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Abstract

In this paper we study certain linear differential equations in Fréchet spaces. We show that they can be solved uniquely, for any given initial condition, a fact not necessarily true in general. Our approach is based on the fact that every Fréchet space $\mathbb{F}$ is a projective limit of Banach spaces and on the replacement of the (non-Fréchet) space $\mathcal{L}(\mathbb{F})$ by an appropriate Fréchet space. Some applications are also included

Introduction

One of the main drawbacks of Fréchet spaces is the lack of a general theory concerning the solvability of differential equations. Even if such an equation can be solved, the solution is not, in general, uniquely determined by the initial conditions, as in the Banach case.

However, if we use the fact that every Fréchet space $\mathbb{F}$ can be thought of as the limit of a projective system $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$ of Banach spaces, then a certain type of linear equations can be uniquely solved. To be more precise, we define the space

$$H(\mathbb{F}) := \{(f_i)_{i\in\mathbb{N}} : f_i \in \mathcal{L}(\mathbb{E}_i) \text{ and } \varprojlim f_i \text{ can be defined}\}$$

which is Fréchet, whereas the space of continuous linear mappings $\mathcal{L}(\mathbb{F})$ (used in the classical case) is not. Then, the main result of the paper shows that the equation

$$x^{(n)} = A_0 \cdot x + A_1 \cdot \dot{x} + \ldots + A_{n-1} \cdot x^{(n-1)} + B\,,$$

where $A_i : I = [0,1] \longrightarrow \mathcal{L}(\mathbb{F})$, $0 \le i \le n-1$, and $B : I \longrightarrow \mathbb{F}$ are continuous mappings, admits a unique solution for any given initial data, if each A_i can be

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factored in the form $A_i = \varepsilon \circ A_i^*$, where $A_i^* : I \longrightarrow H(\mathbb{F})$ are also continuous and $\varepsilon : H(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F})$ with $\varepsilon((f_i)_{i\in\mathbb{N}}) = \varprojlim f_i$.

In the second part of the paper we prove that the linear differential equations studied by N. Papaghiuc ([7]) and R. S. Hamilton ([3]) are special cases of our main result. Regarding the former, in particular, we note that they are precisely the 1-order linear differential equations described above. Moreover our method reduces Papaghiuc's approach (which is a complicated variation of the classical proof within the Fréchet case) to an almost trivial argument. By the same token, Hamilton's equations fit in the same scheme.

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1. – Preliminaries

Let $\mathbb{F}$ be an arbitrarily chosen Fréchet space. It is known then (cf. [8]) that $\mathbb{F}$ can be thought of as the limit of a projective system of Banach spaces. More precisely: if $\{p_\nu\}_{\nu\in\mathbb{N}}$ is the family of seminorms of $\mathbb{F}$, with $p_1 \leq p_2 \leq \ldots$, then each quotient space $\mathbb{F}/\operatorname{Ker}p_i$ is a normed space. Setting

$$f_{ji} : \mathbb{F}/\operatorname{Ker}p_j \longrightarrow \mathbb{F}/\operatorname{Ker}p_i : x + \operatorname{Ker}p_j \longmapsto x + \operatorname{Ker}p_i \quad (j \geq i),$$

we obtain the projective system $\{\mathbb{F}/\operatorname{Ker}p_i; f_{ji}\}_{i,j\in\mathbb{N}}$. Moreover, taking the completions $\mathbb{E}_i$ of $\mathbb{F}/\operatorname{Ker}p_i$ and ρ_{ji} of f_{ji}, we obtain the projective system of Banach spaces $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$. It is then proved that

$$\mathbb{F} \equiv \varprojlim(\mathbb{F}/\operatorname{Ker}p_i) \overset{\lambda}{\equiv} \varprojlim \mathbb{E}_i ,$$

where the first identification is given by $x \equiv (x + \operatorname{Ker}p_i)$, $x \in \mathbb{F}$, and the second by the map $\lambda = \varprojlim \lambda_i$, with $\lambda_i : \mathbb{F}/\operatorname{Ker}p_i \longrightarrow \mathbb{E}_i$ the isometric embeddings.

For a projective system of differential mappings between Fréchet spaces we have the following result (cf. [2; Lemma 1.2]).

Proposition 1.1. *Let $\mathbb{F} = \varprojlim \mathbb{E}_i$ be a Fréchet space as above and $\{f_i : \mathbb{E}_i \longrightarrow \mathbb{E}_i, i \in \mathbb{N}\}$ a projective system of C^k-differential mappings. Then the corresponding limit $f := \varprojlim f_i : \mathbb{F} \longrightarrow \mathbb{F}$ is also C^k and $df((x_i)) = \varprojlim(df_i(x_i))$, for any $(x_i) \in \mathbb{F}$.*

We note that concerning the differentiability of mappings between Fréchet spaces, we adopt the definition of J. Leslie ([5], [6]).

2. – The main result

As we noticed in the introduction, *linear* differential equations (l.d.e.) in Fréchet spaces cannot be solved, in general. Even in case there are solutions, these are not uniquely determined by the initial conditions.

In this section we prove that a special type of l.d.e. in Fréchet spaces admits a unique solution, for any given initial condition.

A basic reason of the difficulty to solve a l.d.e. in a Fréchet space $\mathbb{F}$ is the fact that $\mathcal{L}(\mathbb{F})$ is not always Fréchet. It is merely a Hausdorff locally convex topological vector space, whose topology is determined by the family of seminorms $\{|\cdot|_d\}_{d\in D}$, where

$$D := \{(p, B) : p \text{ is a seminorm of } \mathbb{F} \text{ and } B \subseteq \mathbb{F} \text{ bounded}\}$$
$$|f|_{(p,B)} = \sup\{p(f(x)),\ \ x \in B\}$$

(for details cf. e.g. [4]).

In order to overcome the previous difficulty, we define the space

$$H(\mathbb{F}) := \{(f_i)_{i\in\mathbb{N}} : f_i \in \mathcal{L}(\mathbb{E}_i) \text{ and } \rho_{ji} \circ f_j = f_i \circ \rho_{ji}, \text{ for any } j \geq i\},$$

where $\{\mathbb{E}_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ is the projective system of Banach spaces with corresponding limit $\mathbb{F}$ as in Section 1.

PROPOSITION 2.1. (i) *$H(\mathbb{F})$ is a Fréchet space.*
(ii) *The mapping $\varepsilon : H(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F})$ with $\varepsilon((f_i)) = \varprojlim f_i$, is continuous linear.*

PROOF. (i) For every $i \in \mathbb{N}$, we define

$$H_i(\mathbb{F}) := \{(f_1, \dots, f_i), \text{ where } f_j \in \mathcal{L}(\mathbb{E}_j) \text{ and } \rho_{jk} \circ f_j = f_k \circ \rho_{jk},\ \ i \geq j \geq k\}.$$

Each $H_i(\mathbb{F})$ is a Banach space as a closed subspace of $\prod_{j=1}^{i} \mathcal{L}(\mathbb{E}_j)$. Moreover, $\{H_i(\mathbb{F})\}_{i\in\mathbb{N}}$ form a projective system, with connecting morphisms the natural projections. By setting

$$h_i : H(\mathbb{F}) \longrightarrow H_i(\mathbb{F}) : (f_1, f_2, \dots) \longmapsto (f_1, f_2, \dots, f_i);\ \ i \in \mathbb{N},$$

we readily verify that the injection

$$h := \varprojlim h_i : H(\mathbb{F}) \longrightarrow \varprojlim H_i(\mathbb{F})$$

can be defined. h is also a surjection, since for any $a = ((f_1^i, \dots, f_i^i)) \in \varprojlim H_i(\mathbb{F})$ we have that $f_k^j = f_k^i$, $j \geq i \geq k$. Setting $f_1 := f_1^1 = f_1^2 = \dots$, $f_2 := f_2^2 = f_2^3 = \dots$ and so on, we check that $(f_i) \in H(\mathbb{F})$ and $h((f_i)) = a$. Hence $H(\mathbb{F}) \overset{h}{\simeq} \varprojlim H_i(\mathbb{F})$ and $H(\mathbb{F})$ becomes a Fréchet space.

(ii) The map ε is obviously linear. To prove its continuity we proceed as follows: Let $|\cdot|_d$ be a seminorm of $\mathcal{L}(\mathbb{F})$, where $d = (p, B) \in D$. Then for any $a = (f_i) \in H(\mathbb{F})$, we have that

$$\begin{aligned}|h(a)|_d = |\varprojlim f_i|_d &= \sup\{p((f_i(x_i))), \quad x = (x_i) \in B\} \\ &\leq q_\mu(a) \cdot \sup\{p(x); \quad x \in B\},\end{aligned}$$

where $q_\mu((a_i)) = \sum_{i=1}^{M} \|a_i\|_{\prod_{j=1}^{i} \mathcal{L}(\mathbb{E}_j)}$ is a seminorm of $\varprojlim H_i(\mathbb{F})$. This concludes the proof. □

The space $H(\mathbb{F})$ and Proposition 2.1 enable us to prove the following main result.

THEOREM 2.2. *For a given Fréchet space* $\mathbb{F}$ *we consider the l.d.e.*

$$\dot{x} = A \cdot x + B\,, \tag{I}$$

with $A : I = [0, 1] \longrightarrow \mathcal{L}(\mathbb{F})$ *and* $B : I \longrightarrow \mathbb{F}$ *continuous* maps. *If* A *can be factored in the form* $A = \varepsilon \circ A^*$*, where* $A^* : I \longrightarrow H(\mathbb{F})$ *is continuous, then* (I) *admits a unique solution, for any initial condition* $(t_0, x_0) \in I \times \mathbb{F}$.

PROOF. We consider once again the projective system of Banach spaces $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$ with $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$. Since A^* takes values in $H(\mathbb{F})$ it has the form $A^*(t) = (A_i(t))_{i\in\mathbb{N}}$, $t \in I$, with each $A_i : I \longrightarrow \mathcal{L}(\mathbb{E}_i)$ continuous.

On the other hand, by setting $B_i := \rho_i \circ B : I \longrightarrow \mathbb{E}_i$, $i \in \mathbb{N}$, where $\rho_i : \mathbb{F} \longrightarrow \mathbb{E}_i$ are the canonical projections, we routinely check that $B = \varprojlim B_i$. Let now $f_i : I \longrightarrow \mathbb{E}_i$ be the unique solution of the l.d.e. in $\mathbb{E}_i$:

$$\dot{x} = A_i \cdot x + B_i \tag{I,i}$$

satisfying the initial condition $(t_0, \rho_i(x_0))$. For any $j \geq i$ we see that

$$\begin{aligned}(\rho_{ji} \circ f_j)^{\cdot}(t) = \rho_{ji}(\dot{f}_j(t)) &= \rho_{ji}(A_j(t)(f_j(t))) + \rho_{ji}(B_j(t)) \\ &= A_i(t)((\rho_{ji} \circ f_j)(t)) + B_i(t)\,.\end{aligned}$$

Moreover, $(\rho_{ji} \circ f_j)(t_0) = \rho_i(x_0)$. We obtain in this way, using the uniqueness of the solutions of (I,i), that $\rho_{ji} \circ f_j = f_i$ $(j \geq i)$; hence the C^1-map $f := \varprojlim f_i : I \longrightarrow \mathbb{F}$ exists. This is the required solution of (I), since

$$\begin{aligned}A(t)(f(t)) + B(t) &= (\varprojlim(A_i(t)))((f_i(t))) + \varprojlim B_i(t) \\ &= (A_i(t)(f_i(t)) + B_i(t)) = (\dot{f}_i(t)) = \dot{f}(t)\end{aligned}$$

and $f(t_0) = (\rho_i(x_0)) = x_0$.

Finally, f is unique. Indeed, if $g : I \longrightarrow \mathbb{F}$ is another solution of (I) with $g(t_0) = x_0$ we check, using analogous techniques as before, that $g_i := \rho_i \circ g$ is the solution of (I, i) with $g_i(t_0) = \rho_i(x_0)$. Hence $f_i = g_i$, for every $i \in \mathbb{N}$, and $f = g$. □

Based on the previous theorem we can solve the analogous n-order linear differential equations.

THEOREM 2.3. *Let $\mathbb{F}$ be a Fréchet space. We consider the n-order l.d.e.:*

$$\text{(II)} \qquad x^{(n)} = A_0 \cdot x + A_1 \cdot \dot{x} + \ldots + A_{n-1} \cdot x^{(n-1)} + B\,,$$

where $A_i : I = [0,1] \longrightarrow \mathcal{L}(\mathbb{F})$ and $B : I \longrightarrow \mathbb{F}$ are continuous mappings. If, for any $i = 0, 1, \ldots, n-1$, $A_i = \varepsilon \circ A_i^*$, with $A_i^* : I \longrightarrow H(\mathbb{F})$ continuous, then (II) *admits a unique solution for any given $(t_0, x_0, \ldots, x_{n-1}) \in I \times \mathbb{F}^n$.*

PROOF. As in the proof of Theorem 2.2, $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$. It is well known that (II) reduces to the 1-order l.d.e.

$$\dot{X} = A \cdot X + C$$

where $X(t) = (x(t), \dot{x}(t), \ldots, x^{(n-1)}(t))$, $C : I \longrightarrow \mathbb{F}^n : t \longmapsto (0, \ldots, 0, B(t))$, and

$$A : I \longrightarrow \mathcal{L}(\mathbb{F}^n) : t \longmapsto \begin{pmatrix} 0 & 1_{\mathbb{F}} & 0 & \ldots & 0 \\ 0 & 0 & 1_{\mathbb{F}} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1_{\mathbb{F}} \\ A_0(t) & A_1(t) & A_2(t) & \ldots & A_{n-1}(t) \end{pmatrix}.$$

Since $\mathbb{F}^n$ is the limit of the projective system $\{\mathbb{E}_i^n; \rho_{ji}^n\}_{i,j\in\mathbb{N}}$, it suffices (according to Theorem 2.2) to prove the existence of a continuous map $A^* : I \longrightarrow H(\mathbb{F}^n)$ such that $A = \varepsilon^n \circ A^*$, where $\varepsilon^n : H(\mathbb{F}^n) \longrightarrow \mathcal{L}(\mathbb{F}^n) : (g_i) \longmapsto \varprojlim g_i$. To this end we proceed as follows: we observe that each A_i has the form $A_i(t) = \varprojlim_j (A_i^j(t))$, where $A_i^j : I \longrightarrow \mathcal{L}(\mathbb{E}_j)$ is continuous, and we define $\Delta_i : I \longrightarrow \mathcal{L}(\mathbb{E}_i^n)$ by $\Delta_i(t)(a_0, a_1, \ldots, a_{n-1}) = (a_1, a_2, \ldots, a_{n-1}, A_0^i(t)(a_0) + \ldots + A_{n-1}^i(t)(a_{n-1}))$. Since $\rho_{ji} \circ A_k^j(t) = A_k^i(t) \circ \rho_{ji}$, for any $j \geq i$ and $k = 0, 1, \ldots n-1$, we can prove that $\rho_{ji}^n \circ \Delta_j(t) = \Delta_i(t) \circ \rho_{ji}^n$ $(j \geq i)$. Hence, for every $t \in I$, the continuous linear map $\varprojlim(\Delta_i(t)) : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ exists. We set now

$$A^* : I \longrightarrow H(\mathbb{F}^n) : t \longmapsto (\Delta_1(t), \Delta_2(t), \ldots)\,.$$

To prove that A^* is continuous, it suffices to check the continuity of each $\Delta_i (i \in \mathbb{N})$. Indeed, this is the case since

$$\Delta_i(t) = (\mathrm{pr}_2, \mathrm{pr}_3, \ldots, \mathrm{pr}_n, \sum_{k=0}^{n-1} (\mathrm{Comp} \circ (A_k^i, C_{k+1}))(t))\,,$$

where $\mathrm{Comp} : \mathcal{L}(\mathbb{E}_i, \mathbb{E}_i) \times \mathcal{L}(\mathbb{E}_i^n, \mathbb{E}_i) \longrightarrow \mathcal{L}(\mathbb{E}_i^n, \mathbb{E}_i) : (f, g) \longmapsto f \circ g$, $C_k : I \longrightarrow \mathcal{L}(\mathbb{E}_i^n, \mathbb{E}_i)$ is the constant mapping $C_k(t) := \mathrm{pr}_k$ and $\mathrm{pr}_k : \mathbb{E}_i^n \longrightarrow \mathbb{E}_i$ is the projection to the k-factor.

Moreover

$$
\begin{aligned}
((\varepsilon^n \circ A^*)(t))(x_0, \dots, x_{n-1}) &= (\varprojlim(\Delta_j(t)))(x_0, \dots, x_{n-1}) \\
&= \left(\rho_j(x_1), \dots, \rho_j(x_{n-1}), \sum_{k=0}^{n-1}(A_k^j(t)(\rho_j(x_k)))\right)_{j\in\mathbb{N}} \\
&= \left(\rho_j(x_1), \dots, \rho_j(x_{n-1}), \rho_j\left(\sum_{k=0}^{n-1}(A_k(t)(x_k))\right)\right)_{j\in\mathbb{N}} \\
&= \left(x_1, \dots, x_{n-1}, \sum_{k=0}^{n-1}(A_k(t)(x_k))\right) \\
&= A(t)(x_0, \dots, x_{n-1}).
\end{aligned}
$$

Therefore, $\varepsilon^n \circ A^* = A$ and the proof is now complete. □

3. – Applications

Among other authors, N. Papaghiuc ([7]) and R. S. Hamilton ([3]) have studied particular classes of linear differential equations in Fréchet spaces. Here, we prove that the aforementioned equations are special cases of those studied in the previous section. Especially for the equations of [7] we shall show that they are exactly the equations of Theorem 2.2.

More precisely, in [7] the author, considering a Fréchet space $\mathbb{F}$ with seminorms $\{p_i\}_{i\in\mathbb{N}}$, defines the space

$$
\mathcal{L}_I(\mathbb{F}) = \left\{ f \in \mathcal{L}(\mathbb{F}) : q_i(f) := \sup\left\{ \frac{p_i(f(x))}{p_i(x)},\ p_i(x) \neq 0 \right\} < +\infty \ \ \forall\, i \in \mathbb{N} \right\}
$$

which is also Fréchet. It is then proved that the 1-order l.d.e.

$$
\text{(III)} \qquad \dot{x} = T \cdot x + U\,,
$$

where $T : [0, 1] \longrightarrow \mathcal{L}_I(\mathbb{F})$ and $U : [0, 1] \longrightarrow \mathbb{F}$ are continuous mappings, admits a unique solution for any given initial condition.

If $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$ is the projective system of Banach spaces, described in Section 1, where $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$ holds, next proposition is valid.

Proposition 3.1. *If $f : \mathbb{F} \longrightarrow \mathbb{F}$ is a continuous linear map, the following conditions are equivalent:*
(i) $f \in \mathcal{L}_I(\mathbb{F})$.
(ii) $f = \varprojlim f_i,\ f_i \in \mathcal{L}(\mathbb{E}_i)$.

PROOF. Assuming that $f \in \mathcal{L}_I(\mathbb{F})$, we set $C_i := \sup\left\{\frac{p_i(f(x))}{p_i(x)}, p_i(x) \neq 0\right\}$ and $f_i : \mathbb{E}_i \longrightarrow \mathbb{E}_i : [(x_\nu + \mathrm{Ker} p_i)_\nu] \longmapsto [(f(x_\nu) + \mathrm{Ker} p_i)_\nu]$, $i \in \mathbb{N}$. We check that $f_i \in \mathcal{L}(\mathbb{E}_i)$, $i \in \mathbb{N}$, since $\|f_i(x)\|_{\mathbb{E}_i} \leq C_i \cdot \|x\|_{\mathbb{E}_i} (x \in \mathbb{E}_i)$, $\rho_{ji} \circ f_j = f_i \circ \rho_{ji} (j \geq i)$ and $\rho_i \circ f = f_i \circ \rho_i (i \in \mathbb{N})$, where $\rho_i : \mathbb{F} \longrightarrow \mathbb{E}_i$ are the canonical projections. Hence $f = \varprojlim f_i$.

Conversely, let $f = \varprojlim f_i$ with $f_i \in \mathcal{L}(\mathbb{E}_i)$. Then, for any $x = ([x + \mathrm{Ker} p_i]) \in \mathbb{F}$, we have that

$$p_i(f(x)) = \sum_{j=1}^{i} \|f_j([x + \mathrm{Ker} p_j])\|_{\mathbb{E}_j} \leq \sum_{j=1}^{i} (\|f_j\|_j \cdot p_j(x)) \leq \left(\sum_{j=1}^{i} \|f_j\|_j\right) \cdot p_i(x)$$

if $\|\cdot\|_j$ denotes the norm of the Banach space $\mathcal{L}(\mathbb{E}_j)$. Hence, we prove condition (i). □

COROLLARY 3.2. *Linear differential equations of type* (III) *are precisely equations* (I) *of Theorem 2.2.*

PROOF. Obviously, for a l.d.e. of type (I), $A(t) = \varprojlim(A_i(t))$ holds, where $A_i(t) \in \mathcal{L}(\mathbb{E}_i)$. Hence $A([0, 1)]) \subseteq \mathcal{L}_I(\mathbb{F})$.

Conversely, for any l.d.e. of type (III), in virtue of Proposition 3.1, we have that $T(t) = \varprojlim(T_i(t))$, $t \in I$, where $T_i : [0, 1] \longrightarrow \mathcal{L}(\mathbb{E}_i)$. As a result, $T = \varepsilon \circ T^*$, with

$$T^* : I \longrightarrow H(\mathbb{F}) : t \longmapsto (T_1(t), T_2(t), \dots).$$

Moreover, T^* is continuous since, for any $(t_\mu)_{\mu \in \mathbb{N}} \subseteq I$ with $t_\mu \xrightarrow[\mu]{} t_0$, $\|T_i(t_\mu) - T_i(t_0)\|_i \leq q_i(T(t_\mu) - T(t_0))$, if $\{q_i\}_{i \in \mathbb{N}}$ is the family of seminorms of $\mathcal{L}_I(\mathbb{F})$. This concludes the proof. □

REMARK 3.3. By Corollary 3.2 we see that solutions of (III) are obtained in a direct way, making use of trivial tools, whereas Papaghiuc's approach is a rather lengthy and complicated extension of the classical proof in the Fréchet case.

In the remaining of this paper we deal with the l.d.e. examined by R. S. Hamilton in [3] and we prove that they are also a special case of our study.

DEFINITION 3.4 ([3; p. 129]). Let $\mathbb{F}$ be a Fréchet space. A C^∞-map $f : \mathbb{F} \longrightarrow \mathbb{F}$ is said to be *smooth-Banach* if there exist a Banach space $\mathbb{B}$, and C^∞-mappings $\phi : \mathbb{F} \longrightarrow \mathbb{B}$ and $g : \mathbb{B} \longrightarrow \mathbb{F}$ such that $f = g \circ \phi$.

It is proved that equation

$$\text{(IV)} \qquad \dot{x} = f(t, x)$$

admits unique solutions, for given initial conditions, if f is a smooth-Banach map (cf. [3; Theorem 5.6.3]).

If we restrict the previous considerations to the case of linear differential equations, we prove the following.

PROPOSITION 3.5. *In a Fréchet space* $\mathbb{F}$ *we consider the l.d.e.*

$$\dot{x} = A \cdot x + B\,,$$

where the mapping $f(t,x) := A(t) \cdot x + B(t)$ *is smooth-Banach. Then A satisfies the properties of Theorem 2.2.*

PROOF. Let $(t_0, x_0) \in J \times \mathbb{F}$ be a given initial condition, where J is an open interval of $\mathbb{R}$. Let also $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$ be the projective system of Banach spaces, described in Section 1, such that $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$. Since f is a smooth-Banach map, there exist a Banach space $\mathbb{B}$ and C^∞-mappings $\phi : J \times \mathbb{F} \longrightarrow \mathbb{B}$, $g : \mathbb{B} \longrightarrow \mathbb{F}$ with $f = g \circ \phi$. Then we can write $g = \varprojlim g_i$, with $g_i := \rho_i \circ g : \mathbb{B} \longrightarrow \mathbb{E}_i$, if ρ_i $(i \in \mathbb{N})$ are the canonical projections.

On the other hand, according to [3; Theorem 5.1.3], there exist a seminorm p_k of $\mathbb{F}$ and $\varepsilon, c > 0$ such that

$$(*) \qquad \begin{array}{l} \text{If } \; p_k(x) < \varepsilon, \quad p_k(y) < \varepsilon, \quad |t_i - t_0| < \varepsilon \quad (i = 1, 2), \quad \text{then} \\ \|\phi(t_1, x) - \phi(t_2, y)\|_{\mathbb{B}} \leq c \cdot (p_k(x-y) + |t_1 - t_2|) \end{array}$$

where $\|\cdot\|_{\mathbb{B}}$ is the norm of $\mathbb{B}$. We set $C_k := \{a \in \mathbb{E}_k : \|a\|_{\mathbb{E}_k} < \varepsilon\}$ and, for any $i \geq k$,

$$\phi_i : (t_0 - \varepsilon, t_0 + \varepsilon) \times \rho_{ik}^{-1}(C_k) \longrightarrow \mathbb{B} : (t, [(x_\nu + V_i)_\nu]) \longmapsto \lim_\nu \phi(t, x_\nu)$$

This is a well defined mapping, in virtue of condition $(*)$. It is also uniformly continuous, as a result of a routine checking.

Furthermore, the commutativity of the following diagrams

$$\begin{array}{ccc} (t_0 - \varepsilon, t_0 + \varepsilon) \times \rho_{jk}^{-1}(C_k) & \xrightarrow{\ \phi_j\ } & \mathbb{B} \\ {\scriptstyle id_{\mathbb{R}} \times \rho_{ji}} \big\downarrow & \nearrow_{\phi_i} & \\ (t_0 - \varepsilon, t_0 + \varepsilon) \times \rho_{ik}^{-1}(C_k) & & \end{array} \qquad (j \geq i)$$

$$\begin{array}{ccc} (t_0 - \varepsilon, t_0 + \varepsilon) \times \rho_{k}^{-1}(C_k) & \xrightarrow{\ \phi\ } & \mathbb{B} \\ {\scriptstyle id_{\mathbb{R}} \times \rho_{i}} \big\downarrow & \nearrow_{\phi_i} & \\ (t_0 - \varepsilon, t_0 + \varepsilon) \times \rho_{ik}^{-1}(C_k) & & \end{array} \qquad (i \in \mathbb{N})$$

implies that $\phi|_{(t_0-\varepsilon, t_0+\varepsilon)\times\rho_k^{-1}(C_k)} = \varprojlim \phi_i$.

Setting, for any $i \geq k$, $A_i : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{L}(\mathbb{E}_i)$ with $(A_i(t))(x) := g_i(\phi_i(t, x)) - B_i(t)$, where $B_i := \rho_i \circ B$, we observe that $A_i(t)$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, is determined globally on $\mathbb{E}_i$ since $A_i(t)(\lambda \cdot x) = \lambda \cdot A_i(t)(x)$, for any $x, \lambda \cdot x \in \rho_{ik}^{-1}(C_k)$.

On the other hand, the fact that each g_i ($i \in \mathbb{N}$) is smooth, allows one to use once again [3; Theorem 5.1.3]. Taking also into account the uniform continuity of ϕ_i, we prove that A_i is continuous, $i \geq k$.

Finally, we check that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{E}_j & \xrightarrow{A_j(t)} & \mathbb{E}_j \\ \rho_{ji}\downarrow & & \downarrow\rho_{ji} \\ \mathbb{E}_i & \xrightarrow{A_i(t)} & \mathbb{E}_i \end{array} \quad (j \geq i), \qquad \begin{array}{ccc} \mathbb{F} & \xrightarrow{A(t)} & \mathbb{F} \\ \rho_i\downarrow & & \downarrow\rho_i \\ \mathbb{E}_i & \xrightarrow{A_i(t)} & \mathbb{E}_i \end{array} \quad (i \in \mathbb{N}).$$

Indeed, for the first we observe that, if $x = [(x_\nu + V_j)_\nu] \in \mathbb{E}_j$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \cdot x \in \rho_{jk}^{-1}(C_k)$ and $\frac{1}{n} \cdot \rho_{ji}(x) \in \rho_{ik}^{-1}(C_k)$. Hence

$$\begin{aligned} (\rho_{ji} \circ A_j(t))(x) &= n \cdot \rho_{ji}\left(g_j\left(\phi_j\left(t, \left[\left(\frac{1}{n} \cdot x_\nu + V_j\right)_\nu\right]\right)\right) - B_j(t)\right) \\ &= n \cdot \lim_\nu g_i\left(\phi\left(t, \frac{1}{n} \cdot x_\nu\right)\right) - n \cdot B_i(t) \\ &= n \cdot \lim_\nu \rho_i\left(g\left(\phi\left(t, \frac{1}{n} \cdot x_\nu\right)\right)\right) - n \cdot B_i(t) \\ &= n \cdot \lim_\nu \rho_i\left(A(t)\left(\frac{1}{n} \cdot x_\nu\right)\right) = \lim_\nu \rho_i(A(t)(x_\nu)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (A_i(t) \circ \rho_{ji})(x) &= n \cdot A_i(t)\left(\frac{1}{n} \cdot [(x_\nu + V_i)_\nu]\right) \\ &= n \cdot \lim_\nu g_i\left(\phi\left(t, \frac{1}{n} \cdot x_\nu\right)\right) - n \cdot B_i(t) \\ &= n \cdot \lim_\nu \rho_i\left(A(t)\left(\frac{1}{n} \cdot x_\nu\right)\right) \\ &= \lim_\nu \rho_i(A(t)(x_\nu)). \end{aligned}$$

Similarly we prove the commutativity of the second diagram. Therefore, the continuous map

$$A^* : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow H(\mathbb{F}) : t \longmapsto (A_i(t))_{i \geq k}$$

can be defined and $A = \varepsilon \circ A^*$, which completes the proof. □

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