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Controllability, Penalty and Stiffness

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To the Memory of Ennio de Giorgi

Abstract. We consider problems of controllability (approximate or exact) for linear parabolic operators, with distributed controls. All what is presented here applies, with slight technical changes, to boundary controls, to operators of Petrowsky type, or to hyperbolic operators or to Schroedinger operators. We penalize the state equation. Let $\frac{1}{\varepsilon}$, ε small, be the penalization factor. We consider the OS (Optimality System) of the penalized problem, i.e. the set of equations which characterize the solution of the penalized problem. We then consider the asymptotic expansion in powers of ε of the solution of the OS. It has an interesting structure. It corresponds to a sequence of controllability problems, where, roughly speaking, the cost function for the problem of order N in the expansion contains the Lagrange multiplier of the OS of problem of order $N-1$. This type of structure is completely general. We also show how these questions are related to stiff problems of Calculus of Variations.

1. – Introduction

In a bounded domain Ω of $\mathbb{R}^d$ ($d = 1, 2, 3$ in most of the applications —but not all of them!—) we consider a partial differential operator P of parabolic type

$$P = \frac{\partial}{\partial t} + A$$

where A is a second order elliptic operator with non necessarily smooth coefficients. Let Ω_0 be an open set contained in Ω. Actually Ω_0 can be arbitrarily "tiny" in Ω.

We denote by m the characteristic function of Ω_0. The *state* $y(x, t; v)$ of the system we want to control is given by the solution of

$$\begin{aligned} &Py = mv \quad \text{in} \quad Q = \Omega \times (0, T) \\ &y|_{t=0} = 0 \\ &y = 0 \qquad \text{on} \quad \Sigma = \Gamma \times (0, T), \quad \Gamma = \partial\Omega, \end{aligned} \tag{1.1}$$

where $v = v(x,t) =$ control which satisfies

$$v \in L^2(Q_0), \quad Q_0 = \Omega_0 \times (0,T) \tag{1.2}$$

(and where in (1.1) $mv = 0$ outside Ω_0). Indeed it is classical that (1.1) admits a unique solution, which satisfies to

$$y, \frac{\partial y}{\partial x_i} \in L^2(Q), \ \frac{\partial y}{\partial t} \in L^2(0,T; H^{-1}(\Omega)) \tag{1.3}$$

where $H^{-1}(\Omega) =$ dual of $H_0^1(\Omega)$, with the usual notations of Sobolev spaces. Actually if the coefficients of A are smooth enough, then

$$\frac{\partial^2 y}{\partial x_i \partial x_j}, \quad \frac{\partial y}{\partial t} \in L^2(Q)\,. \tag{1.4}$$

The *controllability* problem is to drive the system, i.e. to choose v, so that, if possible

(1.5) $y(x,T;v) =$ state at time T, belongs to an arbitrarily small neighborhood of a given function $y^T \in L^2(\Omega)$

(this is the *approximate controllability* problem), or, whenever it is possible, so that

$$y(x,T;v) = y^T\,; \tag{1.6}$$

this is the *exact controllability* problem. □

REMARK 1.1. It is known (cf. J. L. Lions [1], [2]) that system (1.1) is *approximately controllable*, i.e. there always exists v such that (1.5) holds true. It is also known that there is *no exact controllability*, and that (1.6) is possible *only if y^T belongs to a rather "small" subspace of $L^2(\Omega)$.* □

REMARK 1.2. If the coefficients of A are smooth enough, the fact that *in general* (1.6) is impossible follows from the fact that the solution y of (1.1) is such that $y(x,T;v)$ satisfies some regularity properties *not* enjoyed by any function y^T in $L^2(\Omega)$. Actually if the coefficients of A are *very* irregular functions of x and t, such that one can still define a weak solution of (1.1), it *could* happen that (1.6) becomes possible for every y^T in $L^2(\Omega)$.

This *open question* is related to the Nash-de Giorgi regularity theorems and also to the *uniqueness* properties of solutions of

$$Pw = 0 \quad \text{in} \quad Q$$

which are zero on $\Omega_0 \times (0,T)$. □

We consider now the problem

$$\inf_{v} . \frac{1}{2} \int\int_{Q_0} v^2 dx\, dt\,, \text{ among all v's such that } y(T; v) \in y^T + \beta B \tag{1.7}$$

where $B =$ unit ball in $L^2(\Omega)$ and where β is given arbitrarily small. A variant of (1.7) is to consider

$$\inf_{v} \frac{1}{2} \int\int_{Q_0} v^2 dx\, dt + \frac{\alpha}{2} \| y(T; v) - y^T \|^2 \tag{1.8}$$

where α is given > 0 "large enough" and where

$$\| \quad \| = \text{norm in } L^2(\Omega)\,.$$

REMARK 1.3. Problem (1.8) is simpler than (1.7) in many respects:
1) it always admits a solution;
2) it is linear quadratic, whereas (1.7) is not, because of the constraints.

We *cannot* in general let α go to $+\infty$ in (1.8). Indeed it would imply, if the process was convergent, that in the limit $\alpha = +\infty$, one has (1.6), which is in general impossible. If α is "large", the last term in (1.8) is a *penalty term*. We return to this point below. □

REMARK 1.4. Formulation (1.8) is used in numerical methods. Cf. R. Glowinski and J. L. Lions [1], where one can find how to choose α such that (1.7) is approximately satisfied. □

We now introduce the *penalty of the state equation*. It is very often the case that we are not absolutely certain that the state equation $Py = mv$ is *exactly* satisfied. Hence the idea to *relax* this equation. We then introduce the problem

$$\inf_{v,y} . \frac{1}{2} \|v\|^2_{Q_0} + \frac{\alpha}{2} \|y(T) - y^T\|^2 + \frac{k}{2} \|Py - mv\|^2_Q \tag{1.9}$$

where:

$\| \quad \|_R =$ norm in $L^2(R)$,
$k > 0$ is *large*, it is the *penalty coefficient*,
v and y and independant,

y satisfies

$$\begin{aligned} y &= 0 \quad \text{at} \quad t = 0\,, \\ y &= 0 \quad \text{on} \quad \Sigma\,. \end{aligned} \tag{1.10}$$

REMARK 1.5. We could as well consider other cases than (1.10). For instance

$$y|_{t=0} = y^0, \quad y = g \quad \text{on} \quad \Sigma \tag{1.11}$$

and we could "penalize" these conditions, by adding to (1.9) the terms

$$\frac{k_1}{2}\|y(0)-y^0\|^2+\frac{k_2}{2}\|y-g\|_\Sigma^2\,. \tag{1.12}$$

In any case we assume that $Py \in L^2(Q)$ so that everything makes sense if (1.10) holds true. Things are getting technically more complicated if one uses (1.12). □

REMARK 1.6. One can show that if we denote by v_k, y_k the unique solution of (1.9), then, as $k \to \infty$,

$$\begin{aligned} &v_k \to v \quad \text{in} \quad L^2(Q_0) \\ &y_k \to y \quad \text{in the space corresponding to (1.3)} \end{aligned} \tag{1.13}$$

where v, y is the unique solution of (1.8). □

REMARK 1.7. One can also introduce the penalized analog of (1.7), namely

$$\begin{aligned} &\inf_{v,y}.\frac{1}{2}\|v\|_{Q_0}^2+\frac{k}{2}\|Py-mv\|_Q^2\,, \\ &\text{where } y \text{ satisfies (1.10) and } y(T)\in y^T+\beta B\,. \end{aligned} \tag{1.14}$$

□

REMARK 1.8. The idea of *penalizing the state equation* has been introduced in J. L. Lions [3] to obtain the *Optimality System* (OS) of control problems not of the controllability type (i.e. with other functionals).

It has been used in J. L. Lions [4] in cases P is *non linear* and *non well set* (*singular problems*) but, again, with other functionals.

A very general formulation was given by E. de Giorgi and Ambrosio again not in the framework of controllability. □

Our main goal in this paper is to introduce the (OS) of problem (1.9), that we denote by $(\mathrm{OS})_\varepsilon$ where $\varepsilon = \frac{1}{k}$. This is made in Section 2. We then consider (Section 3) *the asymptotic expansion* in ε of the solution of $(\mathrm{OS})_\varepsilon$.

We shall obtain the following result, described here in a formal fashion (precise statement is given in Section 3):

$$(\mathrm{OS})_\varepsilon = (\mathrm{OS})_0 + \varepsilon(\mathrm{OS})_1 + \varepsilon^2(\mathrm{OS})_2 + \dots \tag{1.15}$$

where

$$\begin{aligned} (\mathrm{OS})_0 &= \text{optimality system for problem (1.8)}, \\ (\mathrm{OS})_m &= \text{optimality system for a control problem of type (1.8)} \end{aligned}$$

where the state equation contains the Lagrange multiplier of $(\mathrm{OS})_{m-1}$ $(m \geq 1)$ and where y^T is replaced by 0.

REMARK 1.9. The expansion (1.15) (of course made precise) is very general. The structure of P is not essential. For instance, one can have

$$P = \frac{\partial}{\partial t} + A \quad \text{where} \quad A \text{ is elliptic of order } 2m, m > 1,$$
$$P = \text{system of parabolic operators}$$

(of course subject to appropriate boundary conditions). One can also have:

$$P = \text{Hyperbolic operator of say, second order,}$$

but one has to add then appropriate initial conditions, and if y is the state, one wants then $y(T)$ and $\frac{\partial y}{\partial t}(T)$ close, or equal to, given "target" functions.

We can also take

$$P = \frac{\partial^2}{\partial t^2} + A, \quad A \quad \text{elliptic of order } 2m, m > 1\,,$$

a Petrowsky type operator. One could also consider (using then complex L^2 spaces)

$$P = \text{Schroedinger operator, with } \Omega \subset \mathbb{R}^d, d \text{ arbitrarily large}\,. \qquad \square$$

REMARK 1.10. One can also replace the distributed control v *by boundary controls*. The structure of (1.15) remains true. Only technical details, using J. L. Lions and E. Magenes [8], are then different. □

The expansion (1.15) shows "how far" problem (1.9) is from (1.8). One can also obtain informations on this question by using *duality arguments* as presented in Section 4.

Problem (1.9) is also related to *stiff problems*, as explained in Section 5. We present in Section 3 a remark on the expansion (1.15) when we take $\alpha = k$ in (1.9). Everything then becomes much more complicated, because we deal in fact with *exact* controllability, an impossible problem in general, as we said before.

2. – Controllability and penalty. The optimality system

Let us denote by v, y *the* solution of problem (1.9). Let us *define*

$$p = -k(Py - mv)\,. \tag{2.1}$$

Then the Euler equations of (1.9), are, in variational form

$$(v, \hat{v})_{Q_0} + (p, m\hat{v})_Q = 0, \quad \forall\, \hat{v} \in L^2(Q_0)\,, \tag{2.2}$$

$$\begin{gathered} \alpha(y(T) - y^T, \hat{y}(T)) - (P_y, P\hat{y})_Q = 0\,, \\ \forall\, \hat{y} \text{ smooth enough and satisfying (1.10).} \end{gathered} \tag{2.3}$$

Equation (2.2) is equivalent to

$$mv + mp = 0 \tag{2.4}$$

and (2.3) is equivalent to

$$\begin{aligned} &P^* p = 0 && \text{in} \quad Q\,, \\ &p(T) = \alpha(y(T) - y^T)\,, && \\ &p = 0 && \text{on} \quad \Sigma \end{aligned} \tag{2.5}$$

where P^* is the adjoint of P, i.e. $P^* = -\frac{\partial}{\partial t} + A^*$, $A^* =$ adjoint of A.
Using (2.4) in (2.1) gives

$$p = -k(Py + mp)$$

i.e., if we set

$$\varepsilon = \frac{1}{k}\,, \tag{2.6}$$

$$\begin{aligned} &Py + mp + \varepsilon p = 0\,, \\ &P^* p = 0\,, \\ &y(0) = 0, \quad p(T) = \alpha(y(T) - y^T)\,, \\ &y = p = 0 \quad \text{on} \quad \Sigma\,. \end{aligned} \tag{2.7}$$

This is the $(\mathrm{OS})_\varepsilon =$ *Optimality System* for (1.9).

REMARK 2.1. If we take $\alpha = k$ then (2.7) becomes

$$\begin{aligned} &Py + mp + \varepsilon p = 0\,, \\ &P^* p = 0\,, \\ &y(0) = 0, \quad \varepsilon p(T) = y(T) - y^T\,, \\ &y = p = 0 \quad \text{on} \quad \Sigma\,, \end{aligned} \tag{2.8}$$

whose expansion in powers of ε is much more complicated than the one for (2.7). □

3. – Asymptotic expansion

Let us first proceed formally for $(\mathrm{OS})_\varepsilon$. We write

$$y = y_\varepsilon = y_0 + \varepsilon y_1 + \dots \qquad (3.1)$$
$$p = p_\varepsilon = p_0 + \varepsilon p_1 + \dots$$

and we make an identification. We obtain

$$\begin{aligned} &P y_0 + m p_0 = 0\,, \\ &P^* p_0 = 0 \\ &y_0(0) = 0\,, \quad p_0(T) = \alpha(y_0(T) - y^T)\,, \\ &y_0 = p_0 = 0 \quad \text{on} \quad \Sigma \end{aligned} \qquad (3.2)$$

$$\begin{aligned} &P y_1 + m p_1 + p_0 = 0\,, \\ &P^* p_1 = 0 \\ &y_1(0) = 0\,, \quad p_1(T) = \alpha y_1(T)\,, \\ &y_1 = p_1 = 0 \quad \text{on} \quad \Sigma \end{aligned} \qquad (3.3)$$

and so on. We observe that (3.2) is the (OS) of problem (1.8), denoted by $(\mathrm{OS})_0$.

Let us verify now that (3.3) is the (OS) of the following problem. The *state equation* is given by

$$\begin{aligned} &P y + p_0 = m v\,, \\ &y(0) = 0, \quad y = 0 \quad \text{on} \quad \Sigma \end{aligned} \qquad (3.4)$$

and the *cost function* is given by

$$J_1(v) = \frac{1}{2}\|v\|^2_{Q_0} + \frac{\alpha}{2}\|y(T; v)\|^2\,. \qquad (3.5)$$

The unique solution v_1, y_1 of

$$\inf J_1(v), \quad v \in L^2(Q_0) \qquad (3.6)$$

is given by the solution of (3.3) and by

$$v_1 = -m p_1\,. \qquad (3.7)$$

In other words:

(3.8) One adds to the left hand side of the state equation the Lagrange multiplier p_0 of the first (OS) and one replaces y^T by 0.

This rule is general for $y_2, p_2, \dots$. The couple y_N, p_N satisfies

$$(3.9)\qquad \begin{aligned} &P y_N + m p_N + p_{N-1} = 0\,,\\ &P^* p_N = 0\\ &y_N(0) = 0\,, \quad p_N(T) = \alpha y_N(T)\,,\\ &y_N = p_N = 0 \quad \text{on} \quad \Sigma\,. \end{aligned}$$

This is the (OS) of

$$(3.10)\qquad \begin{aligned} &P y + p_{N-1} = m v\,,\\ &y(0) = 0, y = 0 \quad \text{on} \quad \Sigma \end{aligned}$$

and the solution v_N of $\inf J_N(v)$ is given by

$$(3.11)\qquad v_N = -m p_N\,.$$

We can write *formally*

$$(3.12)\qquad (\mathrm{OS})_\varepsilon = (\mathrm{OS})_0 + \varepsilon(\mathrm{OS})_1 + \dots + \varepsilon^N (\mathrm{OS})_N + \dots$$

where $(\mathrm{OS})_0$ is (3.2) and $(\mathrm{OS})_N$ is (3.9). □

We now justify (3.12). Let us introduce

$$(3.13)\qquad \begin{aligned} &z = y_\varepsilon - (y_0 + \varepsilon y_1 + \dots + \varepsilon^N y_N)\,,\\ &q = p_\varepsilon - (p_0 + \varepsilon p_1 + \dots + \varepsilon^N p_N)\,. \end{aligned}$$

Using (2.7), (3.2), (3.3), ... we obtain

$$(3.14)\qquad \begin{aligned} &P z + m q + \varepsilon q = -\varepsilon^{N+1} p_N\,,\\ &P^* q = 0\,,\\ &z(0) = 0, \quad q(T) = \alpha z(T)\,,\\ &z = q = 0 \quad \text{on} \quad \Sigma\,. \end{aligned}$$

We multiply $(3.14)_1$ by q and integrate over Q. We obtain, after noticing that $(Pz, q)_Q = (z(T), q(T)) + (z, P^* q)_Q = \alpha \|z(T)\|^2$, that

$$(3.15)\qquad \alpha\|z(T)\|^2 + \|q\|^2_{Q_0} + \varepsilon\|q\|^2_Q = -\varepsilon^{N+1}(p_N, q)_Q$$

so that

$$(3.16)\qquad \|q\|_Q \le \varepsilon^N \|p_N\|_Q\,,$$

$$(3.17)\qquad \|q\|_{Q_0} \le \varepsilon^{N+1/2} \|p_N\|_Q\,.$$

Therefore (3.14), can be written

$$Pz = f, \|f\|_Q \le \varepsilon^{N+1/2}\|p_N\|_Q(1+2\varepsilon^{1/2}). \tag{3.18}$$

This equation, together with $z(0)=0, z=0$ on Σ, gives estimates on z which depend on the regularity of the coefficients of A.

If the coefficients are only L^∞, with $A = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$, then, for suitable constants C,

$$\|z\|_Q + \Sigma\|\frac{\partial z}{\partial x_i}\|_Q \le C\varepsilon^{N+1/2}, \tag{3.19}$$

$$\|z(T)\| \le C\varepsilon^{N+1/2}. \tag{3.20}$$

Estimates (3.20) and (3.14) give in turn

$$\|q\|_Q + \Sigma\|\frac{\partial q}{\partial x_i}\|_Q \le C\varepsilon^{N+1/2}. \tag{3.21}$$

These estimates give a precise meaning to the expansion (3.12).

REMARK 3.1. Let us consider now the situation of Remark 2.1, where $\alpha = k$.

If we use again an expansion like (3.1), we obtain

$$\begin{aligned} &Py_0 + mp_0 = 0, \\ &P^*p_0 = 0 \\ &y_0(0) = 0, \quad y_0(T) = y^T, \qquad y_0 = p_0 = 0 \quad \text{on} \quad \Sigma \end{aligned} \tag{3.22}$$

$$\begin{aligned} &Py_1 + mp_1 + p_0 = 0, \\ &P^*p_1 = 0 \\ &y_1(0) = 0, \quad p_0(T) = y_1(T), \quad y_1 = p_1 = 0 \quad \text{on} \quad \Sigma \end{aligned} \tag{3.23}$$

and so on. *Formally* (3.22) is the OS of the following problem (of the *exact* controllability type). The state equation is given by

$$Py = mv \tag{3.24}$$

and one looks for

(3.25) $\inf.\frac{1}{2}\|v\|^2_{Q_0}$, when y is subject to usual initial and boundary conditions, *and* to $y(T) = y^T$.

But this problem admits a (unique) solution iff y^T belongs to a subspace of $L^2(\Omega)$ defined as follows. One considers

$$
\begin{aligned}
&P^*\varphi = 0 && \text{in} \quad Q\,, \\
&\varphi(T) = f \in L^2(\Omega), \varphi = 0 && \text{on} \quad \Sigma
\end{aligned}
\tag{3.26}
$$

and one *defines*

$$
[f] = \|\varphi\|_{Q_0}\,. \tag{3.27}
$$

Because of the uniqueness theorem of Mizohata's type (cf. S. Mizohata [1]) (3.27) is a *norm* on $L^2(\Omega)$. One introduces (cf. J. L. Lions[1], [2], [5] for a systematic use of this technique, the so-called HUM = Hilbert Uniqueness Method)

$$
F = \text{ completion of } L^2(\Omega) \text{ for the norm (3.27).} \tag{3.28}
$$

This is a "very large" space, *not* contained in the space of distributions on Ω.

Then (3.25) admits a solution iff

$$
y^T \in F' = \text{dual space of } F. \tag{3.29}
$$

□

System (3.23) is *formally* the OS of the following problem. The state equation is given by

$$
\begin{aligned}
&Py + p_0 = mv\,, \\
&y(0) = 0, y = 0 \quad \text{on} \quad \Sigma
\end{aligned}
\tag{3.30}
$$

and one looks for

$$
\inf.\frac{1}{2}\|v\|^2_{Q_0}\,, \qquad y(T) = p_0(T) \tag{3.31}
$$

a problem which admits a (unique) solution because p_0 (i.e. p_0 in Q and $p_0(T)$) satisfy suitable conditions following from (3.29). □

4. – Duality

Problem (1.9) can be formulated in the following equivalent form:

$$
\begin{aligned}
\inf_{\substack{v\\f\\g\\y}}.\sup_{\substack{p\\q}} \Big[&\frac{1}{2}\|v\|^2_{Q_0} + \frac{\alpha}{2}\|f - y^T\|^2 + \frac{k}{2}\|g - mv\|^2_Q \\
&- (p, Py - g)_Q - (q, y(T) - f) \Big]
\end{aligned}
\tag{4.1}
$$

Indeed the supremum with respect to $p \in L^2(Q), q \in L^2(\Omega)$ is $+\infty$ *except* if $g = Py$ and $f = y(T)$ and *then* the infimum with respect to v, f, g reduces to the inf. in (1.9). Now, if we commute the inf. and the sup. operations in (4.1), a commutation which is valid here, one obtains the equivalent problem

$$\sup_{\substack{p\\q}} \inf_{\substack{v\\f\\g\\y}} \left[\frac{1}{2}\|v\|^2_{Q_0} + \frac{\alpha}{2}\|f - y^T\|^2 + \frac{k}{2}\|g - mv\|^2_Q - (p, Py - g)_Q \right. \tag{4.2}$$
$$\left. - (q, y(T), f)\right] .$$

The Euler equations for the problem of infimum with respect to v, f, g are, in variational form

$$(v, \hat{v})_{Q_0} + (\rho, m\hat{v})_Q = 0 \tag{4.3}$$

where we have set

$$\rho = -k(g - mv), \tag{4.4}$$
$$\alpha(f - y^T, \hat{f}) + (q, \hat{f}) = 0, \tag{4.5}$$
$$(-\rho, \hat{g}) + (p, \hat{g}) = 0, \tag{4.6}$$
$$-(p, P\hat{y}) - (q, \hat{y}(T)) = 0. \tag{4.7}$$

Then the inf. is equal to

$$-\frac{\alpha}{2}(f - y^T, y^T) + \frac{1}{2}(q, f) + \frac{1}{2}(p, q)_Q . \tag{4.8}$$

The equations (4.3), ..., (4.7) are equivalent to

$$\begin{aligned} & mv + mp = 0, \\ & g - mv + \varepsilon\rho = 0 \qquad \text{where} \quad \varepsilon = \frac{1}{k}, \\ & \alpha(f - y^T) + q = 0, \\ & \rho = p, \\ & P^*p = 0, p = 0 \qquad \text{on} \quad \Sigma, \\ & p(T) + q = 0. \end{aligned} \tag{4.9}$$

Using (4.9) in (4.8) gives (since $g + mp + \varepsilon p = 0$)

$$\inf_{v,f,g,y} = -\frac{1}{2\alpha}\|q\|^2 - \frac{1}{2}\|p\|^2_{Q_0} - \frac{\varepsilon}{2}\|p\|^2_Q + (q, y^T). \tag{4.10}$$

Therefore (4.2) shows that if we set (cf. (1.9))

$$j_\varepsilon = \inf_{v,y}\left[\frac{1}{2}\|v\|^2_{Q_0} + \frac{\alpha}{2}\|y(T) - y^T\|^2 + \frac{k}{2}\|Py - mv\|^2_Q\right] \tag{4.11}$$

then

$$j_\varepsilon = -\inf_q\left[\frac{1}{2}\|p\|^2_{Q_0} + \frac{\varepsilon}{2}\|p\|^2_Q + \frac{1}{2\alpha}\|q\|^2 - (q, y^T)\right], \tag{4.12}$$

where

$$P^*p = 0,\ p(T) = -q,\ p = 0 \quad \text{on} \quad \Sigma.$$

We can change p into $-p$, so that finally j_k is given by (4.12) where

$$P^*p = 0,\ p(T) = q,\ p = 0 \quad \text{on} \quad \Sigma. \tag{4.13}$$

Formulas (4.12), (4.13) show that, by duality, penalization becomes a *regularization of the dual problem, by the addition of* $\frac{\varepsilon}{2}\|p\|^2_Q$. (This type of remark is well known.)

REMARK 4.1. The dual problem of (1.8) equals

$$j_0 = -\inf_q.\left[\frac{1}{2}\|p\|^2_{Q_0} + \frac{1}{2\alpha}\|q\|^2 - (q, y^T)\right]. \tag{4.14}$$

The dual problem of (3.5), (3.6) is given by

$$j_1 = -\inf_q.\left[\frac{1}{2}\|p\|^2_{Q_0} - (p, p_0)_Q + \frac{1}{2\alpha}\|q\|^2\right] \tag{4.15}$$

and, in general

$$j_N = -\inf_q.\left[\frac{1}{2}\|p\|^2_{Q_0} - (p, p_{N-1})_Q + \frac{1}{2\alpha}\|q\|^2\right]. \tag{4.16}$$

This gives the expansion (formally)

$$j_\varepsilon = j_0 + \varepsilon j_1 + \ldots + \varepsilon^N j_N + \ldots \tag{4.17}$$

whose justification corresponds to the estimates (3.19), (3.20), (3.21). □

REMARK 4.2. The dual formulation of (1.14) is

$$-\inf_q.\left[\frac{1}{2}\|p\|^2_{Q_0} + \frac{\varepsilon}{2}\|p\|^2 + \beta\|q\| - (q, y^T)\right]. \tag{4.18}$$

(The functional is not differentiable this time). □

REMARK 4.3. If one takes $\beta = 0$ in (4.18), it corresponds to "exact controllability" for the penalized (or regularized) problem, namely

$$-\inf_q \left[\frac{1}{2}\|p\|_{Q_0}^2 + \frac{\varepsilon}{2}\|p\|^2 - (q, y^T)\right], \tag{4.19}$$

a problem which still admits a unique solution, which can be expanded as before (with slight changes).

The situation changes radically if one takes $\varepsilon = 0$ in (4.19). One sees clearly on (4.19) (with $\varepsilon = 0$) that

$$q \to (q, y^T)$$

should be a continuous linear form for the norm $\|p\|_{Q_0}$, i.e. that y^T should belong to F' (cf. (3.29)). □

5. – Stiffness

There is still another way to study (1.9), by making an *explicit* computation of the inf. with respect to v, y being fixed. We find that v should satisfy

$$mv - km(Py - mv) = 0. \tag{5.1}$$

Then the $\inf_v$ in (1.9) equals

$$\frac{\alpha}{2}\|y(T) - y^T\|^2 + \frac{k}{2}(Py - mv, Py)_Q. \tag{5.2}$$

It follows from (5.1) that $mv = \frac{k}{1+k} mPy$ so that (5.2) equals to

$$\frac{\alpha}{2}\|y(T) - y^T\|^2 + \frac{k}{2}\|Py\|_Q^2 - \frac{k^2}{2(1+k)}\|mPy\|_Q^2. \tag{5.3}$$

Let us set

$$\frac{1}{1+k} = \delta, \ 1 - m = m_0, m = m_1.$$

Then (5.3) equals

$$k\left[\frac{1}{2}\|m_0 Py\|^2 + \frac{\delta}{2}\|m_1 Py\|^2 + \frac{\alpha\delta}{2(1+k)}\|y(T) - y^T\|^2\right]$$

so that problem (1.9) is equivalent to finding (up to the multiplicative factor k)

$$\inf_y \left[\frac{1}{2}\|m_0 Py\|^2 + \frac{\delta}{2}\|m_1 Py\| + \frac{\alpha\delta}{2(1+k)}\|y(T) - y^T\|^2\right]. \tag{5.4}$$

This problem enters the family of stiff problems, i.e. problems of calculus of variations where the coefficients are $0(1)$ on part of the domain and $0(\delta)$ on the other part of the domain. Cf. J. L. Lions [6].

REFERENCES

[1] R. Glowinski – J. L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta Numerica (1994), 269-378, (1995), 159-333.

[2] J. L. Lions, *Exact Controllability for distributed systems. Some trends and some problems*, Applied and Industrial Mathematics, R. Spigler ed., Kluwer (1991) (Venice 1989), 59-84.

[3] J. L. Lions, "Contrôlabilité exacte, Perturbation et Stabilisation des Systèmes distribués", Vol. 1 et 2, Masson, Paris, 1988.

[4] J. L. Lions, "Contrôle Optimal des Systèmes gouvernés par des équations aux dérivées partielles", Dunod, Paris, 1968.

[5] J. L. Lions, "Contrôle des systèmes singuliers", MMI, Gauthier Villars, 1983.

[6] J. L. Lions, *Exact controllability, stabilization and perturbation for distributed systems*, SIAM Rev. **30** (1988), 1-68.

[7] J. L. Lions, "Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal", Lectures Notes in Mathematics, Springer, 1973.

[8] J. L. Lions – E. Magenes, "Problèmes aux limites non homogènes et applications", Vol. 1 et 2 Dunod, Paris, 1968.

[9] S. Mizohata, *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques*, Mem. Coll. Sc. Univ. Kyoto A **31** (1958), 219-239.

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