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Poincaré inequality for some measures in Hilbert spaces and application to spectral gap for transition semigroups


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1. – Introduction and setting of the problem

Let $H$ be a separable Hilbert space (norm $| \cdot |$, inner product $(\cdot, \cdot)$), and let $\nu$ be a Borel measure on $H$. This paper is devoted to prove, under suitable assumptions on $\nu$, an estimate of this kind (Poincaré inequality):

\[ \int_H |\varphi(x) - \int_H \varphi(y) \nu(dy)|^2 \nu(dx) \leq C \int_H |D\varphi(x)|^2 \nu(dx), \]

where $C$ is a suitable positive constant.

Estimate (1.1) can be used to study the spectral gap for a transition semigroup corresponding to a differential stochastic equation:

\[ \begin{cases} 
  dX(t) = (AX(t) + F(X(t)))dt + Q^{1/2}dW(t), & t \geq 0, \\
  X(0) = x, 
\end{cases} \]

Here $A : D(A) \subset H \rightarrow H$ and $Q : H \rightarrow H$, are linear operators, $F : H \rightarrow H$ is nonlinear, and $W(t), t \geq 0$ is an $H$-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [5].

Assume that problem (1.2) has unique solution $X(t, x)$, then the corresponding transition semigroup $P_t, t \geq 0$, is defined by

\[ P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(H), \]

where $B_b(H)$ is the Banach space of all bounded and Borel functions from $H$ into $\mathbb{R}$. We want to prove, under suitable assumptions, an estimate

\[ \int_H \left| P_t \varphi(x) - \int_H \varphi(y) \nu(dy) \right|^2 \nu(dx) \leq C e^{-\alpha t} \int_H |\varphi(x)|^2 \nu(dx), \]

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for all \( \varphi \in L^2(H, \nu) \), where \( \nu \) is an invariant measure for the semigroup, and \( C, \omega \) are positive constants.

Estimate (1.4) implies that the spectrum \( \sigma(L) \) of the infinitesimal generator \( L \) of \( P_t \) in \( L^2(H, \nu) \) has the following property

\[
\sigma(L) \setminus \{0\} \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda < \omega \}.
\]

This spectral gap property is important in the applications, it has been studied in the literature, mainly when the semigroup \( P_t \) is symmetric, see [8], [9], [5].

The content of the paper is the following. In Section 2 we prove a Poincaré inequality when \( \nu = \mu_R \) is a Gaussian measure of mean 0 and covariance operator \( R \in \mathcal{L}_1^+(H) \), the space of all nonnegative, symmetric, linear operators from \( H \) into \( H \) of trace class. In this case estimate (1.1) is a natural generalization of a well known result when \( H \) is finite dimensional. Then we consider in Section 3 the case when \( \nu \) is absolutely continuous with respect to a Gaussian measure \( \mu_R \). Finally section Section 4 is devoted to the spectral gap property.

2. – Poincaré inequality for Gaussian measures

We are given a Gaussian measure \( \mu_R \), on \( H \) with mean 0 and covariance operator \( R \in \mathcal{L}_1^+(H) \). We denote by \( \{e_k\} \) a complete orthonormal system in \( H \) consisting of eigenvectors of \( R \) and by \( \{\lambda_k\} \) the corresponding sequence of eigenvalues:

\[
\text{Re}_k = \lambda_k e_k, \quad k \in \mathbb{N}.
\]

We shall assume that sequence \( \{\lambda_k\} \) is nonincreasing and that \( \lambda_k > 0 \) for all \( k \in \mathbb{N} \). For any \( k \in \mathbb{N} \) we shall denote by \( D_k \) the derivative in the direction of \( e_k \), and we shall set \( x_k = \langle x, e_k \rangle \) for any \( x \in H \). It is well known that \( D_k \) is a closable operator on \( L^2(H, \mu) \), see e.g. [7]. The Sobolev space \( W^{1,2}(H, \mu_R) \) is the Hilbert space of all \( \varphi \in L^2(H, \mu_R) \cap \text{dom}(D_k), k \in \mathbb{N}, \) such that

\[
\|\varphi\|_{W^{1,2}(H, \mu_R)}^2 := \int_H |\varphi(x)|^2 \mu_R(dx) + \sum_{k=1}^{\infty} \int_H |D_k \varphi(x)|^2 \mu_R(dx) < +\infty.
\]

We denote by \( \mathcal{E}(H) \) the linear space spanned by all exponential functions \( \psi(x) = e^{i_h x}, \ x \in H \). Obviously

\[
\mathcal{E}(H) \subset C^\infty(H) \cap L^2(H, \mu_R).
\]

and \( \mathcal{E}(H) \) is dense in \( L^2(H, \mu_R) \).

We denote by \( T_t, t \geq 0 \), the Ornstein-Uhlenbeck semigroup:

\[
T_t \varphi(x) = \int_H \varphi(e^{-t/2}x + y) \mu_{(1-e^{-t})R}(dy), \ t \geq 0, \ \varphi \in L^2(H, \mu_R).
\]
It is well known that $T_t$, $t \geq 0$, is a strongly continuous semigroup of contractions on $L^2(H, \mu_R)$ having as unique invariant measure $\mu_R$:

$$\int_H T_t \varphi(x) \mu_R(dx) = \int_H \varphi(x) \mu_R(dx), \quad t \geq 0, \quad \varphi \in L^2(H, \mu_R).$$

We denote by $\mathcal{L}$ the infinitesimal generator of $T_t$, $t \geq 0$. $\mathcal{L}$ is defined as the closure of the linear operator $\mathcal{L}_0$:

$$\mathcal{L}_0 \varphi(x) = \frac{1}{2} \text{Tr}[RD^2 \varphi(x)] - \frac{1}{2} (x, D\varphi(x)), \quad \varphi \in \mathcal{E}(H), \quad x \in H.$$ 

We recall also that, for any $\varphi \in D(\mathcal{L})$ we have, see [1], [6],

$$\int_H \mathcal{L} \varphi(x) \varphi(x) \mu_R(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \mu_R(dx).$$

Now we prove the result

**Theorem 2.1.** The following estimate holds

$$\int_H |\varphi(x) - \overline{\varphi}|^2 \mu_R(dx) \leq \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx), \quad \varphi \in W^{1,2}(H, \mu_R),$$

where

$$\overline{\varphi} = \int_H \varphi(x) \mu_R(dx).$$

**Proof.** For any $\varphi \in D(\mathcal{L})$ we have, in view of (2.4)

$$\frac{d}{dt} \int_H |T_t \varphi(x)|^2 \mu(dx) = 2 \int_H \mathcal{L} T_t \varphi(x) T_t \varphi(x) \mu(dx)$$

$$= -\int_H |R^{1/2} D T_t \varphi(x)|^2 \mu(dx).$$

To estimate $|R^{1/2} D T_t \varphi(x)|^2$ note that, in view of (2.1),

$$\langle Q^{1/2} D T_t \varphi(x), h \rangle = e^{-t/2} \int_H \langle D\varphi(e^{-t/2}x + y), h \rangle \mu_{R(1-e^{-t})}(dy),$$

for all $h \in H$. It follows, using Hölder’s inequality

$$|\langle Q^{1/2} D T_t \varphi(x), h \rangle|^2 \leq e^{-t} |h|^2 T_t (|R^{1/2} D\varphi|^2)(x), \quad h \in H.$$ 

Therefore, due to the arbitrariness of $h$,

$$|R^{1/2} D T_t \varphi(x)|^2 \leq e^{-t} T_t (|R^{1/2} D\varphi|^2)(x).$$
By integrating on $H$ with respect to $\mu_R$, and taking into account the invariance of $\mu_R$, we have

$$\int_H |R^{1/2}D_t\varphi(x)|^2\mu_R(dx) \leq e^{-t} \int_H |R^{1/2}D\varphi(x)|^2\mu_R(dx).$$

Now, comparing with (2.7) we find

$$\frac{d}{dt} \int_H |T_t\varphi(x)|^2\mu_R(dx) \geq -e^{-t} \int_H |R^{1/2}D\varphi(x)|^2\mu_R(dx).$$

Integrating in $t$ find

$$\int_H |T_t\varphi(x)|^2\mu_R(dx) \geq \int_H |\varphi(x)|^2\mu_R(dx) - (1 - e^{-t}) \int_H |R^{1/2}D\varphi(x)|^2\mu_R(dx).$$

Finally, letting $t$ tend to $+\infty$, and using the fact that, as easily checked,

$$\lim_{t \to +\infty} P_t\varphi(x) = \bar{\varphi}, \ x \text{ a.e. in } H,$$

we get

$$(\bar{\varphi})^2 \geq \int_H |\varphi(x)|^2\mu_R(dx) - \int_H |R^{1/2}D\varphi(x)|^2\mu_R(dx),$$

that is equivalent to (2.5).

\[\square\]

3. – Poincaré inequality for non Gaussian measures

Here we are given, besides a Gaussian measure $\mu = \mu_R$, with $R \in \mathcal{L}_1^+(H)$ and $\ker R = \{0\}$, a function $U : H \to \mathbb{R}$, such that

HYPOTHESIS 1.

(i) $U$ is convex and of class $C^2$.

(ii) $DU$ is Lipschitz continuous.

We set

$$\alpha(x) = ke^{-2U(x)}, \ x \in H,$$

where $k$ is chosen such that

$$\int_H \alpha(x)\mu(dx) = 1.$$

Finally we consider the Borel probability measure on $H$

$$\nu(dx) = \alpha(x)\mu(dx).$$
We are going to prove a Poincaré estimate for measure \( \nu \). We notice that assumptions on \( \alpha \) could be considerably weakened. It will be enough to assume convexity of \( U \) (that implies dissipativity of \(-DU\)), and some additional properties similar to [5]. But we prefer to make Hypothesis 1 for the sake of simplicity.

It is useful to introduce a differential stochastic equation having \( \nu \) as invariant measure:

\[
\begin{aligned}
\left\{ \begin{array}{l}
dZ = (AZ - DU(Z))dt + dW(t) \\
Z(0) = x \in H,
\end{array} \right.
\end{aligned}
\]  

where \( A \) is the negative self-adjoint operator in \( H \) defined as

\[
A = -\frac{1}{2} R^{-1},
\]

and \( W \) is a cylindrical \( H \)-valued Wiener process in some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Problem (3.2) has a unique solution \( Z(t, x) \), and measure \( \nu \) is invariant, see [5]. The corresponding transition semigroup is defined in \( L^2(H, \nu) \) by

\[
(3.3) \quad N_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in L^2(H, \nu), \quad t \geq 0.
\]

Its infinitesimal generator \( \mathcal{N} \) is defined by, see [4]

\[
(3.4) \quad D(\mathcal{N}) = \left\{ \varphi \in W^{2,2}(H; \nu) \cap W^{1,2}_A(H; \nu) : \int_H (D^2U(x)D\varphi(x), D\varphi(x))_\nu(dx) < +\infty \right\},
\]

where \( W^{1,2}_A(H; \nu) \) is the linear space of all \( \varphi \in W^{1,2}(H, \nu) \) such that \( \langle AD\varphi, D\varphi \rangle \in L^2(H, \nu) \).

Finally in [5] it is proved that \( \nu \) is strongly mixing

\[
(3.5) \quad \lim_{t \to \infty} N_t \varphi(x) = \int_H \varphi(y)_\nu(dy), \quad \varphi \in L^2(H, \nu).
\]

We can now prove

**Theorem 3.1.** The following estimate holds

\[
(3.6) \quad \int_H |\varphi(x) - \bar{\varphi}|^2 \nu(dx) \leq \frac{1}{\|R\|} \int_H |D\varphi(x)|^2 \nu(dx), \quad \varphi \in W^{1,2}(H, \nu),
\]

where

\[
(3.7) \quad \bar{\varphi} = \int_H \varphi(x)_\nu(dx).
\]
PROOF. For any \( \varphi \in D(N) \) we have, see \([4]\),

\[
\frac{d}{dt} \int_H |N_t \varphi(x)|^2 v(dx) = 2 \int_H N N_t \varphi(x) N_t \varphi(x) v(dx) = -\int_H |D N_t \varphi(x)|^2 v(dx).
\] (3.8)

We want now to estimate \( |D N_t \varphi(x)|^2 \). To this purpose we note that \( X(t, x) \) is differentiable with respect to \( x \) and

\[
\|X^*(t, x)\| \leq e^{-\frac{1}{2\|\mathcal{R}\|^t}}, t \geq 0.
\] (3.9)

It follows

\[
D N_t \varphi(x) = \mathbb{E}[X^*_t(t, x) D \varphi(X(t, x))].
\] (3.10)

Now by (3.10) and the Hölder’s estimate, it follows,

\[
|D N_t \varphi(x)|^2 \leq e^{-\frac{1}{2\|\mathcal{R}\|^t}} N_t (|D \varphi|^2)(x).
\] (3.11)

By integrating on \( H \) with respect to \( v \), and taking into account the invariance of \( v \), we have

\[
\int_H |D N_t \varphi(x)|^2 v(dx) \leq e^{-\frac{1}{2\|\mathcal{R}\|^t}} \int_H |D \varphi(x)|^2 v(dx).
\]

By substituting in (3.8) we find

\[
\frac{d}{dt} \int_H |N_t \varphi(x)|^2 v(dx) \geq -e^{-\frac{1}{2\|\mathcal{R}\|^t}} \int_H |D \varphi(x)|^2 v(dx).
\]

Integrating in \( t \) we have

\[
\int_H |N_t \varphi(x)|^2 v(dx) \geq \int_H |\varphi(x)|^2 v(dx) - \|R\|(1 - e^{-\frac{1}{2\|\mathcal{R}\|^t}}) \int_H |D \varphi(x)|^2 v(dx).
\]

Finally, letting \( t \) tend to \(+\infty\), and using (3.5) we get

\[
(\bar{\varphi})^2 \geq \int_H |\varphi(x)|^2 v(dx) - \|R\| \int_H |D \varphi(x)|^2 v(dx),
\]

and the conclusion follows. \( \square \)
4. – Spectral gap

4.1. – Gaussian case

We are here concerned with the Ornstein-Uhlenbeck process \( X(\cdot, x) \) solution of the following differential stochastic equation under the following assumptions.

**HYPOTHESIS 2.**

(i) \( A \) is the infinitesimal generator of a strongly continuous semigroup \( e^{tA} \) on \( H \).
(ii) \( Q \) is bounded, symmetric, and nonnegative.
(iii) For all \( t > 0 \) the operator \( e^{tA}Qe^{tA^*} \) is of trace class and its kernel is equal to \([0]\). Moreover

\[
\int_0^{+\infty} \text{Tr}[e^{tA}Qe^{tA^*}]dt < +\infty.
\]

If Hypothesis 2 holds the linear operator

\[
Q_\infty x = \int_0^{+\infty} e^{tA}Qe^{tA^*}x\,dt, \quad x \in H,
\]

is well defined and it is of trace-class. Moreover problem (4.1) has a unique mild solution given by, see [5]

\[
X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s).
\]

The corresponding transition semigroup \( P_t, t \geq 0 \), is defined by

\[
P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))] = \int_H \varphi(e^{tA}x + y)\mu_{Q_t}(dy), \quad \varphi \in B_b(H),
\]

where

\[
Q_t x = \int_0^t e^{sA}Qe^{sA^*}x\,ds, \quad x \in H.
\]

Finally the measure \( \mu_{Q_\infty} \) is invariant, and so the semigroup \( P_t, t \geq 0 \), can be uniquely extended to a strongly continuous semigroup of contractions on \( L^2(H, \mu) \), that we still denote by \( P_t, t \geq 0 \). Its infinitesimal generator will be denoted by \( \mathcal{L} \).
THEOREM 4.1. Assume, besides Hypothesis 2 that

\[(4.4) \quad Q^{1/2}(H) \subset Q^{1/2}_\infty(H).\]

Then for any \(\varphi \in W^{1,2}(H, \mu)\) we have

\[(4.5) \quad \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq e^{-\frac{1}{\|Q^{-1/2}Q^{1/2}_\infty\|^t}} \int_H |\varphi(x)|^2 \mu(dx),\]

where

\[\bar{\varphi} = \int_H \varphi(y)\mu(dy)\]

PROOF. By the Poincaré inequality (2.5), with \(R = Q_\infty\), it follows

\[\int_H |\varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq \|Q^{-1/2}Q^{1/2}_\infty\| \int_H |Q^{1/2}D\varphi(x)|^2 \mu(dx).\]

We also recall that, for any \(\varphi \in D(\mathcal{L})\) we have, see [1], [6],

\[\int_H \mathcal{L}\varphi(x)\varphi(x)\mu(dx) = -\frac{1}{2} \int_H |Q^{1/2}D\varphi(x)|^2 \mu(dx).\]

This implies

\[(4.6) \quad \int_H \mathcal{L}\varphi(x)\varphi(x)\mu(dx) \leq \frac{1}{2\|Q^{-1/2}Q^{1/2}_\infty\|^t} \int_H |\varphi(x) - \bar{\varphi}|^2 \mu(dx).\]

Let now consider the space

\[Y = \{\varphi \in L^2(H, \mu) : \bar{\varphi} = 0\}.

\(Y\) is obviously an invariant subspace of \(P_t, t \geq 0\); denote by \(\mathcal{L}_Y\) the part of \(\mathcal{L}\) in \(Y\). By (4.6) it follows

\[(4.7) \quad \int_H \mathcal{L}_Y\varphi(x)\varphi(x)\mu(dx) \leq \frac{1}{2\|Q^{-1/2}Q^{1/2}_\infty\|^t} \int_H |\varphi(x)|^2 \mu(dx), \varphi \in D(\mathcal{L}_Y).\]

It is easy to check that this inequality yields (4.5). \(\square\)
Another condition implying the spectral gap property holds when the semigroup \( P_t, \ t \geq 0; \) is strong Feller.

**HYPOTHESIS 3.** For any \( t > 0 \) we have

\[
e^{tA}(H) \subset Q_i^{1/2}(H).
\]

When Hypothesis 3 is fulfilled we set

\[
\Gamma(t) = Q_i^{1/2} Q_i^{-1/2} e^{tA}, \ t > 0.
\]

We recall that \( \| \Gamma(t) \| \) is nonincreasing in \( t \) and \( \lim_{t \to 0} \| \Gamma(t) \| = +\infty \). Moreover for any \( \varphi \in L^2(H, \mu) \) and any \( t > 0 \), one has \( P_t \varphi \in W^{1,2}(H, \mu) \) and the following estimate holds, see \([5]\),

\[
(4.8) \quad \int_H |DP_t \varphi(x)|^2 \mu(dx) \leq \| \Gamma(t) \|^2 \int_H |\varphi(x)|^2 \mu(dx).
\]

**THEOREM 4.2.** Assume, besides Hypotheses 2 and 3, that there exist \( M, \omega > 0 \) such that

\[
\| Q_i^{1/2} e^{tA} \| \leq M e^{-\omega t}, \ t \geq 0.
\]

The there exists \( M_1 > 0 \) such that the following estimate holds

\[
(4.9) \quad \int_H |P_t \varphi(x) - \varphi|^2 \mu(dx) \leq M_1 e^{-2\omega t} \int_H |\varphi(x)|^2 \mu(dx).
\]

**PROOF.** Replacing in (2.5) \( \varphi \) with \( P_t \varphi \), and taking into account that \( \overline{P_t \varphi} = \overline{\varphi} \) by the invariance of \( \mu \), we have

\[
\int_H |P_t \varphi(x) - \varphi|^2 \mu(dx) \leq \int_H |Q_i^{1/2} DP_t \varphi(x)|^2 \mu(dx), \ \varphi \in W^{1,2}(H, \mu).
\]

Since

\[
DP_t \varphi(x) = e^{tA^*} P_t D \varphi(x),
\]

it follows

\[
\int_H |P_t \varphi(x) - \varphi|^2 \mu(dx) \leq \| Q_i^{1/2} e^{tA^*} \|^2 \int_H |DP_t \varphi(x)|^2 \mu(dx)
\]

\[
\leq M^2 e^{-2\omega t} \int_H |D \varphi(x)|^2 \mu(dx).
\]

By replacing \( \varphi \) with \( P_t \varphi \), and taking into account (4.8), we find

\[
\int_H |P_{t+1} \varphi(x) - \varphi|^2 \mu(dx) \leq M^2 e^{-2\omega t} \int_H |DP_t \varphi(x)|^2 \mu(dx)
\]

\[
\leq M^2 e^{-2\omega t} \| \Gamma(1) \|^2 \int_H |\varphi(x)|^2 \mu(dx).
\]

By replacing \( t + 1 \) with \( t \) the conclusion follows. \( \square \)
4.2. – Non Gaussian case

We are here concerned with the solution $X(\cdot, x)$ of the following differential stochastic equation

\begin{equation}
\begin{cases}
    dX(t) = (AX(t) + F(X))dt + dW(t), \quad t \geq 0, \\
    X(0) = x,
\end{cases}
\end{equation}

under the following assumptions.

**HYPOTHESIS 4.**

(i) $A$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$ on $H$ and there exists $\omega > 0$ such that $\|e^{tA}\| \leq e^{-\omega t}$, $t \geq 0$.

(ii) For all $t > 0$ the operator $e^{tA}e^{tA^*}$ is of trace class, and $\int_0^\infty \text{Tr}[e^{tA}e^{tA^*}]dt < +\infty$.

(iii) $F : H \to H$ is uniformly continuous and bounded together with its Fréchet derivative.

If Hypothesis 4 holds the linear operator

$$Q_\infty x = \int_0^{+\infty} e^{tA}e^{tA^*} x \, dt, \quad x \in H,$$

is well defined and it is of trace-class. Moreover problem (4.10) has a unique mild solution, see [5]. The corresponding transition semigroup $P_t$, $t \geq 0$, is defined as before by

\begin{equation}
P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(H).
\end{equation}

We set $\mu = \mu_{Q_\infty}$, and denote by $\mathcal{E}_A(H)$ the vector space generated by all functions of the form $\varphi(x) = e^{(h,x)}$, $h \in D(A^*)$.

We denote by $\mathcal{L}$ the infinitesimal generator of $P_t$, $t \geq 0$. $\mathcal{L}$ is defined as the closure of the linear operator $\mathcal{L}_0$:

\begin{equation}
\mathcal{L}\varphi(x) = \frac{1}{2} \text{Tr}[D^2\varphi(x)] \\
+ \langle x, A^*D\varphi(x) \rangle + \langle F(x), D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H), \quad x \in H.
\end{equation}

We need an integration by parts formula.

**LEMMA 4.3.** Assume that Hypotheses 1 and 4 hold. Let $\alpha$ be defined by (3.1), and let $\varphi, \psi \in \mathcal{E}_A(H)$. Then the following identity holds.

\begin{equation}
\int_H [D_k \varphi(x)\psi(x) + \varphi(x)D_k \psi(x)] \nu(dx) = \int_H \left( \frac{x_k}{\lambda_k} - D_k \log \alpha(x) \right) \varphi(x)\psi(x) \nu(dx).
\end{equation}
Proof. Denote by $J$ the left hand side of (4.13). Taking into account a well known result on Gaussian measures, we have

$$J = \int_{\mathcal{H}} \left[ D_k \varphi(x) \psi(x) \alpha(x) + \varphi(x) D_k \psi(x) \alpha(x) \right] \mu(dx)$$

$$= \int_{\mathcal{H}} \left[ -\varphi(x) D_k (\psi(x) \alpha(x)) \psi(x) D_k \psi(x) \alpha(x) \right] \mu(dx)$$

$$+ \int_{\mathcal{H}} \frac{x_k}{\lambda_k} \alpha(x) \varphi(x) \psi(x) \mu(dx)$$

$$= \int_{\mathcal{H}} \left( \frac{x_k}{\lambda_k} - D_k \alpha(x) \right) \varphi(x) \psi(x) \mu(dx).$$

The conclusion follows. \(\square\)

Proposition 4.4. Assume that Hypotheses 1 and 4 hold. Let $\alpha$ be defined by (3.1) and $L$ by (4.12). Then for any $\varphi, \psi \in \mathcal{E}(\mathcal{H})$ we have

$$\int_{\mathcal{H}} L \varphi(x) \psi(x) \nu(dx) = \int_{\mathcal{H}} \langle A Q_\infty D \psi(x), D \varphi(x) \rangle \nu(dx)$$

$$+ \int_{\mathcal{H}} \langle A Q_\infty D \log \alpha(x) + F(x), D \psi(x) \rangle \varphi(x) \nu(dx),$$

and

$$
\int_{\mathcal{H}} L \varphi(x) \varphi(x) \nu(dx) = -\frac{1}{2} \int_{\mathcal{H}} |D \varphi(x)|^2 \nu(dx)$$

$$+ \int_{\mathcal{H}} \langle A Q_\infty D \log \alpha(x) + F(x), D \varphi(x) \rangle \varphi(x) \nu(dx).$$

Notice that

$$Q_\infty(\mathcal{H}) \subset D(A),$$

see [3], so that $AQ_\infty$ is a well defined bounded operator.

Proof. We first compute the integral

$$J = \int_{\mathcal{H}} \langle Ax, D \varphi(x) \rangle \psi(x) \nu(dx).$$

We denote by $\{e_k\}$ a complete orthonormal system in $\mathcal{H}$ consisting of eigenvectors of $Q_\infty$ and by $\{\lambda_k\}$ the corresponding sequence of eigenvalues:

$$Q_\infty e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We assume for simplicity that $\{e_k\} \subset D(A)$, this extra assumption can be easily removed by approximating $A$ with its Yosida approximations. We have

$$\langle Ax, D \varphi(x) \rangle = \sum_{h,k=1}^{\infty} a_{h,k} x_k D_h \varphi(x),$$
where $a_{h,k} = \langle A e_k, e_h \rangle$, and $x_k = \langle x, e_k \rangle$. We proceed here as in [6]. By integration by parts formula (4.11) we have

$$
\int_H x_k D_h \varphi(x) \psi(x) \nu(dx) = \int_H \lambda_k D_h D_k \varphi(x) \psi(x) \nu(dx) + \int_H \lambda_k D_h \varphi(x) D_k \psi(x) \nu(dx) + \int_H \lambda_k D_k \log \alpha(x) D_h \varphi(x) \psi(x) \nu(dx).
$$

It follows

$$
J = \int_H \text{Tr}[AQ_\infty D^2 \varphi(x)] \psi(x) \nu(dx) + \int_H \langle AQ_\infty \partial x, D\varphi(x) \rangle \nu(dx)
$$

Now, taking into account (4.12), a simple computation yields (4.14). Finally (4.15) follows as in [6], recalling the Lyapunov equation

$$
AQ + QA^* + Q_\infty = 0.
$$

**THEOREM 4.5.** Assume that Hypotheses 1 and 4 hold. Assume in addition that $\alpha$, defined by (3.1), can be chosen such that

$$
F(x) = -AQ_\infty D \log \alpha(x), \ x \in H.
$$

Then $\nu$ is an invariant measure for $P_t, t \geq 0$, and for all $\varphi \in L^2(H, \mu)$ we have

$$
\int_H |P_t \varphi(x) - \bar{\varphi}|^2 \nu(dx) \leq e^{-\frac{1}{\alpha_\infty} t} \int_H |\varphi(x)|^2 \nu(dx),
$$

where

$$
\bar{\varphi} = \int_H \varphi(y) \nu(dy).
$$

**PROOF.** First notice that if (4.16) holds, then setting $\psi(x) = 1, \ x \in H$, we have by (4.14)

$$
\int_H \mathcal{L}\varphi(x) \nu(dx) = 0, \ \varphi \in D(\mathcal{L}).
$$

This implies that $\nu$ is invariant for $P_t, t \geq 0$. Now by (4.15) it follows

$$
\int_H \mathcal{L}\varphi(x) \varphi(x) \nu(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \nu(dx), \ \varphi \in D(\mathcal{L}).
$$

Consequently, by (3.6) we have

$$
\int_H \mathcal{L}\varphi(x) \varphi(x) \nu(dx) \leq -\frac{1}{2\|Q_\infty\|} \int_H |\varphi(x) - \bar{\varphi}| \nu(dx).
$$

Arguing as in the proof of Theorem 4.1, we arrive at (4.17).
REFERENCES


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