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Mean Curvature of a Measure and Related Variational Problems

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In memoria di Ennio De Giorgi

Abstract. We introduce the notion of generalized mean curvature of a measure. We then focus attention on functionals depending on curvatures, investigating their weak lower semicontinuity. A crucial role in this study is played by the dimension of the tangent spaces to a measure.

1. - Introduction

The notion of mean curvature of a manifold is an important tool for many problems in Analysis and in Geometry. When we deal with the classical case of a smooth manifold, the definition of mean curvature can be given in terms of second order derivatives of parametrizations, or in terms of first derivatives of tangent fields. However, when we deal with minimization problems involving manifolds as unknown, minimizing sequences of classical smooth manifolds may tend to more irregular objects, so that several theories have been developed to take into account singularities, and to define for these weak objects too some generalized notions of tangential derivatives and mean curvature. This is for instance the case of the theory of rectifiable and general varifolds, for which we refer to the treatises [4], [10].

Here, we try to consider as generalized manifolds even weaker objects as measures. The definition of tangent space associated to a measure μ has been given in the paper [1], together with the corresponding notion of tangential operator D_μ; these tools revealed crucial for some shape optimization problems, considered in [2].

In this paper, in a similar way, we give the definition of generalized mean curvature $H(\mu)$ associated to a measure μ, and we prove a lower semicontinuity result for a class of functionals of the form

$$F(\mu) = \int_{\mathbb{R}^n} f(x, \mu, H(\mu))\, d\mu \ .$$

For instance, the classical

$$F(M)=\int_M (1+|h(M)|^2)\,d\mathcal{H}^k \ ,$$

where $h(M)$ is the mean curvature vector of the smooth k-manifold M, becomes in this weaker framework

$$F(\mu)=\int_{\mathbb{R}^n} (1+|h(\mu)|^2)\,d\mu \ ,$$

where the generalized pointwise curvature $h(\mu)$ is defined as the absolute continuous part of $H(\mu)$ with respect to μ. By means of a suitable notion of boundary $\partial\mu$ of a measure μ we shall be able to consider minimization problems with "Dirichlet boundary conditions"

$$\min\{F(\mu)\ :\ \partial\mu=\lambda\}\ ,$$

where λ is a given measure. In the smooth case this would correspond to the problem

$$\min\{F(M)\ :\ \partial M=\Gamma\ ,\ \nu_M=\gamma\}\ ,$$

where Γ is a prescribed boundary, and γ an assigned normal field.

We give here a brief outline of the paper.

In Section 2 we introduce the notion of curvature of a measure and the space $\mathcal{M}_{BC}$ of measures with bounded curvature; after showing some examples, we prove a monotonicity lemma for measures with bounded curvature whose tangent space has constant dimension.

In Section 3 we study the lower semicontinuity for a class of functionals depending on curvatures, and we show how this problem is related to the lower semicontinuity of the dimension of tangent spaces with respect to the weak convergence of measures.

The study of the lower semicontinuity for the dimension of tangent spaces is treated in Section 4 under two different groups of assumptions: the former is similar to known results about k-varifolds, while the latter involves a more interesting and rather curious condition on the logarithm of the densities.

Section 5 is devoted to some examples and applications.

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2. – The curvature of a measure

Let μ be a positive and finite Borel regular measure on $\mathbb{R}^n$. In the following the notation $\mathcal{M}^+$ will be used to denote the class of positive and finite Borel regular measures on $\mathbb{R}^n$, while $\mathcal{M}^d$ will denote the space of Borel regular $\mathbb{R}^d$-valued measures ν with finite total variation $|\nu|$ on $\mathbb{R}^n$ (in the case $d = 1$ we simply write $\mathcal{M}$ instead of $\mathcal{M}^1$). Then, since throughout the paper we are dealing with measures on $\mathbb{R}^n$, the domain of integration is always omitted whenever it coincides with all $\mathbb{R}^n$.
We are going to define a notion of curvature for μ. To this aim we look at the classical divergence theorem for a C^2 k-manifold $M \subset \mathbb{R}^n$ with boundary ∂M and for a vector field $X \in C^1(M, \mathbb{R}^n)$:

$$(2.1)\qquad \int_M \operatorname{div}_M X \, d\mathcal{H}^k = -\int_{\partial M} X \cdot \eta \, d\mathcal{H}^k - \int_M X \cdot h \, d\mathcal{H}^k \ ,$$

where h is the mean curvature vector of M, and η is the inward co-normal versor of ∂M (see for example [10]).
In view of (2.1) it seems reasonable to give the following definition for the curvature of a measure μ. We recall that, following Bouchitté, Buttazzo and Seppecher [1], the tangent space to μ can be defined, for μ-a.e. x, as

$$(2.2)\qquad T_\mu(x) := \mu - \operatorname{ess} \bigcup \{\Phi(x) : \Phi \in X_\mu\} \ ,$$

where

$$X_\mu := \{\Phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ |\Phi| \in L^1_\mu \ , \ \operatorname{div}(\Phi\mu) \in \mathcal{M}\} \ .$$

For μ-a.e. x we denote by $P_\mu(x, \cdot) : \mathbb{R}^n \to T_\mu(x)$ the orthogonal projection of $\mathbb{R}^n$ on $T_\mu(x)$.

DEFINITION 2.1. The curvature of μ is defined as the vector-valued distribution

$$H(\mu) := \operatorname{div}(P_\mu \mu) \ .$$

In other words $H(\mu)$ is defined by

$$(2.3)\qquad \langle H(\mu), X \rangle = -\int \operatorname{div}_\mu X \, d\mu \qquad \forall X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \ ,$$

where $\operatorname{div}_\mu X = \sum_{i=1}^n \left(P_\mu(\nabla X^i)\right)_i$.

It is now quite natural to consider the class of all measures μ such that $H(\mu)$ is a vector measure.

DEFINITION 2.2. We denote by $\mathcal{M}_{BC}$, which stays for measures with bounded curvature, the class

$$\mathcal{M}_{BC} := \left\{\mu \in \mathcal{M}^+ \ : \ H(\mu) \in \mathcal{M}^n\right\} \ .$$

REMARK 2.3. The curvature $H(\mu)$ of a measure $\mu \in \mathcal{M}_{BC}$ is not necessarily absolutely continuous with respect to μ (see for instance Examples 2.5 and 2.7 below). In particular, by the Radon-Nikodym theorem, when μ belongs to $\mathcal{M}_{BC}$, we can write

$$H(\mu) = h(\mu)\mu + \partial\mu \ ,$$

where $h(\mu) \in L^1_\mu(\mathbb{R}^n, \mathbb{R}^n)$ is the density of $H(\mu)$ with respect to μ, and $\partial\mu$ is the singular part of $H(\mu)$ with respect to μ. In the sequel the notation $h(\mu)$ will be always used to indicate the density $\frac{dH(\mu)}{d\mu}$; we shall call $h(\mu)$ the *pointwise curvature* of μ and $\partial\mu$ the *boundary* of μ.

REMARK 2.4. Definition 2.2 seems to be a quite natural generalization of the notion of function with bounded variation. Actually one immediately sees that if u is a real function defined on an open subset Ω of $\mathbb{R}^n$ with a Lipschitz boundary, u belongs to $BV(\Omega)$ if and only if the measure $\mu = u\mathcal{L}^n \llcorner \Omega$ belongs to $\mathcal{M}_{BC}$ and in this case the curvature $H(\mu)$ coincides with the measure $Du - u\nu\mathcal{H}^{n-1} \llcorner \partial\Omega$, where Du is the gradient of u in the sense of distributions and ν is the exterior normal versor to $\partial\Omega$. However, due to the nonlinearity of the mapping $\mu \mapsto H(\mu)$, the natural metric on $\mathcal{M}_{BC}$ defined by

$$\delta(\mu, \nu) := |\mu - \nu| + |H(\mu) - H(\nu)| \ ,$$

cannot be associated to a norm.

EXAMPLE 2.5. In the case $\mu = \mathcal{H}^k \llcorner M$, with M a C^2 k-manifold with boundary in $\mathbb{R}^n$, by (2.1) one immediately obtains

$$H(\mu) = \eta\mathcal{H}^{k-1} \llcorner \partial M + h\,\mathcal{H}^k \llcorner M\,,$$

where h denotes the mean curvature vector of M and η the co-normal versor of ∂M.

EXAMPLE 2.6. When the tangent space to μ defined in (2.2) is reduced to zero μ-almost everywhere, $H(\mu)$ is of course zero. This is for instance the case if μ is a finite sum of Dirac masses, or else $\mu = \mathcal{H}^\alpha \llcorner C$ with C an α-dimensional Cantor set contained in $[0, 1]$ with $\mathcal{H}^\alpha(C) \in (0, +\infty)$ (see [5]).

EXAMPLE 2.7. Let us consider two cases in which there are corner points, which give rise to a "boundary effect". Let μ_1 and μ_2 be the restrictions of the one dimensional Hausdorff measure $\mathcal{H}^1$ on the structures represented respectively in Figures 1a and 2b below.
In both cases the boundary effect consists in the sum of three oriented Dirac masses. In fact it can be easily verified, by using directly Definition 2.1, that we have

$$H(\mu_1) = v_A\delta_A + v_B\delta_B + v_C\delta_C$$

and

$$H(\mu_2) = v_D\delta_D + v_E\delta_E + v_F\delta_F \ ,$$

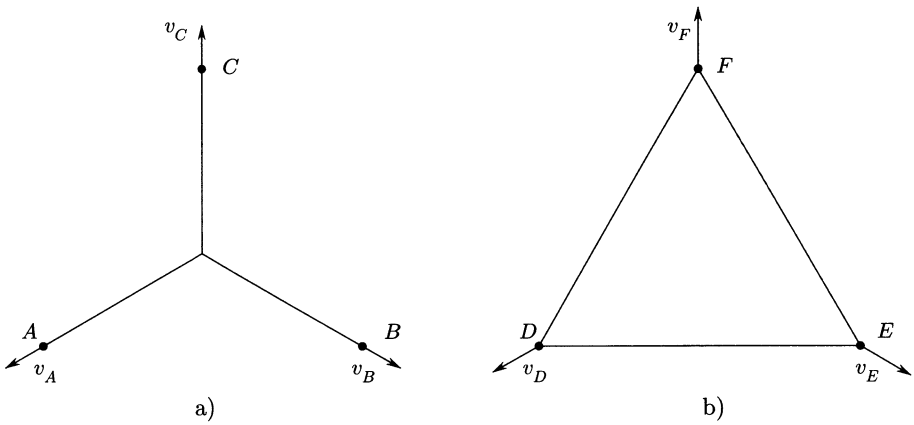

Fig. 1.

with $|v_A| = |v_B| = |v_C| = 1$ and $|v_D| = |v_E| = |v_F| = \sqrt{3}$.

EXAMPLE 2.8. Let $V = \underline{v}(M, \theta)$ be a k-rectifiable varifold, and let $\mu_V = \theta \mathcal{H}^k \llcorner M$ be the associated weight measure. A function $h \in L^1_{\mu_V}(M, \mathbb{R}^n)$ is said to be the generalized mean curvature of V if

$$\int \operatorname{div}_M X \, d\mu_V = -\int X \cdot h \, d\mu_V \qquad \forall X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \; .$$

Moreover it is well known that the left hand side is equal to the first variation of mass

$$\langle \delta V, X \rangle := \frac{d}{dt} \underline{M}(\varphi_{t_\sharp} V)_{|_{t=0}} \; ,$$

where $\underline{M}(\varphi_{t_\sharp} V)$ denotes the mass of the image varifold $\varphi_{t_\sharp} V$, being φ_t a one-parameter group of smooth diffeomorphisms of $\mathbb{R}^n$ into itself with X as initial velocity (see [10] for the details).
Now, let us suppose that the first variation of V is a bounded measure. In this case T_{μ_V} coincides with the approximate tangent space to μ_V (see [5] for a proof). Then μ_V belongs to $\mathcal{M}_{BC}$, $H(\mu_V)$ is equal to $-\delta V$, and V has generalized mean curvature h if and only if $H(\mu_V)$ is absolutely continuous with respect to μ_V, with density h.

The following lemma is a useful result for the measures with bounded curvature whose tangent space has constant dimension. It is proved with the same technique used in the literature for the case of rectifiable varifolds.

LEMMA 2.9. *If $\mu \in \mathcal{M}_{BC}$ is such that $\dim T_\mu(x) = k$ for μ-a.e. x, then the k-dimensional density of μ at x*

$$\theta_k(\mu, x) := \lim_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}$$

exists and it is finite μ-a.e. (here $\omega_k = \pi^{k/2}\Gamma(1+k/2)$, being $\Gamma(t) = \int_o^{+\infty} s^{t-1}e^{-s}ds$ the Euler function). If moreover θ_k is strictly positive μ-a.e., then there exists a k-rectifiable subset $M \subset \mathbb{R}^n$ such that

$$\mu = \theta_k \mathcal{H}^k \llcorner M \ .$$

Proof. For any fixed $\xi \in \mathbb{R}^n$, we choose in (2.3) a vector field $X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$ of the form

$$X(x) = \gamma(r)(x-\xi) \ ,$$

where $r = |x-\xi|$, and $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth nonincreasing function such that for a positive ρ we have $\gamma(r) = 0$ for $r \geq \rho$ and $\gamma(r) = 1$ for $r \leq \rho/2$. We have

$$DX(x) = \gamma(r)\,\mathrm{Id} + r\gamma'(r)\left(\frac{x-\xi}{r} \otimes \frac{x-\xi}{r}\right) \ ;$$

thus, if we compute $\mathrm{div}_\mu X$ taking into account the assumption on the dimension of T_μ, we get

$$\mathrm{div}_\mu X(x) = k\gamma(r) + r\gamma'(r)\left(1 - |D^\perp r|^2\right) \ ,$$

where $D^\perp r = Dr - D_\mu r$. Substituting this into (2.3), we obtain

$$\langle H(\mu), X\rangle = -k\int \gamma(r)\,d\mu - \int r\gamma'(r)\,d\mu + \int r\gamma'(r)|D^\perp r|^2\,d\mu \ .$$

In particular we can take $\gamma(r) = \Phi(r/\rho)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a smooth nonincreasing function such that $\Phi(t) = 0$ for $t \geq 1$ and $\Phi(t) = 0$ for $t \leq 0$. Then the last equality can be rewritten as

$$\begin{aligned}\langle H(\mu), X\rangle &= -k\int \Phi\left(\frac{r}{\rho}\right)d\mu + \int \rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)d\mu - \int \rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)|D^\perp r|^2 d\mu \\ &= -k\int \Phi\left(\frac{r}{\rho}\right)d\mu + \rho\frac{\partial}{\partial\rho}\int \Phi\left(\frac{r}{\rho}\right)d\mu - \rho\frac{\partial}{\partial\rho}\int \Phi\left(\frac{r}{\rho}\right)|D^\perp r|^2 d\mu.\end{aligned}$$

Now we let Φ increase to the characteristic function of $(-\infty, 1)$, so that $\Phi(r/\rho)$ increases to $\chi_{B_\rho(\xi)}(r)$; we obtain

$$\langle H(\mu), \chi_{B_\rho(\xi)}(r)(x-\xi)\rangle = -k\mu(B_\rho(\xi)) + \rho\frac{\partial}{\partial\rho}\mu(B_\rho(\xi)) - \rho\frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} |D^\perp r|^2 d\mu,$$

and dividing by ρ^{k+1}:

$$\begin{aligned}\frac{1}{\rho^k}\left\langle H(\mu), \chi_{B_\rho(\xi)}(r)\frac{(x-\xi)}{\rho}\right\rangle &= -\frac{k}{\rho^{k+1}}\mu(B_\rho(\xi)) + \frac{1}{\rho^k}\frac{\partial}{\partial\rho}\mu(B_\rho(\xi)) \\ &\quad - \frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^k}\,d\mu \\ &= \frac{\partial}{\partial\rho}\left[\frac{\mu(B_\rho(\xi))}{\rho^k}\right] - \frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^k}\,d\mu \ .\end{aligned}$$

Setting $f(\xi, \rho) := \mu(B_\rho(\xi))/\rho^k$ we have

$$\frac{\partial}{\partial \rho} f(\xi, \rho) \geq -\frac{1}{\rho^k} |H(\mu)|(B_\rho(\xi)) ,$$

which implies

$$\frac{1}{f(\xi, \rho)} \frac{\partial}{\partial \rho} f(\xi, \rho) \geq -\frac{|H(\mu)|(B_\rho(\xi))}{\mu(B_\rho(\xi))} . \tag{2.4}$$

If we choose ξ as a density point of the total variation measure of $H(\mu)$ with respect to μ, the limit as $\rho \to 0$ of the right hand side of the above inequality exists and is finite, so that μ-a.e. we have

$$\frac{1}{f(\xi, \rho)} \frac{\partial}{\partial \rho} f(\xi, \rho) \geq -c(\xi)$$

for a suitable nonnegative constant $c = c(\xi)$. This means that, for μ-a.e. $\xi \in \mathbb{R}^n$ the function

$$\rho \mapsto \exp[c(\xi)\rho] f(\xi, \rho)$$

is nondecreasing, and in particular that for μ-a.e. $\xi \in \mathbb{R}^n$ the function $f(\xi, \cdot)$ admits a finite limit when ρ tends to zero.
The last statement follows from known rectifialility results, which can be found for instance on a fundamental paper by Preiss [8], and on a more recent paper by Fragalà and Mantegazza [5]. □

3. – Lower semicontinuous functionals of curvatures

In view of considering minimum problems involving curvatures, we study in this paragraph the lower semicontinuity of functionals $F : \mathcal{M}_{BC} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$F(\mu) = \begin{cases} J(\mu, H(\mu)) & \text{if } \mu \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

where J is defined on the product space $\mathcal{M}_{BC} \times \mathcal{M}^n$ and $\mathcal{A}$ is a subclass of measures in $\mathcal{M}_{BC}$. If J is coercive and weakly lower semicontinuous with respect to the weak convergence of measures, the existence of a minimum for J follows straightforward by direct methods of the calculus of variations as soon as the set $\{(\mu, H(\mu)) : \mu \in \mathcal{A}\}$ is weakly closed in $\mathcal{M}_{BC} \times \mathcal{M}^n$. Since this property does not hold if we choose $\mathcal{A} = \mathcal{M}_{BC}$ (see Example 3.6), the main point of this section will be to exhibit some suitable subclasses $\mathcal{A}$ for which the direct methods can be applied.
As a preliminary tool, we need an upper semicontinuity theorem for the dimension of the tangent space; actually we prove that if a sequence of measures $\{\mu_h\}_h$ is bounded in $\mathcal{M}_{BC}$, when we pass to the weak limit, the dimension of the tangent space cannot decrease. The proof is obtained by using standard convex analysis arguments; it is based on the following lemma.

LEMMA 3.1. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$. Let d be a positive integer and $\{f_h\}_h$ be a sequence belonging to $L^p_{\mu_h}(\mathbb{R}^n, \mathbb{R}^d)$ for some $p \in (1, +\infty]$. Suppose there exists a positive constant M such that*

$$\|f_h\|_{L^p_{\mu_h}} \leq M \qquad \textit{for every } h\ .$$

Then the sequence $\{f_h \mu_h\}_h$ is bounded in $\mathcal{M}^d$, and the weak limit of any convergent subsequence is absolutely continuous with respect to μ, with a density f in $L^p_\mu(\mathbb{R}^n, \mathbb{R}^d)$. Moreover for every convex and lower semicontinuous function $\Phi : \mathbb{R}^d \to [0, +\infty)$ we have

$$\liminf_{h\to+\infty} \int \Phi(f_h)\, d\mu_h \geq \int \Phi(f)\, d\mu\ .$$

PROOF. By Hölder inequality, we have

$$\int |f_h|\, d\mu_h \leq \left(\int |f_h|^p\, d\mu_h\right)^{1/p} (\mu_h(\mathbb{R}^n))^{1/p'}\ ,$$

so that the sequence $\{f_h\, \mu_h\}_h$ is uniformly bounded in variation, hence weakly relatively compact in $\mathcal{M}^d$. Possibly passing to a subsequence, we can assume that $\{f_h\, \mu_h\}_h \rightharpoonup \nu$. Applying this convergence to a test function $\varphi \in C^0_c(\mathbb{R}^n, \mathbb{R}^d)$ and using Fenchel's inequality, we get

$$\begin{aligned}\liminf_{h\to+\infty} \frac{1}{p} \int |f_h|^p d\mu_h &\geq \liminf_{h\to+\infty} \left(\int f_h\, \varphi\, d\mu_h - \frac{1}{p'} \int |\varphi|^{p'} d\mu_h\right)\\ &\geq \langle \nu, \varphi\rangle - \frac{1}{p'} \int |\varphi|^p d\mu\ .\end{aligned}$$

As the left hand side of the above inequality is bounded, we deduce that

$$\sup\left\{ < \nu, \varphi > \ :\ \varphi \in C^0_c(\mathbb{R}^n, \mathbb{R}^d)\ ,\ \|\varphi\|_{L^{p'}} \leq 1\right\} < +\infty\ .$$

Thus ν, extended to $L^{p'}_\mu(\mathbb{R}^n, \mathbb{R}^d)$, can be identified with an element f belonging to $L^p_\mu(\mathbb{R}^n, \mathbb{R}^d)$ (i.e. $\nu = f\, \mu$).
To obtain the last statement, we write

$$\int \Phi(f_h)\, d\mu_h = \int \Psi(f_h, 1)\, d\mu_h\ ,$$

where $\Psi : \mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ is the one homogeneous convex function defined by

$$\Psi(z, t) := \begin{cases} t\, \Phi(z/t) & \text{if } t > 0\\ \Phi^\infty(z) & \text{if } t = 0\ .\\ +\infty & \text{if } t < 0\end{cases}$$

Here $\Phi^\infty(z) = \lim_{t\to 0} t\,\Phi(z/t)$ denotes the recession functional of Φ. By the lower semicontinuity of Φ, we obtain that Ψ is lower semicontinuous with respect to the pair (z, t). Then, by Lemma 3.2 below:

$$\begin{aligned}\liminf_{h\to+\infty} \int \Phi(f_h)\, d\mu_h &= \liminf_{h\to+\infty} \int \Psi(f_h, \mu_h)\, d\mu_h \\ &\geq \int \Psi(f, \mu)\, d\mu = \int \Phi(f) d\mu \ . \end{aligned}$$ □

LEMMA 3.2. *Let $\Psi(x, z) : \mathbb{R}^n \times \mathbb{R}^d \to [0, +\infty)$ be a lower semicontinuous function such that, for every $x \in \mathbb{R}^n$, $\Psi(x, \cdot)$ is convex and homogeneous of degree* 1. *Then the functional defined by*

$$\mathcal{M}^d \ni \lambda \mapsto \int \Psi(x, \lambda) := \int \Psi\left(x, \frac{d\lambda}{d|\lambda|}\right) d|\lambda| \,,$$

is weakly lower semicontinuous. Note that $\int \Psi(x, \lambda) = \int \Psi\left(x, \frac{d\lambda}{d\theta}\right) d\theta$ for every λ such that $\lambda \ll \theta$.

PROOF. We refer to Reshetnyak [9]. □

REMARK 3.3. The assumption $\Phi \geq 0$ in Lemma 3.1 can be dropped if we assume that the sequence $\{\mu_h\}_h$ converge tightly to μ. Indeed, we may substitute Φ by $(\Phi - g)$ where g is an affine minorant of Φ and then notice that the tight convergence of $\{\mu_h\}_h$ implies

$$\lim_{h\to+\infty} \int g(f_h) d\mu_h = \int g(f) d\mu \ .$$

We can now state the upper semicontinuity result for the dimension of T_μ.

THEOREM 3.4. *Let $\{\mu_h\}_h$ be a bounded sequence in $\mathcal{M}_{BC}$, i.e. there exist positive constants C_1 and C_2 such that*

$$|\mu_h| \leq C_1 \,, \qquad |H(\mu_h)| \leq C_2 \ . \tag{3.2}$$

Up to subsequences, let μ be the weak limit of $\{\mu_h\}_h$ and let $f\mu$, with $f \in L^\infty_\mu(\mathbb{R}^n)$, be the weak limit of $\{(\dim T_{\mu_h})\mu_h\}_h$. Then we have

$$\dim T_\mu(x) \geq f(x) \qquad \textit{for } \mu\textit{-a.e. } x \ . \tag{3.3}$$

PROOF. We apply Lemma 3.1, by taking $p = +\infty$, and as f_h the functions $P_h : \mathbb{R}^n \to \mathbb{R}^{n^2}$ which map μ_h-a.e. point $x \in \mathbb{R}^n$ into the matrix of the orthogonal projection of $\mathbb{R}^n$ on $T_{\mu_h}(x)$. By the first part of Lemma 3.1, if we pass to a subsequence we can suppose that

$$P_h \mu_h \rightharpoonup A\mu \ , \tag{3.4}$$

with A belonging to $L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^{n^2})$. Now, let us denote by $\mathcal{S}_1(n\times n)$ the space of symmetric matrices M with spectrum $\sigma(M)$ contained in the closed interval $[0,1]$. We take as a function $\Phi : \mathbb{R}^{n^2} \to \mathbb{R}\cup\{+\infty\}$ the function

$$\Phi(M) = \begin{cases} 0 & \text{if } M \in \mathcal{S}_1(n\times n) \\ +\infty & \text{otherwise.} \end{cases}$$

Since by the second part of Lemma 3.1, we must have $\Phi(A) = 0$ μ-a.e., we get that A takes its values in the symmetric matrices with the spectrum contained in $[0,1]$, so that

$$\text{rank}(A) \geq \text{tr}(A) \qquad \mu\text{-a.e.} \tag{3.5}$$

Next, let us observe that (3.4) yields the weak convergence of $H(\mu_h)$ to $\operatorname{div} A\mu$ in the sense of distributions. Then by the hypothesis (3.2) on the boundedness of the curvatures in $\mathcal{M}^n$, we must also have the weak convergence of the measures $H(\mu_h)$ to $\operatorname{div} A\mu$. In particular $\operatorname{div} A\mu$ turns out to be a measure, and this gives the crucial information

$$\text{rank}(A) \leq \dim T_\mu \qquad \mu\text{-a.e.} \tag{3.6}$$

Moreover, by taking the trace in (3.4), we obtain

$$(\dim T_{\mu_h})\mu_h \rightharpoonup (\text{tr}\, A)\mu \ . \tag{3.7}$$

Taking into account (3.5), (3.6), and (3.7), we get (3.3); indeed we have

$$(\dim T_\mu)\mu \geq (\text{rank}\, A)\mu \geq (\text{tr}\, A)\mu = f\mu \ . \qquad \square$$

REMARK 3.5. In particular, when $\dim T_\mu(x) = k$ for μ_h-a.e. x, with k a fixed integer, by the above theorem we get that the dimension of T_μ cannot be smaller than k on a set of positive measure μ.

EXAMPLE 3.6. When the limit measure μ is identically zero, Theorem 3.4 gives evidently no information. In fact it is possible to approximate the zero measure with a sequence of measures bounded in $\mathcal{M}_{BC}$ and having a tangent space of positive dimension. For instance, we can take in $\mathbb{R}^2$

$$\mu_h = \mathcal{H}^1 \llcorner \partial B_{r_h} \ ,$$

where B_{r_h} are the balls of center the origin and radius r_h tending to zero. Then μ_h strongly converges to zero, the dimension of T_{μ_h} is equal one μ_h-a.e., and the total variation of $H(\mu_h)$ is equal 2π for every h.

EXAMPLE 3.7. In (3.3) we can have the strict inequality $\dim T_\mu(x) > f(x)$ for μ-a.e. x. This happens for instance if we take in $\mathbb{R}$

$$\mu_h = \frac{1}{h}\sum_{i=1}^{h} \delta_{i/h} \ ,$$

where $\delta_{i/h}$ is the Dirac mass at i/h. Then μ_h have zero curvature and weakly converge to the Lebesgue measure on $[0,1]$, which has a one-dimensional tangent space.

We are now in a position to state our result of weak convergence for the curvatures. We deduce it as a direct consequence of Theorem 3.4.

THEOREM 3.8. *Let $\{\mu_h\}_h$ be a bounded sequence in $\mathcal{M}_{BC}$ such that $\{\mu_h\}_h$ weakly converge to μ and $\{(\dim T_{\mu_h})\mu_h\}_h$ weakly converge to $f\mu$. Then the condition*

$$\dim T_\mu(x) \leq f(x) \qquad \mu\text{-a.e.} \tag{3.8}$$

is necessary and sufficient to have

$$P_{\mu_h}\mu_h \rightharpoonup P_\mu \mu \,. \tag{3.9}$$

In this case we have also

$$H(\mu_h) \rightharpoonup H(\mu)\,. \tag{3.10}$$

PROOF. By the same proof as in Theorem 3.4, after passing to a subsequence, we can suppose that $P_{\mu_h}\mu_h \rightharpoonup A\mu$, where the function A belongs to $L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^{n^2})$, takes values in $\mathcal{S}_1(n \times n)$, and satisfies

$$(\dim T_\mu)\mu \geq (\operatorname{rank} A)\mu \geq (\operatorname{tr} A)\mu = f\mu \ .$$

Then, by assumption (3.8), we infer

$$\dim T_\mu = \operatorname{rank} A = \operatorname{tr} A \qquad \mu\text{-a.e.}$$

In particular, if we recall that the columns of the matrix A belong to X_μ and that $\sigma(A) \subset [0, 1]$, the above equality means that, for μ-a.e. x, $A(x)$ is the orthogonal projection on $T_\mu(x)$ i.e. $A = P_\mu$. Then, since the selected subsequence was arbitrary, we get (3.9). Finally (3.10) follows, because the sequence $\{H(\mu_h)\}_h$ is bounded in $\mathcal{M}^n$ and, by (3.9), converges to $H(\mu)$ in the distributional sense.

The converse implication is trivial, since (3.9) implies

$$(\operatorname{tr} P_{\mu_h})\mu_h \rightharpoonup (\operatorname{tr} P_\mu)\mu = f\mu\,,$$

so that $f(x) = \dim T_\mu(x)$ for μ-a.e. x. □

REMARK 3.9. Theorem 3.8 reduces the problem of weak convergence of curvatures to the check of the lower semicontinuity condition (3.8) for the dimension of the tangent space. Such condition seems to play a crucial role and brings to the fore a quite involved geometrical problem: does it exist any criterion to decide whether (3.8) is satisfied or not? Section 4 partially answers to this question, by providing some conditions which are sufficient to have (3.8).

A relevant consequence of Theorem 3.8 is the lower semicontinuity of a large class of functionals of the form (3.1).

COROLLARY 3.10. *Let J be a functional defined on the product space $\mathcal{M}^+ \times \mathcal{M}^n$ and let $\mathcal{A}$ be a subclass of measures in $\mathcal{M}_{BC}$. Assume that*

i) *J is weakly lower semicontinuous and coercive i.e.*

$$\lim_{|\lambda|+|\mu|\to+\infty} J(\lambda,\mu) = +\infty\,,$$

ii) *For every $M > 0$, the set $\{(\mu,\nu) \mid \mu \in \mathcal{A}\,,\ |H(\mu)| \leq M\,,\ \nu \geq \dim(T_\mu)\,\mu\}$ is weakly closed in $\mathcal{M}_{BC} \times \mathcal{M}^+$.*

Then the functional $F(\mu)$ defined by (3.1) *is weakly lower semicontinuous and achieves its infimum over $\mathcal{A}$.*

PROOF. Let $\{\mu_h\}_h \subset \mathcal{A}$ be a sequence such that $\mu_h \rightharpoonup \mu$ and set $\nu_h := \dim(T_{\mu_h})\,\mu_h$. Possibly passing to a subsequence, we can assume that $\liminf_{h\to+\infty} F(\mu_h) = \lim_{h\to+\infty} F(\mu_h) < +\infty$. Then by the coercivity assumption i) and by Lemma 2.1 we can find $C > 0$ and $f \in L^\infty_\mu$ such that

$$|\mu_h| + |H((\mu_h)| \leq C \qquad , \qquad \nu_h \rightharpoonup f\,\mu\ .$$

Hence we can use assumption ii) to obtain that μ still belongs to the class $\mathcal{A}$ and that the inequality $f \geq \dim(T_\mu)$ holds μ-almost everywhere. Theorem 3.8 leads to conclude that $H(\mu_h)$ does converge weakly to $H(\mu)$ in $\mathcal{M}^n$ and then, by the lower semicontinuity of J, we have

$$\begin{aligned}\liminf_{h\to+\infty} F(\mu_h) &\geq \liminf_{h\to+\infty} J(\mu_h, H(\mu_h))\\ &\geq J(\mu, H(\mu)) \;=\; F(\mu)\ .\end{aligned}$$

Finally, by condition i), F is coercive and therefore achieves its infimum. □

4. – Lower semicontinuity of the dimension of tangent space

As pointed out in the previous section, the dimension of the tangent space for measures in the space $\mathcal{M}_{BC}$ cannot decrease under a uniform control of the curvatures and, in order to obtain the weak convergence of the curvatures, we need only to check the lower semicontinuity of the dimension of T_μ (see condition (3.8)). To this aim we will use as a main tool Lemma 2.9 and the monotonicity property for measures in $\mathcal{M}_{BC}$.
Since it seems difficult to extend this monotonicity property to measures with varying dimension, we focus here our attention on the case of a sequence of measures $\{\mu_h\}_h$ whose tangent space has constant dimension k. As noticed in Remark 3.5, we are able in this case to prove that the dimension of the limit measure μ must be greater than or equal to k.

A very useful criterion to have the opposite inequality is given in Lemma 4.1. This lemma is deduced from a result proved in [5]; we use the standard notation $\theta_k^*(\mu, x)$ to indicate the k-dimensional upper density of μ at the point x, given by

$$\theta_k^*(\mu, x) := \limsup_{\rho\to 0} \frac{\mu\big(B_\rho(x)\big)}{\omega_k \rho^k} .$$

LEMMA 4.1. *Let k be an integer, $0 \le k \le n$, and let E be a Borel subset of $\mathbb{R}^n$. Then*

$$\theta_k^*(\mu, x) > 0 \quad \mu\text{-a.e. on } E \implies \dim T_\mu(x) \le k \quad \mu\text{-a.e. on } E .$$

PROOF. For any $\alpha \in (k, k+1)$, we have $\theta_\alpha^*(\mu, x) = +\infty \quad \mu$-a.e. on E. Then it is sufficient to apply Lemma 4.1 of [5] to deduce that we have $\dim T_\mu \le [\alpha] = k$ for μ-a.e. $x \in E$. □

Another preliminary lemma deals with a generalization of known results about the Hardy-Littlewood maximal operator. Given two measures λ and μ in $\mathcal{M}^+$, we denote by $M(\lambda, \mu)$ the maximal function of λ with respect to μ, defined by

$$M(\lambda, \mu)(x) := \sup_{\rho>0} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))} .$$

LEMMA 4.2. *Given λ, $\mu \in \mathcal{M}^+$, the following facts hold:*

i) *there exists a constant $c = c(n)$ such that for every $t > 0$*

$$\mu\left(\{x \,:\, M(\lambda, \mu)(x) > t\}\right) \le \frac{c(n)|\lambda|}{t} ;$$

ii) *if $\lambda \ll \mu$, with $\frac{d\lambda}{d\mu} \in L^p_\mu(\mathbb{R}^n)$ for some $p > 1$, then $M(\lambda, \mu) \in L^p_\mu(\mathbb{R}^n)$; moreover there exists a positive constant $c = c(n, p)$ such that*

$$\|M(\lambda, \mu)\|_{L^p_\mu} \le c(n, p) \left\| \frac{d\lambda}{d\mu} \right\|_{L^p_\mu} .$$

PROOF. We refer to the book by Mattila [7], Theorem 2.19. □

We are now in a position to state the main theorem of the section. Let us fix some integer $k \in [0, n]$ and denote by $\mathcal{M}_k$ the subset of measures $\mu \in \mathcal{M}_{BC}$ such that $\dim T_\mu(x) = k$ for μ-a.e. x.

THEOREM 4.3. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}_k$ such that*

$$\mu_h \rightharpoonup \mu , \qquad \sup_h |H(\mu_h)| = C < +\infty.$$

Let θ_h and θ denote the k-dimensional (upper) density of μ_h and μ at x. Then

$$\liminf_{h\to+\infty} \|(\theta_h)^{-1}\|_{L^\infty_{\mu_h}} \ \ge\ \|\theta^{-1}\|_{L^\infty_\mu} .$$

If moreover μ_h satisfies $H(\mu_h) \ll \mu_h$ and $\sup_h \|h(\mu_h)\|_{L^p_{\mu_h}} < +\infty$, for some $p > 1$, then

$$\liminf_{h\to+\infty} \int (\log\theta_h)^- \, d\mu_h \ \geq \ \int (\log\theta)^- d\mu \, .$$

PROOF. We set

$$f_h(x,\rho) := \frac{\mu_h(B_\rho(x))}{\omega_k \rho^k} \quad , \qquad M_h := M(|H(\mu_h)|, \mu_h)$$

and we observe that, by (2.4) (see proof of Lemma 2.9) and by the definition of M_h, we have

$$\frac{1}{f_h(x,\rho)} \frac{\partial}{\partial\rho} f_h(x,\rho) \geq -\frac{|H(\mu_h)|(B_\rho(x))}{\mu_h(B_\rho(x))} \geq -M_h(x) \ .$$

Hence, by monotonicity, for every $\rho > 0$ and for every integer h, the following inequality holds μ_h-a.e.

$$\theta_h(x) \leq \exp(M_h(x)\rho)\, f_h(x,\rho) \ . \tag{4.1}$$

First, let us prove the statement about the lower semicontinuity of $\|\theta^{-1}\|_{L^\infty_\mu}$. Fix $\alpha > 0$. It is enough to prove that if $\theta_h \geq \alpha$ holds μ_h-a.e. (for every h), then $\theta \geq \alpha$ μ-a.e. Let B_{ht} be the subset of $\mathbb{R}^n$ where M_h is greater than t. By Lemma 4.2 i)

$$\mu_h(B_{ht}) \leq \frac{c(n)}{t} |H(\mu_h)| \leq \frac{c(n)C}{t} \ .$$

We set

$$N := \bigcap_{m=1}^{\infty} \bigcup_{h=1}^{\infty} \left[\operatorname{int} \bigcap_{l\geq h} B_{lm} \right]$$

It is easy to check that $\mu(N) = 0$. In fact for every $m \in \mathbb{N}$ we have

$$\mu(N) \leq \mu \left\{ \bigcup_{h=1}^{\infty} \left[\operatorname{int} \bigcap_{l\geq h} B_{lm} \right] \right\} = \lim_{h\to+\infty} \mu \left\{ \operatorname{int} \bigcap_{l\geq h} B_{lm} \right\} \leq$$

$$\leq \liminf_{h\to+\infty} \mu_h \left\{ \bigcap_{l\geq h} B_{lm} \right\} \leq \liminf_{h\to+\infty} \mu_h(B_{lm}) \leq \frac{c(n)C}{m} \ .$$

Let us prove now that $\theta \geq \alpha$ outside N. If $x \notin N$, we can find $\overline{m} \in \mathbb{N}$, a sequence of points $\{x_h\}_h$, and a sequence of integers $\{l(h)\}_h$, with $l(h) \geq h$, such that $|x - x_h| \leq 1/h$ and $x_h \notin B_{l(h)\overline{m}}$. In particular $M_{l(h)}(x_h) \leq \overline{m}$; thus if we take into account (4.1) we get

$$\alpha \leq \theta_{l(h)}(x_h) \ \leq \ \exp(\overline{m}\rho) \ f_{l(h)}(x_h,\rho) \qquad \forall\rho > 0 \ , \forall h \ . \tag{4.2}$$

Now, since the inclusion $\overline{B}_\rho(x_h) \subset \overline{B}_{\rho'}(x)$ holds eventually for every $\rho' > \rho$, the weak convergence of $\{\mu_h\}_h$ to μ implies that

$$\limsup_{h\to+\infty} \mu_h(\overline{B}_\rho(x_h)) \leq \mu(\overline{B}_\rho(x)) .$$

Passing to the limsup in (4.2) and observing that the equality $\mu(B_\rho(x)) = \mu(\overline{B}_\rho(x))$ holds for a dense subset of values of ρ, we infer that $\frac{\mu(B_\rho(x))}{\omega_k \rho^k} \geq \alpha$ for every ρ. The conclusion follows by taking the limsup as ρ tends to 0.
Now, let us prove the second part of the thesis.
Let us assume that the sequence $\{(\log\theta_h(x))^-\}_h$ is bounded in $L^1_{\mu_h}$ (otherwise there is nothing to prove). Then θ_h is strictly positive μ_h-a.e. and by (4.2) we have

$$(\log\theta_h(x))^- \geq (\log f_h)^-(x,\rho) - M_h(x)\rho \ .$$

Integrating this inequality with respect to μ_h, we obtain for every $\rho > 0$

$$\begin{aligned}\liminf_{h\to+\infty}\int(\log\theta_h)^-\,d\mu_h &\geq \liminf_{h\to+\infty}\int(\log f_h)^-\,d\mu_h \\ &\quad - \ \rho\limsup_{h\to+\infty}\int M_h(x)\,d\mu_h \\ &= \text{(I)} - \text{(II)}.\end{aligned} \tag{4.3}$$

Let us consider separately terms (I) and (II). Term (II) can be estimated by using Hölder inequality with some $p > 1$, Lemma 4.2 ii), and the boundedness of the sequence $\{\mu_h\}_h$:

$$\begin{aligned}\rho\int M_h(x)\varphi(x)\,d\mu_h &\leq \rho\int M_h(x)\,d\mu_h \leq \rho\|M_h\|_{L^p_{\mu_h}}|\mu_h|^{1/p'} \\ &\leq \rho c(p)\|h(\mu_h)\|_{L^p_{\mu_h}}|\mu_h|^{1/p'} \leq \Lambda\rho \ ,\end{aligned}$$

where Λ is a suitable positive constant.
As for term (I), we claim that, except possibly for ρ in a countable subset, we have

$$\liminf_{h\to+\infty}\int(\log f_h(x,\rho))^-\,d\mu_h \geq \int(\log f(x,\rho))^-\,d\mu \ , \tag{4.4}$$

where $f(x,\rho) := \frac{\mu(B_\rho(x))}{\omega_k\rho^k}$. Then, putting together the estimates of terms (I) and (II), (4.3) becomes

$$\liminf_{h\to+\infty}\int(\log\theta_h)^-\,d\mu_h \geq \int(\log f(x,\rho))^-\,d\mu \ - \Lambda\rho \ .$$

In view of the definition of θ, we have $\liminf_{\rho\to 0}(\log f(x,\rho))^- \geq (\log\theta(x))^-$. We conclude by letting ρ tend to 0 and by using Fatou's lemma

$$\liminf_{h\to+\infty}\int(\log\theta_h)^-\,d\mu_h \geq \liminf_{\rho\to 0}\int(\log f(x,\rho))^-\,d\mu \geq \int(\log\theta(x))^-\,d\mu\ .$$

It remains to prove claim (4.4). To this aim, we note that we can choose ρ such that we have $\mu\otimes\mu(\{(x,y)\ :\ |x-y|=\rho\})$ equal to zero. Then, observing that the product measures $\{\mu_h\otimes\mu_h\}_h$ weakly converge to $\mu\otimes\mu$ and using Fubini's Theorem, we infer that, for every continuous compactly supported test function φ

$$\begin{aligned}\lim_{h\to+\infty}\int\mu_h(B_\rho(x))\varphi(x)\,d\mu_h &= \lim_{h\to+\infty}\iint_{\{|x-y|<\rho\}}\varphi(x)\,d(\mu_h\otimes\mu_h)\\ &= \iint_{\{|x-y|<\rho\}}\varphi(x)\,d(\mu\otimes\mu)\\ &= \int\mu(B_\rho(x))\varphi(x)\,d\mu\ .\end{aligned}$$

In other words, we have the weak convergence of the sequence $\{f_h(\cdot,\rho)\,\mu_h\}_h$ to $f(\cdot,\rho)\,\mu$. Then we can apply Lemma 3.1 with $f_h(x):=f_h(x,\rho)$, $f(x):=f(x,\rho)$ and the convex function $\Phi(t):=(\log t)^-$. Thus claim (4.4) holds, and this achieves our proof. □

By using Theorem 3.8, Lemma 4.1, and Theorem 4.3, we immediately deduce the following corollary.

COROLLARY 4.4. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}_k$ weakly converging to μ. Assume that for some positive constants C and α, one of the following conditions holds*

i) $\qquad \theta_h\geq\alpha\quad \mu_h\text{-a.e.},\qquad |H(\mu_h)|\leq C$

ii) $\qquad \int(\log\theta_h)^-\,d\mu_h\leq C,\qquad \|h(\mu_h)\|_{L^p_{\mu_h}}\leq C\quad (H(\mu_h)\ll\mu_h)\,.$

Then

$$\mu\in\mathcal{M}_k\quad,\quad P_{\mu_h}\mu_h\rightharpoonup P_\mu\mu\quad,\quad H(\mu_h)\rightharpoonup H(\mu)\ .$$

REMARK 4.5. If condition i) holds, then, by Theorem 4.3, also the k-dimensional density θ of μ will satisfy the inequality $\theta\geq\alpha$ μ-a.e.

5. – Some examples and applications

It may be interesting to apply our results to prove the existence of a solution for minimum problems involving measures. For instance the problem of minimizing some functionals on a certain class of measures arises in a quite natural way in the framework of shape optimization (see [2]). Here we give, as a simple example, an existence result for a minimum problem on $\mathcal{M}_{BC}$ under a " Dirichlet boundary condition".
We denote by $\mathcal{M}_{k,\alpha}$ the subset of $\mathcal{M}_k$ given by all measures $\mu \in \mathcal{M}_k$ with k-dimensional density greater than or equal to α.

LEMMA 5.1. *Let $\Phi : \mathbb{R}^n \to [0, +\infty)$ be a convex, lower semicontinuous, and proper function satisfying $\lim_{|z|\to\infty} (\Phi(z)/|z|) = +\infty$. Let us fix an integer k, $\alpha > 0$ and $\lambda \in \mathcal{M}^n$ such that the set $\{\mu \in \mathcal{M}_{k,\alpha} : \partial\mu = \lambda\}$ is not empty. Then the minimum problem*

$$(\mathcal{P}) \qquad \min\left\{\int (1 + \Phi(h(\mu))\, d\mu \ : \ \mu \in \mathcal{M}_{k,\alpha}\ ,\ \partial\mu = \lambda\right\}$$

admits a solution.

PROOF. We apply Corollary 3.10, with

$$J(\mu, \nu) := \begin{cases} \int (1 + \Phi(f))\, d\mu & \text{if } \nu = f\mu + \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

The functional J is coercive on $\mathcal{M}^+ \times \mathcal{M}^n$. In order to check that it is also weakly lower semicontinuous, it is enough to apply Lemma 3.2 writing J as $J(\mu, \nu) = \int \Psi(\nu - \lambda, \mu)$ where Ψ satisfies $\Psi(z, t) = t\,\Phi(z/t) + t$ if $t > 0$, $\Psi(0, 0) = 0$ and $\Psi(z, t) = +\infty$ otherwise. Since $J(\mu, H(\mu)) < +\infty$ implies that $\partial\mu = \lambda$, we only need to check that, taking $\mathcal{A} = \mathcal{M}_{k,\alpha}$, assumption ii) of Corollary 3.10 is satisfied. This is a straightforward consequence of Corollary 4.4 and Remark 4.5. □

REMARK 5.2. The minimum problem mentioned in the introduction is obtained by taking $\Phi(z) = |z|^2$ in the lemma above. We think that in this case the assumption that all competing measures must have a given constant dimension for their tangent space is superfluous; in other words, we believe that for every measure λ the minimum problem

$$\min\left\{\int (1 + |h(\mu)|^2)\, d\mu \ : \ \partial\mu = \lambda\right\}$$

admits a solution, but at this stage the proof still seems rather difficult.

REMARK 5.3. It is possible to prove that Lemma 5.1 still holds if we replace the admissible class $\mathcal{M}_{k,\alpha}$ by the set of measures μ in $\mathcal{M}_k$ such that

the k-dimensional density $\theta_k(\mu, x)$ is a positive integer. In this case $\theta_k(\mu, x)$ is called the *integer multiplicity* of μ (see [10]).
In particular, when $k = 1$ and $\lambda = v_A\delta_A + v_B\delta_B$, the minimum problem $(\mathcal{P})$ becomes a weak version of the problem of finding the optimal curve, having endpoints A, B and prescribed tangent vectors v_A, v_B, which minimizes a functional depending on the pointwise curvature (see [3], [6]). However, we do not impose here any topological constraint as connection: thus, in some particular cases, when A and B are far enough, we may get solutions given by two disjoint loops starting from the points A and B.

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