K_{30}\sqrt{\varepsilon}\|v\|_{H_1(\Omega)}.$$

In order to estimate the second term in the right-hand side of (34) we use the continuity of functions from  $H_1(\Omega)$  on hyperplanes in  $L_2$  - norm. We have

$$(39) \quad \begin{aligned} I_2^\varepsilon &\equiv |(a_\varepsilon\varepsilon^{-1})^{n-1}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx' - C_0^{n-1}|\partial G_0| \int_\gamma vdx'| \\ &\leq K_{31} \left\{ (a_\varepsilon\varepsilon^{-1})^{n-1} \left| \int_{\gamma_\varepsilon^-} vdx' - \int_\gamma vdx' \right| + |(a_\varepsilon\varepsilon^{-1})^{n-1} - C_0^{n-1}| \int_\gamma |v|dx' \right\} \\ &\leq K_{32} \{ \sqrt{\varepsilon}\|v\|_{H_1(\Omega)} + |a_\varepsilon\varepsilon^{-1} - C_0|\|v\|_{H_1(\Omega)} \} \\ &\leq K_{33} \{ \sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0| \} \|v\|_{H_1(\Omega)}. \end{aligned}$$

From (38) and (39) it follows that (28) is valid. Lemma 1 is proved.

In order to prove Theorem 3 we note that from (26), (27) it follows that

$$(40) \quad \begin{aligned} \|\nabla(u_\varepsilon - v)\|_{L_2(\Omega_\varepsilon)}^2 &\leq K_{34}(\sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0| \\ &+ |G_\varepsilon|^{1/2} + |l_\varepsilon|^{1/2})\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)}. \end{aligned}$$

We assume that  $|l_\varepsilon| \leq d_3\varepsilon$ ,  $|G_\varepsilon| \leq d_4\varepsilon$ . Then we have from (40) and the Friedrichs inequality that

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \leq K_{35} \{ \sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0| \}.$$

Hence, we have the following theorem

**THEOREM 3.** *Let  $u_\varepsilon$  be a solution of the problem (3), (5), the domain  $\Omega_\varepsilon$  be defined by (22),  $v$  be a solution of problem (23),  $a_\varepsilon \varepsilon^{-1} \rightarrow C_0 = \text{const} > 0$  as  $\varepsilon \rightarrow 0$ ,  $|I_\varepsilon| = |\overline{G}_\varepsilon \cap \partial\Omega| \leq d_3\varepsilon$ . Then*

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \leq K_{36}(\sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0|).$$

**5.** Consider now the problem (3), (6) (the Dirichlet boundary condition on cavities). We study the behavior of solutions of the problem

$$-\Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon$$

as  $\varepsilon \rightarrow 0$ .

Let us define the function  $\varphi_\varepsilon^j$  ( $j = 1, \dots, N(\varepsilon)$ ) for  $n \geq 3$ , setting  $\varphi_\varepsilon^j \equiv 0$  for  $|x - P_j| \leq a_\varepsilon^j$ ,  $\varphi_\varepsilon^j \equiv \frac{1}{\ln c_0} \ln\left(\frac{|x - P_j|}{a_\varepsilon^j}\right)$  for  $a_\varepsilon^j \leq |x - P_j| \leq c_0 a_\varepsilon^j$ ,  $\varphi_\varepsilon^j \equiv 1$  for  $|x - P_j| > c_0 a_\varepsilon^j$ . For  $n = 2$  we set  $\varphi_\varepsilon^j = \varphi\left(\frac{|\ln|x - P_j||}{|\ln c_0 a_\varepsilon^j|}\right)$ , where  $\varphi(\xi) = 1$  for  $\xi \leq 1/2$ ,  $\varphi = 0$  for  $\xi \geq 1$ ,  $0 \leq \varphi \leq 1$ ,  $\varphi \in C^\infty(\mathbb{R}^n)$ ,  $c_0$  is a constant,  $c_0 > 0$ ,  $c_0 a_\varepsilon^j \leq 1$ .

We pose

$$\psi_\varepsilon(x) = \sum_{j=1}^{N(\varepsilon)} \psi_\varepsilon^j(x),$$

where  $\psi_\varepsilon^j(x) = 1 - \varphi_\varepsilon^j(x)$  and  $w_\varepsilon = v\psi_\varepsilon - v_\varepsilon$ ,  $v_\varepsilon$  is a solution of the problem

$$\Delta v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \quad v_\varepsilon = v \quad \text{on } S_\varepsilon, \quad v_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon,$$

$v$  is a solution of the problem (9).

It is evident that  $w_\varepsilon$  is a weak solution of the problem

$$(41) \quad \Delta w_\varepsilon = \Delta(v\psi_\varepsilon) \quad \text{in } \Omega_\varepsilon, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Let us estimate  $w_\varepsilon$  and its derivatives. From the integral identity for the problem (41) it follows that

$$(42) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\leq K_{37} \int_{\Omega_\varepsilon} |\nabla(v\psi_\varepsilon)|^2 dx \leq K_{38} \int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla\psi_\varepsilon|^2) dx \\ &\leq K_{39} \left\{ \max_j (a_\varepsilon^j)^n \varepsilon^{-n+1} + \sum_{j=1}^{N(\varepsilon)} \int_{a_\varepsilon^j}^{c_0 a_\varepsilon^j} r^{-3+n} dr \right\} \\ &\leq K_{40} \max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n}, \end{aligned}$$

for  $n \geq 3$ . For  $n = 2$  we have

$$\begin{aligned}
 \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\leq K_{41} \int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla \psi_\varepsilon|^2) dx \\
 (43) \quad &\leq K_{42} \left( \max_j a_\varepsilon^j \varepsilon^{-1} + \sum_{j=1}^{N(\varepsilon)} |\ln c_0 a_\varepsilon^j|^{-2} \int_{c_0 a_\varepsilon^j}^{(c_0 a_\varepsilon^j)^{1/2}} r^{-1} dr \right) \\
 &\leq K_{43} \max_j |\ln c_0 a_\varepsilon^j|^{-1} \varepsilon^{-1}.
 \end{aligned}$$

From (42), (43), the relation  $v_\varepsilon = v - u_\varepsilon$ , estimates for  $|\nabla(v\psi_\varepsilon)|$  and the Friedrichs inequality it follows that

$$(44) \quad \|u_\varepsilon - v\|_{H^1(\Omega_\varepsilon)}^2 \leq K_{44} (\max_j a_\varepsilon^j)^{n-2} \varepsilon^{1-n} \quad \text{for } n \geq 3,$$

$$(45) \quad \|u_\varepsilon - v\|_{H^1(\Omega_\varepsilon)}^2 \leq K_{45} (\max_j |\ln a_\varepsilon^j|)^{-1} \varepsilon^{-1} \quad \text{for } n = 2.$$

The inequalities (44), (45) imply the following theorem.

**THEOREM 4.** *Assume that*

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n} &= 0, \quad \text{if } n \geq 3, \\
 \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \max_j |\ln a_\varepsilon^j|^{-1} &= 0, \quad \text{if } n = 2,
 \end{aligned}$$

*$v$  is a solution of the problem (9),  $u_\varepsilon$  is a solution of problem (3), (6). Then the estimates (44), (45) are valid.*

Here as in Theorems 1 and 2 we do not use that  $P_j \in \gamma$ .

**6.** Let us consider the problem (3), (6), when

$$(46) \quad \begin{cases} \lim_{\varepsilon \rightarrow 0} a_\varepsilon^{n-2} \varepsilon^{1-n} = C_1 = \text{const} > 0, & \text{if } n \geq 3, \\ \lim_{\varepsilon \rightarrow 0} (\varepsilon |\ln a_\varepsilon|)^{-1} = C_2 = \text{const} > 0, & \text{if } n = 2. \end{cases}$$

We assume that  $G_0 = \{x : |x| < a\}$ ,  $Q = \{x : -1/2 < x_j < 1/2, j = 1, \dots, n\}$ ,  $a_\varepsilon^j = a_\varepsilon$ ,  $\Omega_\varepsilon$  has the form given by (22). In this case the limit problem for (3), (6) is

$$(47) \quad \begin{aligned} &-\Delta v = f \quad \text{in } \Omega^- \cup \Omega^+, \\ [v] \Big|_\gamma &= 0, \quad \left[ \frac{\partial v}{\partial x_1} \right] \Big|_\gamma = \mu_1 v \Big|_\gamma, \quad v = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega^- = \Omega \cap \{x : x_1 < 0\}$ ,  $\Omega^+ = \Omega \cap \{x : x_1 > 0\}$ ,  $\mu_1 = (n-2)a^{n-2}\omega(n)C_1$ , if  $n \geq 3$ ,  $\mu_1 = 2\pi C_2$ , if  $n = 2$ ,  $\omega(n)$  is the area of the unit sphere in  $R^n$ .

We assume that for a solution of the problem (47) the following estimates are valid

$$|v(x)| \leq A_1 \quad \text{for } x \in \overline{\Omega}, \quad |\nabla v(x)| \leq A_2 \quad \text{for } x \in \overline{\Omega}^-$$

$$\text{and } |\nabla v(x)| \leq A_3 \quad \text{for } x \in \overline{\Omega}^+.$$

Consider  $w_\varepsilon^j$  as a solution of the problem

$$\Delta w_\varepsilon^j = 0 \quad \text{in } T_{b\varepsilon}^j \setminus T_{aa\varepsilon}^j, \quad w_\varepsilon^j = 0 \quad \text{on } \partial T_{aa\varepsilon}^j,$$

$$(48) \quad w_\varepsilon^j = 1 \quad \text{on } \partial T_{b\varepsilon}^j, \quad \varepsilon/2 > b\varepsilon > aa\varepsilon, \quad b = \text{const} > 0,$$

where  $T_s^j$  is the ball with the center  $P_j$  and with the radius  $s$ . It is easy to see that the solution of the problem (48) has the form:

$$w_\varepsilon^j = \frac{r^{2-n} - (a_\varepsilon a)^{2-n}}{(b\varepsilon)^{2-n} - (a_\varepsilon a)^{2-n}}, \quad \text{if } n \geq 3,$$

$$w_\varepsilon^j = \left( \ln \frac{r}{a_\varepsilon a} \right) \left( \ln \frac{b\varepsilon}{a_\varepsilon a} \right)^{-1}, \quad \text{if } n = 2,$$

where  $r = |x - P_j|$ . We introduce the function  $w_\varepsilon$  such that  $w_\varepsilon = w_\varepsilon^j$  for  $T_{b\varepsilon}^j \setminus T_{aa\varepsilon}^j$ ,  $w_\varepsilon = 0$  for  $x \in T_{aa\varepsilon}^j$ ,  $j = 1, \dots, N(\varepsilon)$ ,  $w_\varepsilon = 1$  for  $x \in \mathbb{R}^n \setminus \sum_{j=1}^{N(\varepsilon)} T_{b\varepsilon}^j$ .

Using the Green formula for functions  $w_\varepsilon$  and  $\varphi \in H_1(\Omega, G_\varepsilon \cup \partial\Omega)$  we obtain

$$\sum_{j=1}^n \int_{\Omega} \frac{\partial w_\varepsilon}{\partial x_j} \frac{\partial \varphi}{\partial x_j} dx = \sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{aa\varepsilon}^i} (\nabla w_\varepsilon^i, \nabla \varphi) dx$$

$$(49) \quad = - \sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{aa\varepsilon}^i} (\Delta w_\varepsilon^i) \varphi dx + \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu} \varphi ds$$

$$= \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu} \varphi ds.$$

Since

$$\frac{\partial w_\varepsilon^i}{\partial \nu} = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \quad \text{for } x \in \partial T_{b\varepsilon}^j, \quad n \geq 3,$$

$$\frac{\partial w_\varepsilon^j}{\partial \nu} = - \frac{1}{b\varepsilon \ln a_\varepsilon} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon}\right)} \quad \text{for } x \in \partial T_{b\varepsilon}^j, \quad n = 2,$$

we derive from (49) that

$$(50) \quad \int_{\Omega} (\nabla w_\varepsilon, \nabla \varphi) dx = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds,$$

if  $n \geq 3$ , and

$$(51) \quad \int_{\Omega} (\nabla w_{\varepsilon}, \nabla \varphi) dx = \frac{1}{b\varepsilon |\ln a_{\varepsilon}|} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_{\varepsilon}} + \frac{\ln a}{\ln a_{\varepsilon}}\right)} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds,$$

if  $n = 2$ .

Applying Lemma 1 proved in Section 4 we get

$$(52) \quad \left| \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds - b^{n-1} \omega(n) \int_{\gamma} \varphi dx' \right| \leq K_{46} \sqrt{\varepsilon} \|\varphi\|_{H_1(\Omega)}.$$

Let us note that for a solution of problem (47) we have the integral identity

$$(53) \quad \int_{\Omega} (\nabla v, \nabla \varphi) dx + (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \varphi dx' = \int_{\Omega} f \varphi dx,$$

if  $n \geq 3$ , and

$$(54) \quad \int_{\Omega} (\nabla v, \nabla \varphi) dx + 2\pi C_2 \int_{\gamma} v \varphi dx_2 = \int_{\Omega} f \varphi dx,$$

in  $n = 2$ ,  $\varphi \in H_1(\Omega, \partial\Omega)$ . From (50), (53) for  $n \geq 3$  it follows that

$$(55) \quad \begin{aligned} \int_{\Omega} (\nabla(\tilde{u}_{\varepsilon} - w_{\varepsilon}v), \nabla \tilde{\varphi}) dx &= \int_{\Omega} (\nabla \tilde{u}_{\varepsilon}, \nabla \tilde{\varphi}) dx - \int_{\Omega} (\nabla(w_{\varepsilon}v), \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega_{\varepsilon}} f \tilde{\varphi} dx - \int_{\Omega} (\nabla w_{\varepsilon}, v \nabla \tilde{\varphi}) dx - \int_{\Omega} w_{\varepsilon} (\nabla v, \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega} f \tilde{\varphi} dx - \int_{\Omega} (\nabla w_{\varepsilon}, \nabla(v \tilde{\varphi})) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx \\ &\quad - \int_{\Omega} (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega} f \tilde{\varphi} dx - \frac{(n-2)(b\varepsilon)^{1-n} (a_{\varepsilon}a)^{n-2}}{1 - (a^{-1}b)^{2-n} (a_{\varepsilon}\varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v \tilde{\varphi} ds \\ &\quad - \int_{\Omega} f \tilde{\varphi} dx + (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \tilde{\varphi} dx' \\ &\quad + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx \\ &= \left\{ (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \tilde{\varphi} dx' \right. \\ &\quad \left. - \frac{(n-2)(b\varepsilon)^{1-n} (a_{\varepsilon}a)^{n-2}}{1 - (a^{-1}b)^{2-n} (a_{\varepsilon}\varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v \tilde{\varphi} ds \right\} \\ &\quad + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx, \end{aligned}$$

where  $\tilde{\varphi} = \varphi$  for  $x \in \Omega_\varepsilon$  and  $\tilde{\varphi} = 0$  for  $x \in \Omega \setminus \Omega_\varepsilon$ ,  $\varphi \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$ .

In a similar way we get for  $n = 2$

$$(56) \quad \int_{\Omega} (\nabla(\tilde{u}_\varepsilon - w_\varepsilon v), \nabla\tilde{\varphi})dx = \left\{ 2\pi C_2 \int_{\gamma} v\tilde{\varphi}dx_2 - \frac{1}{b\varepsilon|\ln a_\varepsilon|} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon}\right)} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds \right\} + \int_{\Omega} (1 - w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx + \int_{\Omega} (\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx.$$

Let us estimate two last integrals in the right-hand side of (55) and (56) . We denote them by  $J_\varepsilon(\varphi)$ . Using Lemma 1, we get

$$(57) \quad \left| \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds - (n-2)a^{n-2}\omega(n)C_1 \int_{\gamma} v\tilde{\varphi}dx' \right| \leq K_{47} \left\{ \left| \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds - b^{n-1}\omega(n) \int_{\gamma} \tilde{\varphi}vdx' \right| + \int_{\gamma} |v\tilde{\varphi}|dx' |a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1| + \alpha(\varepsilon) \int_{\gamma} v\tilde{\varphi}dx' \right\} \leq K_{48} \{ \sqrt{\varepsilon} + |a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1| \} \|\tilde{\varphi}\|_{H_1(\Omega)},$$

for  $n \geq 3$  and  $\alpha(\varepsilon) \leq c_1\varepsilon$  ,and

$$(58) \quad \left| 2\pi C_2 \int_{\gamma} v\tilde{\varphi}dx_2 - \frac{1}{b\varepsilon|\ln a_\varepsilon|} \frac{1}{(1 - \alpha(\varepsilon))} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds \right| \leq K_{49} \{ \sqrt{\varepsilon} + |(\varepsilon|\ln a_\varepsilon|)^{-1} - C_2| \} \|\tilde{\varphi}\|_{H_1(\Omega)},$$

for  $n = 2$ ,  $\alpha(\varepsilon) = \frac{\ln b\varepsilon}{\ln a_\varepsilon} - \frac{\ln a}{\ln a_\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

For  $J_\varepsilon(\tilde{\varphi})$  we have estimates:

$$(59) \quad \left| J_\varepsilon(\tilde{\varphi}) \right| \leq K_{50} \{ \|w_\varepsilon - 1\|_{L_2(\Omega)} \|\tilde{\varphi}\|_{H_1(\Omega)} + \sqrt{\varepsilon} \|\nabla w_\varepsilon\|_{L_2(\Omega)} \|\tilde{\varphi}\|_{L_2(\gamma)} \} \leq K_{51} \sqrt{\varepsilon} \|\tilde{\varphi}\|_{H_1(\Omega)}.$$

Taking  $\tilde{\varphi} = \tilde{u}_\varepsilon - w_\varepsilon v$  in (55) and (56), we obtain

$$(60) \quad \|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \leq K_{52} \{ \sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2}\varepsilon^{1-n}| \},$$

if  $n \geq 3$ , and

$$(61) \quad \|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \leq K_{53} \{\sqrt{\varepsilon} + |C_2 - (\varepsilon |\ln a_\varepsilon|)^{-1}|\},$$

if  $n = 2$ .

From (60) and (61) we derive

$$(62) \quad \|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{54} \{\sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2} \varepsilon^{1-n}|\},$$

if  $n \geq 3$ , and

$$(63) \quad \|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{55} \{\sqrt{\varepsilon} + |C_2 - (\varepsilon |\ln a_\varepsilon|)^{-1}|\},$$

if  $n = 2$ .

Thus we have proved the following theorem.

**THEOREM 5.** *Let conditions (46) be satisfied. Assume that  $\Omega_\varepsilon$  has the form (22),  $u_\varepsilon$  is a solution of the problem (3), (6),  $v$  is a solution of the problem (47). Then estimates (62), (63) hold.*

7. Consider the problem (3), (6) under conditions:

$$(64) \quad a_\varepsilon^{2-n} \varepsilon^{n-1} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad n \geq 3,$$

$$(65) \quad \varepsilon |\ln a_\varepsilon| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad n = 2.$$

We assume that  $a_\varepsilon^j = a_\varepsilon$ , the domain  $\Omega_\varepsilon$  has the form (22). In this case the limit problem for the problem (3), (6) is:

$$(66) \quad -\Delta v^- = f \text{ in } \Omega^-, \quad v^- = 0 \text{ on } \partial\Omega^-,$$

$$(67) \quad -\Delta v^+ = f \text{ in } \Omega^+, \quad v^+ = 0 \text{ on } \partial\Omega^+.$$

We will use the following Lemma 2, proved in [6].

**LEMMA 2.** *Let  $u \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$ . Then*

$$(68) \quad \|u\|_{L_2(\Pi_\varepsilon)} \leq K_{56} a_\varepsilon^{(2-n)/2} \varepsilon^{n/2} \|\nabla u\|_{L_2(\Pi_\varepsilon)},$$

if  $n \geq 3$ , and

$$(69) \quad \|u\|_{L_2(\Pi_\varepsilon)} \leq K_{57} \varepsilon \sqrt{|\ln a_\varepsilon|} \|\nabla u\|_{L_2(\Pi_\varepsilon)},$$

if  $n = 2$ .

From the integral identity for the problem (3), (6) we derive

$$(70) \quad \|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leq K_{58}.$$

Therefore, from (68)-(70) we get

$$(71) \quad \frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \leq K_{59} a_\varepsilon^{2-n} \varepsilon^{n-1},$$

if  $n \geq 3$ , and

$$(72) \quad \frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \leq K_{60} \varepsilon |\ln a_\varepsilon|,$$

if  $n = 2$ . Here  $\tilde{u}_\varepsilon = u_\varepsilon$  for  $x \in \Omega_\varepsilon$  and  $\tilde{u}_\varepsilon = 0$  for  $x \in \Omega \setminus \Omega_\varepsilon$ .

It is easy to prove that

$$(73) \quad \int_{\gamma_\varepsilon^\pm} u^2(x_1, x') dx' \leq K_{61} \left( \frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} u^2(x) dx + \varepsilon \|\nabla u\|_{L_2(\Pi_\varepsilon)}^2 \right),$$

From this inequality and (71), (72), (73) it follows that

$$(74) \quad \|\tilde{u}_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)}^2 \leq K_{62} \{\varepsilon + a_\varepsilon^{2-n} \varepsilon^{n-1}\}$$

for  $n \geq 3$ , and

$$(75) \quad \|\tilde{u}_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)}^2 \leq K_{63} \{\varepsilon + \varepsilon |\ln a_\varepsilon|\}$$

for  $n = 2$ .

We set  $w_\varepsilon^- = u_\varepsilon - v^-$ ,  $w_\varepsilon^+ = u_\varepsilon - v^+$ ,  $\Omega_\varepsilon^+ = \Omega \cap \{x : x_1 > \varepsilon/2\}$ ,  $\Omega_\varepsilon^- = \Omega_\varepsilon \cap \{x : x_1 < -\varepsilon/2\}$ . The functions  $w_\varepsilon^\pm$  are weak solutions of the problems:

$$\Delta w_\varepsilon^\pm = 0 \quad \text{in } \Omega_\varepsilon^\pm, \quad w_\varepsilon^\pm = u_\varepsilon - v^\pm \quad \text{on } \gamma_\varepsilon^\pm, \quad w_\varepsilon^\pm = 0 \quad \text{on } \partial\Omega_\varepsilon^\pm \setminus \gamma_\varepsilon^\pm.$$

From the inequality (see [7], ch 4, sec. 1)

$$\|w_\varepsilon^\pm\|_{L_2(\Omega_\varepsilon^\pm)} \leq K_{64} \{\|u_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)} + \|v^\pm\|_{L_2(\gamma_\varepsilon^\pm)}\},$$

and (71), (72) we have estimates

$$(76) \quad \|\tilde{u}_\varepsilon - v^\pm\|_{L_2(\Omega^\pm)}^2 \leq K_{65} a_\varepsilon^{2-n} \varepsilon^{n-1},$$

for  $n \geq 3$ , and

$$(77) \quad \|\tilde{u}_\varepsilon - v^\pm\|_{L_2(\Omega^\pm)}^2 \leq K_{66} \varepsilon |\ln a_\varepsilon|,$$

for  $n = 2$ .

From the estimates (76), (77) we get the following theorem.

**THEOREM 6.** *Let  $\Omega_\varepsilon$  be a domain of the form (22),  $a_\varepsilon^{2-n} \varepsilon^{n-1} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $n \geq 3$ , and  $\varepsilon |\ln a_\varepsilon| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $n = 2$ ,  $a_\varepsilon^j = a_\varepsilon$ . Let  $u_\varepsilon$  be a solution of the Dirichlet problem (3), (6),  $v^\pm$  be solutions of problems (66), (67). Then estimates (76), (77) are valid.*

The Dirichlet problems in domains with holes were considered in [8].



8. Using Theorems 1-6, proved above, we can get theorems about the spectrum of corresponding eigenvalue problems. We apply here the theorem about the spectrum of a sequence of singularly perturbed operators proved in [5].

THEOREM 7. *Consider the eigenvalue problem*

$$(78) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(79) \quad \frac{\partial u_\varepsilon^k}{\partial \nu} = 0 \quad \text{on } S_\varepsilon, \quad u_\varepsilon^k = 0 \quad \text{on } \Gamma_\varepsilon,$$

and the eigenvalue problem

$$(80) \quad \Delta v^k + \lambda^k v^k = 0 \quad \text{in } \Omega, \quad v^k = 0 \quad \text{on } \partial\Omega.$$

Assume that  $S'_\varepsilon \cap \partial\Omega = \emptyset$ . Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_1 \left( \max_j a_\varepsilon^j \right)^n \varepsilon^{1-n}.$$

If  $S'_\varepsilon \cap \partial\Omega \neq \emptyset$  and  $M(\varepsilon) \leq d_1 \varepsilon^{2-n}$ , then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_2 \left( \max_j a_\varepsilon^j \right)^{n-1} \varepsilon^{2-n}.$$

where  $\lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots$  is a nondecreasing sequence of eigenvalues to problem (78), (79) and  $\lambda^1 \leq \lambda^2 \leq \dots$  is a nondecreasing sequence of eigenvalues to problem (80) and every eigenvalue is counted as many times as its multiplicity.

Here and in what follows constants  $C_j$  do not depend on  $\varepsilon$ . This Theorem is a consequence of Theorem 1.

We note that in Theorem 1, from which Theorem 7 follows, it is not necessary to assume that  $P_j$  belongs to  $\gamma$ . We use only the fact, that the number  $N(\varepsilon)$  of  $P_j$  is such that  $N(\varepsilon) \leq d_0 \varepsilon^{1-n}$  and  $a_\varepsilon^j \leq C_0 \varepsilon$ .

THEOREM 8. *Consider the eigenvalue problem*

$$(81) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u_\varepsilon^k}{\partial \nu} + \beta(x) u_\varepsilon^k = 0 \quad \text{on } S_\varepsilon, \quad u_\varepsilon^k = 0 \quad \text{on } \Gamma_\varepsilon,$$

and the eigenvalue problem

$$(82) \quad \Delta v^k + \lambda^k v^k = 0 \quad \text{in } \Omega, \quad v^k = 0 \quad \text{on } \partial\Omega.$$

Assume that  $|\partial G^j(a_\varepsilon^j)| \leq C_3 (a_\varepsilon^j)^{n-1}$ ,  $\max_j a_\varepsilon^j \varepsilon^{-1} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $\beta(x) \geq \beta_0 = \text{const} > 0$ . Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_4 \left( \left( \max_j a_\varepsilon^j \right) \varepsilon^{-1} \right)^{n-1}.$$

Here  $\lambda_\varepsilon^k$  and  $\lambda^k$  are ordering in the same way as in Theorem 7.

Now let us consider the case when  $\Omega_\varepsilon$  has the structure, given by (22), and the eigenvalues of the problem (81).

**THEOREM 9 (Critical case).** *Let  $u_\varepsilon^k$  be an eigenfunction of the problem (81) and  $u^k$  be an eigenfunction of the problem*

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega \setminus \gamma, v^k = 0 \text{ on } \partial\Omega,$$

$$[v^k] \Big|_\gamma = 0, \left[ \frac{\partial v^k}{\partial x_1} \right] \Big|_\gamma = \mu v^k,$$

where  $\beta(x) = \beta_0 = \text{const} > 0$ ,  $\mu = \beta_0 c_0^{n-1} |\partial \overline{G}_0|$ ,  $a_\varepsilon \varepsilon^{-1} \rightarrow c_0 = \text{const} > 0$  as  $\varepsilon \rightarrow 0$ ,  $|\overline{G}_\varepsilon \cap \partial\Omega| \leq C_4 \varepsilon$ . Let  $\Omega_\varepsilon$  have the structure, defined by (22). Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_5(\varepsilon + |a_\varepsilon \varepsilon^{-1} - c_0|^2).$$

This Theorem is a consequence of Theorem 3.

**THEOREM 10.** *Consider the Dirichlet eigenvalue problem*

$$(83) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \text{ in } \Omega_\varepsilon, u_\varepsilon^k = 0 \text{ on } \partial\Omega_\varepsilon,$$

and the Dirichlet eigenvalue problem

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega, v^k = 0 \text{ on } \partial\Omega.$$

Assume that

$$\max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for } n \geq 3,$$

$$\max_j (|\ln a_\varepsilon| \varepsilon)^{-1} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for } n = 2.$$

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_8 (\max_j a_\varepsilon)^{n-2} \varepsilon^{1-n} \text{ for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_9 (\max_j |\ln a_\varepsilon| \varepsilon)^{-1} \text{ for } n = 2.$$

This Theorem is a consequence of Theorem 4.

**THEOREM 11 (Critical case).** *Let  $\Omega_\varepsilon$  be given by (22),  $G_0 = \{x : |x| < a, 0 < a < 1/2\}$ ,  $a_\varepsilon^j = a_\varepsilon$ ,  $Q = \{x : -1/2 < x_i < 1/2, i = 1, \dots, n\}$ . Assume that  $a_\varepsilon^{n-2} \varepsilon^{1-n} \rightarrow c_0 = \text{const} > 0$  as  $\varepsilon \rightarrow 0$  for  $n \geq 3$  and  $(\varepsilon |\ln a_\varepsilon|)^{-1} \rightarrow c_1 = \text{const} > 0$  as  $\varepsilon \rightarrow 0$  for  $n = 2$ . Let  $u_\varepsilon^k$  be a solution of the eigenvalue problem (83) and  $v^k$  be a solution of the eigenvalue problem*

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega,$$

$$[v^k] \Big|_\gamma = 0, \left[ \frac{\partial v^k}{\partial x_1} \right] \Big|_\gamma = \mu_1 v^k \Big|_\gamma, v^k = 0 \text{ on } \partial\Omega,$$

where  $\mu_1 = (n - 2)a^{n-2} \omega_n c_0$  for  $n \geq 3$  and  $\mu_1 = 2\pi c_1$  for  $n = 2$ .

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{10}(\varepsilon + |c_0 - a_\varepsilon^{n-2}\varepsilon^{1-n}|^2) \quad \text{for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{11}\{\varepsilon + |c_1 - (\varepsilon|\ln a_\varepsilon|)^{-1}|^2\} \quad \text{for } n = 2.$$

This theorem is a consequence of Theorem 5.

**THEOREM 12.** *Let  $u_\varepsilon^k$  be a solution of the eigenvalue problem (83) and let us consider two eigenvalue problems:*

$$\Delta v_+^k + \lambda_+^k v_+^k = 0 \quad \text{in } \Omega^+ = \Omega \cap \{x : x_1 > 0\}, \quad v_+^k = 0 \quad \text{on } \partial\Omega^+,$$

$$\Delta v_-^k + \lambda_-^k v_-^k = 0 \quad \text{in } \Omega^- = \Omega \cap \{x : x_1 < 0\}, \quad v_-^k = 0 \quad \text{on } \partial\Omega^-.$$

*Let  $\{\lambda^k\}$  be a sequence, which is the set  $\{\lambda_-^k\} \cup \{\lambda_+^k\}$ , ordered as a nondecreasing sequence and every eigenvalue is counted as many time as its multiplicity. Assume that  $a_\varepsilon^{2-n}\varepsilon^{n-1} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for  $n \geq 3$  and  $\varepsilon|\ln a_\varepsilon| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for  $n = 2$ . Then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{12}a_\varepsilon^{2-n}\varepsilon^{n-1} \quad \text{for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{13}\varepsilon|\ln a_\varepsilon| \quad \text{for } n = 2.$$

We note that in Theorems 1-12 the Laplace operator can be substituted by any elliptic second order selfadjoint operator.

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