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Some Properties of Reachable Solutions of Nonlinear Elliptic Equations with Measure Data

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Abstract. Let $\mathcal{A}$: $W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$, $1/p+1/q = 1$, be a monotone operator of the form $\mathcal{A}(u) = -\mathrm{div}(A(x, Du))$ on a bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$. Given a measure μ with bounded variation on Ω and a function $F \in L^q(\Omega, \mathbb{R}^N)$, we study some properties of those solutions of the equation $\mathcal{A}(u) = \mu - \mathrm{div}(F)$ which can be approximated by solutions u_n of equations of the form $\mathcal{A}(u_n) = f_n - \mathrm{div}(F_n)$, where f_n are functions in $C_c^\infty(\Omega)$ converging to μ in the weak* topology of measures, and F_n are functions in $C_c^\infty(\Omega, \mathbb{R}^N)$ converging to F strongly in $L^q(\Omega, \mathbb{R}^N)$.

1. – Introduction and statement of the results

Throughout this paper Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, while p and q are two real numbers with $p > 1$, $q > 1$, and $1/p + 1/q = 1$.

Let $A: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, i.e., $A(\cdot, \xi)$ is measurable in Ω for every $\xi \in \mathbb{R}^N$ and $A(x, \cdot)$ is continuous in $\mathbb{R}^N$ for almost every $x \in \Omega$. Assume that there exists two constants $c_0 > 0$ and $c_1 > 0$, and two nonnegative functions $a_0 \in L^1(\Omega)$ and $a_1 \in L^q(\Omega)$, such that for every ξ, $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and for almost every $x \in \Omega$ the following properties hold:

$$\text{(H1)} \qquad \big(A(x, \xi), \xi\big) \geq c_0|\xi|^p - a_0(x)\,,$$

$$\text{(H2)} \qquad |A(x, \xi)| \leq c_1|\xi|^{p-1} + a_1(x)\,,$$

$$\text{(H3)} \qquad \big(A(x, \xi) - A(x, \eta), \xi - \eta\big) > 0\,,$$

where $(\cdot, \cdot)$ and $|\cdot|$ denote the scalar product and the norm in $\mathbb{R}^N$.

Let us consider the operator $\mathcal{A}(u) = -\mathrm{div}(A(x, Du))$ between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,q}(\Omega)$, defined by

$$\langle \mathcal{A}(u), v\rangle = \int_\Omega \big(A(x, Du), Dv\big)\, dx$$

for every $u, v \in W_0^{1,p}(\Omega)$. Here and in the sequel $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. Under our assumptions, the operator $\mathcal{A}(u)$ turns out to be coercive, bounded, continuous, and monotone.

It is well known that for every $f \in L^q(\Omega)$ and for every $F \in L^q(\Omega, \mathbb{R}^N)$ there exists a unique function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \big(A(x, Du), Dv\big)\, dx = \int_\Omega f v\, dx + \int_\Omega \big(F, Dv\big)\, dx \tag{1.1}$$

for every $v \in W_0^{1,p}(\Omega)$ (see, e.g., [17]). In the sequel we shall shortly say that $u \in W_0^{1,p}(\Omega)$ satisfying (1.1) is a solution of the equation $\mathcal{A}(u) = f - \operatorname{div}(F)$ in $W^{-1,q}(\Omega)$, or that u is a solution of the problem

$$\begin{cases} \mathcal{A}(u) = f - \operatorname{div}(F) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,. \end{cases} \tag{1.2}$$

The space $\mathcal{M}_b(\Omega)$ consists of all Radon measures μ on Ω whose total variation $|\mu|$ is bounded on Ω. If $\mu \in \mathcal{M}_b(\Omega)$ and B is a Borel set in Ω, the measure $\mu \llcorner B \in \mathcal{M}_b(\Omega)$ is defined by $(\mu \llcorner B)(E) = \mu(B \cap E)$ for every Borel set $E \subseteq \Omega$.

The aim of this paper is to study some properties of the solutions of the problem

$$\begin{cases} \mathcal{A}(u) = \mu - \operatorname{div}(F) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{1.3}$$

where $\mu \in \mathcal{M}_b(\Omega)$ and $F \in L^q(\Omega, \mathbb{R}^N)$. Notice that, if $p > N$, then the Sobolev embedding theorem implies that $\mathcal{M}_b(\Omega)$ is contained in $W^{-1,q}(\Omega)$, hence every $\mu \in \mathcal{M}_b(\Omega)$ can be written in the form $\mu = -\operatorname{div}(G)$ for some $G \in L^q(\Omega, \mathbb{R}^N)$, and, consequently, (1.3) is a particular case of (1.2). Therefore, in the rest of the paper we shall always assume that $1 < p \leq N$.

If $p = 2$ and $A(x, \xi)$ is linear with respect to ξ, then it is possible to introduce a notion of solution of (1.3) by a duality method, and it is known that this solution is unique (see [19] and [23]). Moreover, it can be easily seen that this solution belongs to the Sobolev space $W_0^{1,r}(\Omega)$ for every $r < \frac{N}{N-1}$, and that this is the unique solution in the sense of distributions of the equation $\mathcal{A}(u) = \mu - \operatorname{div}(F)$ which can be obtained as limit of solutions u_n to the problems

$$\begin{cases} \mathcal{A}(u_n) = f_n - \operatorname{div}(F_n) & \text{in } \Omega\,, \\ u_n = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{1.4}$$

where (f_n) is a sequence of functions in $C_c^\infty(\Omega)$ converging to μ in the weak* topology of measures, and (F_n) is a sequence of functions in $C_c^\infty(\Omega, \mathbb{R}^N)$ converging to F strongly in $L^2(\Omega, \mathbb{R}^N)$.

On the other hand, it is known that uniqueness fails in $W_0^{1,r}(\Omega)$, $r < \frac{N}{N-1}$, when we look for solutions of (1.3) in the sense of distributions in Ω without any additional requirement (see [22] and [21]).

Starting from these results, the problem of existence and uniqueness of the solution of (1.3), when A is a monotone operator and $F = A(\cdot, 0) = 0$, was studied in a large number of papers. A complete answer to this problem, based on the notion of entropy solution, was given in [1] when μ has a density with respect to the Lebesgue measure, and in [4] for measures not charging sets of p-capacity zero. In the case $p = N$ an existence and uniqueness result for solutions in the grand Sobolev space $W_0^{1,N)}(\Omega)$ was obtained in [15] for every $\mu \in \mathcal{M}_b(\Omega)$.

In spite of the different notions of solutions used in these papers, all these existence results are obtained by constructing the solution u as the almost everywhere limit of the solutions u_n of problems (1.4) corresponding to smooth functions f_n and F_n which converge to μ and F in the weak* topology of $\mathcal{M}_b(\Omega)$ and in the strong topology of $L^q(\Omega, \mathbb{R}^N)$ respectively. The same technique was used in [2], [3], [11], and [12] to obtain a solution in the sense of distributions, and in [18] and [10] to prove the existence of a renormalized solution.

Therefore the notion of reachable solutions of (1.3) (or solutions obtained as limit of approximations), considered explicitly in [7] and [5], plays a fundamental rôle in the study of nonlinear elliptic equations with measure data. For the rigorous definition of this notion in our more general context we refer to Definition 2.3 below.

We recall that every reachable solution of (1.3) belongs to the space $\mathcal{T}_0^{1,p}(\Omega)$ of those functions u such that $T_k(u) = \max(-k, \min(u, k))$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. It is known that every $u \in \mathcal{T}_0^{1,p}(\Omega)$ has an approximate gradient Du defined a.e. in Ω, and that $A(x, Du) \in L^1(\Omega, \mathbb{R}^N)$, so that the distribution $\mathcal{A}(u) = -\mathrm{div}(A(x, Du))$ is well defined. Moreover, every reachable solution is a solution of the equation $\mathcal{A}(u) = \mu - \mathrm{div}(F)$ in the sense of distributions in Ω (for all these results see Section 2 below).

The aim of this paper is to study some properties of the reachable solutions of (1.3). The main result is the following theorem, which characterizes the reachable solutions as those solutions in the sense of distributions which satisfy some additional estimates. The proof will be given in Section 4.

THEOREM 1.1. *Let μ be a measure in $\mathcal{M}_b(\Omega)$ and let F be a function in $L^q(\Omega, \mathbb{R}^N)$. A function $u\colon \Omega \to \overline{\mathbb{R}}$ is a reachable solution of* (1.3) *if and only if the following conditions are satisfied:*

(S1) *$u \in \mathcal{T}_0^{1,p}(\Omega)$, and there exists a constant $M > 0$ such that*

$$\int_\Omega |DT_k(u)|^p \, dx \le M(k+1) \qquad \forall k > 0;$$

(S2) *u is a solution in the sense of distributions of $\mathcal{A}(u) = \mu - \mathrm{div}(F)$ in Ω;*

(S3) *there exists a constant* $C > 0$ *such that*

$$\int_\Omega \big(A(x, Du) - F, D(h(u)\varphi)\big)\, dx \le C\|h\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\Omega)}$$

for every $\varphi \in C_c^\infty(\Omega)$ *and for every* $h \in C_c^\infty(\mathbb{R})$.

Notice that the integral in the left-hand side of (S3) is well defined. Namely, if $\mathrm{supp}(h) \subseteq [-k, k]$, then $h(u) = h(T_k(u))$, so that

$$\begin{aligned}\big(A(x, Du) - F, D(h(u)\varphi)\big) = {} & \big(A(x, DT_k(u)) - F, DT_k(u)\big)h'(T_k(u))\varphi \\ & + \big(A(x, Du) - F, D\varphi\big)h(u)\,,\end{aligned}$$

and the functions $\big(A(x, DT_k(u)) - F, DT_k(u)\big)$ and $A(x, Du) - F$ belong to $L^1(\Omega)$ and $L^1(\Omega, \mathbb{R}^N)$ respectively, thanks to (H2), (S1), and Lemma 2.2 below.

If u is a reachable solution, then (S3) implies that for every fixed $h \in C_c^\infty(\mathbb{R})$, the functional

$$\varphi \mapsto \int_\Omega \big(A(x, Du) - F, D(h(u)\varphi)\big)\, dx$$

can be represented by a measure of $\mathcal{M}_{\mathrm{b}}(\Omega)$.

The second part of this paper is devoted to the integral representation of this functional for a more general class of functions h. To this aim we introduce the space $\mathrm{Lip}_0(\mathbb{R})$ of all Lipschitz functions $h: \mathbb{R} \to \mathbb{R}$ whose derivative h' has compact support, and, for every $h \in \mathrm{Lip}_0(\mathbb{R})$, we define the real numbers

$$h(+\infty) = \lim_{t\to+\infty} h(t)\,, \qquad h(-\infty) = \lim_{t\to-\infty} h(t)\,.$$

Let $\mathcal{M}^p_{\mathrm{b},0}(\Omega)$ be the set of all measures $\mu \in \mathcal{M}_{\mathrm{b}}(\Omega)$ such that $\mu(E) = 0$ for every Borel set E of Ω with p-capacity zero (for the definition of the p-capacity see Section 2 below). In Section 5 we shall prove the following result.

THEOREM 1.2. *Let* $\mu \in \mathcal{M}_{\mathrm{b}}(\Omega)$ *and* $F \in L^q(\Omega, \mathbb{R}^N)$. *Then* u *is a reachable solution of* (1.3) *if and only if* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *and there exist two nonnegative measures* α *and* β *in* $\mathcal{M}_{\mathrm{b}}(\Omega)$, *and a measure* $\nu \in \mathcal{M}^p_{\mathrm{b},0}(\Omega)$, *such that* $\mu = \nu + \alpha - \beta$ *and*

$$\begin{aligned}\int_\Omega \big(A(x, Du), D(h(u)\varphi)\big)dx = {} & \int_\Omega h(u)\varphi\, d\nu + h(+\infty)\int_\Omega \varphi\, d\alpha - h(-\infty)\int_\Omega \varphi\, d\beta \\ & + \int_\Omega \big(F, D(h(u)\varphi)\big)\, dx\end{aligned} \tag{1.5}$$

for every $\varphi \in C_c^\infty(\Omega)$ *and for every* $h \in \mathrm{Lip}_0(\mathbb{R})$.

It is known that every $\mu \in \mathcal{M}_b(\Omega)$ can be decomposed in a unique way as $\mu = \mu_0 + \mu_s$, where $\mu_0 \in \mathcal{M}^p_{b,0}(\Omega)$ and μ_s is concentrated on a set of p-capacity zero (see [14], Lemma 2.1). In [10] (see also [9]) it was recently proved that for every $\mu \in \mathcal{M}_b(\Omega)$ there exists a function $u \in \mathcal{T}^{1,p}_0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \big(A(x, Du), D(h(u)\varphi)\big)\, dx = \int_\Omega h(u)\varphi\, d\mu_0 + h(+\infty)\int_\Omega \varphi\, d\mu_s^+ \\ - h(-\infty)\int_\Omega \varphi\, d\mu_s^- + \int_\Omega \big(F, D(h(u)\varphi)\big)\, dx \end{aligned} \tag{1.6}$$

for every $\varphi \in C^\infty_c(\Omega)$ and for every $h \in \mathrm{Lip}_0(\mathbb{R})$, where μ_s^+ and μ_s^- are the positive and the negative part of μ_s respectively.

By Theorem 1.1 every solution of (1.6) is a reachable solution. Conversely, Theorem 1.2 shows that every reachable solution u solves an equation similar to (1.6), where μ_0, μ_s^+, and μ_s^- are replaced by the measures ν, α, and β, which may depend on u.

2. – Reachable solutions

For every set $E \subseteq \Omega$ the p-capacity of E with respect to Ω is defined by

$$C_p(E) = \inf \int_\Omega |Du|^p\, dx\,,$$

where the infimum is taken over all the functions $u \in W^{1,p}_0(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E. We say that a property $\mathcal{P}(x)$ holds C_p-quasi everywhere (shortly C_p-q.e.) in a set $E \subseteq \Omega$, if it holds for all $x \in E$ except for a subset N of E with $C_p(N) = 0$.

A function $u\colon \Omega \to \overline{\mathbb{R}}$ is said to be C_p-quasi continuous if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$, with $C_p(E) < \varepsilon$, such that the restriction of u to $\Omega \setminus E$ is a continuous function with values in $\overline{\mathbb{R}}$. It is well known that every $u \in W^{1,p}_0(\Omega)$ has a C_p-quasi continuous representative, which is uniquely defined (and finite) up to a set of p-capacity zero. In the sequel we shall always identify u with its C_p-quasi continuous representative, so that the pointwise values of a function $u \in W^{1,p}_0(\Omega)$ are defined C_p-quasi everywhere.

A set $E \subseteq \Omega$ is said to be C_p-quasi open if for every $\varepsilon > 0$ there exists an open set U such that $E \subseteq U \subseteq \Omega$ and $C_p(U \setminus E) \leq \varepsilon$. It can be easily seen that, if u is a C_p-quasi continuous function, then for every $k \in \mathbb{R}$ the sets $\{u > k\} = \{x \in \Omega : u(x) > k\}$ and $\{u < k\} = \{x \in \Omega : u(x) < k\}$ are C_p-quasi open.

It is well known that, if a measure μ belongs to $W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$, then every $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ is summable with respect to μ and

$$\langle \mu, u \rangle = \int_\Omega u\, d\mu\,,$$

where, in the right hand side, u denotes the C_p-quasi continuous representative and, consequently, the pointwise values of u are defined μ-almost everywhere.

We recall that the characteristic function of a set E is defined by

$$1_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

We shall need the following approximation result.

LEMMA 2.1. *For every C_p-quasi open set $U \subseteq \Omega$ there exists an increasing sequence (v_n) of nonnegative functions in $W_0^{1,p}(\Omega)$ which converges to 1_U C_p-q.e. in Ω.*

PROOF. See [8], Lemma 1.5. □

For every $k > 0$ we define the truncation function $T_k: \mathbb{R} \to \mathbb{R}$ by

$$T_k(t) = \begin{cases} t, & \text{if } |t| \leq k, \\ k \operatorname{sign}(t), & \text{if } |t| > k. \end{cases}$$

Let us consider the space $\mathcal{T}_0^{1,p}(\Omega)$ of all functions $u: \Omega \to \overline{\mathbb{R}}$ which are almost everywhere finite and such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. It is easy to see that every function $u \in \mathcal{T}_0^{1,p}(\Omega)$ has a C_p-quasi continuous representative with values in $\overline{\mathbb{R}}$, that will always be identified with the function u. Moreover, for every $u \in \mathcal{T}_0^{1,p}(\Omega)$ there exists a measurable function $\Psi: \Omega \to \mathbb{R}^N$ such that $DT_k(u) = \Psi 1_{\{|u| \leq k\}}$ a.e. in Ω (see, e.g., [1] and [16]). This function Ψ, which is unique up to almost everywhere equivalence, will be denoted by Du. It is possible to prove (see [12]) that Du is the approximate gradient of u in the sense of Geometric Measure Theory (see [13], Definition 3.1.2). Moreover Du coincides with the distributional gradient of u whenever $u \in \mathcal{T}_0^{1,p}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and $Du \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

The following result deals with the summability of the functions in $\mathcal{T}_0^{1,p}(\Omega)$.

LEMMA 2.2. *Let u be a function in $\mathcal{T}_0^{1,p}(\Omega)$. Suppose that there exists a constant $M > 0$, independent of k, such that*

$$\int_\Omega |DT_k(u)|^p \, dx \leq M(k+1) \qquad \forall k > 0.$$

Then $u \in L^s(\Omega)$ for every $s < \frac{N(p-1)}{N-p}$ and $|Du|^{p-1} \in L^r(\Omega)$ for every $r < \frac{N}{N-1}$.

PROOF. See [1], Lemmas 4.1 and 4.2, or [16], Lemma 7.43. □

Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in L^q(\Omega, \mathbb{R}^N)$. We say that a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a solution of the equation $\mathcal{A}(u) = \mu - \operatorname{div}(F)$ in the sense of distributions in Ω if

$$\int_\Omega \big(A(x, Du), D\varphi\big) \, dx = \int_\Omega \varphi \, d\mu + \int_\Omega (F, D\varphi) \, dx$$

for every $\varphi \in C_c^\infty(\Omega)$. Note that all integrals in the previous formula make sense, since $A(x, Du) \in L^1(\Omega, \mathbb{R}^N)$ by (H2) and Lemma 2.2.

We say that a sequence (μ_n) of measures in $\mathcal{M}_b(\Omega)$ converges weakly* to $\mu \in \mathcal{M}_b(\Omega)$ if

$$\lim_{n\to\infty} \int_\Omega \varphi \, d\mu_n = \int_\Omega \varphi \, d\mu$$

for every φ in the space $C_0(\Omega)$ of all continuous functions vanishing on $\partial\Omega$.

We are now in a position to introduce the notion of reachable solution.

DEFINITION 2.3. Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in L^q(\Omega, \mathbb{R}^N)$. A function $u\colon \Omega \to \overline{\mathbb{R}}$ is a reachable solution of the problem

$$\begin{cases} \mathcal{A}(u) = \mu - \operatorname{div}(F) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{2.1}$$

if there exist three sequences (f_n), (F_n), and (u_n) such that

(i) $f_n \in C_c^\infty(\Omega)$, and (f_n) converges to μ weakly* in $\mathcal{M}_b(\Omega)$;
(ii) $F_n \in C_c^\infty(\Omega, \mathbb{R}^N)$, and (F_n) converges to F strongly in $L^q(\Omega, \mathbb{R}^N)$;
(iii) $u_n \in W_0^{1,p}(\Omega)$, and $\mathcal{A}(u_n) = f_n - \operatorname{div}(F_n)$ in the sense of $W^{-1,q}(\Omega)$;
(iv) (u_n) converges to u a.e. in Ω.

The notion of solution obtained by approximation, studied in [2], [3], [7], and [5], corresponds to the case $F = F_n = 0$. In Remark 3.6 we shall prove that this class of solutions coincides with the class of reachable solutions whenever $A(x, 0) = 0$, a condition that was always assumed in the previous papers on this subject.

REMARK 2.4. The existence of a reachable solution can be obtained, with minor changes, following the lines of the existence proof in [3]. Moreover, assuming (i)-(iv), one can prove that $|Du_n|^{p-1}$ is bounded in $L^r(\Omega)$ for every $r < \frac{N}{N-1}$ and that $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$ for every $k > 0$. In particular, every reachable solution belongs to $\mathcal{T}_0^{1,p}(\Omega)$, and, assuming (i)-(iv), one can prove also that (Du_n) converges to Du a.e. in Ω, and that $A(x, Du_n)$ converges to $A(x, Du)$ strongly in $L^r(\Omega, \mathbb{R}^N)$ for every $r < \frac{N}{N-1}$.

All the known properties of the reachable solutions needed in this paper are collected in the following theorem.

THEOREM 2.5. *Let $\mu \in \mathcal{M}_b(\Omega)$, let $F \in L^q(\Omega, \mathbb{R}^N)$, and let u be a reachable solution of* (2.1). *Then*

(i) *$u \in \mathcal{T}_0^{1,p}(\Omega)$ and there exists a constant $M > 0$ such that*

$$\int_\Omega |DT_k(u)|^p \, dx \le M(k+1) \qquad \forall k > 0\,;$$

(ii) *$u \in L^s(\Omega)$ for every $s < \frac{N(p-1)}{N-p}$ and $|Du|^{p-1} \in L^r(\Omega)$ for every $r < \frac{N}{N-1}$;*
(iii) *$A(x, Du)$ belongs to $L^r(\Omega, \mathbb{R}^N)$ for every $r < \frac{N}{N-1}$;*
(iv) *u is a solution of $\mathcal{A}(u) = \mu - \operatorname{div}(F)$ in the sense of distributions in Ω.*

Proof. Property (i) can be easily obtained by taking $T_k(u_n)$ as test function in the approximating equations $\mathcal{A}(u_n) = f_n - \operatorname{div}(F_n)$. Property (ii) follows from (i) and Lemma 2.2. Property (iii) follows from (ii) and (H2). Finally, (iv) is proved in [3]. □

3. – The rôle of truncations

In this section we collect some properties of reachable solutions which are based on the behaviour of the truncations $T_k(u)$. We begin with a result which is crucial in the proof of Theorems 1.1 and 1.2.

Theorem 3.1. *Let $\mu \in \mathcal{M}_b(\Omega)$, let $F \in L^q(\Omega, \mathbb{R}^N)$, and let u be a reachable solution of* (2.1). *Then for almost every $k > 0$ there exists a measure μ_k in $W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ such that*

$$\int_{\{|u|\leq k\}} \big(A(x, Du), Dv\big)\, dx = \int_\Omega v\, d\mu_k + \int_{\{|u|\leq k\}} \big(F, Dv\big)\, dx \tag{3.1}$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, there exists a sequence (k_n) of positive numbers tending to $+\infty$ such that (μ_{k_n}) converges to μ in the weak topology of $\mathcal{M}_b(\Omega)$.*

Remark 3.2. For every $k > 0$ let $G_k = F\, 1_{\{|u|\leq k\}} + A(\cdot, 0)\, 1_{\{|u|>k\}}$. Theorem 3.1 shows that for almost every $k > 0$ the truncation $T_k(u)$ is the solution in $W_0^{1,p}(\Omega)$ of the equation $\mathcal{A}(T_k(u)) = \mu_k - \operatorname{div}(G_k)$ in the sense of $W^{-1,q}(\Omega)$, where $\mu_k \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ and (G_k) converges to F strongly in $L^q(\Omega, \mathbb{R}^N)$. In the special case $F = A(\cdot, 0) = 0$ we have $G_k = 0$.

We shall see in Remark 3.6 that, if $F = A(\cdot, 0) = 0$, then we can take $F_n = 0$ in Definition 2.3 (iii). In this case it is proved in [5] that (3.1) holds for every $k > 0$, and that (μ_{k_n}) converges weakly* to μ for every sequence (k_n) tending to $+\infty$.

Proof of Theorem 3.1. The proof follows the lines of [5], with some important modifications due to the presence of the term $-\operatorname{div}(F)$ in the equation.

Since u is a reachable solution of (2.1), there exist three sequences (f_n), (F_n), and (u_n) which satisfy conditions (i)-(iv) of Definition 2.3.

Let $\varphi \in C_c^\infty(\Omega)$ and, for every $\varepsilon, k > 0$, let $h_{k,\varepsilon}$ be the Lipschitz continuous function defined by

$$\begin{cases} h_{k,\varepsilon}(t) = 1\,, & \text{if } |t| \leq k\,, \\ h_{k,\varepsilon}(t) = 0\,, & \text{if } |t| \geq k+\varepsilon\,, \\ |h'_{k,\varepsilon}(t)| = 1/\varepsilon\,, & \text{if } k < |t| < k+\varepsilon\,. \end{cases} \tag{3.2}$$

In the proof the letter c will denote a positive constant, independent of φ, k, ε, n, whose value can change from line to line.

If we use $h_{k,\varepsilon}(u_n)\varphi$ as test function in the equation satisfied by u_n, we obtain

$$\int_\Omega \big(A(x, Du_n) - F_n, D\varphi\big) h_{k,\varepsilon}(u_n)\, dx = \int_\Omega (f^n_{k,\varepsilon} + g^n_{k,\varepsilon})\varphi\, dx\,, \tag{3.3}$$

where

$$\begin{aligned} f^n_{k,\varepsilon} &= f_n h_{k,\varepsilon}(u_n)\,, \\ g^n_{k,\varepsilon} &= \frac{1}{\varepsilon}\big(A(x, Du_n) - F_n, Du_n\big)\mathrm{sign}(u_n) 1_{\{k\le |u_n| < k+\varepsilon\}}\,. \end{aligned} \tag{3.4}$$

By condition (i) in Definition 2.3 the sequence (f_n) is bounded in $L^1(\Omega)$, so that

$$\|f^n_{k,\varepsilon}\|_{L^1(\Omega)} \le c\,. \tag{3.5}$$

In order to obtain a similar estimate for $g^n_{k,\varepsilon}$, we consider the Lipschitz continuous function $\sigma_{k,\varepsilon}$ defined by

$$\begin{cases} \sigma_{k,\varepsilon}(t) = 0\,, & \text{if } |t| \le k\,, \\ \sigma_{k,\varepsilon}(t) = \mathrm{sign}(t)\,, & \text{if } |t| \ge k+\varepsilon\,, \\ \sigma'_{k,\varepsilon}(t) = 1/\varepsilon\,, & \text{if } k < |t| < k+\varepsilon\,. \end{cases}$$

As $\sigma_{k,\varepsilon}(0) = 0$, the function $\sigma_{k,\varepsilon}(u_n)$ belongs to $W^{1,p}_0(\Omega)$. If we use it as test function in the equation satisfied by u_n, we obtain

$$\frac{1}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} \big(A(x, Du_n) - F_n, Du_n\big)\, dx = \int_\Omega f_n \sigma_{k,\varepsilon}(u_n)\varphi\, dx \le c\,.$$

Using (H1) and Young's inequality we get

$$\frac{1}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} |Du_n|^p dx \le c + \frac{c}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} (|F_n|^q + a_0)\, dx\,. \tag{3.6}$$

By (H2) and by Young's inequality we have

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} \big|\big(A(x, Du_n) - F_n, Du_n\big)\big|\, dx \\ &\qquad \le \frac{c}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} \big(|Du_n|^p + a_1^q + |F_n|^q\big)\, dx\,, \end{aligned}$$

which, together with (3.4) and (3.6), gives

$$\int_\Omega |g^n_{k,\varepsilon}|\, dx \le c + \frac{c}{\varepsilon}\int_{\{k\le |u_n|<k+\varepsilon\}} \big(|F_n|^q + b\big)\, dx\,, \tag{3.7}$$

where $b = a_0 + a_1^q$.

Let J_n be the nondecreasing function defined by

$$J_n(k) = \int_{\{|u_n|<k\}} \big(|F_n|^q + b\big)\, dx\,. \tag{3.8}$$

Then for almost every $k > 0$ the derivative $J_n'(k)$ exists and is finite. Moreover,

$$\int_0^k J_n'(t)\, dt \le J_n(k) \le \int_\Omega \big(|F_n|^q + b\big)\, dx \le c\,,$$

hence

$$\int_0^{+\infty} J_n'(t)\, dt \le c\,,$$

which implies by Fatou's lemma

$$\int_0^{+\infty} \liminf_{n\to\infty} J_n'(t)\, dt \le c\,. \tag{3.9}$$

Let us fix $k > 0$ such that $\operatorname{meas}(\{|u| = k\}) = 0$, $J_n'(k)$ exists and is finite for every n, and

$$\liminf_{n\to\infty} J_n'(k) < +\infty\,. \tag{3.10}$$

By (3.9) almost every $k > 0$ satisfies these properties. From (3.7) and (3.8) we obtain

$$\limsup_{\varepsilon\to 0^+} \|g^n_{k,\varepsilon}\|_{L^1(\Omega)} \le c\,(1 + J_n'(k)) < +\infty\,.$$

From this inequality and from (3.5) we infer that there exists a sequence (ε_j) of positive numbers tending to 0 such that, as $j \to \infty$, the sequence $(f^n_{k,\varepsilon_j} + g^n_{k,\varepsilon_j})$ converges in the weak* topology of $\mathcal{M}_b(\Omega)$ to a measure $\mu_{k,n}$ with

$$|\mu_{k,n}|(\Omega) \le c\,(1 + J_n'(k))\,. \tag{3.11}$$

Since $h_{k,\varepsilon}(u_n)$ converges to $1_{\{|u_n|\le k\}}$ a.e. in Ω as $\varepsilon \to 0^+$, by the dominated convergence theorem we deduce from (3.3) that

$$\int_{\{|u_n|\le k\}} \big(A(x, Du_n) - F_n, D\varphi\big)\, dx = \int_\Omega \varphi\, d\mu_{k,n}\,. \tag{3.12}$$

By (3.10) and (3.11) there exists a subsequence (μ_{k,n_j}) of $(\mu_{k,n})$ such that

$$\limsup_{j\to\infty} |\mu_{k,n_j}|(\Omega) \le c\,\big(1 + \liminf_{n\to\infty} J_n'(k)\big)\,.$$

Therefore, a further subsequence, still denoted by (μ_{k,n_j}), converges weakly*, as $j \to +\infty$, to a measure $\mu_k \in \mathcal{M}_b(\Omega)$ with

$$|\mu_k|(\Omega) \le c\left(1 + \liminf_{n\to\infty} J_n'(k)\right). \tag{3.13}$$

Since the sequence $A(x, Du_n)$ converges to $A(x, Du)$ strongly in $L^1(\Omega, \mathbb{R}^N)$ by Remark 2.4 and $1_{\{|u_n|\le k\}}$ converges to $1_{\{|u|\le k\}}$ a.e. in Ω by our choice of k, we can pass to the limit in (3.12) along the sequence (n_j) and we obtain

$$\int_{\{|u|\le k\}} \big(A(x, Du) - F, D\varphi\big)\, dx = \int_\Omega \varphi\, d\mu_k \tag{3.14}$$

for every $\varphi \in C_c^\infty(\Omega)$. As $u \in \mathcal{T}_0^{1,p}(\Omega)$, by (H2) and (3.14) the measure μ_k belongs to $W^{-1,q}(\Omega)$, and (3.14) can be extended to every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ by a standard approximation argument. This concludes the proof of the first statement of the theorem.

By (3.9) there exists a sequence (k_j) of positive numbers tending to $+\infty$ such that $\mathrm{meas}(\{|u| = k_j\}) = 0$ for every j, $J_n'(k_j)$ exists and is finite for every j and n, and

$$\lim_{j\to\infty} \liminf_{n\to\infty} J_n'(k_j) = 0\,.$$

By (3.13) the sequence $|\mu_{k_j}|(\Omega)$ is bounded, so that there exists a subsequence, still denoted by (μ_{k_j}), which converges weakly* to a measure $\lambda \in \mathcal{M}_b(\Omega)$. Since the function $A(x, Du)$ belongs to $L^1(\Omega, \mathbb{R}^N)$ by Theorem 2.5 (iii), we can pass to the limit in (3.14) along the sequence (k_j) and we obtain

$$\int_\Omega \big(A(x, Du) - F, D\varphi\big)\, dx = \int_\Omega \varphi\, d\lambda$$

for every $\varphi \in C_c^\infty(\Omega)$. Since, by Theorem 2.5 (iv), u is a solution in the sense of distributions of the equation $\mathcal{A}(u) = \mu - \mathrm{div}(F)$ in Ω, we conclude that $\lambda = \mu$, hence (μ_{k_j}) converges to μ weakly* in $\mathcal{M}_b(\Omega)$. □

In Proposition 3.4 we shall show that the class of reachable solutions of (2.1) does not change if we replace f_n by $\lambda_n \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ and F_n by $G_n \in L^q(\Omega, \mathbb{R}^N)$. In order to obtain this result we need the following lemma.

LEMMA 3.3. *Let $\lambda \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$. For every $\varepsilon > 0$ there exists $f \in C_c^\infty(\Omega)$ such that*

$$\|f - \lambda\|_{W^{-1,q}(\Omega)} \le \varepsilon \quad \textit{and} \quad \|f\|_{L^1(\Omega)} \le |\lambda|(\Omega)\,.$$

PROOF. We may assume that $|\lambda|(\Omega) > 0$. Let $\mathcal{F}$ be the set of all functions $f \in C_c^\infty(\Omega)$ such that $\|f\|_{L^1(\Omega)} \le |\lambda|(\Omega)$. Suppose, by contradiction, that there exists $\varepsilon > 0$ such that no $f \in \mathcal{F}$ satisfies $\|f - \lambda\|_{W^{-1,q}(\Omega)} \leq \varepsilon$. Then λ does not belong to the closure of $\mathcal{F}$ in $W^{-1,q}(\Omega)$. As $\mathcal{F}$ is convex, by Hahn-Banach theorem, λ can be separated from $\mathcal{F}$. Therefore there exist $u \in W_0^{1,p}(\Omega)$ and $s \in \mathbb{R}$ such that

$$\sup_{f\in\mathcal{F}} \int_\Omega fu\,dx \leq s < \int_\Omega u\,d\lambda\,. \tag{3.15}$$

By the definition of $\mathcal{F}$, the first inequality in (3.15) implies that $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq \frac{s}{|\lambda|(\Omega)}\,.$$

As $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have that $|u| \leq \|u\|_{L^\infty(\Omega)}$ C_p-q.e. in Ω, hence $|\lambda|$-a.e. in Ω. Therefore

$$\int_\Omega u\,d\lambda \leq \|u\|_{L^\infty(\Omega)}|\lambda|(\Omega) \leq |\lambda|(\Omega)\frac{s}{|\lambda|(\Omega)} \leq s\,,$$

which contradicts the second inequality in (3.15). □

PROPOSITION 3.4. *Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in L^q(\Omega, \mathbb{R}^N)$. Then a function $u\colon \Omega \to \overline{\mathbb{R}}$ is a reachable solution of* (2.1) *if and only if there exist three sequences (λ_n), (G_n), and (v_n) such that*

(i) *$\lambda_n \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$, and (λ_n) converges to μ weakly* in $\mathcal{M}_b(\Omega)$;*
(ii) *$G_n \in L^q(\Omega, \mathbb{R}^N)$, and (G_n) converges to F strongly in $L^q(\Omega, \mathbb{R}^N)$;*
(iii) *$v_n \in W_0^{1,p}(\Omega)$, and $\mathcal{A}(v_n) = \lambda_n - \mathrm{div}(G_n)$ in the sense of $W^{-1,q}(\Omega)$;*
(iv) *(v_n) converges to u a.e. in Ω.*

PROOF. If u is a reachable solution, then by Definition 2.3 there exist three sequences (λ_n), (G_n), and (v_n) which satisfy (i)-(iv). Let us prove the converse.

By Lemma 3.3, for every n there exists a sequence (f_n^m) of functions in $C_c^\infty(\Omega)$, with $\|f_n^m\|_{L^1(\Omega)} \leq |\lambda_n|(\Omega)$, such that (f_n^m) converges to λ_n strongly in $W^{-1,q}(\Omega)$ as $m \to \infty$. For every n let (G_n^m) be a sequence of function in $C_c^\infty(\Omega, \mathbb{R}^N)$ which converges to G_n strongly in $L^q(\Omega, \mathbb{R}^N)$ as $m \to \infty$, and let $u_n^m \in W_0^{1,p}(\Omega)$ be the solution of the equation $\mathcal{A}(u_n^m) = f_n^m - \mathrm{div}(G_n^m)$ in $W^{-1,q}(\Omega)$. Under the hypotheses (H1), (H2), and (H3) it is easy to prove that (u_n^m) converges to v_n strongly in $W_0^{1,p}(\Omega)$ as $m \to \infty$. All these properties, together with (i)-(iv), imply by a standard argument that for every n there exists $m(n)$ such that $(f_n^{m(n)})$ converges to μ weakly* in $\mathcal{M}_b(\Omega)$, $(G_n^{m(n)})$ converges to F strongly in $L^q(\Omega, \mathbb{R}^N)$, and $(u_n^{m(n)})$ converges to u in measure. Thus there exists a subsequence, still denoted by $(u_n^{m(n)})$, which converges to u a.e. in Ω, so that u satisfies all conditions of Definition 2.3. □

REMARK 3.5. Let u be a reachable solution of (2.1), and let μ_k and (k_n) be the measures and the sequence introduced in Theorem 3.1, let $\lambda_n = \mu_{k_n}$, and let $G_n = F\, 1_{\{|u|\leq k_n\}} + A(\cdot, 0)\, 1_{\{|u|>k_n\}}$. By Remark 3.2 the truncation $T_{k_n}(u)$ satisfies the equation $\mathcal{A}(T_{k_n}(u)) = \lambda_n - \operatorname{div}(G_n)$ in $W^{-1,q}(\Omega)$, (λ_n) converges to μ weakly* in $\mathcal{M}_b(\Omega)$, and (G_n) converges to F strongly in $L^q(\Omega, \mathbb{R}^N)$. Conversely, by Proposition 3.4, every function $u \in \mathcal{T}_0^{1,p}(\Omega)$, whose truncations satisfy the previous property for a suitable sequence (k_n) tending to $+\infty$, is a reachable solution.

Moreover, the previous argument shows that, if u is a reachable solution of (2.1), then the sequence $(v_n) = (T_{k_n}(u))$, together with other two sequences (λ_n) and (G_n), satisfies conditions (i)-(iv) of Proposition 3.4. In this case $T_k(v_n)$, which is equal to $T_{k\wedge k_n}(u)$, converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ for every $k > 0$. Arguing as in the proof of Proposition 3.4, we can regularize λ_n and G_n, and we obtain that for every reachable solution u of (2.1) there exist three sequences (f_n), (F_n), and (u_n), satisfying conditions (i)-(iv) of Definition 2.3, such that $T_k(u_n)$ converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ for every $k > 0$.

REMARK 3.6. Assume that $A(\cdot, 0) = 0$ and $F = 0$. If u is a reachable solution of (2.1), by Remark 3.5 the sequences $v_n = T_{k_n}(u)$, $\lambda_n = \mu_{k_n}$, and $G_n = 0$ given by Threorem 3.1 satisfy conditions (i)-(iv) of Proposition 3.4. Arguing as in the proof of that proposition, we can construct two sequences (f_n) and (u_n) such that

(i) $f_n \in C_c^\infty(\Omega)$, and (f_n) converges to μ weakly* in $\mathcal{M}_b(\Omega)$;
(ii) $u_n \in W_0^{1,p}(\Omega)$, and $\mathcal{A}(u_n) = f_n$ in the sense of $W^{-1,q}(\Omega)$;
(iii) (u_n) converges to u a.e. in Ω;
(iv) $T_k(u_n)$ converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ for every $k > 0$.

Therefore, if $A(\cdot, 0) = 0$ and $F = 0$, our definition is equivalent to the definition of solution obtained by approximation used in [7] and [5].

4. – Proof of Theorem 1.1

In order to prove Theorem 1.1, we enlarge the class of test functions which are admissible in (S3).

LEMMA 4.1. *If u satisfies* (S1), (S2), *and* (S3), *then*

$$\int_\Omega \big(A(x, Du) - F, D(h(u)\varphi)\big)\, dx \leq C \|h\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\Omega)}$$

for every $\varphi \in C_c^\infty(\Omega)$ and for every Lipschitz function h with compact support in $\mathbb{R}$.

Proof. Let us fix a Lipschitz function h with compact support in $\mathbb{R}$, and let $k > 0$ be such that $\operatorname{supp}(h) \subseteq [-k, k]$. Let us consider a sequence (h_n) of functions in $C_c^\infty(\mathbb{R})$, with $\operatorname{supp}(h_n) \subseteq [-k, k]$, such that (h_n) converges to h uniformly in $\mathbb{R}$, (h_n') converges to h' a.e. in $\mathbb{R}$, and $\|h_n'\|_{L^\infty(\mathbb{R})} \leq \|h'\|_{L^\infty(\mathbb{R})}$. By (S3) we obtain

$$\begin{aligned} &\int_\Omega \big(A(x, DT_k(u)) - F, DT_k(u)\big) h_n'(T_k(u))\varphi \, dx \\ &\quad + \int_\Omega \big(A(x, Du) - F, D\varphi\big) h_n(u) \, dx \leq C\|h_n\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\Omega)} . \end{aligned} \tag{4.1}$$

Let N be a subset of $\mathbb{R}$ with Lebesgue measure zero such that $(h_n'(t))$ converges to $h'(t)$ for every $t \in \mathbb{R} \setminus N$. It is well known that $DT_k(u) = 0$ a.e. in the set $E = \{x \in \Omega : T_k(u)(x) \in N\}$ (see [20] and [6]). Hence, by (H2) and by the dominated convergence theorem, we get

$$\begin{aligned} \lim_{n\to\infty} &\int_\Omega \big(A(x, DT_k(u)) - F, DT_k(u)\big) h_n'(T_k(u))\varphi \, dx \\ &= \lim_{n\to\infty} \int_{\Omega\setminus E} \big(A(x, DT_k(u)) - F, DT_k(u)\big) h_n'(T_k(u))\varphi \, dx \\ &= \int_{\Omega\setminus E} \big(A(x, DT_k(u)) - F, DT_k(u)\big) h'(T_k(u))\varphi \, dx \\ &= \int_\Omega \big(A(x, DT_k(u)) - F, DT_k(u)\big) h'(T_k(u))\varphi \, dx . \end{aligned}$$

The conclusion can be obtained taking the limit in (4.1) as n goes to ∞. □

The following lemma will be used in the proof of Lemmas 4.3 and 5.4.

Lemma 4.2. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ and let $a \in L^q(\Omega)$ with $a \geq 0$ a.e. in Ω. Suppose that there exists a constant M, independent of k, such that*

$$\int_\Omega |DT_k(u)|^p \, dx \leq M(k+1) \qquad \forall k > 0 .$$

Let I and J be the monotone nondecreasing functions defined by

$$I(k) = \int_{\{|u|<k\}} |Du|^p dx , \qquad J(k) = \int_{\{|u|<k\}} a\,|Du| \, dx . \tag{4.2}$$

Then the derivatives $I'(k)$ and $J'(k)$ exist and are finite for almost every $k > 0$, and there exists a sequence (k_n) of positive numbers tending to $+\infty$ such that

$$\limsup_{n\to\infty} I'(k_n) \leq M , \qquad \lim_{n\to\infty} J'(k_n) = 0 . \tag{4.3}$$

PROOF. The existence of the derivative for almost every $k > 0$ follows from Lebesgue's theorem on monotone functions, which yields

$$\text{(4.4)}\qquad \begin{aligned} &\int_0^{k_1} I'(k)\,dk \le I(k_1) \le M(k_1+1)\,,\\ &\int_0^{k_1} J'(k)\,dk \le J(k_1) \le C(k_1+1)^{1/p}\,, \end{aligned}$$

where $C = \|a\|_{L^q(\Omega)} M^{1/p}$. To prove the existence of a sequence (k_n) which tends to $+\infty$ and satisfies (4.3), it is enough to show that for every $k_0 > 0$ and for every $\varepsilon > 0$ there exists $k > k_0$ such that

$$\text{(4.5)}\qquad I'(k) < M + \varepsilon\,, \qquad J'(k) < \varepsilon\,.$$

Let us fix $k_0 > 0$ and $\varepsilon > 0$. Then there exists $k_1 > k_0$ such that

$$\text{(4.6)}\qquad \frac{M}{M+\varepsilon}(k_1+1) + \frac{C}{\varepsilon}(k_1+1)^{1/p} < k_1 - k_0\,.$$

Since by (4.4)

$$\begin{aligned} &\operatorname{meas}(\{k \in [k_0,k_1] : I'(k) \ge M+\varepsilon\}) \le \frac{M}{M+\varepsilon}(k_1+1)\,,\\ &\operatorname{meas}(\{k \in [k_0,k_1] : J'(k) \ge \varepsilon\}) \le \frac{C}{\varepsilon}(k_1+1)^{1/p}\,, \end{aligned}$$

we deduce from (4.6) that

$$\operatorname{meas}(\{k \in [k_0,k_1] : I'(k) < M+\varepsilon\,,\ J'(k) < \varepsilon\}) > 0\,,$$

hence there exists $k > k_0$ which satisfies (4.5). □

The following result is the analogue of Theorem 3.1 for functions satisfying properties (S1), (S2), and (S3).

LEMMA 4.3. *Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in L^q(\Omega,\mathbb{R}^N)$. Suppose that u satisfies* (S1), (S2), *and* (S3). *Then for almost every $k > 0$ there exists a measure μ_k in $W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ such that*

$$\int_{\{|u|\le k\}} \big(A(x,Du),Dv\big)\,dx = \int_\Omega v\,d\mu_k + \int_{\{|u|\le k\}} \big(F,Dv\big)\,dx$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, there exists a sequence (k_n) of positive numbers tending to $+\infty$ such that (μ_{k_n}) converges to μ in the weak topology of $\mathcal{M}_b(\Omega)$.*

PROOF. For every $\varepsilon, k > 0$ let us consider the Lipschitz continuous function $h_{k,\varepsilon}$ defined by (3.2). By Lemma 4.1 we have

$$\int_\Omega \big(A(x, Du) - F, D(h_{k,\varepsilon}(u)\varphi)\big)\, dx \leq C\|\varphi\|_{L^\infty(\Omega)}$$

for every $\varphi \in C_c^\infty(\Omega)$ and for every $\varepsilon, k > 0$. Thus by (H2) we get

$$\begin{aligned} &\int_\Omega \big(A(x, Du) - F, D\varphi\big) h_{k,\varepsilon}(u)\, dx \\ &\leq C\|\varphi\|_{L^\infty(\Omega)} + \int_\Omega |A(x, Du) - F||Du||h'_{k,\varepsilon}(u)||\varphi|\, dx \qquad (4.7) \\ &\leq K\|\varphi\|_{L^\infty(\Omega)} \left(1 + \frac{1}{\varepsilon}\int_{\{k\leq |u| < k+\varepsilon\}} \big(|Du|^p + a\,|Du|\big)\, dx\right), \end{aligned}$$

where $a = a_1 + |F|$ and K is a constant independent of φ, k, ε. Let $I(k)$ and $J(k)$ be the functions defined in (4.2). For almost every $k > 0$ the derivatives $I'(k)$ and $J'(k)$ exist and are finite, and for these values of k we can take the limit as ε goes to zero in (4.7), obtaining

$$\int_{\{|u|\leq k\}} \big(A(x, Du) - F, D\varphi\big)\, dx \leq K\|\varphi\|_{L^\infty(\Omega)}(1 + I'(k) + J'(k)).$$

This implies that for almost every $k > 0$ there exists a measure $\mu_k \in \mathcal{M}_b(\Omega)$ such that

$$\int_{\{|u|\leq k\}} \big(A(x, Du) - F, D\varphi\big)\, dx = \int_\Omega \varphi\, d\mu_k \tag{4.8}$$

for every $\varphi \in C_c^\infty(\Omega)$, and

$$|\mu_k|(\Omega) \leq K(1 + I'(k) + J'(k)) < +\infty. \tag{4.9}$$

Moreover, by (S1), (H2), and (4.8) the measure μ_k belongs to $W^{-1,q}(\Omega)$, and (4.8) can be extended extended to every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by a standard approximation argument. This concludes the proof of the first statement of the lemma.

By Lemma 4.2 there exists a sequence (k_n) tending to $+\infty$ such that

$$I'(k_n) + J'(k_n) \leq M + 1 \qquad \forall n \in \mathbb{N}. \tag{4.10}$$

Thus, by (4.9), we have that $|\mu_{k_n}|(\Omega) \leq K(M + 2)$. Hence there exist a subsequence, still denoted by (μ_{k_n}), which converges weakly* to a measure $\lambda \in \mathcal{M}_b(\Omega)$. On the other hand, taking into account that $A(x, Du) \in L^1(\Omega, \mathbb{R}^N)$ by Theorem 2.5 (iii), from the definition (4.8) of μ_k and from (S2) we obtain for every $\varphi \in C_c^\infty(\Omega)$

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \varphi\, d\mu_{k_n} &= \lim_{n\to\infty} \int_{\{|u|\leq k_n\}} \big(A(x, Du) - F, D\varphi\big)\, dx \\ &= \int_\Omega \big(A(x, Du) - F, D\varphi\big)\, dx = \int_\Omega \varphi\, d\mu, \end{aligned}$$

which implies that $\lambda = \mu$. □

PROOF OF THEOREM 1.1. From Lemma 4.3, Proposition 3.4, and Remark 3.5 we obtain that, if u satisfies (S1), (S2), and (S3), then u is a reachable solution of (2.1).

Conversely, let us suppose that u is a reachable solution of (2.1). Properties (S1) and (S2) follow from Theorem 2.5. Moreover, by Theorem 3.1 there exists a sequence (k_n) of positive numbers tending to $+\infty$ and a sequence (μ_{k_n}) of measures in $W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$, converging to μ weakly* in $\mathcal{M}_b(\Omega)$, such that

$$\text{(4.11)} \qquad \int_{\{|u| \le k_n\}} \big(A(x, Du), Dv\big)\, dx = \int_\Omega v\, d\mu_{k_n} + \int_{\{|u| \le k_n\}} \big(F, Dv\big)\, dx$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

There exists a constant c such that $|\mu_{k_n}|(\Omega) \le c$ for every n. Furthermore, for every $h \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\Omega)$ we can choose $h(T_{k_n}(u))\varphi$ as test function in (4.11). Since $h(u) = h(T_{k_n}(u))$ and $D(h(u)\varphi) = D(h(T_{k_n}(u))\varphi)1_{\{|u| \le k_n\}}$ for n large, we obtain

$$\text{(4.12)} \quad \int_\Omega \big(A(x, Du), D(h(u)\varphi)\big)\, dx = \int_\Omega h(u)\varphi\, d\mu_{k_n} + \int_\Omega \big(F, D(h(u)\varphi)\big)\, dx\,,$$

and hence

$$\begin{aligned}\int_\Omega \big(A(x, Du) - F, D(h(u)\varphi)\big)\, dx &\le \|h\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\Omega)} |\mu_{k_n}|(\Omega) \\ &\le c\, \|h\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\Omega)}\end{aligned}$$

for every $\varphi \in C_c^\infty(\Omega)$ and for every $h \in C_c^\infty(\mathbb{R})$. □

5. – Proof of Theorem 1.2

The proof of Theorem 1.2 relies on a careful description of the measures μ_k used in (3.1).

LEMMA 5.1. *Let $\mu \in \mathcal{M}_b(\Omega)$, let $F \in L^q(\Omega, \mathbb{R}^N)$, and let u be a reachable solution of* (2.1). *For almost every $k > 0$ let $\mu_k \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ be the measure introduced in Theorem* 3.1. *Then there exists a measure $\nu \in \mathcal{M}_{b,0}^p(\Omega)$ such that $\nu \llcorner \{|u| < k\} = \mu_\ell \llcorner \{|u| < k\}$ for every $k > 0$ and for every $\ell \ge k$ for which μ_ℓ is defined.*

PROOF. As a first step we prove that $\mu_\ell \llcorner \{|u| < k\} = \mu_k \llcorner \{|u| < k\}$ for every $\ell \ge k > 0$ for which μ_ℓ and μ_k are defined. Since the set $\{|u| < k\}$ is C_p-quasi open, by Lemma 2.1 there exists an increasing sequence (v_n) of nonnegative functions in $W_0^{1,p}(\Omega)$ which converges to $1_{\{|u|<k\}}$ C_p-q.e. in Ω. For

every $\varphi \in C_c^\infty(\Omega)$, we can choose $v_n\varphi$ as test function in (3.1). As $v_n = 0$ a.e. in $\{|u| \geq k\}$, we obtain

$$\int_\Omega v_n\varphi \, d\mu_k = \int_{\{|u|\leq k\}} \big(A(x, Du) - F, D(v_n\varphi)\big) \, dx$$
$$= \int_{\{|u|\leq \ell\}} \big(A(x, Du) - F, D(v_n\varphi)\big) \, dx = \int_\Omega v_n\varphi \, d\mu_\ell$$

for every $\ell \geq k$. Passing to the limit as n goes to ∞, we get

$$\int_{\{|u|<k\}} \varphi \, d\mu_\ell = \int_{\{|u|<k\}} \varphi \, d\mu_k$$

for every $\varphi \in C_c^\infty(\Omega)$, which yields $\mu_\ell \llcorner \{|u| < k\} = \mu_k \llcorner \{|u| < k\}$. This implies that there exists a unique Borel measure ν such that $\nu \llcorner \{|u| = +\infty\} = 0$ and $\nu \llcorner \{|u| < k\} = \mu_\ell \llcorner \{|u| < k\}$ for every $k > 0$ and for every $\ell \geq k$ for which μ_ℓ is defined. As μ_k vanishes on all sets of p-capacity zero, the same property holds for ν. Finally, by Theorem 3.1 there exists a sequence (k_n) of positive numbers tending to $+\infty$ such that the measures $|\mu_{k_n}|$ are bounded uniformly with respect to n. This implies that the sequence $|\nu|(\{|u| < k_n\})$ is bounded, hence $|\nu|(\Omega) < +\infty$. □

REMARK 5.2. If we apply (4.12) with n large enough, from Lemma 5.1 we obtain that

$$\int_\Omega \big(A(x, Du), D(h(u)\varphi)\big) \, dx = \int_\Omega h(u)\varphi \, d\nu + \int_\Omega \big(F, D(h(u)\varphi)\big) \, dx \tag{5.1}$$

for every $h \in C_c^\infty(\mathbb{R})$ and for every $\varphi \in C_c^\infty(\Omega)$. Moreover, it is easy to see, using a standard approximation argument, that (5.1) holds for every test function $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

LEMMA 5.3. *Let $\mu \in \mathcal{M}_b(\Omega)$, let $F \in L^q(\Omega, \mathbb{R}^N)$, and let u be a reachable solution of* (2.1). *For almost every $k > 0$ let $\mu_k \in W^{-1,q}(\Omega) \cap \mathcal{M}_b(\Omega)$ be the measures introduced in Theorem* 3.1. *Then $|\mu_k|(\{|u| > k\}) = 0$ for every $k > 0$ for which μ_k is defined.*

PROOF. Let us fix $k > 0$ for which μ_k is defined. As u is C_p-quasi continuous, the set $U = \{|u| > k\}$ is C_p-quasi open. Then, for every open subset V of Ω there exists an increasing sequence (v_n) of nonnegative functions in $W_0^{1,p}(\Omega)$ which converges to $1_{U\cap V}$ C_p-q.e. in Ω. If we choose v_n as test function in (3.1), we obtain

$$\int_\Omega v_n \, d\mu_k = \int_{\{|u|\leq k\}} \big(A(x, Du) - F, Dv_n\big) \, dx = 0 \,,$$

where the last equality is due to the fact that $v_n = 0$ a.e. in $\{|u| \leq k\}$. Thus, taking the limit as n goes to ∞, we get $(\mu_k \llcorner U)(V) = 0$ for every open subset V of Ω, which concludes the proof. □

LEMMA 5.4. *Let $\mu \in \mathcal{M}_b(\Omega)$, let $F \in L^q(\Omega, \mathbb{R}^N)$, and let u be a reachable solution of* (2.1). *For almost every $k > 0$ let μ_k be the measure introduced in Theorem* 3.1, *and let $\alpha_k = \mu_k \llcorner \{u = k\}$ and $\beta_k = -\mu_k \llcorner \{u = -k\}$. Then for almost every $k > 0$ we have*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\{k-\varepsilon<u<k\}} \big(A(x, Du) - F, Du\big)\varphi \, dx = \int_\Omega \varphi \, d\alpha_k \,, \tag{5.2}$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\{-k<u<-k+\varepsilon\}} \big(A(x, Du) - F, Du\big)\varphi \, dx = \int_\Omega \varphi \, d\beta_k \,, \tag{5.3}$$

for every $\varphi \in C_0(\Omega)$. Moreover there exist a sequence (k_n) of positive numbers, tending to $+\infty$, and two nonnegative measures α, β in $\mathcal{M}_b(\Omega)$, such that (α_{k_n}) converges to α and (β_{k_n}) converges to β in the weak topology of $\mathcal{M}_b(\Omega)$.*

PROOF. For every $0 < \varepsilon < k$, consider the Lipschitz continuous function $\omega_{k,\varepsilon}$ defined by

$$\begin{cases} \omega_{k,\varepsilon}(t) = 0 & \text{if } t \leq k - \varepsilon \,, \\ \omega_{k,\varepsilon}(t) = 1 & \text{if } t \geq k \,, \\ \omega'_{k,\varepsilon}(t) = 1/\varepsilon & \text{if } k - \varepsilon < t < k \,. \end{cases}$$

Let $k > 0$ be such that μ_k is defined. Since $\omega_{k,\varepsilon}(0) = 0$, for every $\varphi \in C^1(\overline{\Omega})$ we can choose $h_\varepsilon(T_k(u))\varphi$ as test function in (3.1), and we get

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{\{k-\varepsilon<u<k\}} \big(A(x, Du) - F, Du\big)\varphi \, dx \\ &\quad = \int_\Omega \omega_{k,\varepsilon}(u)\varphi \, d\mu_k - \int_{\{k-\varepsilon<u<k\}} \big(A(x, Du) - F, D\varphi\big)\omega_{k,\varepsilon}(u) \, dx \,. \end{aligned} \tag{5.4}$$

By the dominated convergence theorem for every $\varphi \in C^1(\overline{\Omega})$ the right hand side of (5.4) converges to $\int_{\{u \geq k\}} \varphi \, d\mu_k$ as $\varepsilon \to 0^+$.

If we set

$$g_{k,\varepsilon} = \frac{1}{\varepsilon}(A(x, Du) - F, Du) 1_{\{k-\varepsilon<u<k\}} \,,$$

then (5.4) implies that for almost every $k > 0$ and for every $\varphi \in C^1(\overline{\Omega})$ we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega g_{k,\varepsilon} \, \varphi \, dx = \int_{\{u \geq k\}} \varphi \, d\mu_k = \int_\Omega \varphi \, d\alpha_k \,, \tag{5.5}$$

where the last equality follows from Lemma 5.3. On the other hand, by (H3) the negative part of $g_{k,\varepsilon}$ satisfies

$$g^-_{k,\varepsilon} \leq \frac{1}{\varepsilon} |A(\cdot, 0) - F| |Du| \, 1_{\{k-\varepsilon<u<k\}} \leq \frac{1}{\varepsilon} (a_1 + |F|) |Du| \, 1_{\{k-\varepsilon<u<k\}} \,.$$

Let $I(k)$ and $J(k)$ be the function defined in (4.2) with $a = a_1 + |F|$. By Lemma 4.2 for almost every $k > 0$ we have

$$\limsup_{\varepsilon\to 0^+} \|g^-_{k,\varepsilon}\|_{L^1(\Omega)} \le J'(k) < +\infty\,. \tag{5.6}$$

Moreover, there exists a sequence (k_n) of positive numbers tending to $+\infty$ such that (4.10) holds and

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0^+} \|g^-_{k_n,\varepsilon}\|_{L^1(\Omega)} \le \lim_{n\to\infty} J'(k_n) = 0\,. \tag{5.7}$$

By using $\varphi = 1$ in (5.5) we obtain that for every $k > 0$ the integral $\int_\Omega g_{k,\varepsilon} dx$ is bounded as $\varepsilon \to 0^+$. Since $|g_{k,\varepsilon}| = g_{k,\varepsilon} + 2g^-_{k,\varepsilon}$, from (5.6) we obtain that for almost every $k > 0$

$$\limsup_{\varepsilon\to 0^+} \|g_{k,\varepsilon}\|_{L^1(\Omega)} < +\infty\,.$$

Then an easy approximation argument shows that for these values of k (5.5) holds also for every $\varphi \in C_0(\Omega)$, and this proves (5.2). Equality (5.3) can be proved in a similar way.

Finally, by (5.7) we have

$$\lim_{n\to\infty} \alpha^-_{k_n}(\Omega) \le \lim_{n\to\infty} \limsup_{\varepsilon\to 0^+} \|g^-_{k_n,\varepsilon}\|_{L^1(\Omega)} = 0\,. \tag{5.8}$$

On the other hand, by (4.9) and (4.10) we have $|\alpha_{k_n}|(\Omega) \le |\mu_{k_n}|(\Omega) \le K(M+2)$ for every n. Hence there exists a subsequence, still denoted by (k_n), such that (α_{k_n}) converges weakly* to a measure $\alpha \in \mathcal{M}_b(\Omega)$, and (5.8) implies that α is nonnegative. The proof for β is similar. □

Proof of Theorem 1.2 Clearly, if $u \in \mathcal{T}^{1,p}_0(\Omega)$ and u solves (1.5), then u satisfies (S1), (S2) and (S3), and this implies that u is a reachable solution by Theorem 1.1.

Conversely, let us suppose that u is a reachable solution, and let α_k, β_k, and (k_n) be the measures and the sequence introduced in Lemma 5.4. Given $h \in \mathrm{Lip}_0(\mathbb{R})$ and $\varphi \in C^\infty_c(\Omega)$, we put $h(T_{2k_n}(u))\varphi$ as test function in (3.1). Since $T_{2k_n}(u) = u$ in $\{|u| \le k_n\}$, $\mu_{k_n} = \nu \llcorner \{|u| < k_n\} + \alpha_{k_n} - \beta_{k_n}$, $\alpha_{k_n}(\{u \ne k_n\}) = 0$, and $\beta_{k_n}(\{u \ne -k_n\}) = 0$, we obtain

$$\begin{aligned}&\int_{\{|u|\le k_n\}} \big(A(x,Du) - F, Du\big)h'(u)\varphi\, dx \\ &\quad + \int_{\{|u|\le k_n\}} \big(A(x,Du) - F, D\varphi\big)h(u)\, dx \\ &= \int_{\{|u|<k_n\}} h(u)\varphi\, d\nu + h(k_n)\int_\Omega \varphi\, d\alpha_{k_n} - h(-k_n)\int_\Omega \varphi\, d\beta_{k_n}\,.\end{aligned} \tag{5.9}$$

Thanks to Theorem 2.5 (iii) the function $A(\cdot, Du)$ belongs to $L^1(\Omega, \mathbb{R}^N)$. Moreover, since $u \in \mathcal{T}_0^{1,p}(\Omega)$ and h' has compact support, the function $\big(A(\cdot, Du) - F, Du\big)h'(u)$ belongs to $L^1(\Omega)$. Thus, by the dominated convergence theorem, we can pass to the limit as $n \to \infty$ in every integral of (5.9), obtaining that u satisfies (1.5) for every $\varphi \in C_c^\infty(\Omega)$ and for every $h \in \mathrm{Lip}_0(\mathbb{R})$.

If we take $h = 1$ in (1.5), we obtain

$$\int_\Omega \big(A(x, Du) - F, D\varphi\big)\, dx = \int_\Omega \varphi\, d(\nu + \alpha - \beta)\,. \tag{5.10}$$

Since, by Theorem 2.5 (iv), u is a solution of $\mathcal{A}(u) = \mu - \mathrm{div}(F)$ in the sense of distributions in Ω, from (5.10) we obtain that $\nu + \alpha - \beta = \mu$. □

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