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Further Qualitative Properties for Elliptic Equations in Unbounded Domains

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Dedicated to the Memory of Ennio De Giorgi

1. - Introduction and main results

This article is one in a series by the authors to study some qualitative properties of positive solutions of elliptic second order boundary value problems of the type

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \qquad u > 0 \quad \text{in} \quad \Omega \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

in various kinds of *unbounded* domains Ω of $\mathbb{R}^n$. Typically, we are interested in features like monotonicity in some directions and symmetry. In some cases, the positive solutions we consider are supposed to be bounded while in other cases boundedness is not assumed. The function f appearing in (1.1) will always be assumed to be (globally) Lipschitz continuous: $\mathbb{R}^+ \to \mathbb{R}$.

The present paper is devoted to the investigation of three main configurations. We consider a half space $\Omega = \{x = (x_1, \dots, x_n), x_n > 0\}$, infinite cylindrical or slab-like domains $\Omega = \mathbb{R}^{n-1} \times (0, h)$ and also the case when Ω is the whole plane. In the case of the half space, we derive some monotonicity and symmetry results establishing that a bounded solution of (1.1)-(1.2) actually only depends on one variable. This is related to a conjecture of De Giorgi [12], stated later in the introduction, on the classification of solutions to some problems of the type (1.1)-(1.2) in the whole space.

In [4] we considered domains Ω bounded by a Lipschitz graph, i.e.

$$\Omega = \{x = (x_1, \dots, x_n) \text{ in } \mathbb{R}^n \ ; \ x_n > \varphi(x_1, \dots, x_{n-1})\}$$

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with φ a Lipschitz function on $\mathbb{R}^n$. Under certain conditions on f, we proved that if u is bounded and satisfies (1.1), (1.2), then

$$\frac{\partial u}{\partial x_n} > 0 \quad \text{in} \quad \Omega .$$

This work was motivated by the study of regularity in some free boundary problems.

The case when the graph satisfies

$$\lim_{|x'|\to\infty} \varphi(x') = \infty$$

had been treated earlier by Esteban and Lions [13] in the case that φ is smooth with the aid of the method of moving planes. Indeed, they had observed that in this situation one can just apply the usual moving plane method as in the bounded domain case without changes since one makes use of the maximum principle in bounded domains. Use of that method as in [2] allows one in fact to treat nonsmooth φ.

The "sliding method" is used in [4], instead of the moving plane method.

In our paper [5] we take up another class of unbounded domains: infinite cylinders, or more generally, product domains of the form

$$\Omega = \mathbb{R}^{n-j} \times \omega, \qquad \text{where } \omega \text{ is a bounded smooth domain in } \mathbb{R}^j .$$

We denote the variables in Ω by (x, y), $x \in \mathbb{R}^{n-j}$, $y \in \omega \subset \mathbb{R}^j$. It is not assumed that u is bounded. Indeed, we establish that solutions u of (1.1) have at most exponential growth, that is:

$$u(x, y) \leq Ce^{\mu|x|} \qquad \forall x \in \mathbb{R}^{n-j}, \quad \forall y \in \omega . \tag{1.3}$$

Next, in [5] we prove a symmetry result: Assuming either that $j \geq 2$ or that $j = 1$ with $f(0) \geq 0$, we show that if ω is convex in some direction, say y_1, and symmetric with respect to the hyperplane $y_1 = 0$, then any solution u of (1.1) is symmetric about that hyperplane and decreases in y_1 for $y_1 > 0$. In [5] we consider also more general operators.

The case $f(0) < 0$ and $j = 1$, that is, when ω is an interval, say $\omega = (0, h)$, was left open. It turns out to be rather intricate. In [5], we announced that for that case we can prove symmetry, at least for dimension 2, i.e. for a strip. Originally, we had devised another proof – in the spirit of our paper [5] – which was rather involved. This proof was to appear in a paper entitled "Inequalities for second order elliptic equations with applications to unbounded domains, II: symmetry in infinite strips". However, after [5] appeared we found a much simpler argument involving some new uses of the moving plane method in dimension 2. We present this proof here, so this paper replaces II announced above.

In most of what follows, we consider the case $j = 1$. In this case, the proof of monotonicity and symmetry in [5] yields the following statement for $j = 1$:

THEOREM 1.1. *In a slab* $\Omega = \mathbb{R}^{n-1} \times (0, h)$, *in* $\mathbb{R}^n$, $n \geq 2$, *let* u *be a solution of* (1.1) *satisfying*

$$u(x, 0) = 0, \qquad \forall x \in \mathbb{R}^{n-1}. \tag{1.4}$$

Assume that f *is Lipschitz and that*

$$f(0) \geq 0. \tag{1.5}$$

Then,

$$\frac{\partial u}{\partial y}(x, y) > 0, \qquad \forall x \in \mathbb{R}^{n-1}, \quad \forall y \in (0, h/2). \tag{1.6}$$

Furthermore, for every λ *in* $(0, h/2)$,

$$u(x, y) < u(x, 2\lambda - y), \qquad \forall x \in \mathbb{R}^{n-1}, \forall y \in (0, \lambda). \tag{1.7}$$

It is not assumed here that $u = 0$ on the upper boundary $\{y = h\}$ of Ω. If u does vanish there, then one has, as in [5], the immediate consequence:

COROLLARY 1.2. *Under the assumptions of Theorem* 1.1 *if* (1.2) *holds then* u *is also symmetric in* y *about* $\{y = h/2\}$.

Under the additional hypothesis that u is bounded, Theorem 1.1 was first proved by E. N. Dancer [10].

In this paper, in case $n = 2$, we show that the condition (1.5) in Theorem 1.1 may be omitted. This is one of our main results here; it is proved in Section 2 (see Theorem 2.2).

THEOREM 1.1′. *If* $n = 2$, *then Theorem* 1.1 *holds even if the condition* (1.5), $f(0) \geq 0$, *is dropped.*

Of course, in view of this result the Corollary 1.2 then also holds without the assumption $f(0) \geq 0$.

The problem for higher dimensional slabs, $n > 2$, and $f(0) < 0$, is still open.

Another type of unbounded domain is that of half spaces, that is,

$$\Omega = \{x \in \mathbb{R}^n, x_n > 0\} =: \mathbb{R}^n_+ .$$

In this context, we are interested in two properties. The first one, *monotonicity* refers to the property that $\frac{\partial u}{\partial x_n} > 0$, and the second one, *symmetry*, to the property that $u = u(x_n)$ is a function of x_n alone – it does not depend on the variables $x' = (x_1, \ldots, x_{n-1})$. Thus, symmetry results here can be thought of as extensions of the Gidas, Ni and Nirenberg [15] symmetry result for spheres, i.e. equation (1.1) with Ω being a ball) when the radius of the sphere increases to infinity while a point on the boundary is being kept fixed.

As part of Theorems 1.1 and 1.1′ there is a monotonicity result in a half space which we now state explicitely.

Corollary 1.3. *In the half space $\Omega = \mathbb{R}^n_+$, suppose that u is a solution of* (1.1)-(1.2). *Then, when $n = 2$ or when $n \geq 3$ and $f(0) \geq 0$, the function u satisfies*

$$\frac{\partial u}{\partial x_n} > 0 \quad in \quad \Omega\,.$$

Whether this property holds in general in dimension $n \geq 3$ in case $f(0) < 0$, is an open problem.

Let us now turn to symmetry in a half space. In [3] we proved the following.

Theorem 1.4. *Suppose that u is a bounded solution of* (1.1)-(1.2) *in a half space with*

$$M = \sup_{\Omega} u\,.$$

Then, if $f(M) \leq 0$, the function u is symmetric, i.e. $u = u(x_n)$ and it is also monotonic, that is $u_{x_n} > 0$ in Ω. Furthermore, $f(M) = 0$.

Actually, in [3], we consider more general equations of the type

$$\Delta u + g(x_n, u) = 0, \quad u > 0 \text{ in } \Omega = \mathbb{R}^n_+\,. \tag{1.8}$$

Assuming that g is Lipschitz, that $t \to g(t,u)$ is nondecreasing in t, that $f(u) := \lim_{t\nearrow\infty} g(t,u)$ exists and is Lipschitz continuous in u and, lastly, that $g(t,M) \leq 0 \ \forall t$, we prove symmetry – and monotonicity – of solutions u of (1.8) and (1.2). Other related results concerning nonlinear Liouville type results in half spaces are also given in [3]. Tehrani [21] has treated a more general form of (1.8):

$$\Delta u + g(x_n, u, |\nabla u|) = 0\,.$$

Remark. The condition that u is bounded is important. Indeed, in the half plane $\{(x_1,x_2) \in \mathbb{R}^2;\, x_2 > 0\}$ the function $u(x_1,x_2) = x_2 e^{x_1}$ satisfies $\Delta u - u = 0$. So, here $M = \infty$ (and $f(\infty) = -\infty$). Monotonicity – as we have seen – still holds for unbounded solutions but symmetry does not.

Some symmetry results in half space problems had been obtained earlier by S. Angenent [1] (see also for related results the paper by Ph. Clément and G. Sweers [9]). Angenent's motivation stemmed from a uniqueness result for some singular pertubation problems. He considers the particular class of functions f satisfying

$$\begin{cases} f > 0 \text{ on } (0,\mu), \text{ and } f(s) \geq \delta s \text{ for } s \in [0, s_0], \\ \qquad\qquad\qquad\qquad \text{for some } \delta > 0 \text{ and } 0 < s_0 < \mu, \\ f(\mu) = 0, \quad f'(\mu) < 0 \end{cases} \tag{1.9}$$

with solutions such that

$$\sup u \leq \mu\,. \tag{1.10}$$

In [4] we assumed essentially, the same conditions, but in a much more general geometric setting. There, we considered a domain bounded by a Lipschitz graph: $\Omega = \{x = (x_1, \dots, x_n)\,;\, x_n > \varphi(x_1, \dots, x_{n-1})\}$ where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function and we showed that a bounded solution necessarily satisfies $\partial u/\partial x_n > 0$ in Ω. Actually, under these restrictive conditions above on f, in the particular case of a half space, symmetry follows easily from this result by using monotonicity in a cone of directions. Note however that this result does not follow from Theorem 1.4 since, a priori, it may be that $f(M) > 0$. In the end, though, one finds that $f(M) = 0$.

In the present paper we derive still further symmetry results in the half space in low dimensions.

THEOREM 1.5. *In the half space $\Omega = \mathbb{R}^n_+$ with $n = 2$ or 3, let u be a bounded solution of* (1.1)-(1.2). *If $n = 2$, u is symmetric. If $n = 3$ the same conclusion holds, if one assumes in addition that $f(0) \geq 0$ and that f is C^1.*

The assumption here is different from that in Theorem 1.4. but in the proof we show that the conditions of Theorem 1.4 hold.

It is not known whether the restriction on the dimension or the assumption on $f(0)$ can be lifted. We make the following.

CONJECTURE. If u is a solution of (1.1), (1.2) in $\mathbb{R}^n_+$ with

$$M = \sup u < \infty$$

then, necessarily, $f(M) = 0$ (and so the result of [3] applies) and consequently u is symmetric.

We have tried to test this conjecture by looking at a simple model problem, $f(u) = u - 1$,

$$(1.11)\qquad \begin{aligned} 0 < u < \sup u = M < \infty, \qquad \Delta u + u - 1 &= 0 \quad \text{in } \Omega = \mathbb{R}^n_+ \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

For this problem it is not difficult to verify that if there were a solution, then $M > 2$ – and so our conjecture that $f(M) = 0$ – in general, would be wrong. This leads us to the following.

CONJECTURE. There is no solution of (1.11).

We should remark that there is a *nonnegative* solution:

$$u = 1 - \cos x_n\,.$$

We have been able to prove the last conjecture only for $n = 2$ and 3: It follows from

PROPOSITION 1.6. *Set $\Sigma = \{x \in \mathbb{R}^n; 0 < x_n < 2\pi\}$, $n = 2$ or 3. Suppose $u \in C^2(\overline{\Sigma})$, $0 \leq u \leq M$; and u satisfies*

$$\begin{aligned} \Delta u + u - 1 &= 0 \ \textit{in}\ \overline{\Sigma} \\ u &= 0 \ \textit{on}\ \{x_n = 0\}\,. \end{aligned}$$

Then $u(x', 2\pi) \equiv 0\ \forall x' \in \mathbb{R}^{n-1}$.

Thus there is no positive bounded solution of (1.11) in case $n = 2$ or 3. In these dimensions, the proof is very simple; it is given in Section 5. However, we have not been able to rule out such solutions in higher dimensions.

The proof of Theorem 1.5 brings together three main elements. The first one is the monotonicity property of Corollary 1.3. Then, we use Theorem 1.4; the goal of the proof is to establish that if u is a bounded positive solution of (1.1)-(1.2) in $\mathbb{R}^n_+$, with $M = \sup_{\mathbb{R}^n_+} u$, necessarily $f(M) = 0$. But there is another ingredient here which is somewhat surprising. It is a result which concerns linear Schrödinger operators.

In $\mathbb{R}^m$, let $q(x)$ be a potential such that $q \in L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$. We assume that there exists a solution $\psi \in W^{2,p}_{\mathrm{loc}}$, for some $p > m$, of

$$\Delta\psi + q\psi = 0 \text{ in } \mathbb{R}^m . \tag{1.12}$$

The result is the following.

THEOREM 1.7. *Suppose that the solution ψ of* (1.12) *changes sign in $\mathbb{R}^m$ and that*

$$\psi(x) = O\left(|x|^{1-\frac{m}{2}}\right) \textit{ as } |x| \to \infty . \tag{1.13}$$

In particular, when $m = 1, 2$, it suffices to assume that ψ is bounded. Then the operator $L = -\Delta - q$ has negative spectrum. This means that there is a function $\zeta \in C_0^\infty(\mathbb{R}^m)$ such that

$$\int_{\mathbb{R}^m} |\nabla\zeta|^2 - q\zeta^2 < 0 . \tag{1.14}$$

We can only make use of this theorem in dimensions 1 and 2 – condition (1.13) is very restrictive in higher dimensions.

QUESTION. Does Theorem 1.7 hold for $m > 2$ if condition (1.13) is replaced by the condition

$$\psi \in L^\infty(\mathbb{R}^m) , \tag{1.13$'$}$$

assuming also $q \in L^\infty, q$ smooth?

We originally raised this question for all dimensions m, but very recently, Ghoussoub and Gui [14] have constructed a counter–example to this statement in dimension $m \geq 7$. Therefore, the question that really remains open is to know whether this result holds in dimensions $m = 3, 4, 5$ or 6.

REMARK. If the answer were yes, our assertion in Theorem 1.5, for $n = 3$, would also hold for $4 \leq n \leq 7$. Indeed, the proof of Theorem 1.5 in dimension n makes use of Theorem 1.7 in dimension $m = n - 1$.

Theorem 1.7 is derived from a result which turns out to be very useful and which is a variant of a familiar result in bounded domains (see e.g. [6]). Here is the result in all of $\mathbb{R}^m$.

THEOREM 1.8. *Let φ be a positive function in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^m)$, for some $p > m$, satisfying*

$$(\Delta + q)\varphi \leq 0 \tag{1.15}$$

where $q \in L^\infty_{\mathrm{loc}}$. Suppose $\psi \in W^{2,p}_{\mathrm{loc}}$, $\psi \not\equiv 0$ satisfies

$$\psi(\Delta + q)\psi \geq 0 \tag{1.16}$$

and also (1.13). *Then $\psi = C\varphi$ for some constant C and equality holds in* (1.15).

The following result is essentially contained in the preprint [14] by Ghoussoub and Gui. Since the proof is short, we include it in Section 7 as a corollary of Theorem 1.8.

THEOREM 1.9. *In the plane $\mathbb{R}^2$, consider a bounded solution $u \in L^\infty(\mathbb{R}^2)$ of the equation*

$$\Delta u + f(u) = 0 \quad \textit{in all of } \mathbb{R}^2 \tag{1.17}$$

where f is an arbitrary continuously differentiable function. Suppose in addition that u is monotonic in some direction, say

$$\frac{\partial u}{\partial x_1} \geq 0 \quad \textit{in } \mathbb{R}^2 . \tag{1.18}$$

Then, u is a function of one variable only, that is, there exist a and b such that

$$u = u(ax_1 + bx_2) . \tag{1.19}$$

This result gives a complete answer in dimension two – in a somewhat generalized form – to a conjecture of De Giorgi [12]. We recall that this conjecture states that if u is a solution of the model equation

$$\Delta u + u - u^3 = 0 \quad \text{in all of } \mathbb{R}^n \tag{1.20}$$

such that $|u(x)| \leq 1$ and $\frac{\partial u}{\partial x_1} > 0$ in $\mathbb{R}^n$ and, in addition, satisfies $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ then, u is a function of one variable only. That is, there exists $\alpha \in \mathbb{R}^{n-1}$ such that $u = u(\ x_1 + < \alpha, x' >)$ where we write $x = (x_1, x')$ with $x' = (x_2, \ldots, x_n)$ for all $x \in \mathbb{R}^n$. For other results related to the conjecture, see the references [8], [17], [18]. In [14] Ghoussoub and Gui, also proved a related result in dimension three. They assume that in the previous setting, $u \to \pm 1$ as $x_1 \to \pm\infty$ *uniformly* with respect to x_2, x_3. Then, u is a function of x_1 alone.

The paper is organized as follows:

1. Introduction and main results.
2. Monotonicity and symmetry in strips in case f is C^1.
3. Symmetry in half planes and half spaces.
4. On the Schrödinger operator; proofs of Theorems 1.7 and 1.8.
5. The equation $\Delta u + u - 1 = 0$ in $\mathbb{R}^n_+$ for $n = 2$ and 3.
6. The proof of Theorem 1.5 for $n = 2$ when f is merely Lipschitz.
7. Proof of Theorem 1.9.

The new monotonicity and symmetry results are Theorems 1.1′ and 1.5. The result in Theorem 1.9 is proved in the last section.

In $\mathbb{R}^n = \mathbb{R}^{n-j} \times \mathbb{R}^j$ we usually set $X = (x, y)$ and use coordinates $x = (x_1, \dots, x_{n-j})$ in $\mathbb{R}^{n-j}$, and $y = (y_1, \dots, y_j)$ in $\mathbb{R}^j$.

2. – Monotonicity and symmetry in strips

This section is devoted to the proof of Theorem 1.1′. We will make use of a version of the maximum principle, of Phragmèn-Lindelöf type, in an infinite cylinder, or strip, having small cross section.

In $\mathbb{R}^n$, let

$$\Sigma = \mathbb{R}^{n-j} \times \omega$$

be an infinite cylinder, where ω is a smooth bounded domain in $\mathbb{R}^j$ with $j \geq 1$.

THEOREM 2.1 ([5]). *In Σ, consider an operator*

$$L = \Delta + q(X)$$

with $\|q\|_{L^\infty(\Sigma)} \leq b$; let $\mathcal{C}_\mu$ be the class of functions $z \in C^2(\Sigma) \cap C\left(\overline{\Sigma}\right)$ satisfying

$$z(x, y) \leq Ce^{\mu|x|} \qquad in\ \Sigma\,. \tag{2.1}$$

Here, C, μ are positive constants. The maximum principle holds for functions in this class, provided meas $(\omega) := |\omega| < \delta$. That is, for any $z \in \mathcal{C}_\mu$,

$$\left.\begin{array}{ll} Lz \geq 0 & in\ \Sigma \\ z \leq 0 & on\ \partial\Sigma \end{array}\right\} \Longrightarrow z \leq 0\ in\ \Sigma\,.$$

Here δ is a constant depending only on n, b and μ.

As remarked on page 479 in [5], our proof works for a wider class of uniformly elliptic operators. But, recently, J. Busca [7] proved the result for the fully general case

$$L = a_{ij}(X)\partial_i\partial_j + b_i(X)\partial_i + c(X)\,.$$

The constant δ then depends on the ellipticity constant c_0, such that $c_0|\xi|^2 \leq a_{ij}(X)\xi_i\xi_j$, and on upper bounds for c, $\|b_i\|_{L^\infty}$ and $\|a_{ij}\|_{L^\infty}$.

We take up first the proof of Theorem 1.1′. Here

$$\Sigma = \{(x, y)\ ;\ x \in \mathbb{R}, 0 < y < h\}$$

and u satisfies

$$u > 0, \quad \Delta u + f(u) = 0 \text{ in } \Sigma \tag{2.2}$$

$$u = 0 \text{ on } \{y = 0\}\,. \tag{2.3}$$

We recall the statement to be proved:

THEOREM 2.2. *Suppose u satisfies* (2.2), (2.3) *and that f is Lipschitz continuous on $\mathbb{R}^+$. Then*

$$\frac{\partial u}{\partial y} > 0 \quad in \quad \Sigma_{h/2} = \left\{(x, y); 0 < y < \frac{h}{2}\right\} \tag{2.4}$$

and, for any positive $\ell < h/2$,

$$u(x, y) < u(x, 2\ell - y) \qquad \forall x \ and \ \forall y,\ 0 < y < \ell . \tag{2.5}$$

The proof relies on a variant of the moving plane method which was used in [5] to derive Theorem 1.1.

PROOF. First, some notation – by now, standard in the method of moving planes. For λ in $(0, h/2)$ we denote

$$\Sigma_\lambda = \{(x, y); 0 < y < \lambda\}$$

and, for $(x, y) \in \Sigma_\lambda$, we let

$$\begin{aligned} v_\lambda(x, y) &= u(x, 2\lambda - y) \\ w_\lambda(x, y) &= v_\lambda(x, y) - u(x, y) . \end{aligned}$$

As usual, the key property on which we rely is that w_λ satisfies some linear equation

$$\Delta w_\lambda + c_\lambda(x, y) w_\lambda = 0 \quad \text{in} \quad \Sigma_\lambda . \tag{2.6}$$

Indeed, v_λ satisfies the same equation (2.2) as u, and (2.6) is obtained by subtracting one from the other and letting

$$c_\lambda(x, y) = \frac{f(u(x, y)) - f(v_\lambda(x, y))}{u(x, y) - v_\lambda(x, y)}$$

if $u(x, y) \neq v_\lambda(x, y)$ and, say $c_\lambda(x, y) = 0$, if $u(x, y) = v_\lambda(x, y)$. Since we assume that f is (globally) Lipschitz continuous, with some Lipschitz constant b, we have

$$\|c_\lambda\|_{L^\infty(\Sigma_\lambda)} \leq b , \qquad \forall \lambda \in \left(0, \frac{h}{2}\right) . \tag{2.7}$$

To prove Theorem 2.2, it is enough to derive the following property – again this is classical in the moving plane method.

PROPOSITION 2.3. *For all λ, $0 < \lambda < h/2$, $w_\lambda > 0$ in Σ_λ.*

Indeed, this is precisely property (2.5) in Theorem 2.2. Further, since w_λ satisfies the linear equation (2.6) with $w_\lambda(x, \lambda) = 0$, (by construction), once we know that $w_\lambda > 0$ in Σ_λ it follows from the Hopf Lemma that

$$-2\frac{\partial u}{\partial y}(x, \lambda) \equiv \frac{\partial w_\lambda}{\partial y}(x, \lambda) < 0, \qquad \forall x \in \mathbb{R}, \quad \forall \lambda \in (0, h/2). \tag{2.8}$$

i.e. (2.4) holds. Theorem 2.2 is thus proved once Proposition 2.3 is established.
The first step in the proof of Proposition 2.3 is a standard one:

PROPOSITION 2.4. *For sufficiently small λ, $w_\lambda > 0$ in Σ_λ.*

Indeed, from our general result in [5], we know that a solution u of (2.2)-(2.3) with f Lipschitz has at most exponential growth. More precisely, for all h_0, $0 < h_0 < h$, there exist C and μ positive (which may depend on h_0) such that

$$0 < u(x, y) \le Ce^{\mu|x|}, \qquad \forall x \in \mathbb{R}, \forall y \in [0, h_0]. \tag{2.9}$$

Fix h_0, say $h_0 = h/2$. Using (2.9), we may apply Theorem 2.1. In view of equation (2.6) and since, by assumption and construction, $w_\lambda \ge 0$ on $\partial\Sigma_\lambda$, the maximum principle shows that for sufficiently small $\sigma > 0$,

$$w_\lambda \ge 0 \text{ in } \Sigma_\lambda \qquad \forall \lambda \in (0, \sigma).$$

Now, w_λ satisfies the linear equation (2.6) and $w_\lambda \not\equiv 0$ (for instance $w_\lambda(x, 0) > 0$). The classical strong maximum principle then shows that

$$w_\lambda > 0 \text{ in } \Sigma_\lambda \qquad \forall \lambda, 0 < \lambda \le \sigma. \tag{2.10}$$

The next step involves a new idea.

PROPOSITION 2.5. *Suppose that for some μ, $0 < \mu < h/2$, $w_\lambda > 0$ in Σ_λ for every λ in $(0, \mu)$. Then there exists $\varepsilon > 0$, with $\mu + \varepsilon < h/2$, such that $w_\lambda > 0$ in Σ_λ for all λ in $0 < \lambda < \mu + \varepsilon$.*

From this it follows that $\mu := \sup\{\Lambda \in (0, h/2);\ \forall \lambda \in (0, \Lambda),\ w_\lambda > 0$ in $\Sigma_\lambda\}$ is actually equal to $h/2$ and the proof is complete.

Turn to the proof of Proposition 2.5. First, by continuity, $w_\mu \ge 0$ in Σ_μ. But since $\mu < h/2$, we know that $w_\mu > 0$ in Σ_μ – by the strong maximum principle. Therefore, for all λ, $0 < \lambda \le \mu$, $w_\lambda > 0$ in Σ_λ. Note, furthermore, that $w_\lambda > 0$ on $y = 0$ for all $\lambda \in (0, \mu]$. Hence, we can push slightly further the inequality $w_\lambda > 0$ on some given line, say $x = 0$, in the following sense.

For $\lambda \in (0, h/2]$, and $\theta \in (-\pi/2, \pi/2)$, we denote by $T_{\lambda,\theta}$ the line of slope $\tan\theta$ going through the point $(0, \lambda)$, and we let $S_{\lambda,\theta}$ denote the reflection in this line (see fig. 1 below). We claim the following holds.

LEMMA 2.6. *Let $\rho > 0$ be given $(0 < \rho < \mu)$. There exists $\varepsilon > 0$ sufficiently small such that for all θ, $-\varepsilon \le \theta \le \varepsilon$ and for all λ, $\rho \le \lambda \le \mu + \varepsilon$,*

$$u(0, y) < u(S_{\lambda,\theta}(0, y)), \qquad \forall\, y, 0 \le y < \lambda\,.$$

This lemma just reflects the C^1 character of u. We argue by contradiction. If the conclusion in the lemma does not hold, then, there exists a sequence of points $(0, y_n) \to (0, \bar{y})$ with $0 \le \bar{y} \le \mu$, and sequences $\theta_n \to 0$, $\lambda_n \to \lambda$, $\rho \le \lambda \le \mu$ such that

$$u(0, y_n) \ge u(S_{\lambda_n,\theta_n}(0, y_n))\,. \tag{2.11}$$

Thus, in the limit, we get $0 \le \bar{y} \le \lambda$, and

$$u(0, \bar{y}) \ge u(0, 2\lambda - \bar{y})\,.$$

Since $u(0, y) < u(0, 2\lambda - y)$ if $0 < y < \lambda$, it must be the case that $\bar{y} = \lambda$. Let p_n denote the point $S_{\lambda_n,\theta_n}(0, y_n)$. From (2.11), using the theorem of the mean, we see that for some direction ξ_n converging to (0,1) and some point q_n on the line between $(0, y_n)$ and p_n,

$$\xi_n \cdot \nabla u(q_n) \le 0\,.$$

In the limit, this yields

$$\frac{\partial u}{\partial y}(0, \lambda) \le 0$$

which contradicts (2.8) since we know that $w_\lambda > 0$ in Σ_λ. The lemma is proved.

We are now ready to conclude the proof of Proposition 2.5.

Firstly, in a bounded domain D for an operator L of the form

$$L = \Delta + q$$

recall that the maximum principle is said to be satisfied by L in D if for all z such that

$$Lz \le 0 \text{ in } D, \quad z \ge 0 \text{ on } \partial D$$

z satisfies $z \ge 0$ in D. It is well known that if D is narrow enough in one direction this is the case (see [20] or [6]). More precisely, if b is the Lipschitz constant of f, then, there exists $\rho > 0$ such that for any function q with $\|q\|_{L^\infty} \le b$, the operator L satisfies the maximum principle in a bounded domain D such that $D \subset \{(x, y); 0 < y < \rho\}$.

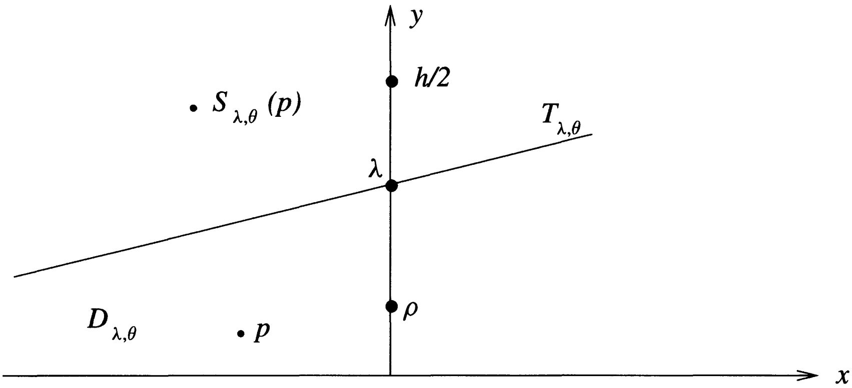

Fig. 1.

Having chosen ρ in this manner, we let $\varepsilon > 0$ be as in Lemma 2.6. The line $T_{\lambda,\theta}$, for $\theta > 0$ cuts out a triangle $D_{\lambda,\theta}$ bounded by it and by parts of the negative x axis and positive y axis (compare fig. 1)

In $D_{\lambda,\theta}$, we consider the functions

$$v_{\lambda,\theta}((x,y)) = u(S_{\lambda,\theta}(x,y))$$

and

$$w_{\lambda,\theta} = v_{\lambda,\theta} - u\,.$$

We will now show that $w_{\lambda,\theta} > 0$ in $D_{\lambda,\theta}$ for all values of λ from ρ up to $\mu + \varepsilon$ and all $\theta \neq 0$ in $(-\varepsilon, \varepsilon)$. Thus, we now perform a *tilted* version of the moving planes with the aim to derive the desired result in the limit as $\theta \to 0$.

Like the function w_λ above, the function $w_{\lambda,\theta}$, too, satisfies a linear equation in $D_{\lambda,\theta}$:

$$\Delta w_{\lambda,\theta} + q_{\lambda,\theta} w_{\lambda,\theta} = 0$$

with $\|q_{\lambda,\theta}\|_{L^\infty(D_{\lambda,\theta})} \le b$.

On the boundary $\partial D_{\lambda,\theta}$, the function $w_{\lambda,\theta}$ is ≥ 0. Dropping the indices λ, θ where not needed, we see indeed that on the line $y = 0$, $u = 0$, so $w = v \ge 0$; on the line $T_{\lambda,\theta}$, $w = 0$ by construction and, lastly, on the y-axis, $w \ge 0$, by Lemma 2.6.

Therefore, for each fixed $\theta \in (-\varepsilon, \varepsilon)\backslash\{0\}$, we can carry out the moving plane method, keeping $\theta \neq 0$ fixed. For $\lambda = \rho$, the maximum principle holds in $D_{\rho,\theta}$ – as we have recalled – hence $w_{\rho,\theta} > 0$ there. Next making use of the maximum principle in domains of small volume, as in the usual moving plane method carried out in [2], we can increase λ. We can continue this as long as $w_{\lambda,\theta} \ge 0$ on the boundary i.e. up to $\lambda = \mu + \varepsilon$. This, we do for all fixed $\theta \neq 0$ for then, $D_{\lambda,\theta}$ is bounded.

Therefore, for all $\theta \in (-\varepsilon, \varepsilon)\backslash\{0\}$, we have, for all $\lambda \in [\rho, \mu + \varepsilon]$,

$$u(x,y) \le u(S_{\lambda,\theta}(x,y)) \text{ in } D_{\lambda,\theta}\,. \tag{2.12}$$

Now, to conclude, we let $\theta > 0$ go to zero, likewise, for $\theta < 0$, let $\theta \to 0$. Going to the limit in (2.12) we infer that

$$u(x, y) \leq u(x, 2\lambda - y) \text{ in } \Sigma_\lambda . \tag{2.13}$$

Now the usual argument (based on equation (2.6) and the strong maximum principle) shows that, actually,

$$u(x, y) < u(x, 2\lambda - y) \text{ in } \Sigma_\lambda$$

for all $\lambda, 0 < \lambda \leq \mu + \varepsilon$.

Thus, as we pointed out earlier, it must be the case that this holds for all λ up to $h/2$ and the proof of Theorem 2.2 is now complete. □

Unfortunately this argument does not readily extend to the case when $j = 1$ and $n > 2$.

3. – Symmetry in half planes and half spaces

In this section, we derive Theorem 1.5 of the introduction. Namely, we consider a half plane or a half space

$$\Omega = \{x \in \mathbb{R}^n, x_n > 0\}, \qquad n = 2 \text{ or } 3 .$$

Let u be a solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega , \\ u > 0 & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{3.1}$$

We require here that $f \in C^1$. The case when f is merely Lipschitz, and $n = 2$, is treated in Section 6.

From the previous section (Theorem 2.2), we know that when $n = 2$,

$$\frac{\partial u}{\partial x_n} > 0 \quad \text{in} \quad \Omega . \tag{3.2}$$

In dimension $n \geq 3$, this is known if one further assumes that $f(0) \geq 0$ (see Theorem 1.1).

Here, we establish the *symmetry* of the bounded solutions. That is, in the half plane, when $n = 2$ or for the half space $n = 3$ and $f(0) \geq 0$, we prove that a bounded solution u of (3.1) – which thus satisfies (3.2) – is necessarily symmetric in the sense that $u(x_1, \ldots , x_n)$ only depends on x_n: $u = u(x_n)$. Set

$$\mu = \sup_{x \in \Omega} u(x) .$$

Our proof makes use of several facts:

– The monotonicity of solutions of (3.1) – proved in Section 2
– The symmetry result Theorem 1.4 proved in [3]
– The Schrödinger operator result Theorem 1.7, proved in the next section.

In view of Theorem 1.4, to prove Theorem 1.5 it is sufficient to prove.

PROPOSITION 3.1. *In dimension $n = 2$ or $n = 3$, suppose that u is a bounded monotonic solution of* (3.1), *i.e.* $\frac{\partial u}{\partial x_n} \geq 0$. *Then, its supremum $\mu = \sup_\Omega u$ satisfies*

$$f(\mu) = 0 \tag{3.3}$$

provided f is in C^1.

We start with the following related but much simpler property.

LEMMA 3.2. *Suppose that z is a bounded solution of the equation*

$$\Delta z + f(z) = 0$$

in all of $\mathbb{R}^p$, ($p \geq 1$). Assume that for any direction $\xi \in \mathbb{R}^p \backslash \{0\}$, $\xi \cdot \nabla u$ does not change sign in $\mathbb{R}^p$. Then, the supremum

$$M := \sup_{\mathbb{R}^p} z$$

satisfies $f(M) = 0$.

PROOF. We argue by induction on p. When $p = 1$, z is a solution of the ODE $\ddot{z} + f(z) = 0$. The assumption means that it is monotonic and, in this case M is either $z(+\infty)$ or $z(-\infty)$. It is classical then that $f(M) = 0$. Next, assume that the result holds for all dimensions up to $p - 1 \geq 1$. By taking the direction $\xi = e_p = (0, 0, \dots, 1)$, we know that $\frac{\partial z}{\partial x_p} \geq 0$ or $\frac{\partial z}{\partial x_p} \leq 0$. Assume for instance that $\frac{\partial z}{\partial x_p} \geq 0$. Then, the limit

$$w(x_1, \dots, x_{p-1}) := \lim_{x_p \nearrow +\infty} \nearrow z(x_1, \dots, x_p)$$

is a solution of

$$\Delta w + f(w) = 0 \text{ in } \mathbb{R}^{p-1},$$

with

$$\sup_{\mathbb{R}^{p-1}} w = \sup_{\mathbb{R}^p} z = M.$$

Clearly, for any direction $\eta \in \mathbb{R}^{p-1} \backslash \{0\}$, $\eta \cdot \nabla w$ does not change sign – for this, one relies on the C^1-convergence of z to w which one derives from the classical elliptic theory. By induction we find that

$$f(M) = 0$$

which concludes the proof. □

PROOF OF PROPOSITION 3.1. Consider the limit

$$z(x_1, \dots, x_{n-1}) = \lim_{x_n \nearrow +\infty} \nearrow u(x_1, \dots, x_n).$$

From standard elliptic theory, we infer as before that $u(x_1, \dots, x_{n-1}, x_n)$ converges uniformly in the C^1 sense on compact sets of $\mathbb{R}^{n-1}$ to z and this function z satisfies

$$\left.\begin{aligned} &\Delta z + f(z) = 0 \\ &0 < z, \quad \sup z = \mu \end{aligned}\right\} \text{ in } \mathbb{R}^{n-1}. \tag{3.4}$$

In view of Lemma 3.2, we will have achieved our goal once we prove that for any direction ξ in $\mathbb{R}^{n-1}\backslash\{0\}$, $\xi \cdot \nabla z$ does not change sign.

We argue by contradiction and suppose that for some direction $\xi \in \mathbb{R}^{n-1}\backslash\{0\}$, $\xi \cdot \nabla z$ does change sign. The directional derivative

$$\psi := \frac{\partial z}{\partial \xi} = \xi \cdot \nabla z$$

satisfies the linearized equation

$$\Delta \psi + f'(z)\psi = 0 \text{ in } \mathbb{R}^{n-1}. \tag{3.5}$$

Now, we use our result on the Schrödinger operator in dimension $n-1=1$ or 2. This is the only point in the proof where the restriction on the dimension comes in. Since z is bounded, it follows from (3.4) that ∇z is bounded too. Hence, ψ is a bounded solution of the Schrödinger equation (3.5) which changes sign. Note that the potential $q_1(x') = f'(z(x'))$, $x' \in \mathbb{R}^{n-1}$, is in $L^\infty(\mathbb{R}^{n-1})$. Here we use the notation $x' = (x_1, \dots, x_{n-1})$ in $\mathbb{R}^{n-1}$. From Theorem 1.7 we infer that the operator $-\Delta - q_1$ has negative spectrum. That is, there exists a function with compact support $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ such that

$$-\delta := \int_{\mathbb{R}^{n-1}} \{|\nabla\zeta(x')|^2 - q_1(x')\zeta(x')^2\}dx' < 0. \tag{3.6}$$

Let $R > 0$ be such that the ball $B'_R = \{x' \in \mathbb{R}^{n-1},\ |x'| < R\}$ contains the support of ζ.

Consider now, some finite cylindrical region

$$\begin{aligned} D = D_{a,h} &= \{x = (x', x_n) \in \Omega;\ |x'| < R \text{ and } a < x_n < a+h\} \\ &= B'_R \times (a, a+h). \end{aligned} \tag{3.7}$$

In D, consider the operator

$$L = -\Delta - f'(u(x', x_n))$$

(where Δ is the Laplace operator for the n variables $x_1, \dots, x_n$). Set

$$q(x', x_n) = f'(u(x', x_n)).$$

We let $\lambda_{a,h}$ be the principle eigenvalue of the operator L with Dirichlet conditions in $D_{a,h}$. That is, there exists $\psi(x)$ such that

$$\begin{cases} L\psi = \lambda_{a,h}\psi, & \psi > 0 \quad \text{in } D_{a,h} \\ \psi = 0 & \text{on } \partial D_{a,h}. \end{cases}$$

We require the following consequence of (3.6).

LEMMA 3.3. *Suppose that for some* $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ *with* $supp(\zeta) \subset B_{R'}$, *inequality* (3.6) *holds. Then, for a and h sufficiently large, the principal eigenvalue of L in* $D_{a,h}$ *satisfies*

$$\lambda_{a,h} < 0\,.$$

PROOF. By the variational characterization of $\lambda_{a,h}$, it suffices to construct a function $\rho(x', x_n) \in H_0^1(D_{a,h})$ such that

$$I := \int_{D_{a,h}} \{|\nabla\rho|^2 - q(x', x_n)\rho^2\}\, dx \; < \; 0\,.$$

To achieve this, we take the function

$$\rho(x', x_n) := \zeta(x') \sin\left[\pi\left(\frac{x_n - a}{h}\right)\right]\,.$$

A direct computation yields

$$\begin{aligned} I &= \int_D [|\nabla_{x'}\zeta|^2 - q_1(x')\zeta^2] \sin^2\left[\pi\left(\frac{x_n - a}{h}\right)\right] dx'dx_n \\ &\quad + \frac{\pi^2}{h^2}\int_D \cos^2\left[\pi\left(\frac{x_n - a}{h}\right)\right] \zeta(x')^2 dx'dx_n \\ &\quad - \int_D (q(x', x_n) - q_1(x'))\zeta(x')^2 \sin^2\left[\pi\left(\frac{x_n - a}{h}\right)\right] dx'dx_n\,. \end{aligned} \tag{3.8}$$

We know that $u(x', x_n)$ converges to $z(x')$ as $x_n \nearrow \infty$ uniformly in $B_{R'}$. Therefore, given $\varepsilon > 0$, for a (as in (3.7)) sufficiently large,

$$|q(x', x_n) - q_1(x')| < \varepsilon \quad \forall\, x' \in B_{R'}\,, \quad \forall\, x_n > a\,.$$

It is here and only here, that we use the condition that f is in C^1. We may suppose

$$\int_{B_{R'}} \zeta(x')^2 dx' = 1\,.$$

Integrating separately (3.8) with respect to x' and x_n, we find

$$I \le (-\delta + \varepsilon)\frac{h}{2} + \frac{\pi^2}{2h}\,.$$

By choosing $\varepsilon = \frac{\delta}{2}$ and h sufficiently large, we conclude that $I < 0$. $\square$

COMPLETION OF THE PROOF OF PROPOSITION 3.1. Starting with the solution u of (3.1) in the half space Ω, we see that $\frac{\partial u}{\partial x_n}$ too is a solution of the linearized equation

$$-(\Delta + q(x', x_n))\frac{\partial u}{\partial x_n} = 0 \text{ in } \Omega. \tag{3.9}$$

We also know that u is monotonic in x_n, that is

$$\frac{\partial u}{\partial x_n} > 0 \text{ in } \Omega. \tag{3.10}$$

From this it follows by a general result (see e.g. [20] or [6]) that in any bounded subdomain Ω' of Ω, the principal eigenvalue of $-\Delta - q(x)$ in Ω' is positive. In particular, in D, we see that

$$\lambda_{a,h} > 0.$$

We have reached a contradiction. Consequently, it must be the case that $\xi \cdot \nabla z$ does not change sign for any ξ in $\mathbb{R}^{n-1}\backslash\{0\}$ and hence, by Lemma 3.2, that $f(\mu) = 0$.

From Theorem 1.4 we then conclude that u is symmetric, that is

$$u = u(x_n). \qquad \square$$

4. – On the Schrödinger operator; proofs of Theorems 1.7 and 1.8

We start with the proof of Theorem 1.8.

Set

$$\sigma = \psi/\varphi. \tag{4.1}$$

Our goal is to prove that σ is a constant. Because

$$\psi \cdot (\Delta\psi + q\psi) \geq 0,$$

we see that

$$\sigma\,\varphi\,[\varphi\Delta\sigma + 2\nabla\varphi\cdot\nabla\sigma + \sigma(\Delta\varphi + q\varphi)] \geq 0.$$

This yields

$$\sigma\,\nabla\cdot(\varphi^2\nabla\sigma) + \sigma^2\varphi(\Delta\varphi + q\varphi) \geq 0.$$

Since $\Delta\varphi + q\varphi \leq 0$, it follows that

$$\sigma\,\nabla\cdot(\varphi^2\nabla\sigma) \geq 0. \tag{4.2}$$

Let ζ be a C^∞ function on $\mathbb{R}^+$ with $0 \le \zeta(t) \le 1$,

$$\zeta = \begin{cases} 1 & \text{for } 0 \le t \le 1\,, \\ 0 & \text{for } t \ge 2\,. \end{cases} \tag{4.3}$$

For $R > 0$ set

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right) \quad \text{in } \mathbb{R}^m\,. \tag{4.4}$$

Multiplying (4.2) by ζ_R^2 and integrating over $\mathbb{R}^m$ we find, by Green's theorem,

$$\begin{aligned}\int \zeta_R^2 \varphi^2 |\nabla\sigma|^2 &\le 2\left|\int \varphi^2\, \zeta_R\, \sigma\, \nabla\zeta_R \cdot \nabla\sigma\right| \\ &\le 2\left[\int_{R<|x|<2R} \varphi^2\, \zeta_R^2\, |\nabla\sigma|^2\right]^{1/2} \left[\int \varphi^2\, \sigma^2\, |\nabla\zeta_R|^2\right]^{1/2}.\end{aligned}$$

Using (4.4), (4.1) and $\psi(x) = O(|x|^{1-m/2})$ as $|x| \to \infty$, we infer that for $R > 1$, for some constant C independent of R,

$$\int \zeta_R^2\, \varphi^2\, |\nabla\sigma|^2 \le C\left[\int_{R<|x|<2R} \varphi^2\, \zeta_R^2\, |\nabla\sigma|^2\right]^{1/2}, \tag{4.5}$$

which implies

$$\int \zeta_R^2 \varphi^2 |\nabla\sigma|^2 \le C\,.$$

Letting $R \to \infty$ we see that

$$\int_{\mathbb{R}^m} \varphi^2 |\nabla\sigma|^2 \le C\,.$$

But then it follows that the right hand side in (4.5) tends to zero as $R \to \infty$, so that

$$\int_{\mathbb{R}^m} \varphi^2 |\nabla\sigma|^2 = 0\,.$$

Hence, σ is constant and the proof is complete. □

We turn now to the

PROOF OF THEOREM 1.7. The proof that the Schrödinger operator in $\mathbb{R}^m$,

$$L = -\Delta - q\,,$$

has some negative spectrum is by contradiction. Let λ_R be the principal eigenvalue of L in the ball

$$B_R = \{\|x\| < R\}$$

in $\mathbb{R}^m$, with corresponding eigenfunction φ_R. That is,

$$\begin{aligned} \varphi_R > 0\,, \quad L\varphi_R &= \lambda_R \varphi_R \ \text{ in } B_R \\ \varphi_R &= 0 \qquad \text{on } \partial B_R\,. \end{aligned}$$

We normalize φ_R by requiring
$$\varphi_R(0) = 1\,.$$

It is well known (see e.g. [6]) that λ_R is decreasing in $\mathbb{R}$. To prove the theorem it suffices to show that $\lambda_R < 0$ for some R, for the variational characterization of λ_R is

$$\lambda_R = \inf_{\zeta\in C_0^\infty(B_R)\setminus\{0\}} \frac{\displaystyle\int_{B_R} |\nabla\zeta|^2 - q\zeta^2}{\displaystyle\int_{B_R} \zeta^2}\,.$$

Suppose that $\bar\lambda = \lim_{R\to\infty}\lambda_R \geq 0$. (It is not difficult to show in general that $\bar\lambda \leq 0$; so that, in this case, $\bar\lambda = 0$.) Clearly $\lambda_R \leq \lambda_1$ for $R \geq 1$. Apply the Krylov-Safonov Harnack inequality (see, e.g. [16]) in the ball B_{2R}. We infer that for some positive constant $\delta_R > 0$

$$\delta_R \leq \varphi_R \leq \delta_R^{-1} \text{ on } \overline{B_{3R/2}}\,.$$

Hence by elliptic theory we find for any $p > n$,

$$\|\varphi_R\|_{W^{2,p}(B_R)} \leq C_R\,,$$

where C_R depends on p as well as on R.

Let $R_j \to \infty$ through an increasing sequence. By elliptic estimates, a subsequence of φ_{R_j} converges in $C^{1,\mu}$, $0 < \mu < 1$ in every compact subset, to a positive function φ satisfying

$$(\Delta + q)\varphi = -\bar\lambda\varphi \leq 0 \text{ in } \mathbb{R}^m\,.$$

We can say nothing about how φ behaves at infinity. (If q were in L^∞ it would follow from the Krylov-Safonov Harnack inequality that φ and $1/\varphi$ grow at most exponentially.)

Now apply Theorem 1.8; we conclude that $\psi = C\varphi$ for some constant C – but this is impossible since ψ changes sign. The proof is thereby complete. □

REMARK. For $m > 1$, Theorem 1.8 need not hold in case Δ is replaced by

$$\Delta + b_i(x)\frac{\partial}{\partial x_i}\,.$$

Here is an example for $m = 2$: the functions

$$\varphi = e^{x_2}, \quad \psi = \sin x_1$$

both satisfy the equation

$$\left(\Delta - 2\frac{\partial}{\partial x_2} + 1\right) z = 0 \text{ in } \mathbb{R}^2$$

but ψ is not a multiple of φ. However the following is true.

THEOREM 4.1. *Theorems* 1.8 *and* 1.7 *hold if the operator* Δ *is replaced by one of the form*

$$M = \sum_{i,j=1}^{m} \partial_i(a_{ij}(x)\partial_j)$$

with $a_{ij} \in C(\mathbb{R}^m)$ *provided for some positive continuous function* $c_0(x)$,

$$c_0(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le |\xi|^2 .$$

The proofs are the same as those of Theorems 1.8 and 1.7. In place of (4.2) in the proof of Theorem (1.8) one has

$$\sigma\,\partial_i(\varphi^2 a_{ij}\sigma_j) \ge 0 \tag{4.2$'$}$$

and one proceeds as before. For the extension of Theorem 1.7 one uses, in place of the Krylov-Safonov Harnack principle, the De Giorgi-Moser one. We refer the reader to [16] and to the original papers by De Giorgi [11] and Moser [19].

5. – The equation $\Delta u + u - 1$ in $\mathbb{R}^n_+$ for $n = 2$ and 3

This section is devoted to a simple proof of Proposition 1.6.

Set

$$\varphi(x') = \int_0^{2\pi} u(x', x_n)\sin x_n\, dx_n .$$

We write u_n and u_{nn} for the derivatives $\frac{\partial u}{\partial x_n}$ and $\frac{\partial^2 u}{\partial x_n^2}$. If Δ' denotes the Laplacian in $\mathbb{R}^{n-1}$, we see that

$$\begin{aligned}\Delta'\varphi(x') &= \int_0^{2\pi} (1 - u(x', x_n) - u_{nn}(x', x_n))\sin x_n\, dx_n \\ &= \int_0^{2\pi} \big\{ -u(x', x_n)\sin x_n + u_n(x', x_n)\cos x_n \big\}\, dx_n\end{aligned}$$

after integration by parts. Integrating by parts again and using the fact that $u(x', 0) = 0$, we find that

$$\Delta'\varphi(x') = u(x', 2\pi) . \tag{5.1}$$

Thus, for $n = 2$ or 3, φ is a subharmonic function in $\mathbb{R}^1$ or $\mathbb{R}^2$ which is bounded in absolute value. But then φ is a constant, and the conclusion follows from (5.1). □

Note that the proof above works if u satisfies, instead of (1.11), a more general equation

$$\Delta u + u - g(x) = 0$$

provided

$$\int_0^{2\pi} g(x', x_n) \sin x_n \, dx_n \geq 0 \qquad \forall x' \in \mathbb{R}^{n-1} .$$

This argument fails if $n > 3$, for there are nonconstant L^∞ subharmonic functions in $\mathbb{R}^j$ for $j > 2$.

6. – The proof of Theorem 1.5 for $n = 2$, with f merely Lipschitz

In this section we consider again problems (1.1), (1.2) when $n = 2$, i.e. in a half plane Ω:

$$\text{(6.1)} \qquad \begin{cases} 0 < u \leq \sup u = M < \infty \\ \Delta u + f(u) = 0 \text{ for } x_2 > 0 \\ u = 0 \text{ on } \{x_2 = 0\} \end{cases}$$

Here, we only assume that f is Lipschitz continuous. By Theorem 1.1′, we know that

$$\text{(6.2)} \qquad u(x_1, x_2) \text{ is nondecreasing in } x_2 .$$

As before, it suffices to prove the following

THEOREM 6.1. *Under conditions* (6.1) *and* (6.2), *necessarily,*

$$\text{(6.3)} \qquad f(M) = 0 .$$

Indeed, it then follows from Theorem 1.4 that $u = u(x_2)$ and $u_{x_2} > 0$ if $x_2 > 0$.

PROOF OF THEOREM 6.1. Since $u_{x_2} \geq 0$ and u_{x_2} satisfies

$$\text{(6.4)} \qquad \Delta u_{x_2} + f'(u) u_{x_2} = 0 \text{ in } \Omega$$

it follows as usual, from the maximum principle, that $u_{x_2} > 0$ if $x_2 > 0$. Set

$$u_{x_2} = \varphi, \quad u_{x_1} = \psi ;$$

ψ also satisfies the equation

$$\Delta \psi + f'(u) \psi = 0 .$$

By standard elliptic estimates,

$$\varphi, |\psi| \leq k \text{ on } \overline{\Omega} \tag{6.5}$$

for some constant k.

We claim that $\psi \equiv C\varphi$ for some constant C.

Postponing the proof of the claim for the moment we see that

$$\psi > 0, \quad \psi < 0, \quad \text{ or } \psi \equiv 0 \text{ in } \Omega.$$

In case $\psi \equiv 0$ it follows that $u = u(x_2)$ and u satisfies

$$0 < u \leq M, \quad \ddot{u} + f(u) = 0 \quad \text{for} \quad x_2 > 0. \tag{6.6}$$

Since u is increasing we have as before, $f(M) = 0$. In case $\psi > 0$, $u(x_1, x_2)$ is increasing in x_1. Letting $x_1 \to +\infty$ we find that

$$v(x_2) = \lim_{x_1 \to \infty} u(x_1, x_2)$$

satisfies (6.6), v is nondecreasing, and $\sup v = M$. Consequently as before, $f(M) = 0$. The case $\psi < 0$ is similar, and we conclude in general that $f(M) = 0$.

Turn now to the

PROOF OF THE CLAIM. We adopt the argument of Section 4 used in proving Theorem 1.8. Set

$$\sigma := \psi / \varphi. \tag{6.7}$$

Then in Ω,

$$\begin{aligned} 0 &= \Delta\psi + f'(u)\psi \\ &= \varphi\Delta\sigma + 2\nabla\varphi \cdot \nabla\sigma + \sigma(\Delta\varphi + f'(u)\varphi) \\ &= \frac{1}{\varphi} \sum_{i=1}^{2} \partial_{x_i}(\varphi^2 \sigma_{x_i}). \end{aligned}$$

Thus σ satisfies an equation in divergence form:

$$\sum_{i=1}^{2} \partial_i (\varphi^2 \, \partial_i \, \sigma) = 0. \tag{6.8}$$

By (6.5),

$$|\sigma|\varphi \leq k.$$

As before in Section 4, let ζ be a nonnegative C^∞ function with $\zeta \leq 1$ on $\mathbb{R}^+$, satisfying

$$\zeta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t \geq 2. \end{cases}$$

In Ω, for $R > 1$, set

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right).$$

Multiply (6.8) by $\sigma\zeta_R^2$, and, for $\varepsilon > 0$, integrate over

$$\Omega_\varepsilon = \{x; x_2 > \varepsilon\}.$$

Integrating by parts we find that

$$0 = \int_{\Omega_\varepsilon} \sigma\, \zeta_R^2\, \nabla\cdot(\varphi^2\, \nabla\sigma) = \int_{\Omega_\varepsilon} -\zeta_R^2\, \varphi^2\, |\nabla\sigma|^2 - 2\int_{\Omega_\varepsilon} \sigma\, \zeta_R\, \varphi^2 \nabla\zeta_R \cdot \nabla\sigma + J$$

where

$$J = -\int_{\{x_2=\varepsilon\}} \sigma\, \zeta_R^2\, \varphi^2\, \sigma_2\, dx_1\,. \tag{6.9}$$

Thus

$$\int_{\Omega_\varepsilon} \zeta_R^2\, \varphi^2\, |\nabla\sigma|^2 \le 2\int_{\Omega_\varepsilon} |\sigma|\, \zeta_R\, \varphi^2\, |\nabla\zeta_R| \cdot |\nabla\sigma| + J\,. \tag{6.10}$$

We show first that $J \to 0$ as $\varepsilon \to 0$. There are two cases to consider:

Case a. $f(0) \ge 0$.
In this case, we can write

$$\Delta u + c(x)u = -f(0) \le 0$$

where $|c(x)| \le b =$ the Lipschitz constant of f. By the Hopf lemma it follows that

$$\varphi = u_{x_2} > 0 \text{ on } \{x_2 = 0\}.$$

Consequently $\sigma = 0$ on $\{x_2 = 0\}$ and

$$\lim_{\varepsilon\searrow 0} J = 0\,.$$

Case b. $f(0) < 0$.
Set $\alpha = -f(0) > 0$. Then $u_{22}(x_1, 0) = \alpha$ and, consequently,

$$\left.\begin{aligned} \varphi(x_1,\varepsilon) &= \varphi(x_1,0) + \alpha\varepsilon + o(\varepsilon) \\ \varphi_2(x_1,\varepsilon) &= \alpha + o(1); \end{aligned}\right\} \text{ as } \varepsilon \to 0\,, \tag{6.11}$$

these hold uniformly for x_1 in compact sets. Thus for $|x_1| \le R$ and ε small (depending on R),

$$\varphi(x_1,\varepsilon) \ge \frac{\alpha}{2}\varepsilon \text{ and } |\varphi_2(x_1,\varepsilon)| \le 2\alpha\,. \tag{6.12}$$

By elliptic estimates we know that for some constant C,

$$|\psi| \le Cx_2, \quad |\psi_2| \le C. \tag{6.13}$$

Therefore, for $|x_1| \le R$ and $x_2 = \varepsilon$ small, we can estimate the integrand in J:

$$\begin{aligned} |\sigma\zeta_R^2\varphi^2\sigma_2| &= \left|\sigma\zeta_R^2\varphi^2\left(\frac{\psi_2}{\varphi} - \frac{\psi\varphi_2}{\varphi^2}\right)\right| \\ &\le |\psi\psi_2| + \psi^2\frac{|\varphi_2|}{\varphi} \\ &\le C^2\varepsilon + C^2\varepsilon^2\frac{4}{\varepsilon} \end{aligned}$$

by (6.12) and (6.13). Recalling the formula (6.9) for J we conclude also in the case b that $J \to 0$ as $\varepsilon \to 0$.

Letting $\varepsilon \to 0$ in (6.10) we obtain:

$$\begin{aligned} I = \int_\Omega \varphi^2\zeta_R^2|\nabla\sigma|^2 &\le 2\int_\Omega \varphi^2\zeta_R|\nabla\sigma|\,|\nabla\zeta_R|\,|\sigma| \\ &\le \frac{C}{R}\int_{\Omega\cap\{R<|x|<2R\}} \varphi^2\zeta_R|\nabla\sigma|\,|\sigma| \end{aligned}$$

for some constant C. Thus

$$I \le \frac{C}{R}\left[\int_{\Omega\cap\{R<|x|<2R\}} \varphi^2\zeta_R^2\,|\nabla\sigma|^2\right]^{1/2}\left[\int_{\Omega\cap\{R<|x|<2R\}} \varphi^2\sigma^2\right]^{1/2} \tag{6.14}$$

by Schwarz inequality. Since $|\varphi\sigma| = |\psi| \le k$ we find that, with a different constant C,

$$I \le Ck\left[\int_{\Omega\cap\{R<|x|<2R\}} \varphi^2\,\zeta_R^2\,|\nabla\sigma|^2\right]^{1/2}. \tag{6.15}$$

and hence

$$I \le C^2k^2.$$

Letting, now, $R \to \infty$ we find that

$$\int_\Omega \varphi^2|\nabla\sigma|^2 \le C^2k^2. \tag{6.16}$$

Returning to (6.15), it follows from (6.16) that the right hand side of (6.15) tends to zero as $R \to \infty$. Thus

$$\int_\Omega \varphi^2|\nabla\sigma|^2 = 0,$$

implying that σ is a constant.

The claim is proved, and so also is Theorem 6.1.

7. – Proof of Theorem 1.9

Consider the solution u of equation (1.17). The derivative $u_1 = \frac{\partial u}{\partial x_1}$ is a solution of the linear equation

$$\Delta u_1 + f'(u(x))u_1 = 0 \quad \text{in} \quad \mathbb{R}^2 . \tag{7.1}$$

Furthermore, it satisfies $u_1 \geq 0$. If it vanishes at some point, by the strong maximum principle, then $u_1 \equiv 0$ everywhere, which means that u is a function of x_2 alone. Otherwise, u_1 is positive in the plane. For any direction $\xi \in S^1$ of the plane, the derivative $\frac{\partial u}{\partial \xi}$ satisfies the same linear equation (7.1):

$$\Delta \frac{\partial u}{\partial \xi} + f'(u(x)) \frac{\partial u}{\partial \xi} = 0 .$$

Therefore, since $u_1 > 0$ in the plane and $f(u(x))$ as well as all derivatives $\frac{\partial u}{\partial \xi}$ (by elliptic theory) are bounded we may apply Theorem 1.8. Thus, we infer that for some constant C_ξ,

$$\frac{\partial u}{\partial \xi} = C_\xi \, u_1 . \tag{7.2}$$

Since C_ξ depends continuously on ξ, when one moves ξ on S^1 all the way from e_1 to $-e_1$, there has to be a direction ξ for which $C_\xi = 0$. Let $\eta \in S^1$ denote a vector orthogonal to that ξ. Since $\frac{\partial u}{\partial \xi} = 0$, we see that u is a function of the single variable $< \eta, x >$ alone. This completes the proof of Theorem 1.9.

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