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On the Convergence of Eigenvalues for Mixed Formulations

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Abstract. Eigenvalue problems for mixed formulation show peculiar features that make them substantially different from the corresponding mixed direct problems. In this paper we analyze, in an abstract framework, necessary and sufficient conditions for their convergence.

1. – Introduction

In a general way, we say that a variational problem is written in *mixed form* if it fits the following abstract setting. We assume that

(1) $$\Phi \text{ and } \Xi \text{ are Hilbert spaces,}$$

(2) $a(\psi, \varphi)$ and $b(\psi, \xi)$ are bilinear forms on $\Phi \times \Phi$ and $\Phi \times \Xi$ respectively,

(3) $$\begin{aligned} & a(\cdot,\cdot) \text{ and } b(\cdot,\cdot) \text{ are continuous, that is} \\ & \exists M_a > 0 \quad \forall \psi, \varphi \in \Phi \quad a(\psi, \varphi) \le M_a ||\psi||_\Phi ||\varphi||_\Phi \\ & \exists M_b > 0 \quad \forall \varphi \in \Phi,\ \forall \xi \in \Xi \quad b(\varphi, \xi) \le M_b ||\varphi||_\Phi ||\xi||_\Xi, \end{aligned}$$

and, to simplify the presentation, we also assume that

(4) $$a(\cdot,\cdot) \text{ is symmetric and positive semidefinite.}$$

Setting $||\varphi||_a := (a(\varphi,\varphi))^{1/2}$ (which in general will only be a seminorm on Φ) this immediately gives

(5) $$\forall \psi, \varphi \in \Phi \quad a(\psi, \varphi) \le ||\psi||_a ||\varphi||_a.$$

Properties (1) to (4) will be assumed to hold throughout all the paper.

For any given pair (f, g) in $\Phi' \times \Xi'$ we consider now the problem

(6) $$\text{find } (\psi, \chi) \text{ in } \Phi \times \Xi \text{ such that} \quad \begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = \langle f, \varphi \rangle & \forall \varphi \in \Phi \\ b(\psi, \xi) = \langle g, \xi \rangle & \forall \xi \in \Xi. \end{cases}$$

It is known that, in order to have existence, uniqueness and continuous dependence from the data for problem (6) it is necessary and sufficient that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the following conditions

$$\text{(7)}\qquad \text{(IS)}\qquad \text{there exists } \beta > 0, \text{ such that } \inf_{\xi\in\Xi}\sup_{\varphi\in\Phi}\frac{b(\varphi,\xi)}{\|\varphi\|_\Phi\|\xi\|_\Xi}\geq\beta$$

$$\text{(8)}\qquad \text{(EK)}\qquad \text{there exists } \alpha > 0, \text{ such that } a(\varphi,\varphi)\geq\alpha\|\varphi\|^2_\Phi\quad\forall\varphi\in\mathbb{K}$$

where the kernel $\mathbb{K}$ is defined as:

$$\mathbb{K}=\{\varphi\in\Phi \text{ such that } b(\varphi,\xi)=0\ \forall\xi\in\Xi\}.$$

EXAMPLE 1. *Stokes problem.* We take $\Phi=(H^1_0(\Omega))^2$, $\Xi=L^2(\Omega)/_{\mathbb{R}}$, $a(\psi,\varphi)=(\nabla\psi,\nabla\varphi)$, $b(\psi,\xi)=-(\operatorname{div}\psi,\xi)$ where, as usual, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or in $(L^2(\Omega))^2$. It is easy to see that (7) and (8) are satisfied. Moreover if we take $g=0$ the solution of (6) is related to the solution of the Stokes problem

$$\text{(9)}\qquad \begin{aligned} -\Delta\underline{u}+\nabla p&=f &&\text{in } \Omega\\ \operatorname{div}\underline{u}&=0 &&\text{in } \Omega\\ \underline{u}&=0 &&\text{on } \partial\Omega \end{aligned}$$

by the relations $\psi=\underline{u}$ and $\chi=p$. Approximations based on this approach are classical and are usually called approximations in the primitive variables.

EXAMPLE 2. *Dirichlet problem with Lagrange multipliers.* We take $\Phi=H^1(\Omega)$, $\Xi=H^{-1/2}(\partial\Omega)$, $a(\psi,\varphi)=(\nabla\psi,\nabla\varphi)$ and $b(\psi,\xi)=\langle\psi,\xi\rangle_{\partial\Omega}$ (duality between $H^{1/2}(\partial\Omega)$ and Ξ). It is easy to see that (7) and (8) are satisfied. Moreover, for every $f\in H^{-1}(\Omega)$ and $g\in H^{1/2}(\partial\Omega)$ the unique solution of (6) is related to the solution of

$$\text{(10)}\qquad \begin{aligned} -\Delta u&=f &&\text{in } \Omega\\ u&=g &&\text{on } \partial\Omega, \end{aligned}$$

by the relations $\psi=u$ and $\chi=-\frac{\partial u}{\partial n}$. Approximations based on this approach where first introduced by Babuška [2].

EXAMPLE 3. *Mixed formulation of second order linear elliptic problems.* We take $\Phi=H(\operatorname{div};\Omega)$, $\Xi=L^2(\Omega)$, $a(\psi,\varphi)=(\psi,\varphi)$, $b(\psi,\xi)=(\operatorname{div}\psi,\xi)$. It is easy to see that (7) and (8) are satisfied. Moreover if we take $f=0$ the solution of (6) is related to the solution of the problem

$$\text{(11)}\qquad \begin{aligned} \Delta u&=g &&\text{in } \Omega\\ u&=0 &&\text{on } \partial\Omega \end{aligned}$$

by the relations $\chi=u$ and $\psi=\nabla u$. Approximations based on this approach where first introduced by Raviart-Thomas [24].

EXAMPLE 4. *Biharmonic problem.* We take $\Phi = H^1(\Omega)$, $\Xi = H_0^1(\Omega)$, $a(\psi, \varphi) = (\psi, \varphi)$ and $b(\psi, \xi) = -(\nabla\psi, \nabla\xi)$. It is easy to see that (7) is satisfied, but (8) is not. However, if Ω is smooth enough and we take $f = 0$ then (6) has a unique solution, related to the solution of

$$\begin{aligned} \Delta^2 u &= -g && \text{in } \Omega \\ u &= \frac{\partial u}{\partial n} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{12}$$

by the relations $\chi = u$ and $\psi = -\Delta u$. Approximations based on this approach where first introduced by Glowinski [16] and analyzed by Ciarlet-Raviart [13] and Mercier [20].

For many other examples of mixed formulations of boundary value problems related to various applications in fluidmechanics and in continuous mechanics we refer, for instance, to [9].

Let us now consider the problem of discretization. Assume that we are given two families of finite dimensional subspaces Φ_h and Ξ_h of Φ and Ξ, respectively. We consider the discretized problem:

$$\begin{aligned} &\text{find } (\psi_h, \chi_h) \text{ in } \Phi_h \times \Xi_h \text{ such that} \\ &\begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = \langle f, \varphi_h \rangle & \forall \varphi_h \in \Phi_h \\ b(\psi_h, \xi_h) = \langle g, \xi_h \rangle & \forall \xi_h \in \Xi_h. \end{cases} \end{aligned} \tag{13}$$

It is known that discrete analogues of (7) and (8) are sufficient to ensure solvability of the discrete problem together with optimal error bounds. More precisely if the spaces Φ_h and Ξ_h satisfy the following conditions

$$\text{(DEK)} \quad \begin{aligned} &\text{there exists } \alpha > 0, \text{ independent of } h, \text{ such that} \\ &a(\varphi_h, \varphi_h) \ge \alpha ||\varphi_h||_\Phi^2 \quad \forall \varphi_h \in \mathbb{K}_h \end{aligned} \tag{14}$$

where the discrete kernel $\mathbb{K}_h$ is defined as

$$\mathbb{K}_h = \{\varphi_h \in \Phi_h \text{ such that } b(\varphi_h, \xi_h) = 0 \ \forall \xi_h \in \Xi_h\}$$

and

$$\text{(DIS)} \quad \begin{aligned} &\text{there exists } \beta > 0, \text{ independent of } h, \text{ such that} \\ &\inf_{\xi_h \in \Xi_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \xi_h)}{||\varphi_h||_\Phi ||\xi_h||_\Xi} \ge \beta, \end{aligned} \tag{15}$$

then we have unique solvability of (13) and the following error estimate

$$||\psi - \psi_h||_\Phi + ||\chi - \chi_h||_\Xi \le C \left(\inf_{\varphi \in \Phi_h} ||\psi - \varphi||_\Phi + \inf_{\xi \in \Xi_h} ||\chi - \xi||_\Xi \right). \tag{16}$$

As we shall see, conditions (14) and (15) are also necessary for having (16), in a suitable sense.

We turn now to the eigenvalue problems. As we can see from the examples above, the eigenvalue problem which is naturally associated with the corresponding boundary value problem in strong form (namely (9), (10), (11) or (12)) does not correspond to taking $(\lambda\psi, \lambda\chi)$ as right-hand side of (6). Instead, according with the different cases, the *natural* eigenvalue problem is obtained by taking $(\lambda\psi, 0)$ or $(0, -\lambda\chi)$ as right-hand side of (6). One expects, as for instance in [21], that (14) and (15), together with suitable compactness properties, are sufficient to ensure good convergence of the eigenvalues. However, when the problem is set in mixed variational form, compactness is more delicate to deal with. In a previous paper [5] we showed that, for the particular case of Example 3, even if the operator mapping g into u in (11) is clearly compact, assumptions (14) and (15) are not sufficient to avoid, for instance, the presence of spurious eigenvalues in the discrete spectrum. Here we address a more general problem, in abstract form, and we look for sufficient (and, possibly, necessary) conditions in order to have good approximation properties for the eigenvalue problems having either $(\lambda\psi, 0)$ or $(0, -\lambda\chi)$ at the right-hand side. As we shall see, in each of the two cases, (14) and (15) might be neither necessary nor sufficient for that.

Our approach turns out to be more similar to the one of [14] rather than the one of [8] or [1]. Important references for the study of eigenvalue problems in mixed form are [21], [3], [23]. As far as the *sufficient* conditions are concerned, we have only little improvements over the previous papers. For instance, our bilinear form $a(\cdot, \cdot)$ is not supposed to be positive definite as in the previous literature. Moreover, previous related papers deal mostly with cases in which the two components of the solution of the direct problem are both convergent, while we accept discretizations that can produce singular global matrices. On the other hand, having assumed symmetry of $a(\cdot, \cdot)$, we do not have to consider adjoint problems as in [14]. However, in practical cases, the actual gain is negligible. The major interest of the paper, in our opinion, consists in showing that our sufficient conditions are, mostly, also *necessary*, thus providing a severe test for assessing whether a given discretization is suitable for computing eigenvalues or not. This justifies, in our opinion, the apparently excessive generality of our abstract approach. Indeed, as we shall see, convergence of discrete eigenvalues does not even imply, for mixed formulations, the nonsingularity of the corresponding global matrices.

Finally we point out that in this paper we do not look for *a priori estimates* for eigenvalues and eigenvectors, but only deal with convergence. This is somehow in agreement with the fact that necessary conditions are a major issue here. However, in most cases, a priori error estimates can be readily deduced checking the last step in the proofs of sufficient conditions and/or applying the general instruments of, say, [7], [21], [3].

An outline of the paper is the following. In Section 2 we state the problem and relate the convergence of the spectrum with the uniform convergence of the resolvent operators. Moreover we point out the role of the discrete conditions (DEK) and (DIS) is order to have existence and boundedness of the different components of the solution of (13).

Section 3 and 4 are devoted to the analysis of the eigenvalue problems associated to (6) when the right-hand side is of the type $(\lambda\psi, 0)$ or $(0, -\lambda\chi)$, respectively. In both cases we state sufficient and necessary conditions for the good approximation of the spectrum. At the end of each section we will show how the known good approximations of the problems in the examples above satisfy our sufficient conditions for convergence of eigenvalues and eigenvectors, and more generally we discuss the validity of other possible approximations in light of our conditions.

2. – Statement of the problems

Let H be a Hilbert space and $T : H \to H$ a selfadjoint compact operator. To simplify the presentation we assume that T is nonnegative.

We are interested in the eigenvalues $\lambda \in \mathbb{R}$ defined by

$$\lambda T u = u, \quad \text{with } u \in H \setminus \{0\}. \tag{17}$$

In the above assumptions it is well-known that there exists a sequence $\{\lambda_i\}$ and an associated orthonormal basis $\{u_i\}$ such that

$$\begin{aligned} &\lambda_i T u_i = u_i, \\ &0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \ldots, \\ &\lim_{i\to\infty} \lambda_i = +\infty. \end{aligned} \tag{18}$$

We also set, for $i \in \mathbb{N}$, $E_i = \text{span}(u_i)$.

The following mapping will be useful. Let $m : \mathbb{N} \to \mathbb{N}$ be the application which to every N associates the dimension of the space generated by the eigenspaces of the first N distinct eigenvalues; that is

$$\begin{aligned} m(1) &= \dim\{\oplus_i E_i : \lambda_i = \lambda_1\}, \\ m(N+1) &= m(N) + \dim\left\{\oplus_i E_i : \lambda_i = \lambda_{m(N)+1}\right\}. \end{aligned} \tag{19}$$

Clearly, $\lambda_{m(1)}, \ldots, \lambda_{m(N)}$ $(N \in \mathbb{N})$ will now be the first N *distinct* eigenvalues of (17).

Assume that we are given, for every $h > 0$, a selfadjoint nonnegative operator $T_h : H \to H$ with finite range. We denote by $\lambda_i^h \in \mathbb{R}$ the eigenvalues of the problem

$$\lambda T_h u = u, \quad \text{with } u \in H \setminus \{0\}. \tag{20}$$

Let H_h be the finite-dimensional range of T_h and $\dim H_h =: N(h)$; then T_h admits $N(h)$ real eigenvalues denoted λ_i^h such that

$$0 \le \lambda_1^h \le \cdots \le \lambda_i^h \le \ldots \lambda_{N(h)}^h. \tag{21}$$

The associated discrete eigenfuntions u_i^h, $i = 1, \dots, N(h)$, give rise to an orthonormal basis of H_h with respect to the scalar product of H. Let $E_i^h := \mathrm{span}(u_i^h)$.

We assume that

$$\lim_{h\to 0} ||T - T_h||_{\mathcal{L}(H)} = 0. \tag{22}$$

It is a classical result in spectrum perturbation theory that (22) implies the following convergence property for eigenvalues and eigenvectors:

$$\begin{aligned} &\forall \epsilon > 0,\ \forall N \in \mathbb{N}\quad \exists h_0 > 0 \text{ such that } \forall h \le h_0 \\ &\max_{i=1,\dots,m(N)} |\lambda_i - \lambda_i^h| \le \epsilon, \\ &\hat{\delta}(\oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h) \le \epsilon, \end{aligned} \tag{23}$$

where $\hat{\delta}(E, F)$, for E and F linear subspaces of H, represents the gap between E and F and is defined by

$$\begin{aligned} &\hat{\delta}(E, F) = \max[\delta(E, F), \delta(F, E)], \\ &\delta(E, F) = \sup_{u\in E,\ ||u||_H=1} \inf_{v\in F} ||u - v||_H. \end{aligned} \tag{24}$$

Viceversa, it is not difficult to prove that (23) is a sufficient condition for (22).

We are interested in having (23) for eigenvalue problems in mixed form.

Let us therefore go back to the abstract framework already used in the introduction, with the assumptions therein. In particular assume, for the moment, that (7) and (8) are satisfied and that (13) has a solution for every (f, g) in $\Phi' \times \Xi'$. Problems (6) and (13) define then, in a natural way, two operators $S(f, g) = (\psi, \chi)$ (solution of (6)) and $S_h(f, g) = (\psi_h, \chi_h)$ (solution of (13)).

It is well-known (see [9]) that (DIS) and (DEK) (cfr. equations (15) and (14)) imply that the discrete operator S_h is bounded from $\Phi_h' \times \Xi_h'$ to $\Phi \times \Xi$, uniformly in h (see (16)). Moreover, the converse holds true, as it is proved in the following Lemma 1. Before it we introduce the following notation: for every $h > 0$ we define

$$||f||_{\Phi_h'} = \sup_{\varphi_h\in\Phi_h} \frac{\langle f, \varphi_h\rangle}{||\varphi_h||_\Phi} \qquad ||g||_{\Xi_h'} = \sup_{\xi_h\in\Xi_h} \frac{\langle g, \xi_h\rangle}{||\xi_h||_\Xi}. \tag{25}$$

LEMMA 1. *If there exists a constant C such that for all $f \in \Phi'$ and $g \in \Xi'$*

$$||S_h(f, g)||_{\Phi\times\Xi} \le C(||f||_{\Phi_h'} + ||g||_{\Xi_h'}) \tag{26}$$

for all $h > 0$, then (DIS) *and* (DEK) *are verified.*

PROOF. Let ψ_h belong to $\mathbb{K}_h$, then $(\psi_h, 0, \tilde{f}, 0)$ satisfies (13) with $\langle \tilde{f}, \varphi_h \rangle := a(\psi_h, \varphi_h)$, for all $\varphi_h \in \Phi_h$. Hence the inequality (26) gives (DEK) with $\alpha = 1/(C^2 M_a)$, M_a being the continuity constant of a (see (3)).

If $\chi_h \in \Xi_h$, then $(0, \chi_h, \tilde{f}, 0)$ satisfies (13), with $\langle \tilde{f}, \varphi_h \rangle := b(\varphi_h, \chi_h)$ for all $\varphi_h \in \Phi_h$. Hence the inequality (26) yields (DIS) with $\beta = 1/C$. □

REMARK 1. In the statement of Lemma 1 we implicitly assumed that the operator S_h was defined for every f and g. However, as it can be clearly seen in the proof, this was not really necessary. Indeed it is sufficient to assume that there exists a constant $C > 0$ such that for every $h > 0$ and for every quadruplet $(\psi_h, \chi_h, f, g) \in \Phi_h \times \Xi_h \times \Phi' \times \Xi'$ satisfying (13), one has

$$||\psi_h||_\Phi + ||\chi_h||_\Xi \leq C(||f||_{\Phi'_h} + ||g||_{\Xi'_h}). \tag{27}$$

This should not surprise, as (13) is always a linear system with a square matrix.□

Consider now the eigenvalue problem. For the sake of simplicity, let us assume for the moment that there exist two Hilbert spaces H_Φ and H_Ξ such that we can identify

$$\begin{aligned} H_\Phi &\equiv H'_\Phi, \\ H_\Xi &\equiv H'_\Xi \end{aligned} \tag{28}$$

and such that

$$\begin{aligned} \Phi \subseteq H_\Phi \subseteq \Phi' \\ \Xi \subseteq H_\Xi \subseteq \Xi' \end{aligned} \tag{29}$$

hold with dense and continuous embedding, in a compatible way.

The restrictions of S and S_h to $H_\Phi \times H_\Xi$ define now two operators from $H_\Phi \times H_\Xi$ into itself.

As a consequence of (16) and Lemma 1, it is immediate to prove the following proposition.

PROPOSITION 1. *Assume that* (DIS) *and* (DEK) *hold. Then S_h converges uniformly to S in $\mathcal{L}(H_\Phi \times H_\Xi)$ if and only if S (from $H_\Phi \times H_\Xi$ into itself) is compact.*

This proposition concludes our convergence analysis for the eigenvalue problems associated to (6) and (13). However in the applications one finds more often eigenvalue problems associated to (6) and (13) when one of the two components of the datum is zero. Let us set these eigenvalue problems in their appropriate abstract framework introducing the following operators:

$$\begin{aligned} &C_\Phi : \Phi' \to \Phi' \times \Xi' \qquad && C_\Xi : \Xi' \to \Phi' \times \Xi' \\ &C_\Phi(f) = (f, 0) && C_\Xi(g) = (0, g) \end{aligned} \tag{30}$$

and their adjoints

$$C^*_\Phi : \Phi \times \Xi \to \Phi \qquad C^*_\Xi : \Phi \times \Xi \to \Xi$$
$$C^*_\Phi(\varphi, \xi) = \varphi \qquad C^*_\Xi(\varphi, \xi) = \xi. \tag{31}$$

We shall say that (6) is a *problem of the type* $\binom{f}{0}$ if the right-hand side in (6) satisfies $g = 0$. Similarly, we shall say that (6) is a *problem of the type* $\binom{0}{g}$ if the right-hand side in (6) satisfies $f = 0$. Correspondingly, we shall study the approximation of the eigenvalues of the following operators:

$$\begin{aligned} T_\Phi &= C^*_\Phi \circ S \circ C_\Phi : \Phi' \to \Phi, \quad \text{for problems of the type } \begin{pmatrix} f \\ 0 \end{pmatrix}, \\ T_\Xi &= C^*_\Xi \circ S \circ C_\Xi : \Xi' \to \Xi, \quad \text{for problems of the type } \begin{pmatrix} 0 \\ g \end{pmatrix}. \end{aligned} \tag{32}$$

Whenever the associated discrete problems are solvable, we can introduce the discrete counterparts of T_Φ and T_Ξ as:

$$\begin{aligned} T^h_\Phi &= C^*_\Phi \circ S_h \circ C_\Phi : \Phi' \to \Phi, \quad \text{for problems of the type } \begin{pmatrix} f \\ 0 \end{pmatrix}, \\ T^h_\Xi &= C^*_\Xi \circ S_h \circ C_\Xi : \Xi' \to \Xi, \quad \text{for problems of the type } \begin{pmatrix} 0 \\ g \end{pmatrix}. \end{aligned} \tag{33}$$

In the remaining part of this section we are going to relate the solvability and boundedness of the discrete operators with either (DIS) or (DEK).

PROPOSITION 2. *If* (DEK) (see (14)) *holds and* $g = 0$, *then problem* (13) *has at least one solution* (ψ_h, χ_h). *Moreover* ψ_h *is uniquely determined by* f *and*

$$||\psi_h||_\Phi \le \frac{1}{\alpha}||f||_{\Phi'_h}, \tag{34}$$

(*where* α *is the constant appearing in* (14)).

PROOF. Let ψ_h be the unique solution of $a(\psi_h, \varphi_h) = \langle f, \varphi_h\rangle$ for all φ_h in $\mathbb{K}_h$. Clearly ψ_h exists, is unique and satisfies (34). Now look for χ_h in Ξ_h such that $b(\varphi_h, \chi_h) = (f, \varphi_h) - a(\psi_h, \varphi_h)$ for all $\varphi_h \in \Phi_h$. As the right-hand side is in the polar set of $\mathbb{K}_h$, the system is compatible and hence has at least one solution. □

PROPOSITION 3. *Assume that there exists a constant* $C > 0$ *such that for every* $h > 0$ *and for every quadruplet* $(\psi_h, \chi_h, f, 0) \in \Phi_h \times \Xi_h \times \Phi' \times \Xi'$ *satisfying* (13) *one has*

$$||\psi_h||_\Phi \le C||f||_{\Phi'_h}, \tag{35}$$

then the operator T^h_Φ *is defined in all* Φ' *and* (DEK) *holds with* $\alpha = 1/(C^2 M_a)$, M_a *being the continuity constant of* a (*see* (3)).

PROOF. With the same proof as in Lemma 1 we see that (DEK) holds true. The solvability of (13) is now a consequence of Proposition 2. □

PROPOSITION 4. *Assume that the following* weak discrete inf-sup condition *holds: for every $h > 0$, there exists a constant $\beta_h > 0$ such that*

$$(36) \qquad (\mathrm{DIS}_h) \qquad \inf_{\xi_h \in \Xi_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \xi_h)}{||\varphi_h||_\Phi ||\xi_h||_\Xi} \geq \beta_h.$$

Then for every $g \in \Xi'$ and $f = 0$ problem (13) *has at least one solution (ψ_h, χ_h) and χ_h is uniquely determined by g.*

PROOF. The assumption (36) implies that, with obvious notation, B_h is surjective. Hence for $g \in \Xi'$ there exists at least one $\psi_g \in \Phi_h$ such that $B_h \psi_g = g$. Then find $\psi_k \in \mathbb{K}_h$ such that $a(\psi_k, \varphi_h) = -a(\psi_g, \varphi_h)$ $\forall \varphi_h \in \mathbb{K}_h$. Finally, take $\chi_h \in \Xi_h$ such that $b(\varphi_h, \chi_h) = -a(\psi_g, \varphi_h) - a(\psi_k, \varphi_h)$ for all $\varphi_h \in \Phi_h$. Such a χ_h exists, by the same argument used in the proof of Proposition 2. Finally observe that $(\psi_g + \psi_k, \chi_h)$ solves (13) with $(0, g)$ as right-hand side.

To see the uniqueness, assume that (ψ_h^i, χ_h^i) $(i = 1, 2)$ are two solutions. Clearly $a(\psi_h^1 - \psi_h^2, \varphi_h) = 0$ for all $\varphi_h \in \mathbb{K}_h$. Taking $\varphi_h = \psi_h^1 - \psi_h^2$ one obtains that $a(\psi_h^1 - \psi_h^2, \psi_h^1 - \psi_h^2) = 0$ and hence as a is symmetric and positive semidefinite, $a(\psi_h^1 - \psi_h^2, \varphi_h) = 0$ for all $\varphi_h \in \Phi_h$ (use (5)). Now $b(\varphi_h, \chi_h^1 - \chi_h^2) = 0$ for all φ_h in Φ_h and (DIS_h) implies $\chi_h^1 = \chi_h^2$. □

PROPOSITION 5. *Assume that there exists a constant $C > 0$ such that for every $h > 0$ and for every quadruplet $(\psi_h, \chi_h, 0, g) \in \Phi_h \times \Xi_h \times \Phi' \times \Xi'$ satisfying* (13) *one has*

$$(37) \qquad ||\chi_h||_\Xi \leq C||g||_{\Xi_h'},$$

then the operator T_Ξ^h is defined in all Ξ' and the weak discrete inf-sup condition (DIS_h) *holds. In general,* (37) *does not imply* (DIS).

PROOF. Remark first that the assumption (37) implies that, with obvious notation, B_h^t is injective, therefore B_h will be surjective and this implies (DIS_h).

In order to see that (DIS) cannot be deduced in general, consider the case when $a \equiv 0$, $\Phi_h = \Xi_h$ and b is h times the scalar product in Φ_h. □

PROPOSITION 6. *Assume that there exists a constant $C > 0$ such that for every $h > 0$ and for every quadruplet $(\psi_h, \chi_h, 0, g) \in \Phi_h \times \Xi_h \times \Phi' \times \Xi'$ satisfying* (13) *one has*

$$(38) \qquad ||\psi_h||_\Phi + ||\chi_h||_\Xi \leq C||g||_{\Xi_h'},$$

then both T_Ξ^h and $C_\Phi^* \circ S_h \circ C_\Xi$ are defined on Ξ' and (DIS) *holds with $\beta = 1/C$.*

Proof. Remark first that, from Proposition 4, problem (13) has at least one solution for every $g \in \Xi'$, but now the estimate (38) ensures that such solution is unique. Hence $C^*_\Phi \circ S_h \circ C_\Xi$ is also well-defined in Ξ'. Let now ξ_h be an element of Ξ_h, and let $g \in \Xi'$ be such that $||g||_{\Xi'_h} = 1$ and $\langle g, \xi_h \rangle = ||\xi_h||_\Xi$. Taking $\psi^*_h = C^*_\Phi \circ S_h \circ C_\Xi g$ we have

$$\frac{b(\psi^*_h, \xi_h)}{||\psi^*_h||_\Phi} = \frac{\langle g, \xi_h \rangle}{||\psi^*_h||_\Phi} = \frac{||\xi_h||_\Xi}{||\psi^*_h||_\Phi} \geq \frac{1}{C}||\xi_h||_\Xi. \tag{39}$$ □

Proposition 7. *If there exists $C > 0$ such that*

$$||C^*_\Xi \circ S_h \circ C_\Phi||_{\mathcal{L}(\Phi'_h, \Xi_h)} \leq C \tag{40}$$

for every $h > 0$, then (DIS) *holds with $\beta = 1/C$.*

Proof. The same proof as in Lemma 1. □

We see from Propositions 3 and 7, that for problems of the type $\binom{f}{0}$ the estimate (35) on ψ_h implies (DEK) and the estimate (40) on χ_h implies (DIS). Analogue properties do not entirely hold for problems of the type $\binom{0}{g}$.

3. – Problems of the type $\begin{pmatrix} f \\ 0 \end{pmatrix}$

In this section, together with (1)-(4), we assume that (EK) and (IS) are verified. We also assume that we are given a Hilbert space H_Φ (that we shall identify with its dual space H'_Φ) such that

$$\Phi \subseteq H_\Phi \subseteq \Phi' \tag{41}$$

with continuous and dense embeddings. We consider the eigenvalue problem

$$\begin{aligned} &\text{find } (\lambda, \psi) \text{ in } \mathbb{R} \times \Phi, \text{ with } \psi \neq 0, \\ &\text{such that there exists } \chi \in \Xi \text{ verifying} \\ &\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = \lambda(\psi, \varphi)_{H_\Phi} & \forall \varphi \in \Phi \\ b(\psi, \xi) = 0 & \forall \xi \in \Xi, \end{cases} \end{aligned} \tag{42}$$

which in the formalism of the previous section can be written

$$\lambda T_\Phi \psi = \psi. \tag{43}$$

We assume that the operator T_Φ is compact from H_Φ to Φ.

Suppose now that we are given two finite dimensional subspaces Φ_h and Ξ_h of Φ and Ξ, respectively. Then the approximation of (42) reads

$$\text{(44)}\qquad \begin{array}{l} \text{find } (\lambda_h, \psi_h) \text{ in } \mathbb{R} \times \Phi_h, \text{ with } \psi_h \neq 0 \text{ such that} \\ \text{there exists } \chi_h \in \Xi_h \text{ verifying} \\ \begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = \lambda_h(\psi_h, \varphi_h)_{H_\Phi} & \forall \varphi_h \in \Phi_h \\ b(\psi_h, \xi_h) = 0 & \forall \xi_h \in \Xi_h, \end{cases} \end{array}$$

that is

$$\text{(45)}\qquad \lambda_h T_\Phi^h \psi_h = \psi_h.$$

We are now looking for necessary and sufficient conditions that ensure the uniform convergence of T_Φ^h to T_Φ in $\mathcal{L}(H_\Phi, \Phi)$ which, as we have seen, implies the convergence of eigenvalues and eigenvectors (see (23)).

To start with, we look for sufficient conditions.

We introduce some notation. Let Φ_0^H and Ξ_0^H be the subspaces of Φ and Ξ, respectively, containing all the solutions $\psi \in \Phi$ and $\chi \in \Xi$, respectively, of problem (6) when $g = 0$; that is, with the formalism of the previous section,

$$\text{(46)}\qquad \begin{aligned} \Phi_0^H &= C_\Phi^* \circ S \circ C_\Phi(H_\Phi) = T_\Phi(H_\Phi) \\ \Xi_0^H &= C_\Xi^* \circ S \circ C_\Phi(H_\Phi). \end{aligned}$$

Notice that the following inclusion holds true:

$$\Phi_0^H \subseteq \mathbb{K}.$$

The spaces Φ_0^H and Ξ_0^H will be endowed with the *natural* norm: that is, for instance,

$$\text{(47)}\qquad \begin{aligned} ||\varphi||_{\Phi_0^H} &:= \inf\{||\eta||_{H_\Phi},\ T_\Phi \eta = \varphi\}; \\ ||\xi||_{\Xi_0^H} &:= \inf\{||\eta||_{H_\Phi},\ C_\Xi^* \circ S \circ C_\Phi \eta = \xi\}. \end{aligned}$$

DEFINITION 1. We say that the *weak approximability* of Ξ_0^H is verified if there exists $\omega_1(h)$, tending to zero as h goes to zero, such that for every $\chi \in \Xi_0^H$

$$\text{(48)}\qquad \sup_{\varphi_h \in \mathbb{K}_h} \frac{b(\varphi_h, \chi)}{||\varphi_h||_\Phi} \leq \omega_1(h) ||\chi||_{\Xi_0^H}.$$

Notice that, in spite of its appearance, (48) is indeed an approximability property. Actually as $\varphi_h \in \mathbb{K}_h$, we have $b(\varphi_h, \chi) = b(\varphi_h, \chi - \chi^I)$ for every $\chi^I \in \Xi_h$, which has, usually, to be used to verify (48).

DEFINITION 2. We say that the *strong approximability of* Φ_0^H is verified if there exists $\omega_2(h)$, tending to zero as h goes to zero, such that for every $\psi \in \Phi_0^H$ there exists $\psi^I \in \mathbb{K}_h$ such that

$$||\psi - \psi^I||_\Phi \leq \omega_2(h)||\psi||_{\Phi_0^H}. \tag{49}$$

THEOREM 1. *Let us assume that* (DEK) *is verified* (*see* (14)). *Assume moreover the* weak approximability *of* Ξ_0^H *and the* strong approximability *of* Φ_0^H. *Then the sequence* T_Φ^h *converges uniformly to* T_Φ *in* $\mathcal{L}(H_\Phi, \Phi)$, *that is there exists* $\omega_3(h)$, *tending to zero as h goes to zero, such that*

$$||T_\Phi f - T_\Phi^h f||_\Phi \leq \omega_3(h)||f||_{H_\Phi}, \quad \textit{for all } f \in H_\Phi. \tag{50}$$

PROOF. Let $f \in H_\Phi$ and let $(\psi, \chi) \in \Phi_0^H \times \Xi_0^H$ be solution of (6): $(\psi, \chi) = S(f, 0)$. As we assumed (DEK) Proposition 2 ensures that T_Φ^h is well defined on Φ'. Recall that $\psi := T_\Phi(f)$. Let $\psi_h := T_\Phi^h(f)$ and let χ^I be such that (ψ_h, χ^I) is a solution of (13) (such χ^I might not be unique). In order to prove the uniform convergence of T_Φ^h to T_Φ, we have to estimate the difference $||\psi - \psi_h||_\Phi$. We do it by bounding the term $||\psi^I - \psi_h||_\Phi$, where ψ^I is given by (49), and then by using the triangular inequality. We have

$$\begin{aligned}
\alpha||\psi^I - \psi_h||_\Phi^2 &\leq a(\psi^I - \psi_h, \psi^I - \psi_h)\\
&= a(\psi^I - \psi, \psi^I - \psi_h) + a(\psi - \psi_h, \psi^I - \psi_h)\\
&\leq M_a||\psi^I - \psi||_\Phi||\psi^I - \psi_h||_\Phi - b(\psi^I - \psi_h, \chi - \chi_h)\\
&\leq \left(M_a||\psi^I - \psi||_\Phi + \sup_{\varphi_h \in \mathbb{K}_h} \frac{b(\varphi_h, \chi - \chi_h)}{||\varphi_h||_\Phi}\right)||\psi^I - \psi_h||_\Phi\\
&= \left(M_a||\psi^I - \psi||_\Phi + \sup_{\varphi_h \in \mathbb{K}_h} \frac{b(\varphi_h, \chi)}{||\varphi_h||_\Phi}\right)||\psi^I - \psi_h||_\Phi.
\end{aligned} \tag{51}$$

The result then follows immediately from the *strong approximability* of Φ_0^H and the *weak approximability* of Ξ_0^H. In particular we can take $\omega_3(h) = (1 + M_a/\alpha)\omega_2(h) + \omega_1(h)/\alpha$. □

In the following theorem we shall see that the assumptions of Theorem 1 are also, in a sense, necessary for the uniform convergence of T_Φ^h to T_Φ in $\mathcal{L}(H_\Phi, \Phi)$.

THEOREM 2. *Assume that the sequence* T_Φ^h *is bounded in* $\mathcal{L}(\Phi', \Phi)$, *and converges uniformly to* T_Φ *in* $\mathcal{L}(H_\Phi, \Phi)$ (*see* (50)). *Then, the ellipticity in the kernel property* (DEK) *holds true. Moreover, both the* strong approximability *of* Φ_0^H *and the* weak approximability *of* Ξ_0^H *are satisfied.*

PROOF. The (DEK) property can be obtained applying Proposition 3. Let ψ be an element of Φ_0^H. Then by definition of Φ_0^H there is $f \in H_\Phi$ such that $\psi = T_\Phi f$. Define $\psi^I := T_\Phi^h f$. Uniform convergence implies the *strong approximability* of Φ_0^H.

In a similar way, let χ be an element of Ξ_0^H. Then by definition of Ξ_0^H, $\chi = C_\Xi^* \circ S \circ C_\Phi f$ for some $f \in H_\Phi$. There might be more than one such f. We choose $\overline{f}$ such that $\|\overline{f}\|_{H_\Phi} \le \frac{3}{2}\inf\{\|f\|_{H_\Phi} : C_\Xi^* \circ S \circ C_\Phi f = \chi\} = \frac{3}{2}\|\chi\|_{\Xi_0^H}$. Let $\psi := T_\Phi \overline{f}$. Correspondingly let $\psi_h := T_\Phi^h \overline{f}$ and let χ_h be such that (ψ_h, χ_h) is a solution of (13) with the same right-hand side (such χ_h might not be unique). Then we obtain

$$\sup_{\varphi_h \in \mathbb{K}_h} \frac{b(\varphi_h, \chi)}{\|\varphi_h\|_\Phi} = \sup_{\varphi_h \in \mathbb{K}_h} \frac{b(\varphi_h, \chi - \chi_h)}{\|\varphi_h\|_\Phi} = \sup_{\varphi_h \in \mathbb{K}_h} \frac{a(\psi - \psi_h, \varphi_h)}{\|\varphi_h\|_\Phi}$$
$$\le M_a \|\psi - \psi_h\|_\Phi \le M_a \omega_3(h) \|f\|_{H_\Phi} \le \frac{3}{2} M_a \omega_3(h) \|\chi\|_{\Xi_0^H},$$

which gives (48) with $\omega_1(h) = \frac{3}{2} M_a \omega_3(h)$, that is the *weak approximability* of Ξ_0^H. □

EXAMPLE 1. We go back to the Example 1 of the Introduction (Stokes problem). Now $\Phi = V = (H_0^1(\Omega))^2$ and $\Xi = Q = L^2(\Omega)/_{\mathbb{R}}$. It is easy to see that if Ω is, for instance, a convex polygon, Ξ_0^H is $H^1(\Omega)/_{\mathbb{R}}$ and Φ_0^H is the subspace of $(H^2(\Omega) \cap H_0^1(\Omega))^2$ made of free divergence functions (see [19]). In particular we can check that $\|\underline{u}\|_{\Phi_0^H} = \|\Delta \underline{u}\|_0 \sim \|\underline{u}\|_2$ and $\|p\|_{\Xi_0^H} = \|\underline{\nabla} p\|_0$ (with standard notation, here and in the following, we denote by $\|\cdot\|_k$ the norm in $H^k(\Omega)$ for $k \in \mathbb{N}$). Let V_h and Q_h be finite dimensional subspaces of V and Q respectively. The weak approximability of Ξ_0^H will surely hold if

$$\inf_{q_h \in Q_h} \|p - q_h\|_0 \le \omega_1(h) \|p\|_1 \quad \text{for all } p \in H^1(\Omega)/_{\mathbb{R}},$$

which is satisfied by all choices of finite element spaces that one may seriously think to use in practice.

The strong approximability of Φ_0^H, which now reads

$$\|\underline{u} - \underline{u}^I\|_1 \le \omega_2(h) \|\underline{u}\|_2 \qquad \text{for all } \underline{u} \in \Phi_0^H, \tag{52}$$

is more delicate, as $\underline{u}^I$ has to be chosen in $\mathbb{K}_h$. If the pair (V_h, Q_h) satisfies the inf-sup condition (DIS) then the property trivially holds. Remark, however, that the typical way to proving the inf-sup condition is to show, following [15], that: *for every $\underline{u}$ in V there exists $\underline{u}^I$ in V_h such that $\|\underline{u}^I\|_V \le C\|\underline{u}\|_V$ (C independent of $\underline{u}$ and h) and* $b(\underline{u} - \underline{u}^I, q_h) = 0\ \forall q_h \in Q_h$, which is more difficult than proving (52) directly. Moreover there are choices of elements that fail to satisfy the inf-sup condition, for which (52) holds true. For instance, we may think to the so-called $Q_1 - P_0$ element, where

$$
(53)\qquad \begin{aligned} V_h &:= \{\underline{v}_h \in (C^0(\Omega))^2 : \ \underline{v}_h|_K \in (Q_1(K))^2 \ \forall K \in \mathcal{T}_h\}, \\ Q_h &:= \{q_h : \ q_h|_K \in P_0(K) \ \forall K \in \mathcal{T}_h\}, \end{aligned}
$$

where, with standard finite element notation, for k integer ≥ 0 $P_k(D)$ denotes the space of polynomials of degree $\leq k$ on a domain D, and $Q_k(D)$ the space of polynomials of degree $\leq k$ separately in each variable. Hence, here $Q_1(K)$ is the set of bilinear polynomials on K, and $P_0(K)$ is the set of the constant functions on K. We may assume, for simplicity, that Ω is a square and that the decomposition $\mathcal{T}_h$ is made by $2N \times 2N$ equal subsquares. It is known that this choice of elements does not satisfy the inf-sup condition: the operator B_h^t has a non trivial kernel (the checkerboard mode), and by discarding it we still have at best (DIS$_h$) with $\beta_h \sim h$ (see [22], [18], [6]). Nevertheless, for $u \in \Phi_0^H \subset \mathbb{K}$, we can construct $\underline{u}^I$ as follows: let $\underline{\hat{u}}$ be the vector in V_h which is bilinear in each square of the $N \times N$ (coarser) grid and agrees with $\underline{u}$ at the vertices of the coarser grid. Let now $\underline{\tilde{u}}$ be the vector in V_h with the following properties. It vanishes at the vertices of the coarser grid; its tangential component vanishes on the midpoints of the edges of the coarser grid; its normal component at the midpoints of each edge e of the coarser grid is chosen in such a way that $\int_e (\underline{u} - \underline{\hat{u}} - \underline{\tilde{u}}) \cdot n = 0$, and finally the values at the center of each element K of the coarser grid are chosen to satisfy $\int_K (\underline{u} - \underline{\hat{u}} - \underline{\tilde{u}}) q_h = 0$ for $q_h = \operatorname{sign}(x - x_c)$ and $q_h = \operatorname{sign}(y - y_c)$ (where (x_c, y_c) is the center of K). It is not difficult to check that $\underline{u}^I = \underline{\hat{u}} + \underline{\tilde{u}}$ satisfies (52) with $\omega_2(h) = O(h)$. For a similar construction see [9], pages 241-242. We have here a first example in which the eigenvalues are approximated correctly even though the global matrix associated to (13) is singular.

EXAMPLE 2. *Dirichlet problem with Lagrange multipliers.* Here $\Phi = V = H^1(\Omega)$ and $\Xi = M = H^{-1/2}(\partial\Omega)$. It is well-known (see e.g. [17]), that if Ω is, for instance, a convex polygon, then $\Phi_0^H = H^2(\Omega) \cap H_0^1(\Omega)$ and $\Xi_0^H = H^{1/2}(\partial\Omega)$. Let now $\{\mathcal{T}_h^\Omega\}$ be a regular sequence of decompositions of Ω (see e.g. [12]), $\{\mathcal{T}_h^\Gamma\}$ be a regular sequence of decompositions of $\partial\Omega$, k_1 and k_2 be integers with $k_1 \geq 1$ and $k_2 \geq 0$. Set

$$
(54)\qquad \begin{aligned} V_h^{k_1} &:= \{\underline{v}_h \in V : \ \underline{v}_h|_K \in P_{k_1}(K) \ \forall K \in \mathcal{T}_h^\Omega\}, \\ M_h^{k_2} &:= \{\mu_h \in M : \ \mu_h|_e \in P_{k_2}(e) \ \forall e \in \mathcal{T}_h^\Gamma\}. \end{aligned}
$$

It is trivial to check that (48) and (49) hold for every choice of $\{\mathcal{T}_h^\Omega\}$, $\{\mathcal{T}_h^\Gamma\}$, k_1 and k_2. In particular (DEK), which is now a sort of Poincaré inequality, only requires that $M_h \subset M$ contains at least a $\overline{\mu}_h$ such that $\langle \overline{\mu}_h, 1 \rangle \neq 0$.

Note that to have (DIS) one must ask rather strict compatibility conditions on $\{\mathcal{T}_h^\Omega\}$, $\{\mathcal{T}_h^\Gamma\}$, k_1 and k_2, see [2]. Therefore, for a general choice, solvability of (13) might fail. Nevertheless, as we have seen, convergence of the eigenvalues is assured under weaker assumptions.

4. – Problems of the type $\begin{pmatrix} 0 \\ g \end{pmatrix}$

In this section, together with (1)-(4), we assume that, for every given $g \in \Xi'$ and $f = 0$, problem (6) has a unique solution (ψ, χ) and that there exists a constant C (independent of g) such that

$$||\psi||_\Phi + ||\chi||_\Xi \leq C||g||_{\Xi'}. \tag{55}$$

It is easy to see that this implies (IS) but not (EK) (see Example 4 of the Introduction). Moreover we assume that we are given a Hilbert space H_Ξ (that we shall identify with its dual space H'_Ξ) such that

$$\Xi \subseteq H_\Xi \subseteq \Xi' \tag{56}$$

with continuous and dense embeddings. For simplicity, we assume that for every $\xi \in \Xi$, we have $||\xi||_{H_\Xi} \leq ||\xi||_\Xi$ (with constant equal to 1).

We consider the eigenvalue problem

$$\begin{aligned} &\text{find } (\lambda, \chi) \text{ in } \mathbb{R} \times \Xi, \text{ with } \chi \neq 0, \text{ such that} \\ &\text{there exists } \psi \in \Phi \text{ satisfying} \\ &\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = 0 & \forall \varphi \in \Phi \\ b(\psi, \xi) = -\lambda(\chi, \xi)_{H_\Xi} & \forall \xi \in \Xi, \end{cases} \end{aligned} \tag{57}$$

which in the formalism of Section 2 can be written

$$\lambda T_\Xi \chi = -\chi. \tag{58}$$

As we shall see, problems of the type $\binom{0}{g}$ are more closely related to the abstract theory of [14] than problems of the previous type $\binom{f}{0}$.

From now on we assume that the operator T_Ξ is compact from H_Ξ into Ξ.

We introduce two finite dimensional subspaces Φ_h and Ξ_h of Φ and Ξ, respectively. Then the approximation of (57) reads

$$\begin{aligned} &\text{find } (\lambda_h, \chi_h) \text{ in } \mathbb{R} \times \Xi_h, \text{ with } \chi_h \neq 0, \text{ such that} \\ &\text{there exists } \psi_h \in \Phi_h \text{ satisfying} \\ &\begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = 0 & \forall \varphi_h \in \Phi_h \\ b(\psi_h, \xi_h) = -\lambda_h(\chi_h, \xi_h)_{H_\Xi} & \forall \xi_h \in \Xi_h, \end{cases} \end{aligned} \tag{59}$$

that is

$$\lambda_h T^h_\Xi \chi_h = -\chi_h. \tag{60}$$

We are now looking for necessary and sufficient conditions that ensure the uniform convergence of T^h_Ξ to T_Ξ in $\mathcal{L}(H_\Xi, \Xi)$, which implies the convergence of eigenvalues and eigenvectors (see (23)).

To start with, we look for sufficient conditions.

We introduce some notation. Let Φ^0_H and Ξ^0_H be the subspaces of Φ and Ξ respectively, containing all the solutions $\psi \in \Phi$ and $\chi \in \Xi$, respectively, of problem (6) when $f = 0$; that is, with the formalism of Section 2,

$$\begin{aligned} \Phi^0_H &= C^*_\Phi \circ S \circ C_\Xi(H_\Xi) \\ \Xi^0_H &= C^*_\Xi \circ S \circ C_\Xi(H_\Xi) = T_\Xi(H_\Xi). \end{aligned} \tag{61}$$

It will also be useful to define the space $\Phi^0_{\Xi'}$ as the image of $C^*_\Phi \circ S \circ C_\Xi$ (from Ξ' to Φ).

As before, the spaces Φ^0_H, Ξ^0_H and $\Phi^0_{\Xi'}$ will be endowed with their natural norms (see for instance (47)).

DEFINITION 3. We say that the *weak approximability of* Ξ^0_H *with respect to* $a(\cdot,\cdot)$ is verified if there exists $\omega_4(h)$, tending to zero as h goes to zero, such that for every $\chi \in \Xi^H_0$ and for every $\varphi_h \in \mathbb{K}_h$

$$b(\varphi_h, \chi) \le \omega_4(h) ||\chi||_{\Xi^0_H} ||\varphi_h||_a. \tag{62}$$

Notice that (62) is indeed an approximation property, as we already pointed out for its counterpart (48).

DEFINITION 4. We say that the *strong approximability* of Ξ^0_H is verified if there exists $\omega_5(h)$, tending to zero as h goes to zero, such that for every $\chi \in \Xi^0_H$ there exists $\chi^I \in \Xi_h$ such that

$$||\chi - \chi^I||_\Xi \le \omega_5(h) ||\chi||_{\Xi^0_H}. \tag{63}$$

Notice that (62) and (63) are (much) weaker forms of assumption H7 of [14].

DEFINITION 5. An operator Π_h from Φ (or from a subspace of it) into Φ_h is called a *Fortin operator* with respect to the bilinear form b and the subspace $\Xi_h \subset \Xi$ if it verifies, for all φ in its domain,

$$b(\varphi - \Pi_h\varphi, \xi_h) = 0 \quad \forall \xi_h \in \Xi_h. \tag{64}$$

The following assumptions for a Fortin operator will be useful.

– There exists a constant C_Π, independent of h such that:

$$||\Pi_h||_{\mathcal{L}(\Phi^0_{\Xi'}, \Phi)} \le C_\Pi. \tag{65}$$

– There exists $\omega_6(h)$, tending to zero as h goes to zero, such that for every $\varphi \in \Phi^0_H$ it holds

$$||\varphi - \Pi_h\varphi||_a \le \omega_6(h) ||\varphi||_{\Phi^0_H}. \tag{66}$$

Notice that (66) is strongly related to assumption H5 of [14]. However, being interested in convergence, we have to assume that $\omega_6(h)$ goes to 0, while [14] only assumes it to be bounded and puts it in the right-hand side of a priori estimates. On the other hand, as we shall see, (66) is actually *necessary* for having convergence of eigenvalues. This was not pointed out in [14] for the very good reason that, first, their interest was in a priori bounds (and not on necessity) and, second, they were dealing with direct problems (and not with eigenvalues). In particular (66) is not necessary for having pointwise convergence of T^h_Ξ to T_Ξ where (DEK) and (DIS) are sufficient. Notice that (as it is also pointed out in Proposition 1 of [14]) (IS) and (DIS) imply (65), but, as we shall see later in this paper, (EK), (DEK), (IS) and (DIS) (all together) imply pointwise convergence but not (66).

THEOREM 3. *Let us assume that there exists a Fortin operator (see (64)) $\Pi_h : \Phi^0_{\Xi'} \to \Phi_h$ satisfying (65) and (66). Assume moreover that the* strong approximability *of Ξ^0_H is verified (see (63)) as well as the* weak approximability of Ξ^0_H with respect to *a* (*see (62)). Then the sequence T^h_Ξ converges to T_Ξ uniformly from H_Ξ into Ξ, that is there exists $\omega_7(h)$, tending to zero as h goes to zero, such that*

$$(67)\qquad ||T_\Xi g - T^h_\Xi g||_\Xi \le \omega_7(h)||g||_{H_\Xi}, \quad \textit{for all } g \in H_\Xi.$$

PROOF. We remark first that, as it is well-known, (7) and (64)-(65) imply (DIS) (see [15] or [9]). Thanks to Proposition 4, T^h_Ξ is then well defined.

Let $g \in H_\Xi$ and let $(\psi, \chi) \in \Phi^0_H \times \Xi^0_H$ be the solution of (6) with $f = 0$. Recall that $\chi = T_\Xi g$. Let $\chi_h := T^h_\Xi g$ and let ψ_h be such that (ψ_h, χ_h) is a solution of (13) (such ψ_h might be not unique). In order to prove the uniform convergence of T^h_Ξ to T_Ξ we have to find a priori estimates for the error $||\chi - \chi_h||_\Xi$. Let $\tilde{g} \in \Xi'$ be such that $\langle \tilde{g}, \chi - \chi_h \rangle = ||\chi - \chi_h||_\Xi$ and $||\tilde{g}||_{\Xi'} = 1$. Take $\tilde{\varphi} := C^*_\Phi \circ S \circ C_\Xi \tilde{g}$, hence $||\tilde{\varphi}||_{\Phi^0_{\Xi'}} \le ||\tilde{g}||_{\Xi'} = 1$ (see (47)). Then we have

$$(68)\qquad \begin{aligned} ||\chi - \chi_h||_\Xi &= \langle \tilde{g}, \chi - \chi_h \rangle = b(\tilde{\varphi}, \chi - \chi_h) \\ &= b(\tilde{\varphi} - \Pi_h \tilde{\varphi}, \chi - \chi_h) + b(\Pi_h \tilde{\varphi}, \chi - \chi_h) \\ &= b(\tilde{\varphi} - \Pi_h \tilde{\varphi}, \chi - \chi^I) - a(\psi - \psi_h, \Pi_h \tilde{\varphi}). \end{aligned}$$

Let us estimate separately the two terms in the right-hand side:

$$(69)\qquad \begin{aligned} b(\tilde{\varphi} - \Pi_h \tilde{\varphi}, \chi - \chi^I) &\le M_b ||\tilde{\varphi} - \Pi_h \tilde{\varphi}||_\Phi ||\chi - \chi^I||_\Xi \\ &\le M_b \left(||\tilde{\varphi}||_\Phi + ||\Pi_h \tilde{\varphi}||_\Phi\right) ||\chi - \chi^I||_\Xi; \\ a(\psi - \psi_h, \Pi_h \tilde{\varphi}) &\le ||\Pi_h \tilde{\varphi}||_a ||\psi - \psi_h||_a. \end{aligned}$$

Using (65) we obtain the following estimate for $\Pi_h \tilde{\varphi}$

$$(70)\qquad ||\Pi_h \tilde{\varphi}||_\Phi \le C_\Pi ||\tilde{\varphi}||_{\Phi^0_{\Xi'}} \le C_\Pi.$$

Putting together (68), (69) and (70) and using (63) we obtain

$$\begin{aligned} ||\chi - \chi_h||_\Xi &\le M_b(1 + C_\Pi)||\chi - \chi^I||_\Xi + C_\Pi||\psi - \psi_h||_a \\ &\le M_b(1 + C_\Pi)\omega_5(h)||\chi||_{\Xi^0_H} + C_\Pi||\psi - \psi_h||_a. \end{aligned} \tag{71}$$

To conclude the proof it remains to estimate $||\psi - \psi_h||_a$. Thanks to the triangular inequality and to (66) we bound only $||\Pi_h\psi - \psi_h||_a$ using also (62) and (64). Notice that $\Pi_h\psi - \psi_h$ belongs to $\mathbb{K}_h$.

$$\begin{aligned} ||\Pi_h\psi - \psi_h||_a^2 &= a(\Pi_h\psi - \psi, \Pi_h\psi - \psi_h) + a(\psi - \psi_h, \Pi_h\psi - \psi_h) \\ &\le ||\psi - \Pi_h\psi||_a||\Pi_h\psi - \psi_h||_a - b(\Pi_h\psi - \psi_h, \chi - \chi_h) \\ &= ||\psi - \Pi_h\psi||_a||\Pi_h\psi - \psi_h||_a - b(\Pi_h\psi - \psi_h, \chi) \\ &\le ||\Pi_h\psi - \psi_h||_a \left(||\psi - \Pi_h\psi||_a + \omega_4(h)||\chi||_{\Xi^0_H}\right), \end{aligned} \tag{72}$$

which, due to (66), gives

$$\begin{aligned} ||\psi - \psi_h||_a &\le 2||\psi - \Pi_h\psi||_a + \omega_4(h)||\chi||_{\Xi^0_H} \\ &\le 2\omega_6(h)||\psi||_{\Phi^0_H} + \omega_4(h)||\chi||_{\Xi^0_H} \end{aligned} \tag{73}$$

and (67) holds with $\omega_7(h) = M_b(1 + C_\Pi)\omega_5(h) + 2C_\Pi\omega_6(h) + C_\Pi\omega_4(h)$. □

REMARK 2. In Theorem 3 we have proved the uniform convergence of T^h_Ξ to T_Ξ in $\mathcal{L}(H_\Xi, \Xi)$. However in Section 2 we have seen that the convergence of the spectrum is equivalent to the uniform convergence of T^h_Ξ to T_Ξ in $\mathcal{L}(H_\Xi)$. Indeed the latter holds under the weaker assumption that there exists a Fortin operator satisfying only (66) as we shall see in the following theorem. □

THEOREM 4. *Let us assume that there exists a Fortin operator (see (64))* $\Pi_h : \Phi^0_{\Xi'} \to \Phi_h$ *satisfying* (66). *Assume moreover that both the* strong approximability *of* Ξ^0_H *(see* (63)*) and the* weak approximability of Ξ^0_H with respect to a *(see* (62)*) are verified. Then the sequence* T^h_Ξ *converges uniformly to* T_Ξ *in* H_Ξ.

PROOF. We observe that (7) and (64) imply the weak discrete inf-sup condition (DIS$_h$). Thanks to Proposition 4, T^h_Ξ is then well defined.

Let $g \in H_\Xi$ and let $(\psi, \chi) \in \Phi^0_H \times \Xi^0_H$ be the solution of (6) with $f = 0$. Recall that $\chi = T_\Xi g$. Let $\chi_h := T^h_\Xi g$ and let ψ_h be such that (ψ_h, χ_h) is a solution of (13) with right-hand side $(0, g)$ (such ψ_h might be not unique). We estimate $||\chi - \chi_h||_{H_\Xi}$. Using a duality argument, let $(\tilde\psi, \tilde\chi) \in \Phi \times \Xi$ be defined by $(\tilde\psi, \tilde\chi) := S(0, \chi - \chi_h)$. Due to the definition (61), $\tilde\psi$ belongs to Φ^0_H with the following estimate $||\tilde{\psi}||_{\Phi^0_H} \leq ||\chi - \chi_h||_{H_\Xi}$ (see (47)). Then

$$\begin{aligned}||\chi - \chi_h||^2_{H_\Xi} &= (\chi - \chi_h, \chi - \chi_h) = b(\tilde{\psi}, \chi - \chi_h)\\ &= b(\tilde{\psi} - \Pi_h\tilde{\psi}, \chi) + b(\Pi_h\tilde{\psi}, \chi - \chi_h)\\ &= -a(\psi, \tilde{\psi} - \Pi_h\tilde{\psi}) - a(\psi - \psi_h, \Pi_h\tilde{\psi})\\ &\leq ||\psi||_a||\tilde{\psi} - \Pi_h\tilde{\psi}||_a + ||\psi - \psi_h||_a||\Pi_h\tilde{\psi}||_a\\ &\leq ||\psi||_a\omega_6(h)||\tilde{\psi}||_{\Phi^0_H} + 2||\psi - \psi_h||_a||\tilde{\psi}||_{\Phi^0_H}\\ &\leq (\omega_6(h)||\psi||_a + 2||\psi - \psi_h||_a)\,||\chi - \chi_h||_{H_\Xi},\end{aligned}$$

having assumed $\omega_6(h) \leq 1$. Hence

$$||\chi - \chi_h||_{H_\Xi} \leq \omega_6(h)||\psi||_a + 2||\psi - \psi_h||_a.$$

The rest of the proof follows the same lines as the one of Theorem 3, using (62) and (66) (see (72) and (73)). □

The remaining part of this section is devoted to see what one can deduce from the uniform convergence of T^h_Ξ to T_Ξ.

THEOREM 5. *Assume that the sequence* T^h_Ξ *is bounded in* $\mathcal{L}(\Xi', \Xi)$. *Then there exists a Fortin operator* (*see* (64)) $\Pi_h : \Phi^0_{\Xi'} \to \Phi_h$ *such that*

$$||\psi - \Pi_h\psi||_a \leq C||\psi||_{\Phi^0_{\Xi'}}. \tag{74}$$

PROOF. Let ψ belong to $\Phi^0_{\Xi'}$. Then by definition $\psi = C^*_\Phi \circ S \circ C_\Xi g$ for some $g \in \Xi'$. There is only one g in this condition, and therefore, by definition, $||\psi||_{\Phi^0_{\Xi'}} = ||g||_{\Xi'}$ (see (47)). Let $\chi \in \Xi$ be such that $(\psi, \chi) = S(0, g)$. Let $\chi_h := T^h_\Xi g$; notice that, by assumption, $||\chi_h||_\Xi \leq C||g||_{\Xi'}$. By Propositions 5 and 4, there exists at least one ψ_h such that $(\psi_h, \chi_h) \in \Phi_h \times \Xi_h$ is a corresponding discrete solution of (13). If such ψ_h is unique, we define $\Pi_h\psi := \psi_h$. Otherwise we still define $\Pi_h\psi$ as the ψ_h having minimum norm in Φ. By construction we have (64) and

$$||\Pi_h\psi||^2_a = \langle g, \chi_h\rangle \leq ||g||_{\Xi'}||T^h_\Xi g||_\Xi \leq C||g||^2_{\Xi'} = C||\psi||^2_{\Phi^0_{\Xi'}}. \tag{75}$$

Let us bound $||\psi - \Pi_h\psi||_a$:

$$\begin{aligned}||\psi - \Pi_h\psi||^2_a &= a(\psi - \Pi_h\psi, \psi - \Pi_h\psi)\\ &= a(\psi, \psi - \Pi_h\psi) - a(\Pi_h\psi, \psi - \Pi_h\psi)\\ &= -b(\psi - \Pi_h\psi, \chi) - a(\psi - \Pi_h\psi, \Pi_h\psi).\end{aligned} \tag{76}$$

The first term in the right-hand side can be handled as follows:

$$
\begin{aligned}
b(\psi - \Pi_h\psi, \chi) &= b(\psi - \Pi_h\psi, \chi - \chi_h) \\
&= \langle g, \chi - \chi_h\rangle - b(\Pi_h\psi, \chi - \chi_h) \\
&= \langle g, \chi - \chi_h\rangle + a(\psi - \Pi_h\psi, \Pi_h\psi).
\end{aligned} \tag{77}
$$

Inserting (77) in (76), we obtain

$$
\begin{aligned}
||\psi - \Pi_h\psi||_a^2 &= -\langle g, \chi - \chi_h\rangle - 2a(\psi - \Pi_h\psi, \Pi_h\psi) \\
&\le ||g||_{\Xi'}||\chi - \chi_h||_{\Xi} + 2||\psi - \Pi_h\psi||_a||\Pi_h\psi||_a \\
&\le ||g||_{\Xi'}\,(||\chi||_{\Xi} + ||\chi_h||_{\Xi}) + 2||\psi - \Pi_h\psi||_a||\Pi_h\psi||_a
\end{aligned} \tag{78}
$$

then the boundedness of T_Ξ^h and (75) imply (74). □

THEOREM 6. *Assume that the sequence T_Ξ^h converges to T_Ξ uniformly from H_Ξ to Ξ, then for all $\chi \in \Xi_H^0$ there is $\chi^I \in \Xi_h$ such that* (63) *holds true.*

PROOF. Let χ belong to Ξ_H^0, then $\chi = T_\Xi g$ for a suitable g in H_Ξ. Let $\chi_h := T_\Xi^h g$ be the corresponding discrete solution, then we define $\chi^I := \chi_h$ and the inequality (63) is an easy consequence of the uniform convergence of $T_\Xi^h g$ to $T_\Xi g$ in Ξ. □

THEOREM 7. *Let us assume that the sequence T_Ξ^h is bounded in $\mathcal{L}(\Xi', \Xi)$ and converges uniformly to T_Ξ in $\mathcal{L}(H_\Xi, \Xi)$. In addition we assume that the following bound holds for the solutions of* (13) *with $f = 0$*

$$
||\psi_h||_\Phi \le C||g||_{\Xi'}. \tag{79}
$$

Then there exists a Fortin operator $\Pi_h : \Phi_{\Xi'}^0 \to \Phi_h$ satisfying (65) *and* (66). *Moreover we have* (DIS) (*see* (15)), *and the weak approximability of Ξ_H^0 with respect to a* (*see* (62)) *holds.*

PROOF. From Proposition 5 we have that $C_\Phi^* \circ S \circ C_\Xi$ is also well defined and (DIS) holds. Let us check (65). For $\psi \in \Phi_{\Xi'}^0$, there exists $g \in \Xi'$ and $\chi \in \Xi$ such that $(\psi, \chi) = S(0, g)$. We set $\Pi_h\psi := C_\Phi^* \circ S_h \circ C_\Xi g$. As we have seen, (64) holds trivially, and now (65) also holds in virtue of (79), with $C_\Pi := C$.

Now let us check (66). Let ψ belong to Φ_H^0; by definition $\psi = C_\Phi^* \circ S \circ C_\Xi g$ for some $g \in H_\Xi$. As in the proof of Theorem 5, g is unique, and $||\psi||_{\Phi_H^0} = ||g||_{H_\Xi}$. Let $\chi := T_\Xi g$; clearly $\chi \in \Xi_H^0$. Let $\chi_h := T_\Xi^h g$. By construction $(\Pi_h\psi, \chi_h)$ solves (13) with the right-hand side $(0, g)$. Moreover by the same computations as above, we arrive (see the first line in (78)) at

$$
||\psi - \Pi_h\psi||_a^2 = -\langle g, \chi - \chi_h\rangle - 2a(\psi - \Pi_h\psi, \Pi_h\psi).
$$

From this we have

$$
\begin{aligned}
||\psi - \Pi_h\psi||_a^2 &= -\langle g, \chi - \chi_h\rangle - 2b(\Pi_h\psi, \chi - \chi_h) \\
&\leq (||g||_{\Xi'} + 2M_b||\Pi_h\psi||_\Phi)\,||\chi - \chi_h||_\Xi \\
&\leq (1 + 2M_bC)||g||_{\Xi'}\omega_7(h)||g||_{H_\Xi} \\
&\leq (1 + 2M_bC)\omega_7(h)||g||_{H_\Xi}^2 \\
&= (1 + 2M_bC)\omega_7(h)||\psi||_{\Phi_H^0}^2,
\end{aligned}
\tag{80}
$$

where we used (79) and the uniform convergence of T_Ξ^h to T_Ξ in $\mathcal{L}(H_\Xi, \Xi)$ (see (67)). The bound (80) gives (66) with $\omega_6(h) = ((1 + 2M_bC)\omega_7(h))^{1/2}$.

Now let us check (62). If $\chi \in \Xi_H^0$, then $\chi = T_\Xi g$ for a suitable g in H_Ξ, let ψ be such that $(\psi, \chi) = S(0, g)$. Next we set $\chi_h := T_\Xi^h g$ and $\psi_h := \Pi_h\psi$. Then we get for every $\varphi_h \in \mathbb{K}_h$

$$
b(\varphi_h, \chi) = b(\varphi_h, \chi - \chi_h) = a(\Pi_h\psi - \psi, \varphi_h) \leq M_a||\Pi_h\psi - \psi||_a||\varphi_h||_a \tag{81}
$$

and (66) (already proved) ends the proof, since $||\psi||_{\Phi_H^0} = ||g||_{H_\Xi} = ||\chi||_{\Xi_H^0}$, by definition. □

EXAMPLE 3. Let us consider the mixed formulation of second order linear elliptic problems. Recall that $\Phi = \Sigma = H(\mathrm{div};\Omega)$ and $\Xi = V = L^2(\Omega)$. As usual we identify $L^2(\Omega)$ with its dual space, so that in our notation we have $\Xi = H_\Xi = \Xi' = L^2(\Omega)$. It is easy to see (using e.g. [17]) that if Ω is, for instance, a convex polygon, then Ξ_H^0 is $H^2(\Omega) \cap H_0^1(\Omega)$ and $\Phi_H^0 \equiv \Phi_{\Xi'}^0 = \underline{\nabla}(\Phi_H^0) \subseteq (H^1(\Omega))^2$.

Let Σ_h and V_h be finite dimensional subspaces of Σ and V respectively. We consider, first, classical approximations of $H(\mathrm{div};\Omega)$; for instance we can choose as Σ_h the spaces of the elements (RT) introduced in [24], or the elements (BDM) and (BDFM) introduced in [11], [10], respectively. For a unified presentation we refer to [9]. Correspondingly V_h will be the space $\mathrm{div}\,\Sigma_h$. For convenience of the reader we recall for instance the definition of BDM spaces. For k integer ≥ 1 we set

$$
\begin{aligned}
\Sigma_h^k &:= \{\underline{\tau}_h \in \Sigma : \ \underline{\tau}_h|_K \in (P_k(K))^2 \ \forall K \in \mathcal{T}_h\} \\
V_h^k &:= \{v_h \in V : \ v_h|_K \in P_{k-1}(K) \ \forall K \in \mathcal{T}_h\},
\end{aligned}
\tag{82}
$$

where, as usual, $\{\mathcal{T}_h\}$ is a regular sequence of triangulations of Ω. A Fortin operator satisfying (65) for all these choices of finite element spaces can be constructed using suitable degrees of freedom. Moreover it is well-known that (66) holds true, see e.g. [9], pag. 132.

Since $V_h = \mathrm{div}\,\Sigma_h$ then $\mathbb{K}_h \subseteq \mathbb{K}$, hence (DEK) and (62) trivially hold. It remains only to verify the strong approximability of Ξ_H^0, that is

$$
||v - v^I||_0 \leq \omega_5(h)||v||_1 \qquad \text{for all } v \in H^1(\Omega), \tag{83}
$$

which also holds thanks to standard approximation properties of piecewise polynomial spaces.

For various reasons, see for instance [4], [26], one might want to approximate Σ_h by continuous functions, using therefore finite element spaces that are not especially fit for mixed formulations. In constructing these new spaces, one might believe that (DEK) and (DIS) should be sufficient in order to well approximate eigenvalues and eigenvectors, once V_h satisfies the *strong approximability in* Ξ^0_H assumption. However, while conditions (62), (64) and (65) can be deduced from (DEK) and (DIS) the bound (66) does not, as it is shown by the following choice of the so-called $P_1 - \mathrm{div}(P_1)$ element on a criss–cross mesh. Let us assume that Ω is a square, which is divided into $2N \times 2N$ sub-squares, each of them partitioned into four triangles K by its diagonals. Then we set

$$\begin{aligned} \Sigma_h &:= \{\underline{\tau}_h \in (C^0(\Omega))^2 : \ \underline{\tau}_h|_K \in (P_1(K))^2 \ \forall K \in \mathcal{T}_h\}, \\ V_h &:= \mathrm{div}(\Sigma_h) \subset \{v_h : \ v_h|_K \in P_0(K) \ \forall K \in \mathcal{T}_h\}. \end{aligned} \tag{84}$$

In a recent paper [5], we proved that the pair (Σ_h, V_h) defined in (84) satisfies (DEK) and (DIS) but the sequence T^h_Ξ does not converge uniformly to T_Ξ in $L^2(\Omega)$. This fact produces in the numerical computations spurious eigenvalues which converge to points belonging to the resolvent set of T_Ξ.

Hence (66), which we have seen to be necessary, has to be checked independently of (DEK) and (DIS). On the other hand, (DEK) is not necessary, and we can obtain convergence of eigenvalues with finite element spaces that fail to satisfy it. For instance, on a quasi-uniform triangulation, one might take $\Sigma_h = \Sigma^3_h$ (see (82)) and $V_h = V^2_h$. Notice that the pair (Σ^3_h, V^3_h), as we have seen, works. Now, however, having chosen a smaller V_h, we obtain a bigger $\mathbb{K}_h$ (not anymore contained in $\mathbb{K}$). This will not jeopardize property (66) (the Π_h operator working for the pair (Σ^3_h, V^3_h) will also work for the pair (Σ^3_h, V^2_h)) but (62) is now at risk. However, by inverse inequality (see e.g. [12])

$$b(\underline{\tau}_h, v) = b(\underline{\tau}_h, v - v^I) \le C||\underline{\tau}_h||_1||v - v^I||_0 \le Ch^{-1}||\underline{\tau}_h||_a \ h^2||v||_2, \tag{85}$$

for $\underline{\tau}_h \in \mathbb{K}_h$ and $v^I = L^2$-projection of v onto V^2_h. Notice that this argument will work for any pair (Σ^k_h, V^r_h) provided $k \ge r > 1$ (in other words, (Σ^2_h, V^1_h) will not work).

EXAMPLE 4. *Biharmonic problem.* In this case we have $\Phi = Z = H^1(\Omega)$ and $\Xi = V = H^1_0(\Omega)$. We take $H_\Xi = L^2(\Omega)$. We always assume that Ω is a convex polygon. In that case we obtain $\Phi^0_{\Xi'} = \{z \in H^1(\Omega) : \ \exists v \in H^2_0(\Omega)$ with $z = \Delta v\} = \{z \in H^1(\Omega) : \ (z, \mu) = 0 \ \forall \mu \in L^2(\Omega)$ with $\Delta\mu = 0\}$ and, with obvious notation, $\Xi^0_{\Xi'} = H^3(\Omega) \cap H^2_0(\Omega)$. For any given polygon, Φ^0_H and Ξ^0_H will be slightly more regular, according to the maximum angle (see e.g. [17]).

For every given regular sequence $\{\mathcal{T}_h\}$ of triangulations of Ω and for every integer $k \ge 2$ we can take as in [16], [20], [13]:

$$\begin{aligned} Z^k_h &:= \{z_h \in Z : \ z_h|_K \in P_k(K) \ \forall K \in \mathcal{T}_h\} \\ V^k_h &:= \{v_h \in V : \ v_h|_K \in P_k(K) \ \forall K \in \mathcal{T}_h\}. \end{aligned} \tag{86}$$

Notice that $V_h^k = Z_h^k \cap H_0^1(\Omega)$. We can now define $\Pi_h w$ in Z_h as the solution of: $(\nabla \Pi_h w, \nabla z_h) = (\nabla w, \nabla z_h)$ for all $z_h \in Z_h^k$. Clearly (64), (65), (66) hold. Similarly (63) holds by taking χ^I (here u^I) as the usual interpolant. On the other hand, to check (62) we have to assume quasi-uniformity of the decomposition and then proceed as in (85): for $z_h \in \mathbb{K}_h$ and $v \in H^3(\Omega) \cap H_0^2(\Omega)$

$$(\nabla z_h, \nabla v) = (\nabla z_h, \nabla v - \nabla v^I) \leq Ch^{-1}||z_h||_a\, Ch^2||v||_3.$$

This shows the utility of the requirement $k \geq 2$. However, a more sophisticated proof, following the arguments of Scholz [25], shows that (62) also holds for $k = 1$.

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