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## Poincaré Inequality for Some Measures in Hilbert Spaces and Application to Spectral Gap for Transition Semigroups

GIUSEPPE DA PRATO

### 1. – Introduction and setting of the problem

Let  $H$  be a separable Hilbert space (norm  $|\cdot|$ , inner product  $\langle \cdot, \cdot \rangle$ ), and let  $\nu$  be a Borel measure on  $H$ . This paper is devoted to prove, under suitable assumptions on  $\nu$ , an estimate of this kind (Poincaré inequality):

$$(1.1) \quad \int_H \left| \varphi(x) - \int_H \varphi(y) \nu(dy) \right|^2 \nu(dx) \leq C \int_H |D\varphi(x)|^2 \nu(dx),$$

where  $C$  is a suitable positive constant.

Estimate (1.1) can be used to study the *spectral gap* for a transition semigroup corresponding to a differential stochastic equation:

$$(1.2) \quad \begin{cases} dX(t) = (AX(t) + F(X(t)))dt + Q^{1/2}dW(t), & t \geq 0, \\ X(0) = x, \end{cases}$$

Here  $A : D(A) \subset H \rightarrow H$  and  $Q : H \rightarrow H$ , are linear operators,  $F : H \rightarrow H$  is nonlinear, and  $W(t)$ ,  $t \geq 0$  is an  $H$ -valued cylindrical Wiener process defined in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , see e.g. [5].

Assume that problem (1.2) has unique solution  $X(t, x)$ , then the corresponding transition semigroup  $P_t$ ,  $t \geq 0$ , is defined by

$$(1.3) \quad P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(H),$$

where  $B_b(H)$  is the Banach space of all bounded and Borel functions from  $H$  into  $\mathbb{R}$ . We want to prove, under suitable assumptions, an estimate

$$(1.4) \quad \int_H \left| P_t \varphi(x) - \int_H \varphi(y) \nu(dy) \right|^2 \nu(dx) \leq C e^{-\omega t} \int_H |\varphi(x)|^2 \nu(dx),$$

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for all  $\varphi \in L^2(H, \nu)$ , where  $\nu$  is an invariant measure for the semigroup, and  $C, \omega$  are positive constants.

Estimate (1.4) implies that the spectrum  $\sigma(\mathcal{L})$  of the infinitesimal generator  $\mathcal{L}$  of  $P_t$  in  $L^2(H, \nu)$  has the following property

$$(1.5) \quad \sigma(\mathcal{L}) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < \omega\}.$$

This *spectral gap* property is important in the applications, it has been studied in the literature, mainly when the semigroup  $P_t$  is symmetric, see [8], [9], [5].

The content of the paper is the following. In Section 2 we prove a Poincaré inequality when  $\nu = \mu_R$  is a Gaussian measure of mean 0 and covariance operator  $R \in \mathcal{L}_1^+(H)$ , the space of all nonnegative, symmetric, linear operators from  $H$  into  $H$  of trace class. In this case estimate (1.1) is a natural generalization of a well known result when  $H$  is finite dimensional. Then we consider in Section 3 the case when  $\nu$  is absolutely continuous with respect to a Gaussian measure  $\mu_R$ . Finally section Section 4 is devoted to the spectral gap property.

## 2. – Poincaré inequality for Gaussian measures

We are given a Gaussian measure  $\mu_R$ , on  $H$  with mean 0 and covariance operator  $R \in \mathcal{L}_1^+(H)$ . We denote by  $\{e_k\}$  a complete orthonormal system in  $H$  consisting of eigenvectors of  $R$  and by  $\{\lambda_k\}$  the corresponding sequence of eigenvalues:

$$Re_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We shall assume that sequence  $\{\lambda_k\}$  is nonincreasing and that  $\lambda_k > 0$  for all  $k \in \mathbb{N}$ . For any  $k \in \mathbb{N}$  we shall denote by  $D_k$  the derivative in the direction of  $e_k$ , and we shall set  $x_k = \langle x, e_k \rangle$  for any  $x \in H$ . It is well known that  $D_k$  is a closable operator on  $L^2(H, \mu)$ , see e.g. [7]. The Sobolev space  $W^{1,2}(H, \mu_R)$  is the Hilbert space of all  $\varphi \in L^2(H, \mu_R) \cap \operatorname{dom}(D_k)$ ,  $k \in \mathbb{N}$ , such that

$$\|\varphi\|_{W^{1,2}(H, \mu_R)}^2 := \int_H |\varphi(x)|^2 \mu_R(dx) + \sum_{k=1}^{\infty} \int_H |D_k \varphi(x)|^2 \mu_R(dx) < +\infty.$$

We denote by  $\mathcal{E}(H)$  the linear space spanned by all exponential functions  $\psi(x) = e^{\langle h, x \rangle}$ ,  $x \in H$ . Obviously

$$\mathcal{E}(H) \subset C^\infty(H) \cap L^2(H, \mu_R).$$

and  $\mathcal{E}(H)$  is dense in  $L^2(H, \mu_R)$ .

We denote by  $T_t$ ,  $t \geq 0$ , the Ornstein-Uhlenbeck semigroup:

$$(2.1) \quad T_t \varphi(x) = \int_H \varphi(e^{-t/2}x + y) \mu_{(1-e^{-t})R}(dy), \quad t \geq 0, \quad \varphi \in L^2(H, \mu_R).$$

It is well known that  $T_t, t \geq 0$ , is a strongly continuous semigroup of contractions on  $L^2(H, \mu_R)$  having as unique invariant measure  $\mu_R$ :

$$(2.2) \quad \int_H T_t \varphi(x) \mu_R(dx) = \int_H \varphi(x) \mu_R(dx), \quad t \geq 0, \quad \varphi \in L^2(H, \mu_R).$$

We denote by  $\mathcal{L}$  the infinitesimal generator of  $T_t, t \geq 0$ .  $\mathcal{L}$  is defined as the closure of the linear operator  $\mathcal{L}_0$ :

$$(2.3) \quad \mathcal{L}_0 \varphi(x) = \frac{1}{2} \text{Tr}[RD^2 \varphi(x)] - \frac{1}{2} \langle x, D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}(H), \quad x \in H.$$

We recall also that, for any  $\varphi \in D(\mathcal{L})$  we have, see [1], [6],

$$(2.4) \quad \int_H \mathcal{L} \varphi(x) \varphi(x) \mu_R(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \mu_R(dx).$$

Now we prove the result

**THEOREM 2.1.** *The following estimate holds*

$$(2.5) \quad \int_H |\varphi(x) - \bar{\varphi}|^2 \mu_R(dx) \leq \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx), \quad \varphi \in W^{1,2}(H, \mu_R),$$

where

$$(2.6) \quad \bar{\varphi} = \int_H \varphi(x) \mu_R(dx).$$

**PROOF.** For any  $\varphi \in D(\mathcal{L})$  we have, in view of (2.4)

$$(2.7) \quad \begin{aligned} \frac{d}{dt} \int_H |T_t \varphi(x)|^2 \mu(dx) &= 2 \int_H \mathcal{L} T_t \varphi(x) T_t \varphi(x) \mu(dx) \\ &= - \int_H |R^{1/2} D T_t \varphi(x)|^2 \mu(dx). \end{aligned}$$

To estimate  $|R^{1/2} D T_t \varphi(x)|^2$  note that, in view of (2.1),

$$\langle R^{1/2} D T_t \varphi(x), h \rangle = e^{-t/2} \int_H \langle D\varphi(e^{-t/2}x + y), h \rangle \mu_{R(1-e^{-t})}(dy),$$

for all  $h \in H$ . It follows, using Hölder's inequality

$$|\langle R^{1/2} D T_t \varphi(x), h \rangle|^2 \leq e^{-t} |h|^2 T_t(|R^{1/2} D\varphi|^2)(x), \quad h \in H.$$

Therefore, due to the arbitrariness of  $h$ ,

$$(2.8) \quad |R^{1/2} D T_t \varphi(x)|^2 \leq e^{-t} T_t(|R^{1/2} D\varphi|^2)(x).$$

By integrating on  $H$  with respect to  $\mu_R$ , and taking into account the invariance of  $\mu_R$ , we have

$$\int_H |R^{1/2} DT_t \varphi(x)|^2 \mu_R(dx) \leq e^{-t} \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Now, comparing with (2.7) we find

$$\frac{d}{dt} \int_H |T_t \varphi(x)|^2 \mu_R(dx) \geq -e^{-t} \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Integrating in  $t$  find

$$\int_H |T_t \varphi(x)|^2 \mu_R(dx) \geq \int_H |\varphi(x)|^2 \mu_R(dx) - (1 - e^{-t}) \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Finally, letting  $t$  tend to  $+\infty$ , and using the fact that, as easily checked,

$$\lim_{t \rightarrow +\infty} P_t \varphi(x) = \bar{\varphi}, \quad x \text{ a.e. in } H,$$

we get

$$(\bar{\varphi})^2 \geq \int_H |\varphi(x)|^2 \mu_R(dx) - \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx),$$

that is equivalent to (2.5).  $\square$

### 3. – Poincaré inequality for non Gaussian measures

Here we are given, besides a Gaussian measure  $\mu = \mu_R$ , with  $R \in \mathcal{L}_1^+(H)$  and  $\ker R = \{0\}$ , a function  $U : H \rightarrow \mathbb{R}$ , such that

HYPOTHESIS 1.

- (i)  $U$  is convex and of class  $C^2$ .
- (ii)  $DU$  is Lipschitz continuous.

We set

$$(3.1) \quad \alpha(x) = k e^{-2U(x)}, \quad x \in H,$$

where  $k$  is chosen such that

$$\int_H \alpha(x) \mu(dx) = 1.$$

Finally we consider the Borel probability measure on  $H$

$$\nu(dx) = \alpha(x) \mu(dx).$$

We are going to prove a Poincaré estimate for measure  $\nu$ . We notice that assumptions on  $\alpha$  could be considerably weakened. It will be enough to assume convexity of  $U$  (that implies dissipativity of  $-DU$ ), and some additional properties similar to [5]. But we prefer to make Hypothesis 1 for the sake of simplicity.

It is useful to introduce a differential stochastic equation having  $\nu$  as invariant measure:

$$(3.2) \quad \begin{cases} dZ = (AZ - DU(Z))dt + dW(t) \\ Z(0) = x \in H, \end{cases}$$

where  $A$  is the negative self-adjoint operator in  $H$  defined as

$$A = -\frac{1}{2} R^{-1},$$

and  $W$  is a cylindrical  $H$ -valued Wiener process in some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Problem (3.2) has a unique solution  $Z(t, x)$ , and measure  $\nu$  is invariant, see [5]. The corresponding transition semigroup is defined in  $L^2(H, \nu)$  by

$$(3.3) \quad N_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in L^2(H, \nu), \quad t \geq 0.$$

Its infinitesimal generator  $\mathcal{N}$  is defined by, see [4]

$$(3.4) \quad D(\mathcal{N}) = \left\{ \varphi \in W^{2,2}(H; \nu) \cap W_A^{1,2}(H; \nu) : \int_H \langle D^2 U(x) D\varphi(x), D\varphi(x) \rangle \nu(dx) < +\infty \right\},$$

where  $W_A^{1,2}(H; \nu)$  is the linear space of all  $\varphi \in W^{1,2}(H, \nu)$  such that  $\langle AD\varphi, D\varphi \rangle \in L^2(H, \nu)$ .

Finally in [5] it is proved that  $\nu$  is strongly mixing

$$(3.5) \quad \lim_{t \rightarrow \infty} N_t \varphi(x) = \int_H \varphi(y) \nu(dy), \quad \varphi \in L^2(H, \nu).$$

We can now prove

**THEOREM 3.1.** *The following estimate holds*

$$(3.6) \quad \int_H |\varphi(x) - \bar{\varphi}|^2 \nu(dx) \leq \frac{1}{\|R\|} \int_H |D\varphi(x)|^2 \nu(dx), \quad \varphi \in W^{1,2}(H, \nu),$$

where

$$(3.7) \quad \bar{\varphi} = \int_H \varphi(x) \nu(dx).$$

PROOF. For any  $\varphi \in D(\mathcal{N})$  we have, see [4],

$$(3.8) \quad \begin{aligned} \frac{d}{dt} \int_H |N_t \varphi(x)|^2 \nu(dx) &= 2 \int_H \mathcal{N} N_t \varphi(x) N_t \varphi(x) \nu(dx) \\ &= - \int_H |DN_t \varphi(x)|^2 \nu(dx). \end{aligned}$$

We want now to estimate  $|DN_t \varphi(x)|^2$ . To this purpose we note that  $X(t, x)$  is differentiable with respect to  $x$  and

$$(3.9) \quad \|X_x^*(t, x)\| \leq e^{-\frac{1}{2\|R\|}t}, \quad t \geq 0.$$

It follows

$$(3.10) \quad DN_t \varphi(x) = \mathbb{E}[X_x^*(t, x) D\varphi(X(t, x))].$$

Now by (3.10) and the Hölder's estimate, it follows,

$$(3.11) \quad |DN_t \varphi(x)|^2 \leq e^{-\frac{1}{\|R\|}t} N_t(|D\varphi|^2(x)).$$

By integrating on  $H$  with respect to  $\nu$ , and taking into account the invariance of  $\nu$ , we have

$$\int_H |DN_t \varphi(x)|^2 \nu(dx) \leq e^{-\frac{1}{\|R\|}t} \int_H |D\varphi(x)|^2 \nu(dx).$$

By substituting in (3.8) we find

$$\frac{d}{dt} \int_H |N_t \varphi(x)|^2 \nu(dx) \geq -e^{-\frac{1}{\|R\|}t} \int_H |D\varphi(x)|^2 \nu(dx).$$

Integrating in  $t$  we have

$$\int_H |N_t \varphi(x)|^2 \nu(dx) \geq \int_H |\varphi(x)|^2 \nu(dx) - \|R\| (1 - e^{-\frac{1}{\|R\|}t}) \int_H |D\varphi(x)|^2 \nu(dx).$$

Finally, letting  $t$  tend to  $+\infty$ , and using (3.5) we get

$$(\bar{\varphi})^2 \geq \int_H |\varphi(x)|^2 \nu(dx) - \|R\| \int_H |D\varphi(x)|^2 \nu(dx),$$

and the conclusion follows.  $\square$

#### 4. – Spectral gap

##### 4.1. – Gaussian case

We are here concerned with the Ornstein-Uhlenbeck process  $X(\cdot, x)$  solution of the following differential stochastic equation

$$(4.1) \quad \begin{cases} dX(t) = AX(t)dt + Q^{1/2}dW(t), & t \geq 0, \\ X(0) = x, \end{cases}$$

under the following assumptions.

HYPOTHESIS 2.

- (i)  $A$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$  on  $H$ .
- (ii)  $Q$  is bounded, symmetric, and nonnegative.
- (iii) For all  $t > 0$  the operator  $e^{tA}Qe^{tA^*}$  is of trace class and its kernel is equal to  $\{0\}$ . Moreover

$$\int_0^{+\infty} \text{Tr}[e^{tA}Qe^{tA^*}]dt < +\infty.$$

If Hypothesis 2 holds the linear operator

$$Q_\infty x = \int_0^{+\infty} e^{tA}Qe^{tA^*}x dt, \quad x \in H,$$

is well defined and it is of trace-class. Moreover problem (4.1) has a unique mild solution given by, see [5]

$$(4.2) \quad X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s).$$

The corresponding transition semigroup  $P_t, t \geq 0$ , is defined by

$$(4.3) \quad P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))] = \int_H \varphi(e^{tA}x + y)\mu_{Q_t}(dy), \quad \varphi \in B_b(H),$$

where

$$Q_t x = \int_0^t e^{sA}Qe^{sA^*}x ds, \quad x \in H.$$

Finally the measure  $\mu_{Q_\infty}$  is invariant, and so the semigroup  $P_t, t \geq 0$ , can be uniquely extended to a strongly continuous semigroup of contractions on  $L^2(H, \mu)$ , that we still denote by  $P_t, t \geq 0$ . Its infinitesimal generator will be denoted by  $\mathcal{L}$ .



THEOREM 4.1. Assume, besides Hypothesis 2 that

$$(4.4) \quad Q^{1/2}(H) \subset Q_\infty^{1/2}(H).$$

Then for any  $\varphi \in W^{1,2}(H, \mu)$  we have

$$(4.5) \quad \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq e^{-\frac{1}{\|Q^{-1/2} Q_\infty^{1/2}\|} t} \int_H |\varphi(x)|^2 \mu(dx),$$

where

$$\bar{\varphi} = \int_H \varphi(y) \mu(dy)$$

PROOF. By the Poincaré inequality (2.5), with  $R = Q_\infty$ , it follows

$$\int_H |\varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq \|Q^{-1/2} Q_\infty^{1/2}\| \int_H |Q^{1/2} D\varphi(x)|^2 \mu(dx).$$

We also recall that, for any  $\varphi \in D(\mathcal{L})$  we have, see [1], [6],

$$\int_H \mathcal{L}\varphi(x) \varphi(x) \mu(dx) = -\frac{1}{2} \int_H |Q^{1/2} D\varphi(x)|^2 \mu(dx).$$

This implies

$$(4.6) \quad \int_H \mathcal{L}\varphi(x) \varphi(x) \mu(dx) \leq \frac{1}{2\|Q^{-1/2} Q_\infty^{-1/2}\|} \int_H |\varphi(x) - \bar{\varphi}|^2 \mu(dx).$$

Let now consider the space

$$Y = \left\{ \varphi \in L^2(H, \mu) : \bar{\varphi} = 0 \right\}.$$

$Y$  is obviously an invariant subspace of  $P_t$ ,  $t \geq 0$ ; denote by  $\mathcal{L}_Y$  the part of  $\mathcal{L}$  in  $Y$ . By (4.6) it follows

$$(4.7) \quad \int_H \mathcal{L}_Y \varphi(x) \varphi(x) \mu(dx) \leq \frac{1}{2\|Q^{-1/2} Q_\infty^{-1/2}\|} \int_H |\varphi(x)|^2 \mu(dx), \quad \varphi \in D(\mathcal{L}_Y).$$

It is easy to check that this inequality yields (4.5).  $\square$

Another condition implying the spectral gap property holds when the semi-group  $P_t$ ,  $t \geq 0$ ; is strong Feller.

HYPOTHESIS 3. For any  $t > 0$  we have

$$e^{tA}(H) \subset Q_t^{1/2}(H).$$

When Hypothesis 3 is fulfilled we set

$$\Gamma(t) = Q_t^{1/2} Q_t^{-1/2} e^{tA}, \quad t > 0.$$

We recall that  $\|\Gamma(t)\|$  is nonincreasing in  $t$  and  $\lim_{t \rightarrow 0} \|\Gamma(t)\| = +\infty$ . Moreover for any  $\varphi \in L^2(H, \mu)$  and any  $t > 0$ , one has  $P_t \varphi \in W^{1,2}(H, \mu)$  and the following estimate holds, see [5],

$$(4.8) \quad \int_H |DP_t \varphi(x)|^2 \mu(dx) \leq \|\Gamma(t)\|^2 \int_H |\varphi(x)|^2 \mu(dx).$$

THEOREM 4.2. Assume, besides Hypotheses 2 and 3, that there exist  $M, \omega > 0$  such that

$$\|Q_\infty^{1/2} e^{tA}\| \leq M e^{-\omega t}, \quad t \geq 0.$$

Then there exists  $M_1 > 0$  such that the following estimate holds

$$(4.9) \quad \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq M_1 e^{-2\omega t} \int_H |\varphi(x)|^2 \mu(dx).$$

PROOF. Replacing in (2.5)  $\varphi$  with  $P_t \varphi$ , and taking into account that  $\overline{P_t \varphi} = \bar{\varphi}$  by the invariance of  $\mu$ , we have

$$\int_H |P_t \varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq \int_H |Q_\infty^{1/2} DP_t \varphi(x)|^2 \mu(dx), \quad \varphi \in W^{1,2}(H, \mu).$$

Since

$$DP_t \varphi(x) = e^{tA*} P_t D\varphi(x),$$

it follows

$$\begin{aligned} \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \mu(dx) &\leq \|Q_\infty^{1/2} e^{tA*}\|^2 \int_H |DP_t \varphi(x)|^2 \mu(dx) \\ &\leq M^2 e^{-2\omega t} \int_H |D\varphi(x)|^2 \mu(dx). \end{aligned}$$

By replacing  $\varphi$  with  $P_1 \varphi$ , and taking into account (4.8), we find

$$\begin{aligned} \int_H |P_{t+1} \varphi(x) - \bar{\varphi}|^2 \mu(dx) &\leq M^2 e^{-2\omega t} \int_H |DP_1 \varphi(x)|^2 \mu(dx) \\ &\leq M^2 e^{-2\omega t} \|\Gamma(1)\|^2 \int_H |\varphi(x)|^2 \mu(dx). \end{aligned}$$

By replacing  $t + 1$  with  $t$  the conclusion follows. □

#### 4.2. – Non Gaussian case

We are here concerned with the solution  $X(\cdot, x)$  of the following differential stochastic equation

$$(4.10) \quad \begin{cases} dX(t) = (AX(t) + F(X))dt + dW(t), & t \geq 0, \\ X(0) = x, \end{cases}$$

under the following assumptions.

HYPOTHESIS 4.

- (i)  $A$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$  on  $H$  and there exists  $\omega > 0$  such that  $\|e^{tA}\| \leq e^{-\omega t}$ ,  $t \geq 0$ .
- (ii) For all  $t > 0$  the operator  $e^{tA}e^{tA^*}$  is of trace class, and  $\int_0^\infty \text{Tr}[e^{tA}e^{tA^*}]dt < +\infty$ .
- (iii)  $F : H \rightarrow H$  is uniformly continuous and bounded together with its Fréchet derivative.

If Hypothesis 4 holds the linear operator

$$Q_\infty x = \int_0^{+\infty} e^{tA}e^{tA^*} x dt, \quad x \in H,$$

is well defined and it is of trace-class. Moreover problem (4.10) has a unique mild solution, see [5]. The corresponding transition semigroup  $P_t, t \geq 0$ , is defined by as before by

$$(4.11) \quad P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(H).$$

We set  $\mu = \mu_{Q_\infty}$ , and denote by  $\mathcal{E}_A(H)$  the vector space generated by all functions of the form

$$\varphi(x) = e^{\langle h, x \rangle}, \quad h \in D(A^*).$$

We denote by  $\mathcal{L}$  the infinitesimal generator of  $P_t, t \geq 0$ .  $\mathcal{L}$  is defined as the closure of the linear operator  $\mathcal{L}_0$ :

$$(4.12) \quad \begin{aligned} \mathcal{L}\varphi(x) = & \frac{1}{2} \text{Tr}[D^2\varphi(x)] \\ & + \langle x, A^* D\varphi(x) \rangle + \langle F(x), D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H), \quad x \in H. \end{aligned}$$

We need an integration by parts formula.

LEMMA 4.3. *Assume that Hypotheses 1 and 4 hold. Let  $\alpha$  be defined by (3.1), and let  $\varphi, \psi \in \mathcal{E}_A(H)$ . Then the following identity holds.*

$$(4.13) \quad \begin{aligned} & \int_H [D_k \varphi(x) \psi(x) + \varphi(x) D_k \psi(x)] \nu(dx) \\ & = \int_H \left( \frac{x_k}{\lambda_k} - D_k \log \alpha(x) \right) \varphi(x) \psi(x) \nu(dx). \end{aligned}$$

PROOF. Denote by  $J$  the left hand side of (4.13). Taking into account a well known result on Gaussian measures, we have

$$\begin{aligned} J &= \int_H [D_k \varphi(x) \psi(x) \alpha(x) + \varphi(x) D_k \psi(x) \alpha(x)] \mu(dx) \\ &= \int_H [-\varphi(x) D_k(\psi(x) \alpha(x)) + \varphi(x) D_k \psi(x) \alpha(x)] \mu(dx) \\ &\quad + \int_H \frac{x_k}{\lambda_k} \alpha(x) \varphi(x) \psi(x) \mu(dx) \\ &= \int_H \left( \frac{x_k}{\lambda_k} - D_k \alpha(x) \right) \varphi(x) \psi(x) \mu(dx). \end{aligned}$$

The conclusion follows. □

PROPOSITION 4.4. *Assume that Hypotheses 1 and 4 hold. Let  $\alpha$  be defined by (3.1) and  $\mathcal{L}$  by (4.12). Then for any  $\varphi, \psi \in \mathcal{E}(H)$  we have*

$$(4.14) \quad \int_H \mathcal{L}\varphi(x) \psi(x) \nu(dx) = \int_H \langle A Q_\infty D\psi(x), D\varphi(x) \rangle \nu(dx) + \int_H \langle A Q_\infty D \log \alpha(x) + F(x), D\psi(x) \rangle \varphi(x) \nu(dx),$$

and

$$(4.15) \quad \int_H \mathcal{L}\varphi(x) \varphi(x) \nu(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \nu(dx) + \int_H \langle A Q_\infty D \log \alpha(x) + F(x), D\varphi(x) \rangle \varphi(x) \nu(dx).$$

Notice that

$$Q_\infty(H) \subset D(A),$$

see [3], so that  $AQ_\infty$  is a well defined bounded operator.

PROOF. We first compute the integral

$$J = \int_H \langle Ax, D\varphi(x) \rangle \psi(x) \nu(dx).$$

We denote by  $\{e_k\}$  a complete orthonormal system in  $H$  consisting of eigenvectors of  $Q_\infty$  and by  $\{\lambda_k\}$  the corresponding sequence of eigenvalues:

$$Q_\infty e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We assume for simplicity that  $\{e_k\} \subset D(A)$ , this extra assumption can be easily removed by approximating  $A$  with its Yosida approximations. We have

$$\langle Ax, D\varphi(x) \rangle = \sum_{h,k=1}^{\infty} a_{h,k} x_k D_h \varphi(x),$$

where  $a_{h,k} = \langle Ae_k, e_h \rangle$ , and  $x_k = \langle x, e_k \rangle$ . We proceed here as in [6]. By integration by parts formula (4.11) we have

$$\begin{aligned} \int_H x_k D_h \varphi(x) \psi(x) \nu(dx) &= \int_H \lambda_k D_h D_k \varphi(x) \psi(x) \nu(dx) \\ &\quad + \int_H \lambda_k D_h \varphi(x) D_k \psi(x) \nu(dx) \\ &\quad + \int_H \lambda_k D_k \log \alpha(x) D_h \varphi(x) \psi(x) \nu(dx). \end{aligned}$$

It follows

$$\begin{aligned} J &= \int_H \text{Tr}[AQ_\infty D^2 \varphi(x)] \psi(x) \nu(dx) + \int_H \langle AQ_\infty D \psi(x), D \varphi(x) \rangle \nu(dx) \\ &\quad + \int_H \langle AQ_\infty D \log \alpha(x), D \varphi(x) \rangle \psi(x) \nu(dx). \end{aligned}$$

Now, taking into account (4.12), a simple computation yields (4.14). Finally (4.15) follows as in [6], recalling the Lyapunov equation

$$AQ + QA^* + Q_\infty = 0. \quad \square$$

**THEOREM 4.5.** *Assume that Hypotheses 1 and 4 hold. Assume in addition that  $\alpha$ , defined by (3.1), can be chosen such that*

$$(4.16) \quad F(x) = -AQ_\infty D \log \alpha(x), \quad x \in H.$$

Then  $\nu$  is an invariant measure for  $P_t$ ,  $t \geq 0$ , and for all  $\varphi \in L^2(H, \mu)$  we have

$$(4.17) \quad \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \nu(dx) \leq e^{-\frac{1}{Q_\infty} t} \int_H |\varphi(x)|^2 \nu(dx),$$

where

$$\bar{\varphi} = \int_H \varphi(y) \nu(dy)$$

**PROOF.** First notice that if (4.16) holds, then setting  $\psi(x) = 1$ ,  $x \in H$ , we have by (4.14)

$$\int_H \mathcal{L}\varphi(x) \nu(dx) = 0, \quad \varphi \in D(\mathcal{L}).$$

This implies that  $\nu$  is invariant for  $P_t$ ,  $t \geq 0$ . Now by (4.15) it follows

$$\int_H \mathcal{L}\varphi(x) \varphi(x) \nu(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \nu(dx), \quad \varphi \in D(\mathcal{L}).$$

Consequently, by (3.6) we have

$$\int_H \mathcal{L}\varphi(x) \varphi(x) \nu(dx) \leq -\frac{1}{2\|Q_\infty\|} \int_H |\varphi(x) - \bar{\varphi}|^2 \nu(dx).$$

Arguing as in the proof of Theorem 4.1, we arrive at (4.17). □

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