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A general-weighted Sturm-Liouville problem

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A General-Weighted Sturm-Liouville Problem

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1. In this paper we describe the periodic, anti-periodic, Dirichlet and Neumann spectrum of the differential equation

$$y'' = q(x)y + \lambda m(x)y \tag{1}$$

where $m,\ q \in C(\mathbb{R})$ are of period 1 and $q(x) \geq 0,\ x \in \mathbb{R},\ \ q \not\equiv 0$. We refer to m as the *potential.*

Equation (1) with $q \equiv \frac{1}{4}$ plays a crucial role in the study of a recently derived shallow water equation (see [1], [4], [5]).

It is well-known that for the similar Hill's equation

$$z'' + [\lambda + Q(x)]z = 0 \tag{2}$$

with $Q \in C(\mathbb{R})$ of period 1, the following holds (see [3], [6], [7], [8]).

There is a simple periodic ground state λ_0 followed by alternately anti-periodic and periodic pairs

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

of simple or double eigenvalues accumulating at ∞. There is precisely one simple eigenvalue of the Dirichlet spectrum in each interval $[\lambda_{2n-1}, \lambda_{2n}]$, $n = 1, 2, \dots$ and no others. All elements of the Neumann spectrum are likewise simple real eigenvalues, one in $(-\infty, \lambda_0)$ and one in each of the intervals $[\lambda_{2n-1}, \lambda_{2n}]$, $n = 1, 2, \dots$

Our aim is to prove an analogous result for (1).

THEOREM 1. *Assume that $m \not\equiv 0$.*

1) *If $m \leq 0$, there is a simple periodic ground state $\lambda_0 > 0$ followed by alternately anti-periodic and periodic pairs*

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

of simple or double eigenvalues accumulating at ∞. There is precisely one simple eigenvalue of the Dirichlet spectrum in each interval $[\lambda_{2n-1}, \lambda_{2n}]$, $n = 1, 2, \dots$

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and no others. All elements of the Neumann spectrum are likewise simple real eigenvalues, one in $(-\infty, \lambda_0)$ *and one in each of the intervals* $[\lambda_{2n-1}, \lambda_{2n}]$, $n = 1, 2, \ldots$

2) *If* $m \geq 0$, *the pattern of* 1) *is simply reflected in* $\lambda = 0$.

In the case when m changes sign the picture is slightly different:

THEOREM 2. *If* $m \in C(\mathbb{R})$ *of period* 1 *changes sign, there are two simple periodic ground states* $\lambda_0^+ > 0$, $\lambda_0^- < 0$, *and* λ_0^+ *is followed,* λ_0^- *preceded by alternately anti-periodic and periodic pairs*

$$\ldots < \lambda_2^- \leq \lambda_1^- < \lambda_0^- < \lambda_0^+ < \lambda_1^+ \leq \lambda_2^+ < \ldots$$

of simple or double eigenvalues accumulating at ∞ *and* $-\infty$ *respectively. There is precisely one simple eigenvalue of the Dirichlet spectrum in each interval of the form* $[\lambda_{2n-1}^+, \lambda_{2n}^+]$ *or* $[\lambda_{2n}^-, \lambda_{2n-1}^-]$ *for* $n = 1, 2, \ldots$ *and no others. All elements of the Neumann spectrum are likewise simple real eigenvalues, one in* $(0, \lambda_0^+]$, *one in* $[\lambda_0^-, 0)$, *and one in each of the intervals* $[\lambda_{2n-1}^+, \lambda_{2n}^+]$, $[\lambda_{2n}^-, \lambda_{2n-1}^-]$ *for* $n = 1, 2, \ldots$

Information about the Dirichlet spectrum of (1) is already provided in [11]. The present paper gives the complete spectral picture in the case when q is non-negative. Such information is not available in the literature, cf. [2] and the references therein.

2. Equation (1) has two solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ determined by the conditions $y_1(0, \lambda) = 1$, $y_1'(0, \lambda) = 0$; $y_2(0, \lambda) = 0$, $y_2'(0, \lambda) = 1$. These *normalized solutions* are defined for all values of $x \in \mathbb{R}$.

The spectral problem is to determine the values of λ for which (1) has a nontrivial periodic solution of period 1 ($y(0) = y(1)$ and $y'(0) = y'(1)$) - this is the *periodic spectrum* - and the values of λ for which it has a nontrivial anti-periodic solution ($y(0) = -y(1)$ and $y'(0) = -y'(1)$) - this is the *anti-periodic spectrum*. The *Dirichlet spectrum* is determined by solving (1) with the boundary conditions $y(0) = y(1) = 0$; it comprises the roots of $y_2(1, \lambda) = 0$. The *Neumann spectrum* is determined by solving with $y'(0) = y'(1) = 0$; it comprises the roots of $y_1'(1, \lambda) = 0$.

The discriminant of (1) is

$$\Delta(\lambda) = \frac{1}{2}[y_1(1, \lambda) + y_2'(1, \lambda)], \quad \lambda \in \mathbb{C}.$$

FLOQUET'S THEOREM [8]. *Equation* (1) *has a nontrivial periodic solution of period* 1 *if and only if* $\Delta(\lambda) = 1$, *and a nontrivial anti-periodic solution if and only if* $\Delta(\lambda) = -1$ *(double roots corresponding to double eigenvalues). If* $|\Delta(\lambda)| \neq 1$, *then* (1) *has two linearly independent solutions* f_1, f_2 *such that* $f_1(x+1) = \alpha f_1(x)$ *and* $f_2(x+1) = \frac{1}{\alpha} f_2(x)$ *for* $x \in \mathbb{R}$ *with* $\alpha = \Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 1}$. *If* $\Delta(\lambda) \in (-1, 1)$, *then* $\alpha \in \mathbb{C} - \mathbb{R}$ *and all solutions of* (1) *are bounded; if* $|\Delta(\lambda)| > 1$, *then* $\alpha \in \mathbb{R}$ *and every nontrivial solution is unbounded.*

Observe that the periodic, anti-periodic, Dirichlet and Neumann spectra are all real.

Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue and let y be the corresponding eigenfunction, $y = f + ig$ with $f, g : \mathbb{R} \to \mathbb{R}$. Multiplying (1) by $\bar{y} = f - ig$, we obtain after integration

$$\int_0^1 \Big([f'(x)]^2 + [g'(x)]^2 dx + q(x)[f^2(x) + g^2(x)]\Big) dx$$
$$= -\lambda \int_0^1 m(x)[f^2(x) + g^2(x)]dx$$

(integration by parts helps). Since $q \geq 0$ and $y \not\equiv 0$, we see that the left-hand side of the above equality is not zero. This proves that $\lambda \in \mathbb{R}$.

LEMMA 1. *There is a neighborhood of* $\lambda = 0$ *disjoint from the periodic, anti-periodic, Dirichlet and Neumann spectrum. Moreover, if* $m \geq 0$, *then the periodic, anti-periodic, Dirichlet and Neumann spectra lie in* $(-\infty, 0)$, *while if* $m \leq 0$ *they lie in* $(0, \infty)$.

PROOF. No element of the periodic, anti-periodic, Dirichlet and Neumann spectrum can satisfy

$$q(x) + \lambda m(x) \geq 0, \quad x \in \mathbb{R}.$$

This can be seen multiplying the differential equation for the eigenfunction y by y itself and then integrating by parts. □

Applying Picard's iterative method to (1), we can easily prove:

LEMMA 2. *The functions* $\Delta(\lambda)$, $y_2(1, \lambda)$ *and* $y_1'(1, \lambda)$ *are entire analytic functions of the complex variable* λ.

3. Assume that m has no zeros and is of class C^2. Then the Liouville substitution

$$z(t) = |m(x)|^{\frac{1}{4}}\, y(x) \quad \text{where} \quad t = \frac{\int_0^x \sqrt{|m(s)|}ds}{\int_0^1 \sqrt{|m(s)|}ds}$$

transforms (1) into

$$\frac{d^2z}{dt^2} \mathrel{\substack{+\\=\\-}} \left(-\lambda - \frac{q(x)}{m(x)} - \frac{m''(x)}{4m^2(x)} + \frac{5[m'(x)]^2}{16m^3(x)}\right) \left[\int_0^1 \sqrt{|m(s)|}ds\right]^2 z$$

with the upper/lower sign according as m is negative or positive.

The Liouville transformation is not applicable unless m is differentiable, at least in the sense that $\frac{d^2m}{dt^2}$ is bounded and continuous almost everywhere [8], but the methods from [8] regarding problem (2) can be easily adapted to (1) to obtain some information about the periodic spectrum:

PROPOSITION. *Assume that $m < 0$ is continuous of period 1. Then there is a simple periodic ground state $\lambda_0 > 0$ followed by alternately anti-periodic and periodic pairs*

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \dots$$

of simple or double eigenvalues accumulating at ∞.

When m changes sign or has zeros, the methods developped in [3], [6], [7], [8] for (2) cannot be adapted to (1).

LEMMA 3. *Suppose $m \not\equiv 0$ is continuous of period 1. Then the Dirichlet spectrum has infinitely many elements with no finite accumulation point. If $m \le 0$, it forms a sequence of stricly positive numbers accumulating at ∞; if $m \ge 0$, it forms a sequence of strictly negative numbers accumulating at $-\infty$; and if m changes sign, it accumulates both to $-\infty$ and to ∞.*

PROOF. The operator $[-\partial_x^2 + q(x)]$ acting on the space $\{f \in H^2[0,1];\ f(0) = f(1) = 0\}$ with values in $L^2[0,1]$, is invertible with bounded, symmetric, positive compact inverse G (this can be easily seen by using Green's function for (1) and the fact that $q \ge 0$).

G has a positive square root A, cf. [10]: it is a bounded positive linear operator and it is compact since G is compact (see [9]).

We know by Lemma 1 that if μ is an element of the Dirichlet spectrum, then $\mu \neq 0$ and $y_2(1,\mu) = 0$. We write (1) in the form

$$(-\partial_x^2 + q)\psi = -\mu m \psi \tag{3}$$

with $\psi(x) = y_2(x,\mu)$ for $x \in \mathbb{R}$. Let $\phi \in L^2[0,1]$ be such that $A\phi = \psi$. Applying A to both sides of (3) we get

$$\lambda\phi = AmA\phi, \quad \lambda = -\frac{1}{\mu}.$$

Conversely, one can see that if $\lambda \neq 0$ is an eigenvalue of AmA, then $\mu = -\frac{1}{\lambda}$ is in the Dirichlet spectrum.

It is an easy exercise to show that AmA is a self-adjoint, compact bounded linear operator on $L^2[0,1]$. If $m \ge 0$, then AmA is positive and if $m \le 0$, then $-AmA$ is positive: indeed,

$$\langle \phi, AmA\phi \rangle = \langle A\phi, mA\phi \rangle = \int_0^1 m(x)[(A\phi)(x)]^2 dx,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2[0,1]$.

Write now the Schur Representation of the operator AmA on $L^2[0,1]$ (see [10])

$$AmA\phi = \sum_{k \ge 1} \lambda_k \langle \phi, e_k \rangle e_k, \quad \phi \in L^2[0,1],$$

where λ_k are the eigenvalues and e_k the corresponding orthonormal eigenvectors.

Let us show that we cannot have a finite number of eigenvalues.

If this would be true, then

$$AmA\phi = \sum_{k\geq 1}^{n} \lambda_k \langle \phi, e_k \rangle e_k, \quad \phi \in L^2[0,1],$$

for some $n \geq 1$ - the operator AmA being compact, all eigenspaces corresponding to nonzero eigenvalues are finite dimensional.

Let $[a,b] \subset [0,1]$ be such that $m(x) > 0$ or $m(x) < 0$ on $[a,b]$ and choose $\phi_1, \ldots, \phi_{n+1} \in L^2[0,1]$ so that $A\phi_j \equiv 0$ on $[0,1]-[a,b]$ for $j = 1, \ldots, n+1$, and $A\phi_1, \ldots, A\phi_{n+1}$ are linearly independent in $L^2[0,1]$ (this is possible since the functions $f \in C^\infty[0,1]$ with $f(0) = f(1) = 0$ belong to the range of A). The system

$$\left\langle \sum_{j=1}^{n+1} a_j\phi_j, e_k \right\rangle = \sum_{j=1}^{n+1} a_j \langle \phi_j, e_k \rangle = 0, \quad k = 1, \ldots, n,$$

has a solution $(a_1, \ldots, a_{n+1}) \neq (0, \ldots, 0)$. Define $\phi = \sum_{j=1}^{n+1} a_j\phi_j \in L^2[0,1]$. Then $AmA\phi \equiv 0$ on $[0,1]$,

$$0 = \langle\ \phi, AmA\phi\rangle = \langle A\phi, mA\phi\rangle = \int_0^1 m(x)(A\phi)^2(x)dx = \int_a^b m(x)(A\phi)^2(x)dx$$

and $(A\phi)(x) = 0$ a.e. on $[a,b]$, which is impossible by construction.

We proved that AmA has infinitely many eigenvalues $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \neq 0$ for $n \geq 1$. AmA being compact, we have $\lim_{n\to\infty} \lambda_n = 0$ and since $\lambda \neq 0$ is an eigenvalue of AmA if and only if $\mu = -\frac{1}{\mu}$ is in the Dirichlet spectrum of (1), this proves Lemma 3 if m does not change sign (then clearly all eigenvalues of AmA have the sign of m).

In order to complete the proof, we have to show that if m changes sign we have infinitely many eigenvalues of AmA on both sides of zero.

Suppose m changes sign and AmA has no negative eigenvalues. Let $[a,b] \subset [0,1]$ be such that $m(x) < 0$ on $[a,b]$ and let $\phi \in L^2[0,1]$ be such that $A\phi \equiv 0$ on $[0,1]-[a,b]$ but $A\phi \not\equiv 0$ on $[0,1]$ (take $A\phi$ of class C^∞). Then

$$\langle \phi, AmA\phi\rangle \geq 0$$

since we do not have negative eigenvalues, but

$$\langle \phi, AmA\phi\rangle = \langle A\phi, mA\phi\rangle = \int_0^1 m(x)(A\phi)^2(x)dx = \int_a^b m(x)(A\phi)^2(x)dx$$

and so, since $m(x) < 0$ on $[a,b]$, we must have $(A\phi)(x) = 0$ a.e. on $[a,b]$ which is likewise impossible by construction.

If AmA would have only a finite number of negative eigenvalues then there are $\lambda_1, \dots, \lambda_n$ negative eigenvalues and corresponding orthonormal eigenvectors $e_1, \dots, e_n$ with

$$\left\langle AmA\phi - \sum_{j=1}^{n} \lambda_j \langle \phi, e_j \rangle e_j, \phi \right\rangle \geq 0, \quad \phi \in L^2[0,1].$$

We choose $\phi_1, \dots, \phi_{n+1} \in L^2[0,1]$ with $A\phi_j \equiv 0$ on $[0,1] - [a,b]$ for $j = 1, \dots, n+1$, and $A\phi_1, \dots, A\phi_{n+1}$ linearly independent in $L^2[0,1]$. If $(a_1, \dots, a_{n+1}) \neq (0, \dots, 0)$ is a solution of the system

$$\left\langle \sum_{j=1}^{n+1} a_j \phi_j, e_k \right\rangle = \sum_{j=1}^{n+1} a_j \langle \phi_j, e_k \rangle = 0, \quad k = 1, \dots, n,$$

then $\phi := \sum_{j=1}^{n+1} a_j \phi_j$ will satisfy

$$\langle \phi, AmA\phi \rangle = \int_a^b m(x)(A\phi)^2(x)dx \geq 0$$

and, since $m(x) < 0$ on $[a,b]$, $(A\phi)(x) = 0$ a.e. on $[a,b]$ which is impossible by construction.

Similarly we prove that it is impossible for AmA to have only finitely many positive eigenvalues and the proof of Lemma 3 is completed. □

REMARK. Statements analogous to Lemma 3 hold also for the periodic and anti-periodic spectra.

Let us now consider the differential equation

$$-y'' + q(x)y + \lambda m(x)y = \gamma y. \tag{4}$$

One can easily see that the associated $\Delta(\lambda, \gamma)$ and $y_2(1, \lambda, \gamma)$ will be analytic functions of λ and γ.

For fixed $\lambda \in \mathbb{R}$, (4) is a standard Hill equation, and so we have (see [8]) that for any $\lambda \in \mathbb{R}$ the roots of $\Delta(\lambda, \gamma) =^{+}_{-} 1$ can be ordered like that: there is a simple real root $\gamma_0(\lambda)$ of $\Delta(\lambda, \gamma) = 1$ followed by alternately (simple or double) real roots of $\Delta(\lambda, \gamma) = -1$ and respectively $\Delta(\lambda, \gamma) = 1$:

$$\gamma_0(\lambda) < \gamma_1(\lambda) \leq \gamma_2(\lambda) < \gamma_3(\lambda) \leq \gamma_4(\lambda) < \dots$$

accumulating at ∞. It is easy to check that, at $\lambda = 0$, all eigenvalues of (4) —periodic, anti-periodic, Dirichlet and Neumann— are strictly positive.

Since the periodic and anti-periodic spectra are formed by the roots of $\Delta(\lambda, \gamma) = 1$, respectively $\Delta(\lambda, \gamma) = -1$, we deduce by Hurwitz's theorem that the curves $(\lambda, \gamma_{2k}(\lambda))$ and $(\lambda, \gamma_{2k+1}(\lambda))$ are continuous for all $k \geq 0$. The elements of the periodic and anti-periodic spectrum of (1) are the λ's corresponding to the intersection of these curves with the λ-axis in the (λ, γ)-plane. Between any two curves $(\lambda, \gamma_{2k-1}(\lambda))$ and $(\lambda, \gamma_{2k}(\lambda))$ (with $k \geq 1$) lies the continuous deformation $(\lambda, \gamma_k^D(\lambda))$ of an element of the Dirichlet spectrum and the continuous deformation $(\lambda, \gamma_k^N(\lambda))$ of an element of the Neumann spectrum. There is also a continuous curve $(\lambda, \gamma_0^N(\lambda))$ below $(\lambda, \gamma_0(\lambda))$ representing the continuous deformation of the first Neumann eigenvalue for (4) with $\lambda \in \mathbb{R}$ fixed.

LEMMA 4. *The functions $\lambda \mapsto \gamma_k^N(\lambda)$, $k \geq 0$, are differentiable on $\mathbb{R}$. If $\gamma_k(\lambda^*)$ is a simple periodic or anti-periodic eigenvalue ($k \geq 0$), then the function $\lambda \mapsto \gamma_k(\lambda)$ is differentiable in a small neighborhood of $\lambda = \lambda^*$. Moreover, if f is any of those functions and if $\lambda^* \neq 0$ is a simple eigenvalue which is also a root of f, then $\frac{\partial f}{\partial \lambda}(\lambda^*) < 0$ if $\lambda^* > 0$ and $\frac{\partial f}{\partial \lambda}(\lambda^*) > 0$ if $\lambda^* < 0$.*

PROOF. Let us prove the differentiability of the function $\lambda \mapsto \gamma_k(\lambda)$ near $\lambda = \lambda^*$, where $\gamma_k(\lambda^*)$ is a simple periodic eigenvalue. Let ε, $\delta > 0$ be small enough so that $\Delta(\lambda, \gamma) \neq 1$ for $|\lambda - \lambda^*| < \delta$ and $\gamma \in C_\varepsilon = \{\gamma_k(\lambda^*) + \varepsilon e^{i\theta}\,;\ 0 \leq \theta \leq 2\pi\}$. Then, for $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$, we have

$$\frac{1}{2\pi i} \int_{C_\varepsilon} \frac{\gamma \frac{\partial}{\partial \gamma} \Delta(\lambda, \gamma)}{\Delta(\lambda, \gamma) - 1}\, d\gamma = \gamma_k(\lambda)\,,$$

so $\frac{\partial \gamma_k}{\partial \lambda}(\lambda)$ exists for $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ (we can differentiate with respect to λ under the integral; recall the analyticity of $\Delta(\lambda, \gamma)$ in both variables).

Assume now that $f(\lambda^*) = 0$ where $\lambda^* \neq 0$ is a simple periodic or anti-periodic eigenvalue or an element of the Neumann spectrum (thus again simple) and let y_p be the eigenfunction corresponding to $f(\lambda)$ for λ very close to λ^* (in order for $f(\lambda)$ to be again a simple eigenvalue).

Differentiating with respect to λ in

$$-y_p'' + q(x) y_p + \lambda m y_p = f(\lambda) y_p$$

we find

$$-\frac{\partial}{\partial \lambda} y_p'' + q(x) \left(\frac{\partial}{\partial \lambda} y_p\right) + \lambda m \left(\frac{\partial}{\partial \lambda} y_p\right) + m y_p = f(\lambda) \left(\frac{\partial}{\partial \lambda} y_p\right) + \left(\frac{\partial}{\partial \lambda} f(\lambda)\right) y_p\,.$$

Multiplying by y_p and adding this to the differential equation satisfied by y_p multiplied on its turn by $-(\frac{\partial}{\partial \lambda} y_p)$, we obtain

$$y_p'' \left(\frac{\partial}{\partial \lambda} y_p\right) - y_p \left(\frac{\partial}{\partial \lambda} y_p''\right) + m y_p^2 = \left(\frac{\partial f}{\partial \lambda}(\lambda^*)\right) y_p^2$$

if we evaluate at $\lambda = \lambda^*$.

Integrating on [0, 1], we find

$$\int_0^1 m(x)y_p^2(x,\lambda^*,0)dx = \left(\frac{\partial f}{\partial \lambda}(\lambda^*)\right)\int_0^1 y_p^2(x,\lambda^*,0)dx\,.$$

The identity

$$\lambda^*\int_0^1 m(x)y_p^2(x,\lambda^*,0)dx = -\int_0^1 q(x)y_p^2(x,\lambda^*,0)dx - \int_0^1 [y_p'(x,\lambda^*,0]^2dx$$

completes the proof. □

Let $\alpha(\lambda)\neq 0$ be a zero of $y_2(x,\lambda)$. This moves continuously with λ and since it can never become a double zero (otherwise $y_2(x,\lambda)\equiv 0$ on $\mathbb{R}$) we deduce that the zeros of $y_2(x,\lambda)$ cannot coalesce and then disappear.

LEMMA 5. *The zeros of $y_2(x,\lambda)$ on $(0,\infty)$ move to the left (remaining on $(0,\infty)$) for $\lambda>0$ increasing and for $\lambda<0$ decreasing. The zeros of $y_2(x,\lambda)$ on $(-\infty,0)$ move to the right (remaining on $(-\infty,0)$) for $\lambda>0$ increasing and for $\lambda<0$ decreasing.*

PROOF. Assume that $\alpha>0$ is a zero of $y_2(x,\lambda)$ for some $\lambda>0$. Differentiating (1) with respect to λ we obtain

$$\frac{\partial}{\partial\lambda}y_2'' = q\left(\frac{\partial}{\partial\lambda}y_2\right) + \lambda m\left(\frac{\partial}{\partial\lambda}y_2\right) + my_2\,.$$

Multiplying this by $(-y_2)$ and adding it to the differential equation satisfied by y_2 multiplied by $\frac{\partial}{\partial\lambda}y_2$, we get

$$-y_2\left(\frac{\partial}{\partial\lambda}y_2''\right) + \left(\frac{\partial}{\partial\lambda}y_2\right)y_2'' = -my_2^2\,.$$

Since $\frac{\partial}{\partial\lambda}y_2(0,\lambda) = y_2(0,\lambda) = y_2(\alpha,\lambda) = 0$, integration on $[0,\alpha]$ produces

$$y_2'(\alpha,\lambda)\left(\frac{\partial}{\partial\lambda}y_2(\alpha,\lambda)\right) = -\int_0^\alpha m(x)y_2^2(x,\lambda)dx\,.$$

Multiplying (1) by y_2 and integrating on $[0,\alpha]$, we get

$$-\lambda\int_0^\alpha m(x)y_2^2(x,\lambda)dx = \int_0^\alpha q(x)y_2^2(x,\lambda)dx + \int_0^\alpha [y_2'(x,\lambda)]^2dx\,,$$

whence

$$y_2'(\alpha,\lambda)\left(\frac{\partial y_2}{\partial\lambda}(\alpha,\lambda)\right) > 0\,.$$

When $y_2'(\alpha,\lambda)<0$, $y_2(\alpha,\lambda)$ is decreasing while λ is increasing and thus the zero of $y_2(x,\lambda)$ is moving to the left; if $y_2'(\alpha,\lambda)>0$, $y_2(\alpha,\lambda)$ is increasing with λ and thus the zero of $y_2(x,\lambda)$ is moving again to the left.

The other cases are handled in a similar way. □

PROOF OF THEOREM 1. Assume that $m \leq 0$.

By Lemma 3, the periodic, anti-periodic, Dirichlet and Neumann spectra are formed by infinitely many positive elements accumulating at ∞.

From Lemma 4 we know that the curves $\big(\lambda, \gamma_k^N(\lambda)\big)$, $k \geq 0$, can intersect the λ-axis only in one point (crossing in the (λ, γ)-plane from the upper half-plane to the lower half-plane). Since the curves $\big(\lambda, \gamma_{2k}(\lambda)\big)$ and $\big(\lambda, \gamma_{2k+1}(\lambda)\big)$ are disjoint (on the first $\Delta(\lambda, \gamma) = 1$ and on the second $\Delta(\lambda, \gamma) = -1$), it follows from the fact that the curve $\big(\lambda, \gamma_0^N(\lambda)\big)$ is below $\big(\lambda, \gamma_0(\lambda)\big)$ and the curve $\big(\lambda, \gamma_k^N(\lambda)\big)$ is between $\big(\lambda, \gamma_{2k-1}(\lambda)\big)$ and $\big(\lambda, \gamma_{2k}(\lambda)\big)$ for $k \geq 1$, that all curves $\big(\lambda, \gamma_k^N(\lambda)\big)$ with $k \geq 0$ must intersect the λ-axis in exactly one point. Therefore, each curve $\big(\lambda, \gamma_k(\lambda)\big)$, $k \geq 0$, and $\big(\lambda, \gamma_k^D(\lambda)\big)$, $k \geq 1$, intersects at least once the positive λ-semiaxis.

We claim that they do this just once so proving the statement of Theorem 1 (the case $m \geq 0$ is similar).

Observe that at each point μ_k where the curve $\big(\lambda, \gamma_k^D(\lambda)\big)$, $k \geq 1$, intersects the λ-axis, $y_2(x, \mu_k)$ has exactly $(k+1)$ zeros on $[0, 1]$: at $x = 0$, $x = 1$ and $(k-1)$ times in $(0, 1)$; this follows by continuous deformation from the case of $y_2(x, 0, \gamma_k(0))$ (for the latter, see [8]). Now, by Lemma 5, it is impossible to have more than one point of intersection.

Since (by Lemma 1 and Lemma 4) at any point λ^* with $\gamma_0(\lambda^*) = 0$ we have $\frac{\partial \gamma_0}{\partial \lambda}(\lambda^*) < 0$, we conclude that the curve $\big(\lambda, \gamma_0(\lambda)\big)$ intersects the λ-axis in exactly one point.

Let us now consider the more delicate case of the curves $\big(\lambda, \gamma_k(\lambda)\big)$, $k \geq 1$. Fix $k \geq 1$ and assume that we deal with the deformation of a periodic eigenvalue (the anti-periodic case is similar). We know that the curve $\big(\lambda, \gamma_k(\lambda)\big)$ has to intersect the positive λ-semiaxis. Let $\lambda^* > 0$ be such that $\gamma_k(\lambda^*) = 0$. If $\frac{\partial \Delta}{\partial \lambda}(\lambda^*, 0) \neq 0$, we know by Lemma 4 that the curve $\big(\lambda, \gamma_k(\lambda)\big)$ crosses the λ-axis from the upper half-plane to the lower half-plane.

If $\frac{\partial \Delta}{\partial \lambda}(\lambda^*, 0) = 0$ then λ^* is a double eigenvalue of (1) and, in particular, it is in the Dirichlet spectrum. If we prove that for $\lambda > \lambda^*$ very close to λ^*, the curve $\big(\lambda, \gamma_k(\lambda)\big)$ is in the lower half-plane, we are done: in view of Lemma 4 and the fact that $\big(\lambda, \gamma^D_{\left[\frac{k+1}{2}\right]}(\lambda)\big)$ crosses just at λ^* the λ-axis, it can never cross again.

Assume the contrary. Then, for $\lambda > \lambda^*$ small, the curve $(\lambda, \gamma_k(\lambda))$ is distinct from the curve $\big(\lambda, \gamma^D_{\left[\frac{k+1}{2}\right]}(\lambda)\big)$ so that we can find a sequence $\{\lambda_n\}_{n \geq 1}$ converging to λ^* with

$$\lambda_n > \lambda^*, \qquad \frac{\partial \gamma_k}{\partial \lambda}(\lambda_n) \geq 0, \quad n \geq 1,$$

and such that $\gamma_k(\lambda_n)$ is a simple periodic eigenvalue for $n \geq 1$ (going through the first part of the proof of Lemma 4 we can see that the function $\lambda \mapsto \gamma_k(\lambda)$ is differentiable in a small neighborhood of λ_n). For all $n \geq 1$, let now y_n be a periodic eigenfunction of (4) corresponding to $(\lambda_n, \gamma_k(\lambda_n))$. Since $y_n \not\equiv 0$

is periodic, we can normalize the family of functions $\{y_n\}_{n\geq 1}$ by imposing the condition

$$\int_0^1 y_n^2(x)dx = 1, \quad n \geq 1.$$

Repeating the arguments from the last part of the proof of Lemma 4, we find

$$\int_0^1 m(x)y_n^2(x, \lambda_n, \gamma_k(\lambda_n))dx = \left(\frac{\partial \gamma_k}{\partial \lambda}(\lambda_n)\right) \int_0^1 y_n^2(x, \lambda_n, \gamma(\lambda_n))dx, \quad n \geq 1,$$

and so

$$\begin{aligned} -\left(\int_0^1 q(x)y_n^2(x, \lambda_n, \gamma_k(\lambda_n))dx + \int_0^1 [y_n'(x, \lambda_n, \gamma_k(\lambda_n))]^2 dx\right) \\ + \gamma_k(\lambda_n) \int_0^1 y_n^2(x, \lambda_n, \gamma_k(\lambda_n))dx \\ = \lambda_n \left(\frac{\partial \gamma_k}{\partial \lambda}(\lambda_n)\right) \\ \times \int_0^1 y_n^2(x, \lambda_n, \gamma(\lambda_n))dx \geq 0, \quad n \geq 1, \end{aligned}$$

that is (taking into account the normalization),

$$\begin{aligned} & -\left(\int_0^1 q(x)y_n^2(x)dx + \int_0^1 [y_n'(x)]^2 dx\right) + \gamma_k(\lambda_n) \\ & = \lambda_n \left(\frac{\partial \gamma_k}{\partial \lambda}(\lambda_n)\right) \geq 0, \quad n \geq 1. \end{aligned} \tag{5}$$

Define now

$$K = \inf_{n\geq 1} \left\{ \int_0^1 q(x)y_n^2(x)dx + \int_0^1 [y_n'(x)]^2 dx \right\}$$

and let us prove that $K > 0$.

Indeed, if $K = 0$, we can find a sequence $n_j \to \infty$ such that

$$\int_0^1 q(x)y_{n_j}^2(x)dx + \int_0^1 [y_{n_j}'(x)]^2 dx \to 0 \quad \text{as} \quad n_j \to \infty.$$

Using Schwarz's inequality, one can check that we would have

$$y_{n_j}' \rightharpoonup 0 \quad \text{weakly in} \quad L^2[0, 1].$$

Recalling the normalization, for every n_j there is a point $x_{n_j} \in [0, 1]$ with $|y_{n_j}(x_{n_j})| \leq 1$. Using now the mean-value theorem and Schwarz's inequality, the Arzela-Ascoli theorem yields the existence of a continuous function $y : [0, 1] \to \mathbb{R}$ and of a subsequence of $\{y_{n_j}\}$, denoted by $\{y_{n_j}\}$ again, such that $y_{n_j} \to y$ uniformly on $[0, 1]$; in particular,

$$y_{n_j} \to y \quad \text{strongly in} \quad L^2[0, 1].$$

Viewing both convergences in the sense of distributions, we see that $y' = 0$ a.e. on $[0, 1]$ and therefore y has to be a constant: due to the normalization $\int_0^1 y_{n_j}^2(x)dx = 1$ for $n_j \geq 1$, we must have that the constant is either 1 or -1. We now have

$$\int_0^1 q(x)y_{n_j}^2(x)dx \to \int_0^1 q(x)y^2(x)dx = \int_0^1 q(x)dx \quad \text{as} \quad n_j \to \infty,$$

and on the other hand we must have

$$\int_0^1 q(x)y_{n_j}^2(x)dx \to 0 \quad \text{as} \quad n_j \to \infty.$$

Since $\int_0^1 q(x)dx \neq 0$, we obtained a contradiction.

We proved that $K > 0$ but by (5) this is impossible since $\lim_{n\to\infty} \gamma_k(\lambda_n) = 0$. The proof is complete. □

PROOF OF THEOREM 2. Let m change sign. We repeat the arguments from before proving that all curves $\big(\lambda, \gamma_k(\lambda)\big)_{k\geq 0}$, $\big(\lambda, \gamma_k^N(\lambda)\big)_{k\geq 0}$ and $\big(\lambda, \gamma_k^D(\lambda)\big)_{k\geq 1}$ intersect the λ-axis in the (λ, γ)-plane exactly twice: once on the positive semiaxis and once on the negative semiaxis. □

4. In this section we will study the oscillation properties of the solutions of the differential equation (1).

THEOREM 3. *Assume that $m \not\equiv 0$ is continuous of period* 1.

1) *If $m \leq 0$, then for $\lambda \leq \lambda_0$ all nontrivial solutions of* (1) *have only a finite number of zeros on $\mathbb{R}$ and for $\lambda > \lambda_0$ every solution of* (1) *has infinitely many zeros.*

2) *If $m \geq 0$, then for $\lambda \geq \lambda_0$ all nontrivial solutions of* (1) *have only a finite number of zeros on $\mathbb{R}$ and for $\lambda < \lambda_0$ every solution of* (1) *has infinitely many zeros.*

3) *If m changes sign, then for $\lambda \in [\lambda_0^-, \lambda_0^+]$ all nontrivial solutions of* (1) *have only a finite number of zeros on $\mathbb{R}$ but for $\lambda > \lambda_0^+$ and $\lambda < \lambda_0^-$ every solution of* (1) *has infinitely many zeros.*

Let us interpret y_1 and y_2 as Cartesian coordinates in the plane and put

$$y_1(x) = \rho(x) \cos\phi(x), \qquad y_2(x) = \rho(x) \sin\phi(x), \quad x \in \mathbb{R},$$

where $\rho > 0$, $\phi(0) = 0$ and $\rho(0) = 1$. Then

$$\rho^2(x) = y_1^2(x) + y_2^2(x), \quad x \in \mathbb{R}, \tag{6}$$

and since the Wronskian $y_1y_2' - y_1'y_2 \equiv 1$ on $\mathbb{R}$, we find

$$\phi(x) = \int_0^x \frac{dt}{\rho^2(t)}, \quad x \in \mathbb{R}. \tag{7}$$

Observe that $\phi(x)$ is a strictly increasing function of x; if $|\phi(x)| \to \infty$ as $x \to \infty$ or $x \to -\infty$, both y_1 and y_2 have infinitely many zeros, otherwise both of them will have a finite number of zeros on $\mathbb{R}$.

For fixed $\lambda \in \mathbb{R}$, every solution of (1) will have infinitely many zeros on $\mathbb{R}$ if a single nontrivial solution has this property (the zeros of two linearly independent solutions of a second order linear differential equation separate each other, cf. [7]).

LEMMA 6. *If there is a solution of* (1) *which has a finite number of zeros on* $\mathbb{R}$, *then for all functions* $f \in C^1(\mathbb{R})$ *of period* 1 *we have*

$$\int_0^1 (q(x) + \lambda m(x))\, f^2(x)dx \geq -\int_0^1 [f'(x)]^2 dx\,.$$

PROOF. Let us first prove that if (1) has a solution with finitely many zeros on $\mathbb{R}$, there must be a non-vanishing solution y to (1) such that

$$y(x+1) = \alpha y(x), \quad x \in \mathbb{R}\,, \tag{8}$$

for some $\alpha > 0$.

By Floquet's theorem, there is a solution y satisfying (8) for some $\alpha \in \mathbb{C}$. If $\alpha \in \mathbb{C} - \mathbb{R}$, all solutions of (1) are bounded and formulas (6)-(7) show that $\phi(x) \to \infty$ as $x \to \infty$. Then y_1 has infinitely many zeros on $\mathbb{R}$ and so does any other solution of (1). Thus $\alpha \in \mathbb{R}$ and we can take y satisfying (8) to be real too. In this case we must have $\alpha > 0$ since otherwise y would have infinitely many changes of sign; moreover, y cannot vanish for some $x = x_0$ because it would also vanish for $x = x_0 + n$ with $n \geq 1$.

Assume now that there is a solution of (1) with a finite number of zeros on $\mathbb{R}$ and let $f \in C^1(\mathbb{R})$ be of period 1. Choose a solution y of (1) which is strictly positive and satisfies (8) for some $\alpha > 0$.

Let $P(x) = \ln y(x)$, $x \in \mathbb{R}$. Then $P \in C^2(\mathbb{R})$,

$$P'' = q + \lambda m - (P')^2\,,$$

and therefore

$$\int_0^1 (q(x) + \lambda m(x))\, f^2(x)dx = \int_0^1 P''(x) f^2(x)dx$$

$$+ \int_0^1 [P'(x)]^2 f^2(x)dx = \int_0^1 \frac{\partial}{\partial x}\Big(P'(x)f^2(x)\Big)\,dx$$

$$+ \int_0^1 \big[P'(x)f(x) - f'(x)\big]^2 dx - \int_0^1 [f'(x)]^2 dx.$$

Since P' and f are of period 1, the proof is complete. □

PROOF OF THEOREM 3. Assume that $m \leq 0$.

Let us first prove that the eigenfunction corresponding to the periodic eigenvalue λ_0 has no zeros on $\mathbb{R}$.

Recall the deformation argument from the previous section. Since $\gamma_0(\lambda)$ is not a Dirichlet eigenvalue, we can normalize the corresponding eigenfunction $y_p(x, \lambda, \gamma_0(\lambda))$ by $y_p(0, \lambda, \gamma_0(\lambda)) = 1$. Then

$$y_p(x, \lambda, \gamma_0(\lambda)) = y_1(x, \lambda, \gamma_0(\lambda)) + \frac{1 - y_1(1, \lambda, \gamma_0(\lambda))}{y_2(1, \lambda, \gamma_0(\lambda))} y_2(x, \lambda, \gamma_0(\lambda)), \quad x \in \mathbb{R}.$$

Since the eigenfunction corresponding to the first periodic eigenvalue for Hill's equation has no zeros (see [8]), the functions $y_p(x, \lambda, \gamma_0(\lambda))$ have no zeros for $\lambda \in [0, \lambda_0)$ and since the curve $(\lambda, \gamma_0(\lambda))$ joins continuously $(0, \gamma_0(0))$ to $(\lambda_0, 0)$, we find (knowing that we can have only simple zeros) that $y_p(x, \lambda_0, 0)$ has no zeros on $[0, 1]$ and thus no zeros on $\mathbb{R}$, being periodic.

We prove now that for $\lambda > \lambda_0$ every solution of (1) has infinitely many zeros by showing that the inequality in Lemma 6 is violated by $f := y_p(x, \lambda_0)$. Indeed,

$$\begin{aligned}
\int_0^1 (q(x) + \lambda m(x))\, y_p^2(x, \lambda_0) dx &= \int_0^1 (q(x) + \lambda_0 m(x))\, y_p^2(x, \lambda_0) dx \\
&\quad + (\lambda - \lambda_0) \int_0^1 m(x) y_p^2(x, \lambda_0) dx \\
&= \int_0^1 y_p''(x, \lambda_0) y_p(x, \lambda_0) dx \\
&\quad + (\lambda - \lambda_0) \int_0^1 m(x) y_p^2(x, \lambda_0) dx \\
&= -\int_0^1 [y_p'(x, \lambda_0)]^2 dx \\
&\quad + (\lambda - \lambda_0) \int_0^1 m(x) y_p^2(x, \lambda_0) dx \\
&< -\int_0^1 [y_p'(x, \lambda_0)]^2 dx
\end{aligned}$$

since

$$\lambda_0 \int_0^1 m(x) y_p^2(x, \lambda_0) dx = -\int_0^1 q(x) y_p^2(x, \lambda_0) dx - \int_0^1 [y_p'(x, \lambda_0)]^2 dx < 0$$

and $\lambda_0 > 0$ by Theorem 1.

To complete the proof in the case $m \leq 0$, we have to show that for $\lambda \leq \lambda_0$ every nontrivial solution of (1) has finitely many zeros on $\mathbb{R}$.

Assume that for some $\lambda < \lambda_0$ there is a nontrivial solution of (1) with infinitely many zeros on $\mathbb{R}$. Then the solution $y(x, \lambda)$ of (1) with $y(0, \lambda) = y_p(0, \lambda_0) = 1$, $y'(0, \lambda) = y_p'(0, \lambda_0)$ has also infinitely many zeros. Let $x_0 > 0$ be the smallest positive one. Multiplying the differential equation for $y(x, \lambda)$ by $-y_p(x, \lambda)$ and adding it to the differential equation for $y_p(x, \lambda_0)$ multiplied by $y(x, \lambda)$, we find

$$y_p''(x, \lambda_0)y(x, \lambda) - y_p(x, \lambda_0)y''(x, \lambda) = (\lambda_0 - \lambda)\, m(x)y_p(x, \lambda_0)y(x, \lambda)$$

and an integration on $[0, x_0]$ yields

$$-y'(x_0, \lambda)y_p(x_0, \lambda_0) = (\lambda_0 - \lambda)\int_0^{x_0} m(x)y_p(x, \lambda_0)y(x, \lambda)dx\,.$$

Since $y'(x_0, \lambda) \leq 0$, $\lambda < \lambda_0$, $m \leq 0$ and $y_p(x, \lambda_0)y(x, \lambda) \geq 0$ on $[0, x_0]$, we would have $y_p(x_0, \lambda_0) \leq 0$. Since $y_p(0, \lambda_0) = 1$, $y_p(x, \lambda_0)$ has a zero on $(0, x_0]$ but this is impossible. The proof is complete if $m \leq 0$; a similar argument works in the case $m \geq 0$.

Assume now that m changes sign.

The fact that all solutions of (1) have infinitely many zeros for $\lambda > \lambda_0^+$ and $\lambda < \lambda_0^-$ can be proved as above (as well as the fact that the eigenfunctions corresponding to λ_0^- and λ_0^+ have no zeros).

Assume that for some $\lambda \in (\lambda_0^-, \lambda_0^+)$ there is a nontrivial solution of (1) with infinitely many zeros. Then $y_2(x, \lambda)$ will have infinitely many zeros on $\mathbb{R}$. This is not possible for $\lambda = 0$. Assume that $\lambda > 0$. If $y_2(x, \lambda)$ has infinitely many zeros on $[0, \infty)$, we deduce by Lemma 5 that $y_2(x, \lambda_0^+)$ will have infinitely many zeros on $[0, \infty)$ and therefore also $y_p(x, \lambda_0^+)$, which is impossible. If $y_2(x, \lambda)$ has infinitely many zeros on $(-\infty, 0]$, by Lemma 5 we deduce that $y_2(x, \lambda_0^+)$ will have infinitely many zeros on $(-\infty, 0]$ and therefore also $y_p(x, \lambda_0^+)$, which is likewise impossible. □

We know by Theorem 3 that all eigenfunctions corresponding to periodic or anti-periodic eigenvalues different from the ground state (or states) have infinitely many zeros on $\mathbb{R}$. A more detailed description is provided by

THEOREM 4. *Assume that $m \not\equiv 0$ is continuous of period* 1. *The eigenfunction (or eigenfunctions) corresponding to λ_n, λ_n^- or λ_n^+ has exactly $[\frac{n+1}{2}]$ zeros on the interval* $[0, 1]$.

PROOF. Let us assume that m changes sign. We proved that the eigenfunction corresponding to λ_0^+ has no zeros on $[0, 1]$. We know that $y_2(x, \mu_1^+)$ has exactly two zeros on $[0, 1]$: at $x = 0$ and at $x = 1$ (here μ_1^+ is the first positive Dirichlet eigenvalue). Let $y_p(x, \lambda_1^+)$ be an eigenfunction corresponding to $\lambda_1^+ \leq \mu_1^+$. Since $y_p(x, \lambda_1^+)$ has infinitely many zeros on $\mathbb{R}$, $y_p(x, \lambda_1^+)$ has at least one zero in $[0, 1)$. If it has two zeros, $y_2(x, \lambda_1^+)$ has at least one zero on $(0, 1)$ - the zeros of two linearly independent solutions of a second order linear differential equation separate each other, cf. [7]; by Lemma 5 $y_2(x, \mu_1^+)$ has also at least one zero on $(0, 1)$, which we know is not the case.

Let now $y_p(x, \lambda_2^+)$ be an eigenfunction corresponding to λ_2^+. Again, it must have at least one zero on $[0, 1)$; if it would have two we find that $y_2(x, \lambda_2^+)$ has a zero on $(0, 1)$ and thus at least three zeros on $[0, 1]$ and therefore, by Lemma 5, $y_2(x, \mu_2^+)$ would have at least two zeros on $(0, 1)$, which is not the case.

The rest will be plain. □

5. To complete the spectral picture for (1), we note the following asymptotics for the Dirichlet eigenvalues (see [2]):

$$\lim_{n\to\infty} \frac{n^2\pi^2}{\mu_n} = \left[\int_0^1 \sqrt{|m(x)|}\,dx\right]^2 \quad \text{if} \quad m \le 0,$$
$$\lim_{n\to\infty} \frac{n^2\pi^2}{\mu_n} = -\left[\int_0^1 \sqrt{|m(x)|}\,dx\right]^2 \quad \text{if} \quad m \ge 0,$$

while if m changes sign,

$$\lim_{n\to\infty} \frac{n^2\pi^2}{\mu_n^+} = \left[\int_0^1 \sqrt{m_-(x)}\,dx\right]^2, \quad \lim_{n\to\infty} \frac{n^2\pi^2}{\mu_n^-} = -\left[\int_0^1 \sqrt{m_+(x)}\,dx\right]^2,$$

where m_+ and m_- are the positive, respectively the negative parts of m.

COMMENT. *The condition that q is non-negative is essential. Richardson* [11] *shows that if we drop this condition the behaviour of the Dirichlet spectrum becomes very complicated* (*considering again the problem* (4), *we see that the curves* $\left(\lambda, \gamma_n^D(\lambda)\right)_{n\ge1}$ *intersect the λ-axis in several points, the number of points of intersection depending on $n \ge 1$ in a way that is even not monotonic*); *the beauty of the whole picture is lost.*

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