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Homoclinic orbits for a class of infinite dimensional hamiltonian systems

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Homoclinic Orbits for a Class of Infinite Dimensional Hamiltonian Systems

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1. – Introduction

The purpose of this paper is to derive some results concerning the existence of positive periodic and of homoclinic solutions to the following hamiltonian-type system:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u = H_v(u,v) \ \text{ in } \ \Omega \times \mathbb{R}, \quad u = 0 \ \text{ in } \partial\Omega \times \mathbb{R}, \\
&-\frac{\partial v}{\partial t} - \Delta v = H_u(u,v) \ \text{ in } \ \Omega \times \mathbb{R}, \quad v = 0 \ \text{ in } \partial\Omega \times \mathbb{R},
\end{aligned} \tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$ and H is a given function, satisfying

$$H_u(0,0) = H_v(0,0) = 0\,.$$

Problem (1.1) can be interpreted as an *unbounded* hamiltonian system ([1]). Though the corresponding initial value problem for (1.1) is *not* well-posed, one can nevertheless look for solutions existing for all $t \in \mathbb{R}$ subject to some boundedness or integrability conditions. Among these solutions there are the stationary ones satisfying the system:

$$
\begin{aligned}
-\Delta u &= H_v(u,v) \ \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega\,, \\
-\Delta v &= H_u(u,v) \ \text{ in } \Omega, \quad v = 0 \text{ in } \partial\Omega\,.
\end{aligned} \tag{1.2}
$$

System (1.2) has been recently studied by several authors ([3], [5], [10], [13], [16], [17]). In particular, it has been proved in [16], [27] (see also [22]) that

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if $H \in C^1(\mathbb{R}^2, \mathbb{R})$, Ω is star-shaped, $N \geq 3$ and the following condition is satisfied:

$$\frac{N}{N-2}(H(u,v) - H(0,0)) \leq auH_u(u,v) + (1-a)vH_v(u,v) \tag{1.3}$$

for all $u, v \in \mathbb{R}^2$ and some $a \in \mathbb{R}$, then problem (1.2) has no positive solutions of class $C^2(\overline{\Omega})$. In case H is given by

$$H(u,v) = \frac{1}{q+1}|u|^{q+1} + \frac{1}{p+1}|v|^{p+1}, \tag{1.4}$$

with $p, q > 0$, condition (1.3) is equivalent to the condition

$$\frac{N-2}{N} < \frac{1}{q+1} + \frac{1}{p+1}. \tag{1.5}$$

If $p = q$, it is easy to show that $u = v$. In this case condition (1.5) reduces to the Pohozaev [18] subcritical condition

$$p = q < \frac{N+2}{N-2}.$$

By analogy with the case $p = q$, we shall call problems (1.1) and (1.2) superlinear or superhomogeneous (in the context of hamiltonian systems superquadratic [11]) if $pq > 1$. Observe that this is equivalent to the condition

$$\frac{1}{q+1} + \frac{1}{p+1} < 1.$$

The existence of positive classical solutions to system (1.2) for more general functions H has been established by several authors using different methods ([3], [5], [10], [13], [17]). For the special H given in (1.4) another approach is given in Section 2.

In this paper we investigate the problem of existence of positive classical periodic and homoclinic solutions to system (1.1) with H given by (1.4) and $pq > 1$.

For this kind of problem a stronger criticality condition (1.6) is used. Work is in progress about the necessity of (1.6) when Ω is star-shaped. We have:

THEOREM 1.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary. Let $p, q > 0$ satisfy*

$$\frac{N}{N+2} < \frac{1}{p+1} + \frac{1}{q+1} < 1. \tag{1.6}$$

Then there exists $T_0 > 0$ *such that the system of equations* (P_T)

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = |v|^p \operatorname{sign}(v) \;\; \textit{in} \;\; \Omega \times (-T, T)\,, \\ &-\frac{\partial v}{\partial t} - \Delta v = |u|^q \operatorname{sign}(u) \;\; \textit{in} \;\; \Omega \times (-T, T)\,, \end{aligned} \tag{1.7}$$

with boundary condition in x:

$$u(x,t) = v(x,t) = 0 \;\; \textit{for all} \;\; t \in [-T, T] \;\; \textit{and} \;\; x \in \partial\Omega\,, \tag{1.8}$$

and periodic conditions in t:

$$u(x,-T) = u(x,T), x \in \overline{\Omega}, \quad v(x,-T) = v(x,T), x \in \overline{\Omega}\,, \tag{1.9}$$

possesses at least one classical positive solution, if $T > T_0$.

The second theorem we prove is:

THEOREM 1.2. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with smooth boundary. Let* $p, q > 0$ *satisfy* (1.6). *Then the system of equations* (P_∞)

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = |v|^p \operatorname{sign}(v) \;\; \textit{in} \;\; \Omega \times \mathbb{R}\,, \\ &-\frac{\partial v}{\partial t} - \Delta v = |u|^q \operatorname{sign}(u) \;\; \textit{in} \;\; \Omega \times \mathbb{R}\,, \end{aligned} \tag{1.10}$$

with boundary conditions

$$u(x,t) = v(x,t) = 0 \;\; \textit{for all} \;\; x \in \partial\Omega \;\; \textit{and} \;\; t \in \mathbb{R} \tag{1.11}$$

together with

$$\lim_{|t|\to\infty} u(x,t) = \lim_{|t|\to\infty} v(x,t) = 0 \tag{1.12}$$

uniformly in $x \in \Omega$, *possesses at least one positive classical solution.*

Observe that if $p = 1$ and if (u, v) is a solution of (P_∞), then u satisfies the equation

$$-u_{tt} + \Delta^2 u = u^q \;\; \text{in} \;\; \Omega \times \mathbb{R}\,, \tag{1.13}$$

together with Navier boundary conditions

$$u = \Delta u = 0 \;\; \text{in} \;\; \partial\Omega \times \mathbb{R}\,, \tag{1.14}$$

and $\lim_{|t|\to\infty} u(x,t) = 0$, uniformly in $x \in \Omega$. In that case condition (1.6) reduces to $q < \frac{N+6}{N-2}$.

The paper is organized as follows. In Section 2 we establish an abstract critical point theorem which is used for the existence of both strong stationary and of periodic solutions problem (P_T). In Section 3 we prove that these solutions are classical and that, for large enough T, the periodic solutions are not stationary.

In Section 4 we prove the convergence of subharmonics to a nontrivial homoclinic solution by using an estimate from below and compactness arguments. In the Appendix we collect some imbedding and regularity results which are used in the paper.

As a final remark, we point out that the methods used is this paper can be applied to more general situations. As an example one can replace the laplacian operator by a selfadjoint second-order operator. The results of this paper were announced in [4].

2. – A critical point theorem

In this section we establish a critical point theorem and we study an abstract equation. The results will be used in the next sections.

Let (Σ, m, μ) be a σ-finite, positive measure space. For $1 \leq r < \infty$ let $L^r(\Sigma) = L^r(\Sigma, m, \mu)$ be the usual Lebesgue space with the norm

$$\|u\|_r = \left(\int_\Sigma |u|^r d\mu\right)^{1/r} . \tag{2.1}$$

We shall denote by r' the Hölder conjugate exponent of r. The space $L^{r'}(\Sigma)$ will be identified with the dual space of $L^r(\Sigma)$.

We consider functionals defined on a real Banach space $(B, \|\cdot\|)$, continuously embedded in $L^{q+1}(\Sigma)$ for some $q > 0$ and we denote by i the inclusion map. We also consider a bounded linear operator A. We assume that:

(H1) $A : B \to L^{p^*+1}(\Sigma)$, *for some* $p > 0$, *is an isomorphism.*

Here and in what follows we use the notation

$$p^* = \frac{1}{p} \quad \text{for} \quad p > 0 . \tag{2.2}$$

We introduce the functionals

$$F(u) = \frac{1}{p^* + 1}\|Au\|_{p^*+1}^{p^*+1}, \quad u \in B , \tag{2.3}$$

$$G(u) = \frac{1}{q + 1}\|u\|_{q+1}^{q+1}, \qquad u \in B , \tag{2.4}$$

$$I(u) = F(u) - G(u), \qquad u \in B . \tag{2.5}$$

When considering positivity property of critical points, we need to introduce the following assumption and functionals:

(H1$_+$) *If $v \in L^{p^*+1}(\Sigma)$ satisfies $v \geq 0$ then $A^{-1}v \geq 0$,*

$$G^+(u) = G(u^+), \qquad u \in B \quad \text{and}$$
$$I^+(u) = F(u) - G(u^+), \quad u \in B\,.$$

We use the standard notation

$$u^+(x) = \max\{u(x), 0\}, \quad x \in \Sigma\,,$$

and sign $(u) = 1$ if $u > 0$, sign $(u) = -1$ if $u < 0$ and sign $(u) = 0$ if $u = 0$. We have

LEMMA 2.1. *Under assumption* (H1), *the functionals F, G, I, G^+ and I^+ are continuously Fréchet differentiable and we have*

$$DG(u)v = \int_\Sigma |u|^q \text{ (sign) (u)v d}\mu \tag{2.6}$$

$$DG_+(u)v = \int_\Sigma (u^+)^q v\, d\mu \tag{2.7}$$

$$DF(u)v = \int_\Sigma |Au|^{p^*} \text{ sign } (Au)Av\, d\mu \tag{2.8}$$

for all $u \in B$ and $v \in B$.

PROOF. The function $s \to |s|^r$ is of class C^1 for $r > 1$, hence the functional $g_r : L^r(\Sigma) \to \mathbb{R}$ given by

$$g_r(u) = \int_\Sigma |u|^r d\mu$$

is continuously Fréchet differentiable and

$$Dg_r(u)v = r \int_\Sigma |u|^{r-1}\text{sign}(u)v\, d\mu$$

for all $v \in L^r(\Sigma)$. See [9] Theorem 2.8. The boundedness of the imbedding i and the operator A gives the continuous differentiability of G and F and the formulae (2.6) and (2.8). Since the function $s \to (s^+)^r, r > 1$, is also of class C^1 the same holds for G_+ and hence formula (2.7) follows. □

One of the main assumptions in critical point theory is the Palais-Smale condition which we recall for the sake of completeness.

We say that a functional $I \in C^1(B, \mathbb{R})$ satisfies the Palais-Smale condition (P.S.) if for every $c \in \mathbb{R}$ and every sequence $\{u_n\} \subset B$ such that

$$I(u_n) \to c \quad \text{as} \quad n \to \infty \tag{2.9}$$

and

$$I'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.10}$$

the sequence $\{u_n\}$ is relatively compact in B.

In order that the functional I (respectively I^+) satisfy the (P.S.) condition we introduce the next assumption:

(H2) *The space B is compactly imbedded in $L^{q+1}(\Sigma)$.*

Proposition 2.1. *Under assumptions* (H1) *and* (H2) *the functionals I and I_+ satisfy the* (P.S.) *condition if*

(H3) $p \cdot q > 1$.

Proof. Suppose that we have a sequence $\{u_n\}$ in B satisfying (2.9) and (2.10). First we show that the sequence is bounded in B. We have

$$\begin{aligned} I(u_n) - \frac{1}{q+1} I'(u_n)u_n &= \left(\frac{1}{p^*+1} - \frac{1}{q+1}\right) \|Au_n\|_{p^*+1}^{p^*+1} \\ &= \frac{pq-1}{(p+1)(q+1)} \|Au_n\|_{p^*+1}^{p^*+1}. \end{aligned}$$

Using (2.9), (2.10) and $pq > 1$, it follows that there exist $c_1, c_2 > 0$ such that

$$\|Au_n\|_{p^*+1}^{p^*+1} \leq c_1 + c_2 \|u_n\|.$$

From (H1) and $p^* > 0$, we obtain $\sup_{n \geq 1} \|u_n\| < \infty$.

Next we show that the sequence $\{u_n\}$ has a convergent subsequence in B. Since A is an isomorphism and $L^{q+1}(\Sigma)$ is reflexive, B is also reflexive and then there exists a subsequence, still denoted by $\{u_n\}$, and $u \in B$ such that $\{u_n\}$ converges weakly to u in B. By (H2), $\{u_n\}$ strongly converges to u in $L^{q+1}(\Sigma)$. Next since $DG(u_n) = i \text{ o } Dg_{q+1}(u_n)$, it follows that $DG(u_n)$ converges to $DG(u)$ in B'. On the other hand, from (2.10) we get

$$DF(u_n) \to DG(u) \;\text{ in } B'. \tag{2.11}$$

Now we use the convexity of F. For $v \in B$ we have

$$\begin{aligned} 0 &\leq (DF(u_n) - DF(v))(u_n - v) \\ &= (DI(u_n) + DG(u_n) - DF(v))(u_n - v), \end{aligned}$$

hence

$$(DG(u) - DF(v))(u - v) \geq 0\,.$$

By setting $v = u - th, t > 0, h \in B$, dividing by t and letting t go to zero we obtain $(DG(u) - DF(u))(h) \geq 0$, for every $h \in B$, hence

$$DG(u) = DF(u)\,. \tag{2.12}$$

By (2.11) and (2.12) we have $DF(u_n) \to DF(u)$ and $DF(u_n)u_n \to DF(u)u$. In other words

$$\lim_{n\to\infty} \|Au_n\|_{p^*+1} = \|Au\|_{p^*+1}\,.$$

Since $\{Au_n\}$ converges weakly to Au in $L^{p^*+1}(\Sigma)$, it follows that $\{Au_n\}$ converges strongly to Au in $L^{p^*+1}(\Sigma)$ and by (H1) $\{u_n\}$ converges strongly to u in B. Hence the functional I satisfies the (P.S.) condition.

A similar argument can be carried out for I^+. □

REMARK 2.1. When $pq < 1$ the functional I also satisfies the (P.S.) condition. This follows from the coercivity of I and the argument used in the above Proposition. Let us see that the functional I is coercive, i.e. for any sequence $\{u_n\} \subset B$ such that $\sup_{n\geq 1} I(u_n) < \infty$, we have $\sup_{n\geq 1} \|u_n\| < +\infty$. Indeed for $u \in B$,

$$\begin{aligned} I(u) &= \frac{1}{p^*+1}\|Au\|_{p^*+1}^{p^*+1} - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &\geq \frac{\|A^{-1}\|^{-(p^*+1)}}{p^*+1}\|u\|^{p^*+1} - \frac{c}{q+1}\|u\|^{q+1}\,. \end{aligned} \tag{2.13}$$

Since $p \cdot q < 1$, by (2.2), we have $q < p^*$ and then the coercivity follows.

For the sake of completeness, we recall the following theorem, due to Ambrosetti and Rabinowitz, for critical points of a functional. See Theorems 2.2 and 9.12 in [13].

THEOREM 2.1. *Suppose* $(D, \|\cdot\|)$ *is an infinite dimensional Banach space and assume* $J : D \to \mathbb{R}$ *is a* C^1 *functional that satisfies the* (P.S.) *condition and* $J(0) = 0$. *Suppose:*

(J1) *There exist* $\alpha > 0$, $\rho > 0$ *such that*

$$J(u) \geq \alpha \quad \forall u \in D, \quad \|u\| = \rho\,, \tag{2.14}$$

and there exists $e \in D$ *such that*

$$J(e) \leq 0, \quad \|e\| > \rho\,.$$

Then J possesses a critical point $\bar{u}$ such that

$$J(\bar{u}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \alpha > 0\,,$$

where $\Gamma = \{\gamma \in C([0,1], D) \quad / \quad \gamma(0) = 0, \gamma(1) = e\}$.

Suppose moreover that J is even, that is $J(u) = J(-u)$ for all $u \in D$, and

(J2) *For any finite dimensional subspace $W \subset D$, there exists a constant $R = R(W)$ such that $J(u) \leq 0$, for all $u \in W$, $\|u\| \geq R$.*

Then J possesses an infinite sequence of pairs of critical points whose critical values form an unbounded sequence.

As a consequence of this theorem we obtain the following:

THEOREM 2.2. *Let I be the functional defined in* (2.5). *Assume that* (H1), (H2) *and* (H3) *hold. Then I possesses an infinite sequence of pairs of critical points whose critical values form an unbounded sequence.*

If moreover (H1_+) *holds then I possesses a nonnegative nontrivial critical point $\bar{u}$, whose critical value satisfies*

$$I(\bar{u}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I^+(\gamma(t)) > 0\,, \tag{2.15}$$

where $\Gamma = \{\gamma \in C([0,1]; B) | \gamma(0) = 0, \gamma(1) = e\}$ *with $e \in B$ such that $e^+ \neq 0$ and $I^+(e) \leq 0$. Moreover, if u is a nontrivial critical point of I^+, then $I^+(u) \geq I^+(\bar{u})$.*

PROOF. We will apply Theorem 2.1. As a consequence of Lemma 2.1 and Proposition 2.1 it follows that I is C^1 and satisfies the (P.S.) condition.

Next we consider the geometric condition (J1). From (2.13) and the fact that $p^* < q$ it follows that there exists $\alpha > 0$ and $\rho > 0$ such that (2.14) holds. On the other hand for $z \in B, z \neq 0$ we have

$$J(tz) = t^{p^*+1}F(z) - t^{q+1}G(z) \quad \text{for all} \quad t \in \mathbb{R}^+\,.$$

Since $p^* < q$ there exists $\bar{t}$ such that $\bar{t} > \rho$ and $J(\bar{t}z) < 0$. Set $e = \bar{t}z$.

Next we show that (J2) holds. If W is a finite dimensional subspace of B then we observe that $\|\cdot\|$ and $\|\cdot\|_{q+1}$ define equivalent norms in W, that is there exist constants $c_1(W), c_2(W) > 0$ such that

$$c_1(W)\|w\|_{q+1} \leq \|w\| \leq c_2(W)\|w\|_{q+1} \quad \text{for all} \quad w \in W\,. \tag{2.16}$$

From (2.16) we obtain that for $w \in W$, we have

$$I(w) \leq \frac{\|A\|^{p^*+1}}{p^*+1}\|w\|^{p^*+1} - \frac{(c_2(W))^{-(q+1)}}{q+1}\|w\|^{q+1}$$

and since $p^* < q$ we have the existence of $R(W) > 0$ such that $\|w\| \geq R(W)$ and $w \in W$ implies $I(w) \leq 0$.

Since I is even the first part of the theorem is proved.

In order to find nonnegative critical points of I we consider the functional I^+. We observe that the geometric condition (J1) is satisfied; we only choose z so that $z^+ \neq 0$. The application of Theorem 2.1 gives then the existence of a critical point u of I^+.

The function u satisfies

$$\int_\Sigma |Au|^{p^*} \operatorname{sign}(Au) Av \, d\mu = \int_\Sigma (u^+)^q v \, d\mu \tag{2.17}$$

for all $v \in B$. From (H1) we have

$$\int_\Sigma |Au|^{p^*} \operatorname{sign}(Au) w d\mu = \int_\Sigma (u^+)^q A^{-1} w \, d\mu$$

for all $w \in L^{p^*+1}(\Sigma)$. Using $(H1_+)$ we conclude that

$$\int_\Sigma |Au|^{p^*} \operatorname{sign}(Au) w \, d\mu \geq 0$$

for all $w \in L^{p^*+1}(\Sigma)$, $w \geq 0$. Hence $|Au|^{p^*} \operatorname{sign}(Au) \geq 0$ and $Au \geq 0$. Using $(H1_+)$ again we find $u \geq 0$. Going back to (2.17) we see that

$$\int_\Sigma |Au|^{p^*} \operatorname{sign}(Au) Av \, d\mu = \int_\Sigma |u|^q \operatorname{sign}(u) v d\mu$$

for all $v \in B$ so that u is a critical point of I also.

It remains to prove the last statement of Theorem 2.2.

Let u be a nontrivial critical point of I^+. It is sufficient to find some path $\gamma \in \Gamma$ such that

$$I^+(u) \geq \max_{t \in [0,1]} I^+(\gamma(t)) \,. \tag{2.18}$$

Observe that if u is a nontrivial critical point of I^+, then the function $s \mapsto I^+(su)$ is strictly increasing on $[0, 1]$, strictly decreasing on $[1, \infty)$ and vanishes for some $s > 1$. In particular it has a global maximum at $s = 1$.

If u is a multiple of e, we choose the path $\gamma(t) = te$, with $t \in [0, 1]$. Then $u = \bar{t}e$ for some $\bar{t} \in (0, 1)$ and (2.18) hold with equality.

If u is not a multiple of e, we consider the two dimensional space W spanned by e and u. There exists $s_1, s_2 > 1$ such that $I^+(s_1 e) < 0$, $I^+(s_2 u) < 0$ and $I^+((1-t)s_1 e + t s_2 u) < 0$ for $t \in [0, 1]$.

We define γ as the union of three paths $\gamma_1, \gamma_2, \gamma_3$. The path γ_1 is the segment joining 0 to $s_2 u$, γ_2 the segment joint $s_2 u$ to $s_1 e$ and γ_3 the segment joining $s_1 e$ to e. Clearly I^+ is negative on γ_2 and γ_3, so

$$\max_{t \in [0,1]} I^+(\gamma(t)) = \max I^+|_{\gamma_1} = I^+(u) \,.$$

This completes the proof of Theorem 2.2. □

In what follows we consider certain systems of equations and we prove an existence result for these systems using Theorem 2.2.

Consider a second real Banach space $(\tilde{B}, \|\cdot\|)$ continuously imbedded in $L^{p+1}(\Sigma)$, and let $\tilde{A}$ be a bounded linear operator such that:

($\tilde{\mathbf{H}}$1) $\tilde{A} : \tilde{B} \to L^{q^*+1}(\Sigma)$ *is an isomorphism, with* $q > 0$.

In order to consider the positivity property of solutions we introduce:

($\tilde{\mathbf{H}}\mathbf{1}_+$) *If* $v \in L^{q^*+1}(\Sigma)$ *satisfies* $v \geq 0$ *then* $\tilde{A}^{-1}v \geq 0$.

Finally we introduce a duality assumption between A and $\tilde{A}$:

(D) $\int_\Sigma Auv\, d\mu = \int_\Sigma u\tilde{A}v\, d\mu$ *for all* $u \in B, v \in \tilde{B}$.

We consider the following system of equations (S) for $u \in B, v \in \tilde{B}$

$$Au = |v|^p \text{ sign } (v)$$
$$\tilde{A}v = |u|^q \text{sign}(u) .$$

THEOREM 2.3. *Assume that* (H1), (H2), (H3), ($\tilde{\text{H}}$1) *and* (D) *hold. Then system* (S) *possesses infinitely many solutions. Moreover, if* (H1$_+$) *and* ($\tilde{\text{H}}$1$_+$) *hold then* (S) *possesses at least one nontrivial nonnegative solution.*

PROOF. The functional I given in (2.5) possesses infinitely many critical points according to Theorem 2.2. Let $u \in B$ be one of those solutions. We define the function v by

$$v = \tilde{A}^{-1}(|u|^q \text{ sign } (u)) , \tag{2.19}$$

then $v \in \tilde{B}$ and obviously

$$\tilde{A}v = |u|^q \text{ sign } (u) . \tag{2.10}$$

On the other hand, since u is a critical point of I we have

$$\int_\Sigma |Au|^{p^*} \text{ sign } (Au)Aw\, d\mu = \int_\Sigma |u|^q \text{ sign } (u)w\, d\mu$$

for all $w \in B$. Using (2.23) and (D) we obtain

$$\int_\Sigma |Au|^{p^*} \text{ sign } (Au)Awd\mu = \int_\Sigma v\, Aw\, d\mu$$

for all $w \in B$, hence

$$|Au|^{p^*} \text{ sign } (Au) = v ,$$

and

$$Au = |v|^p \text{ sign } (v) .$$

This proves the first part of the theorem.

If (H1$_+$) holds, from Theorem 2.2 it follows that there exists a nonnegative critical point u of I. By using ($\tilde{\text{H}}$1$_+$) in (2.19) we obtain $v \geq 0$. This completes the proof. □

3. – Proof of Theorem 1.1

We first prove the existence of strong solutions of (P_T) by using Theorems 2.2 and 2.3. Next, in order to prove that the strong solutions are classical, we apply regularity theory and a bootstrap argument. Finally we show that for T large enough the solution is not stationary.

In what follows, we shall use the notation and the framework of Section 2. Let $T > 0$ and let Σ be the measure space

$$\Sigma = [-T, T] \times \overline{\Omega}$$

with the Lebesgue measure in $\mathbb{R}^{N+1}$. For $1 < r < \infty$ and a Banach space E, let $L^r_T(E)$ be the space $L^r([-T, T], E)$. We define the Banach space

$$B_r = W^{1,r}_T(L^r(\Omega)) \cap L^r_T(W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega))$$

equipped with the norm

$$\|u\| = \left[\int_{-T}^{T}\int_{\Omega}\left\{|u(t,x)|^r + \left|\frac{\partial u}{\partial t}(t,x)\right|^r \right.\right.$$
$$\left.\left. + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}(t,x)\right|^r + \sum_{i,j=1}^{N}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)\right|^r\right\} dx\, dt\right]^{\frac{1}{r}}.$$

Here $W^{1,r}_T(L^r(\Omega))$ denotes the space of functions defined in $[-T, T]$ with values in $L^r(\Omega)$ with derivative with respect to t in $L^r(\Sigma)$, satisfying the periodic boundary condition $u(-T, \cdot) = u(T, \cdot)$. We shall identify $L^r_T(L^r(\Omega))$ with $L^r(\Sigma)$ and as a consequence we observe that $B_r \subset L^r(\Sigma)$.

In the space B_r we define the linear operators $A_r, \tilde{A}_r : B_r \to L^r(\Sigma)$ by

$$(A_r u)(t,x) = \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x), \qquad u \in B_r \tag{3.1}$$

$$(\tilde{A}_r u)(t,x) = -\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x), \quad u \in B_r, \tag{3.2}$$

where the Laplacian Δ acts on the space variable x. In the rest of the section it will be notationally convenient to write $A_r = A$ and $\tilde{A}_r = \tilde{A}$. The following known result is of fundamental importance in what follows.

LEMMA 3.1. *The operators A and $\tilde{A}$ are isomorphisms from B_r into $L^r(\Sigma)$.*

Given $p > 0$, we consider $A : B_{p^*+1} \to L^{p_*+1}(\Sigma)$. It follows from the definitions given above and Lemma 3.1 that hypothesis (H1) is satisfied.

Hypothesis (H1$_+$) follows from the parabolic maximum principle. See Ladyzhenskaya and Ural'tseva [15], Protter and Weinberger[19].

We recall that on p and q we are assuming that

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{N}{N+2}, \quad p,q>0 .$$

Then, taking $r=p^*+1$ and $s=q+1$, we can apply Lemma A.1 in the Appendix to conclude that the space B_{p^*+1} is compactly embedded in $L^{q+1}(\Sigma)$ and then hypothesis (H2) is satisfied.

We also consider $\tilde{A}: B_{q^*+1} \rightarrow L^{q*+1}(\Sigma)$ and observe, as above, that $\tilde{A}$ satisfies ($\tilde{H}1$) and ($\tilde{H}1_+$). From the identity

$$\int_{-T}^{T}\int_{\Omega}\Big(\frac{\partial u}{\partial t}-\Delta u\Big)(t,x)v(t,x)dx\,dt=\int_{-T}^{T}\int_{\Omega}u(t,x)\Big(-\frac{\partial v}{\partial t}-\Delta v\Big)(t,x)dx\,dt$$

for all $u \in B_{p^*+1}$, $v \in B_{q^*+1}$, it follows that hypothesis (D) holds.

We are now in a position to apply Theorem 2.3 in Section 2 and to conclude that problem (P_T) possesses at least one nonnegative nontrivial strong solution. If $(u,v) \in B_{p^*+1} \times B_{q^*+1}$ is one of such solutions then

$$\frac{\partial u}{\partial t}-\Delta u=|v|^p \operatorname{sign}(v) \quad \text{a.e. in } \Sigma, \tag{3.3}$$

$$-\frac{\partial v}{\partial t}-\Delta v=|u|^q \operatorname{sign}(u) \quad \text{a.e. in } \Sigma. \tag{3.4}$$

Next we will show that these strong solutions are classical solutions of (P_T). For this purpose we will use a bootstrap argument.

According to Lemma A.1 we have the following two basic imbeddings:

E1) If $r<1+\frac{N}{2}$ then $B_r \subset L^{r\eta}(\Sigma)$, where $\eta<\eta(r)=\frac{N+2}{N+2-2r}$.

E2) If $r=1+\frac{N}{2}$ then $B_r \subset L^{r+\xi}(\Sigma) \quad \forall \xi \geq 0$.

We have three cases: a. $1+p^*>1+\frac{N}{2}$, b. $1+p^*=1+\frac{N}{2}$ and c. $1+p^*<1+\frac{N}{2}$. Next we analyze each case.

Case a. From Lemma A.3 b) we have that u belongs to $C^{0,\alpha}(\overline{\Sigma})$ for some $1>\alpha>0$. Then we consider equation (3.4) and we apply part c) of Lemma A.3 to obtain that $v \in C^{(1,\tilde{\alpha})(2,\tilde{\alpha})}(\overline{\Sigma})$, for some $1>\tilde{\alpha}>0$. Then we go back to equation (3.3) to conclude the same for u. Thus u and v are classical solutions.

Case b. According to imbedding E2) above $u \in L^{1+p^*+\xi}(\Sigma)$ for all $\xi \geq 0$. Then we use equation (3.4) and Lemma A.3 b) to find that $v \in C^{0,\alpha}(\overline{\Sigma})$. Next we use equation (3.3) and Lemma A.3 c) to conclude that $u \in C^{(1,\tilde{\alpha})(2,\tilde{\alpha})}(\overline{\Sigma})$, for some $0<\tilde{\alpha}<1$. Finally Lemma A.3 c) with equation (3.4) allows us to conclude that the same holds for v. Thus u and v are classical solutions.

Case c. In this case we need the following lemma.

LEMMA 3.2. *If* $\bar{p} \geq p$, $\frac{\bar{p}+1}{p} < 1 + \frac{N}{2}$ *and* $u \in B_{\frac{\bar{p}+1}{p}}$ *then* $u \in B_{\frac{\tilde{p}+1}{p}}$, *for some* $\tilde{p}$, *where* $\tilde{p} - \bar{p} \geq \frac{1}{2}(p+1)^2(q+1)\delta$ *with* $\delta = \frac{1}{p+1} + \frac{1}{q+1} - \frac{N}{N+2}$.

PROOF. (See Lemma 2.2 in [3]) We recall that under our assumption, we have $pq > 1$ and $\delta > 0$. It will be convenient to write δ as

$$\delta = \frac{1 - pq}{(p+1)(q+1)} + \frac{2}{N+2} > 0\,. \tag{3.5}$$

If $u \in B_{\frac{\bar{p}+1}{p}}$ then, using imbedding E1) above we have $u \in L^{r\eta}(\Sigma)$ with $r = \frac{\bar{p}+1}{p}$ and $\eta < \eta(r)$. We shall further restrict η later. Then $|u|^q \text{sign}(u) \in L^{\frac{r}{q}\eta}(\Sigma)$.

From equation (3.4) and Lemma A.3 a) we find that $v \in B_{\frac{r}{q}\eta}$. We note that since (3.5) holds and $\bar{p} \geq p$, we have

$$r\eta(r) \geq \frac{(N+2)(p+1)}{N(p+1) - (N+2)} > q+1$$

and then, we can choose η, such that $\frac{r}{q}\eta > 1$.

If $\frac{r}{q}\eta(r) > 1 + \frac{N}{2}$, then we can further assume $\frac{r}{q}\eta \geq 1 + \frac{N}{2}$. Thus, we can use embedding E2) to obtain $v \in L^{1+\tilde{p}}(\Sigma)$ for all $\tilde{p} \geq p$.

If $\frac{r}{q}\eta(r) \leq 1 + \frac{N}{2}$, then embedding E1) implies $v \in L^{1+\tilde{p}}(\Sigma)$ where $\tilde{p}+1 < \tilde{s}\eta(\tilde{s})$ with $\tilde{s} = \frac{r}{q}\eta$. We observe that $\tilde{p}$ depends on η, and next we see how to choose it adequately. If we put $\hat{p} + 1 = s\eta(s)$ with $s = \frac{r}{q}\eta(r)$ then we have

$$\begin{aligned}\frac{1}{1+\hat{p}} - \frac{1}{\bar{p}+1} &= \frac{pq-1}{\bar{p}+1} - \frac{2}{N+2}(q+1)\\ &\leq (q+1)\Big(\frac{pq-1}{(p+1)(q+1)} - \frac{2}{N+2}\Big) = -(q+1)\delta\end{aligned}$$

then

$$\hat{p} - \bar{p} \geq (\hat{p}+1)(\bar{p}+1)(q+1)\delta \geq (p+1)^2(q+1)\delta\,.$$

By taking η closer to $\eta(r)$ if necessary, we can achieve

$$\tilde{p} - \bar{p} \geq \frac{1}{2}(p+1)^2(q+1)\delta.$$

Thus, $|v|^p \text{sign}\,(v) \in L^{\frac{\tilde{p}+1}{p}}(\Sigma)$. Then using equation (3.3) and Lemma A.3 a) we conclude that $u \in B_{\frac{\tilde{p}+1}{p}}$.

Now we complete Case c. We apply Lemma 3.2 several times up to obtain $u \in B_{\frac{1+\tilde{p}}{p}}$ with $\frac{1+\tilde{p}}{p} \geq 1 + \frac{N}{2}$. Then we can proceed as in cases a. or b. to conclude that u and v are classical solutions.

Finally we show that for T large enough the solution is not stationary. First we prove that there exists a constant $c_1 > 0$ such that if (u, v) is a positive stationary solution of (P_T), then we have

$$\int_\Omega |\Delta u|^{p^*+1} dx \geq c_1 . \tag{3.6}$$

In order to do that, we establish the existence of a positive stationary strong solution by means of Theorem 2.2 and 2.3. We choose $\Sigma = \overline{\Omega}$ with the Lebesgue measure in $\mathbb{R}^N$, $B = W^{2,1+p^*}(\Omega) \cap W_0^{1,1+p^*}(\Omega)$, $\tilde{B} = W^{2,1+q^*}(\Omega) \cap W_0^{1,1+q^*}(\Omega)$, A and $\tilde{A}$ are the Laplacian operator defined respectively on B and $\tilde{B}$. Observe that the imbedding of B in $L^{q+1}(\Omega)$ is compact provided that condition (1.5) holds. This is certainly true since p and q satisfy the stronger condition (1.6).

It is easy to verify that all the conditions of Theorem 2.2 and 2.3 are satisfied, so we obtain a strong positive solution $(\bar{u}, \bar{v}) \in B \times \tilde{B}$ of

$$\begin{aligned} -\Delta u &= |v|^p \operatorname{sign}(v) \quad \text{in } \Omega , \\ -\Delta v &= |u|^q \operatorname{sign}(u) \quad \text{in } \Omega . \end{aligned} \tag{3.7}$$

From (2.15) we also have

$$\frac{1}{p^*+1} \int_\Omega |\Delta \bar{u}|^{p^*+1} dx - \frac{1}{q+1} \int_\Omega |\bar{u}|^{q+1} dx > 0 .$$

Since $\int_\Omega |\Delta \bar{u}|^{p^*+1} dx = \int_\Omega |\bar{u}|^{q+1} dx$ and $\frac{1}{p^*+1} - \frac{1}{q+1} > 0$, we obtain

$$\Big(\frac{1}{p^*+1} - \frac{1}{q+1}\Big) \int_\Omega |\Delta \bar{u}|^{p^*+1} dx > 0 .$$

From the last part of the conclusion of Theorem 2.2, it follows that

$$\Big(\frac{1}{p^*+1} - \frac{1}{q+1}\Big) \int_\Omega |\Delta u|^{p^*+1} dx \geq \Big(\frac{1}{p^*+1} - \frac{1}{q+1}\Big) \int_\Omega |\Delta \bar{u}|^{p^*+1} dx$$

holds for any strong nontrivial solution (u, v) of (3.7). Therefore (3.6) holds with $c_1 = \int_\Omega |\Delta \bar{u}|^{p^*+1} dx$.

Now let (u_T, v_T) be a periodic positive solution of (P_T) obtained as above by means of Theorems 2.2 and 2.3.

In what follows we establish some bounds for (u_T, v_T), by choosing appropriately e in Theorem 2.2.

LEMMA 3.3. *There exists a constant $c > 0$ independent of $T \geq 1$ such that*

$$0 < \int_{-T}^{T}\int_{\Omega} |Au_T|^{p^*+1} dx\, dt \leq c \tag{3.8}$$

and

$$0 < \int_{-T}^{T}\int_{\Omega} |\tilde{A}v_T|^{q^*+1} dx dt \leq c\,. \tag{3.9}$$

PROOF. We denote by I_T^+ the functional

$$I_T^+(u) = \frac{1}{p^*+1}\int_{-T}^{T}\int_{\Omega} |Au|^{p^*+1} dx\, dt - \frac{1}{q+1}\int_{-T}^{T}\int_{\Omega} |u^+|^{q+1} dx\, dt\,, \tag{3.10}$$

for $u \in B_{p^*+1}$. Since

$$\begin{aligned}\int_{-T}^{T}\int_{\Omega} |Au_T|^{p^*+1} dx\, dt &= \int_{-T}^{T}\int_{\Omega} |u_T|^{q+1} dx\, dt\\ &= \int_{-T}^{T}\int_{\Omega} |Av_T|^{q^*+1} dx dt > 0\,,\end{aligned}$$

it is enough to prove that

$$I_T^+(u_T) \leq c, \quad \text{for } T \geq 1, \text{ for some } c > 0\,.$$

We have

$$I_T^+(u_T) = \inf_{\gamma\in\Gamma_T}\max_{t\in[0,1]} I_T^+(\gamma(t)),$$

where

$$\Gamma_T = \{\gamma \in C([0,1], B_{p^*+1});\, \gamma(0) = 0, \gamma(1) = e\}\,,$$

with some $e \in B_{p^*+1}$ such that $e^+ \neq 0$.

Let e_0 be a nonnegative and nontrivial function in $C^2(\mathbb{R}\times\Omega)$ with compact support in $(-1,1)\times\Omega$. Since $pq > 1$, there exists $\alpha > 0$ such that $I_1^+(\alpha e_0) < 0$. Let

$$e = \alpha e_0\,.$$

We observe that $e|_{[-T,T]\times\overline{\Omega}} \in B_{p^*+1}$ for $T \geq 1$ and $I_T^+(e) = I_1^+(e) < 0$ for $T \geq 1$. Choosing $\bar{\gamma} \in \Gamma_T, T \geq 1$ of the form: $\bar{\gamma}(t) = te,\ t \in [0,1]$, we obtain

$$I_T^+(u_T) \leq \max_{t\in[0,1]} I_T^+(\bar{\gamma}(t)) = \max_{t\in[0,1]} I_1^+(\bar{\gamma}(t)) =: c\,.$$

This completes the proof of the lemma. □

Let (u_T, v_T) be as in Lemma 3.1. If u_T is stationary (equivalently v_T), then

$$I_T^+(u_T) = \Big(\frac{1}{p^*+1} - \frac{1}{q+1}\Big) \int_{-T}^{T} \int_{\Omega} |Au_T|^{p^*+1} dx\, dt$$
$$= \Big(\frac{1}{p^*+1} - \frac{1}{q+1}\Big) 2T \int_{\Omega} |\Delta u_T|^{p^*+1} dx\,.$$

Hence for $T \geq 1$, from (3.6) (3.8) and (H3), we have

$$T \leq \frac{c}{2c_1} := T^*\,.$$

So, for $T > \max(1, T^*) =: T_0$, u_T cannot be stationary. This completes the proof of Theorem 1.1. □

4. – Proof of Theorem 1.2

We shall construct a homoclinic orbit to problem (P_∞) by taking a limit of subharmonics to (P_T) with period $2k$, $k \in \mathbb{N}^+$ as in [23], [25]. We consider as in Section 3, for $k \in \mathbb{N}^+$,

$$\Sigma_k = [-k, k] \times \overline{\Omega}\,,$$
$$B_{r,k} = W_k^{1,r}(L^r(\Omega)) \cap L_k^r(W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)))$$

with $1 < r < \infty$, and the operators

$$A_{r,k}, \tilde{A}_{r,k} : B_{r,k} \to L^r(\Sigma_k)$$

defined as in (3.1) and (3.2) respectively. In this section, since no confusion will arise, we again drop the index in denoting the operators just defined. We also have the functional $I_k^+(u)$ for $u \in B_{p^*+1,k}$, as defined in (3.10). We denote by (u_k, v_k) a classical positive solutions of (P_k) satisfying the upper bounds (3.8) and (3.9).

The next lemma allows us to find a lower bound for u_k.

LEMMA 4.1. *There exists a constant $c > 0$ such that*

$$\max_{t \in [-k,k], x \in \overline{\Omega}} |u_k(t, x)| \geq c\,. \tag{4.1}$$

REMARK 4.1. By our assumption $q - p^* = \frac{pq-1}{p} > 0$.

PROOF. In order to obtain a lower bound for u_k, we use a coercivity inequality for the operator $\frac{\partial}{\partial t} - \Delta$. In [2], [20], and Appendix it is shown that for $r \in (1, +\infty)$ there exists a constant M_r such that

$$
\begin{aligned}
&\left(\int_{-1}^{1}\int_{\Omega}\left|\lambda\frac{\partial w}{\partial t}\right|^r dx\,dt\right)^{1/r} + \left(\int_{-1}^{1}\int_{\Omega}|\Delta w|^r dx\,dt\right)^{1/r} \\
&\leq M_r\left(\int_{-1}^{1}\int_{\Omega}\left|\lambda\frac{\partial w}{\partial t} - \Delta w\right|^r dx\,dt\right)^{1/r}
\end{aligned}
\tag{4.2}
$$

for every $\lambda > 0$ and $w \in B_{r,1}$. For $\tilde{w} \in B_{r,k}$, we define $w \in B_{r,1}$ by $w(t, x) = \tilde{w}(tk, x)$, $k \in \mathbb{N}$. Substituting w in (4.2), we obtain

$$
\begin{aligned}
&\left(\int_{-k}^{k}\int_{\Omega}\left|\frac{\partial \tilde{w}}{\partial t}\right|^r dx\,dt\right)^{1/r} + \left(\int_{-k}^{k}\int_{\Omega}|\Delta \tilde{w}|^r dx\,dt\right)^{1/r} \\
&\qquad\qquad \leq M_r\left(\int_{-k}^{k}\int_{\Omega}\left|\frac{\partial \tilde{w}}{\partial t} - \Delta \tilde{w}\right|^r dx\,dt\right)^{1/r},
\end{aligned}
\tag{4.3}
$$

where the constant M_r is the same as the one given in (4.2) and it is independent of k and $\tilde{w}$. On the other hand, from elliptic theory (See Lemma 9.17 in [12]) we have

$$
\left(\int_{\Omega}|z|^r dx\right)^{1/r} \leq C_r\left(\int_{\Omega}|\Delta z|^r dx\right)^{1/r}, \quad \text{for} \quad z \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega), \tag{4.4}
$$

where C_r is independent of z.

Combining (4.3) and (4.4) with $r = p^* + 1$ we obtain a constant C such that, for all $k \in \mathbb{N}$

$$
\int_{-k}^{k}\int_{\Omega}|u_k|^{p^*+1} dx\,dt \leq C\int_{-k}^{k}\int_{\Omega}|Au_k|^{p^*+1} dx\,dt\,. \tag{4.5}
$$

Now we are in a position to find the lower bound announced in (4.1). From (4.3) and recalling that u_k is continuous we have

$$
\begin{aligned}
\int_{-k}^{k}\int_{\Omega}|Au_k|^{p^*+1} dx\,dt &= \int_{-k}^{k}\int_{\Omega}|u_k|^{q+1} dx\,dt \\
&\leq \max_{t\in[-k,k],x\in\overline{\Omega}}|u_k(t, x)|^{q-p^*}\left(\int_{-k}^{k}\int_{\Omega}|u_k|^{p^*+1} dx\,dt\right) \\
&\leq C\max_{t\in[-k,k],x\in\overline{\Omega}}|u_k(t, x)|^{q-p^*}\left(\int_{-k}^{k}\int_{\Omega}|Au_k|^{p^*+1} dx\,dt\right).
\end{aligned}
\tag{4.6}
$$

From (3.8) we have

$$\int_{-k}^{k}\int_{\Omega} |Au_k|^{p^*+1} dx\, dt > 0\,,$$

hence, from (4.6), we conclude that

$$\max_{t\in[-k,k],x\in\overline{\Omega}} |u_k(t,x)| \geq c > 0\,. \qquad \square$$

Next we will extend the functions u_k and v_k to all $\mathbb{R}$ by periodicity; we keep calling them u_k and v_k.

Given $M \in \mathbb{N}$ and $1 < r < \infty$ we define the spaces

$$B_r^M = W^{1,r}([-M, M], L^r(\Omega)) \cap L^r([-M, M], W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega))$$

equipped with the norm defined in Section 3. We note that the only difference between B_r^M and B_r is that the first does not impose a periodicity condition in the variable t.

Under our hypothesis (1.6) on p and q, we see from Lemma A.1 that for these spaces we also have the appropriate imbeddings, that is $B_{p^*+1}^M$ is compactly embedded in $L^{q+1}([-M, M]\times\Omega)$ and $B_{q^*+1}^M$ is compactly embedded in $L^{p+1}([-M, M] \times \Omega)$.

On the spaces B_r^M we have the equivalent norms

$$\|u\|_{B_{p^*+1}^M}^{p^*+1} = \int_{-M}^{M}\int_{\Omega} \{|Au|^{p^*+1} + |u|^{p^*+1}\} dx\, dt \tag{4.7}$$

and

$$\|v\|_{B_{q^*+1}^M}^{q^*+1} = \int_{-M}^{M}\int_{\Omega} \{|\tilde{A}u|^{q^*+1} + |u|^{q^*+1}\} dx\, dt \tag{4.8}$$

for $B_{p^*+1}^M$ and $B_{q^*+1}^M$ respectively. The operators A and $\tilde{A}$ are defined in $B_{p^*+1}^M$ and $B_{q^*+1}^M$ as in (3.1) and (3.2). The fact that (4.7) and (4.8) define equivalent norms is a consequence of (4.2) and (4.4) applied to this situation.

From Lemma 3.3 and (4.5) we obtain that the sequence $\{u_k\}$ and $\{v_k\}$, when restricted to $[-M, M]$, are bounded sequences in $B_{p^*+1}^M$ and $B_{q^*+1}^M$ respectively. Since these spaces are reflexive, there exist $u^M \in B_{p^*+1}^M$ and $v^M \in B_{q^*+1}^M$ and subsequences of $\{u_k\}, \{v_k\}$, which we still denote by $\{u_k\}, \{v_k\}$, such that $u_k \rightharpoonup u^M$ and $v_k \rightharpoonup v^M$. Using the compact imbeddings mentioned above we also obtain that $u_k \to u^M$ in $L^{q+1}([-M, M] \times \Omega)$ and $v_k \to v^M$ in $L^{p+1}([-M, M] \times \Omega)$.

Since (u_k, v_k) are subharmonics to (P_T) we have

$$Au_k = v_k^p \quad \text{in} \quad [-M, M] \times \Omega\,, \tag{4.9}$$

$$\tilde{A}v_k = u_k^q \quad \text{in} \quad [-M, M] \times \Omega\,. \tag{4.10}$$

We recall that the functions u_k and v_k are nonnegative. Since the nonlinear function $s \mapsto |s|^\sigma$ sign (s) induces a homeomorphism between $L^r(\sum)$ and $L^{r/\sigma}(\sum)$, see [14], the above results on convergence imply that $u_k^q \to (u^M)^q$ in $L^{q^*+1}([-M, M] \times \Omega)$ and $v_k^p \to (v^M)^p$ in $L^{p^*+1}([-M, M] \times \Omega)$. Using equations (4.9) and (4.10) we finally conclude that the sequences $\{u_k\}$ and $\{v_k\}$ strongly convergent to u^M and v^M in $B^M_{p^*+1}$ and $B^M_{q^*+1}$ respectively. The functions u^M, v^M satisfy (4.9) and (4.10) also.

We can perform the above analysis for every $M \in \mathbb{N}$. Through a diagonal procedure we can extract subsequences of $\{u_k\}, \{v_k\}$, which we still denote by $\{u_k\}, \{v_k\}$ and we can find functions u, v defined in $\mathbb{R} \times \Omega$ that belong to $B^M_{p^*+1}$ and $B^M_{q^*+1}$ for all M respectively, so that

$$u_k \to u \quad \text{in} \quad B^M_{p^*+1}, \quad v_k \to v \quad \text{in} \quad B^M_{q^*+1}\,, \tag{4.11}$$

for all $M \in \mathbb{N}$, and satisfy

$$\frac{\partial u}{\partial t} - \Delta u = v^p \quad \text{in} \quad \mathbb{R} \times \Omega\,, \tag{4.12}$$

$$-\frac{\partial v}{\partial t} - \Delta v = u^q \quad \text{in} \quad \mathbb{R} \times \Omega\,. \tag{4.13}$$

These functions u and v are the solutions we are looking for, however they could be trivial. In order to exclude this case, we need a more careful analysis.

We have the following lemma. Let us define as in the Appendix $B_r^{T_1,T_2} = W^{1,r}([T_1, T_2], L^r(\Omega)) \cap L^r([T_1, T_2], W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega))$ with the norm (see (4.7))

$$\|u\|^r_{B_r^{T_1,T_2}} = \int_{T_1}^{T_2} \int_\Omega \{|Au|^r + |u|^r\} dx\, dt\,. \tag{4.14}$$

LEMMA 4.2. *Let $T_1 < T_2$. Assume that $u \in B^{T_1,T_2}_{p^*+1}$ and $v \in B^{T_1,T_2}_{q^*+1}$ satisfy the system*

$$\frac{\partial u}{\partial t} - \Delta u = v^p \quad in \quad (T_1, T_2) \times \Omega\,, \tag{4.15}$$

$$-\frac{\partial v}{\partial t} - \Delta v = u^q \quad in \quad (T_1, T_2) \times \Omega\,. \tag{4.16}$$

Then, for every $\varepsilon > 0$ there exists a constant C depending on ε and $T_2 - T_1$ and an exponent $s \geq 1$ such that

$$\|u\|_{C^{0,\alpha}([T_1+\varepsilon, T_2-\varepsilon] \times \overline{\Omega})} \leq C \|u\|^s_{B_r^{T_1,T_2}}\,, \tag{4.17}$$

for some $\alpha > 0$.

PROOF. The proof is based on a bootstrap argument as in Lemma 3.2, and taking into account of the inequalities associated with each imbedding and a regularity result.

As in Lemma 3.2, we have three cases: a. $1+p^* > 1+\frac{N}{2}$, b. $1+p^* = 1+\frac{N}{2}$ and c. $1+p^* < 1+\frac{N}{2}$. Now we analyze each case separately.

Case a: Since $u \in B^{T_1,T_2}_{p^*+1}$, equation (3.3) implies $v^p \in L^{p^*+1}(\Sigma)$ and then we can apply Lemma A.2 b) to conclude that $u \in C^{0,\alpha}(\overline{\Sigma}_\varepsilon)$ and

$$\|u\|_{C^{0,\alpha}(\overline{\Sigma}_\varepsilon)} \leq C\|v^p\|_{L^{p^*+1}(\Sigma)} \leq C\|u\|_{B^{T_1,T_2}_{p^*+1}},$$

where Σ denotes $(T_1, T_2) \times \Omega$, and Σ_ε denotes $(T_1+\varepsilon, T_2-\varepsilon) \times \Omega$.

Case b: Here $p^* = \frac{N}{2}$. From the imbedding E2) we find

$$\|u\|_{L^{p^*+1+\xi}(\Sigma)} \leq C\|u\|_{B^{T_1,T_2}_{p^*+1}}, \tag{4.18}$$

for $\xi > \max\{1, (q-1)(\frac{N}{2}+1)\}$. Then $u^q \in L^r(\Sigma)$ with $r = r(\xi) > 1+\frac{N}{2}$, and using equation (3.4) together with Lemma A.2 b), we find

$$\|v\|_{C^{0,\alpha}(\overline{\Sigma}_{\frac{\varepsilon}{2}})} \leq C\|u\|^q_{L^{\frac{r}{q}}(\Sigma)}. \tag{4.19}$$

From here, $v^p \in L^r(\overline{\Sigma}_{\frac{\varepsilon}{2}})$, and then from equation (3.3) and Lemma A.2.b) again, we find

$$\|u\|_{C^{0,\alpha}(\overline{\Sigma}_\varepsilon)} \leq C\|u\|^p_{L^{\frac{r}{p}}(\Sigma_{\frac{\varepsilon}{2}})}. \tag{4.20}$$

Putting together (4.18), (4.19) and (4.20) we find

$$\|u\|_{C^{0,\alpha}(\overline{\Sigma}_\varepsilon)} \leq C\|u\|^s_{B^{T_1,T_2}_{p^*+1}},$$

with $s = pq$.

Case c: In this case we need a lemma in the line of Lemma 3.2.

LEMMA 4.3. *If $\bar{p} \geq p$, $\frac{\bar{p}+1}{p} < 1+\frac{N}{2}$ and $u \in B^{T_1,T_2}_{\frac{\bar{p}+1}{p}}$ then, given $\varepsilon > 0$ there exists a constant C such that*

$$\|u\|_{B^{T_1+\varepsilon,T_2-\varepsilon}_{(\tilde{p}+1)/p}} \leq C\|u\|^s_{B^{T_1,T_2}_{(\bar{p}+1)/p}} \tag{4.21}$$

where $s = pq$, and $\tilde{p} - \bar{p} \geq \frac{1}{2}(p+1)^2(q+1)\delta$ with $\delta = \frac{1}{p+1} + \frac{1}{q+1} - \frac{N}{N+2}$.

PROOF. Using the ideas of the proof of Lemma 3.2 and the corresponding inequalities as in cases a. and b. we obtain (4.21). We omit the details. □

Now we complete case c. We apply Lemma 4.3 several times up to obtain $u \in B_{\frac{1+\tilde{p}}{p}}$ with $\frac{1+\tilde{p}}{p} \geq 1 + \frac{N}{2}$. Then we can proceed as in cases a. or b. to conclude (4.17). □

From Lemma 3.3, for all large enough k we have that

$$\int_{-M}^{M} \int_{\Omega} |Au_k|^{p^*+1} dx\, dt \leq C\,. \tag{4.22}$$

From (4.2) and (4.4) we obtain, after a suitable rescaling and for k sufficiently large, that

$$\int_{-M}^{M} \int_{\Omega} |u_k|^{p^*+1} dx\, dt \leq C\,. \tag{4.23}$$

Using this estimate and Lemma 4.2 we find that the sequences $\{u_k\}$, and $\{v_k\}$ are bounded in $C^{0,\alpha}([M+1, M-1] \times \overline{\Omega})$. Consequently we can apply the Arzelà Ascoli theorem to conclude that there exists a subsequence converging uniformly in $[M+1, M-1] \times \Omega$. Through a diagonal procedure we can construct a uniformly convergent subsequence over any set of the form $I \times \Omega$ with I a compact subset of $\mathbb{R}$.

Now, from Lemma 4.1, for every $k \in \mathbb{N}$ there exists $t_k \in [-k, k]$ and $x_k \in \Omega$ such that

$$u_k(t_k, x_k) \geq c > 0\,. \tag{4.24}$$

As the equations are autonomous, we can make a translation

$$\bar{u}_k(t, x) = u_k(t + t_k, x), \quad \bar{v}_k(t, x) = v_k(t + t_k, x)\,, \tag{4.25}$$

so that $\bar{u}_k, \bar{v}_k$ also satisfy the equations and the estimates given in Lemma 3.3 as well.

Thus we can perform the procedure described above in order to obtain the limit functions u and v. The estimate (4.24) becomes

$$u_k(0, x_k) \geq c > 0\,. \tag{4.26}$$

Since the sequence $\{u_k\}$ converges uniformly on $\{0\} \times \Omega$ we find that

$$u(0, x) \geq c \tag{4.27}$$

in some point $x \in \Omega$, showing in this way that $u \not\equiv 0$. Equation (4.13) implies that $v \not\equiv 0$ also.

Finally we would like to study the behavior of u and v for t near $+\infty$ and $-\infty$. We first note that the functions u and v are classical solution of (4.12) and (4.13), following the arguments given in Section 3, replacing Lemma A.3 by Lemma A.2 when appropriate. This implies in particular that

$$u(t,x) = v(t,x) = 0 \quad \forall t \in \mathbb{R} \quad \forall x \in \partial\Omega\,. \tag{4.28}$$

On the other hand, using (4.11) and taking limit in (4.22) and (4.23), as k tends to infinity and later as M tends to infinity, we obtain

$$\int_{-\infty}^{\infty}\int_{\Omega} |Au|^{p^*+1} + |u|^{p^*+1} dx\,dt \le C\,. \tag{4.29}$$

From (4.29), we find that

$$\lim_{|R|\to\infty} \int_{R-1}^{R+1}\int_{\Omega} |Au|^{p^*+1} + |u|^{p^*+1} dx\,dt = 0\,, \tag{4.30}$$

and then we can apply Lemma 4.3 to conclude that

$$\sup_{(t,x)\in[R-\frac{1}{2},R+\frac{1}{2}]\times\overline{\Omega}} |u(t,x)| \le C\|u\|^s_{B^{R-1,R+1}_{p^*+1}}\,, \tag{4.31}$$

from where, together with (4.30), we obtain that $u(t,x)$ converges to 0 as t goes to ∞ or $-\infty$, uniformly in $x \in \Omega$. This completes the proof of Theorem 1.2. □

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Appendix

We devote this appendix to the proof of some results that were used in the text. We start with the proof of Lemma 3.1 using the approach of Dore-Venni [7] as extended by Prüss-Sohr [20].

Proof of Lemma 3.1. Let B_r, A_r, $\tilde{A}_r$ be as in Section 3, defined by (3.1) and (3.2). We want to show that A_r and $\tilde{A}_r$ are isomorphisms from B_r into $L^r(\Sigma)$ and that (4.2) holds. We recall that $E = L^r_T(L^r(\Omega))$, equipped with its usual norm, is a Banach space having the UMD property for $r \in (1, \infty)$, Dore-Venni [7]. We define the operators L_1^+ and L_1^- by

$$D(L_1^+) = D(L_1^-) = W_T^{1,r}(L^r(\Omega)),\qquad L_1^{\pm}u = \pm\frac{d}{dt}u \tag{5.1}$$

for $u \in D(L_1^{\pm})$. As in Dore-Venni [7], one verifies that L_1^+ and L_1^- are positive (even m-accretive) operators in E and that they satisfies

$$\|(L_1^{\pm})^{is}\| \le M_r(1+|s|)e^{|s|\pi/2}, \quad s \in \mathbb{R}. \tag{5.2}$$

On the other hand, the operator L_2 defined by

$$D(L_2) = L_T^r(W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)),\qquad L_2u = -\Delta u, \tag{5.3}$$

for $u \in D(L_2)$ is also positive (even m-accretive) in E, and it satisfies

$$\|(L_2)^{is}\| \le M_r(1+s^2), \quad s \in \mathbb{R}. \tag{5.4}$$

It follows from [7], [20] that there is $C > 0$ such that

$$\|L_1^{\pm}u\|_E + \|L_2u\|_E \le C\|L_1^{\pm}u + L_2u\|_E,$$

for every $u \in D(L_1^{\pm}) \cap D(L_2) = B_r$. Note that we even obtain that

$$\|\lambda L_1^{\pm}u\|_E + \|L_2u\|_E \le C\|\lambda L_1^{\pm}u + L_2u\|_E,$$

for every $u \in B_r$ and $\lambda \in \mathbb{R}$, see Clément [2], and Prüss [19]. This proves Lemma 3.1 and (4.2). □

Next we consider some imbedding results.

LEMMA A.1. *Given $T_1, T_2 > 0$ and $\Omega \subset \mathbb{R}^N$ bounded with smooth boundary. If $m_1 > 0, m_2 > 0$ and $s > r > 1$, then*

$$W^{m_1,r}([T_1,T_2], L^r(\Omega)) \cap L^r([T_1,T_2], W^{m_2,r}(\Omega)) \subset\subset L^s(\Sigma)$$

provided

$$0 < \frac{1}{r} - \frac{1}{s} < \frac{m_1m_2}{m_2 + m_1N}.$$

We use $\subset\subset$ to denote compact imbedding.

Proof. We shall only prove a special case of Lemma A.1 which is needed in this paper, namely the case where $m_1 = 1$ and $m_2 = 2$ and where $W^{2,r}(\Omega)$ is replaced by $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$. Without loss of generality we may assume $T_1 = 0$ and $T_2 = T$. We recall that $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ is the domain of the generator of a bounded analytic semigroup in $L^r(\Omega)$, namely the domain of the Laplacian operator which we denote here by B. We are interested in the imbedding of

$$W^{1,r}([0,T]; L^r(\Omega)) \cap L^r([0,T]; D(B)) \quad \text{in} \quad L^r(\Sigma),$$

for this matter we shall use some results of Di Blasio, (p. 60, [17]). Since $D(B) \hookrightarrow D(\theta, p)$, for every $\theta \in (0,1]$, it follows from Di Blasio, (p. 60 [17]) that for every $\theta \in [0,1)$ we have:

$$\begin{aligned} &W^{1,r}([0,T]; L(\Omega)) \cap L^r([0,T]; D(B)) \\ &\hookrightarrow W^{\theta,r}([0,T]; L^r(\Omega)) \cap L^r([0,T]; D_B(\theta,r)) \\ &\hookrightarrow W^{\theta,r}([0,T]; D_B(\theta-\varepsilon,r)), \end{aligned} \tag{5.5}$$

with $0 < \varepsilon < \theta < 1$. Now we choose θ and ε such that $0 < \varepsilon < \theta < 1$, $\varepsilon > \frac{1}{r} - \frac{1}{s}$, $2(\theta - \varepsilon) \neq 1$ and $D_B(\theta - \varepsilon, r) \subset\subset L^s(\Omega)$. Note that $D_B(\theta - \varepsilon, r) \subset W^{2(\theta-\varepsilon),r}(\Omega) \subset\subset L^s(\Omega)$, provided that $2(\theta - \varepsilon) > \frac{N}{r} - \frac{N}{s}$, by Sobolev imbedding. Next we show that ε, θ can be chosen such that all conditions are satisfied. Observe that by the condition $0 < \frac{1}{r} - \frac{1}{s} < \frac{2}{N+2}$, there exists $\bar{\xi} \in (0,1)$ such that

$$\frac{1}{r} - \frac{1}{s} = \bar{\xi}\frac{2}{N+2}.$$

We first choose $\varepsilon = \xi\frac{2}{N+2}$, with $\xi \in (\bar{\xi}, 1)$, so that $0 < \varepsilon < 1$ and $\varepsilon > \frac{1}{r} - \frac{1}{s}$. Observe that $\bar{\xi}\frac{N}{N+2} + \xi\frac{2}{N+2} < \xi < 1$. Hence there exists $\theta \in (\xi, 1)$ such that $2(\theta - \varepsilon) \neq 1$. For such θ, we have

$$1 > \theta > \xi > \bar{\xi}\frac{N}{N+2} + \xi\frac{2}{N+2} = \frac{N}{2}\left(\frac{1}{r} - \frac{1}{s}\right) + \varepsilon > 0,$$

so that $\varepsilon < \theta < 1$ and $2(\theta - \varepsilon) > \frac{N}{r} - \frac{N}{s}$. Finally we observe that both of the imbeddings $W^{\varepsilon,r}([0,T]) \subset L^s([0,T])$ and $D_B(\theta - \varepsilon, r) \subset L^s(\Omega)$ are compact, therefore the imbedding $W^{\varepsilon,r}([0,T]; D_B(\theta - \varepsilon, r)) \subset L^s(\Sigma)$ is also compact. This completes the proof of the lemma. □

Next we recall two regularity results (see Ladyzhenskaya and Ural'tseva [15]). For this purpose we define

$$B_r^{T_1,T_2} :\equiv W^{1,r}([T_1,T_2], L^r(\Omega)) \cap L^r([T_1,T_2], W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)).$$

and we consider $B_r^{T_1,T_2}$ equipped with the norm given by (see (4.7))

$$\|u\|^r_{B_r^{T_1,T_2}} = \int_{T_1}^{T_2} \int_{\Omega} \{|Au|^r + |u|^r\} dx\, dt\,.$$

LEMMA A.2. (Interior Estimate) *Assume that Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $T_1 < T_2$ and set $\Sigma = (T_1, T_2) \times \Omega$. Assume that $u \in B_r^{T_1,T_2}$ satisfies*

$$\frac{\partial u}{\partial t} - \Delta u = f \quad in \quad \Sigma\,,$$

where $f \in L^{r_0}(\Sigma)$ with $r \le r_0$. Then, given $\epsilon > 0$ and setting $\Sigma_\varepsilon = (T_1 + \varepsilon, T_2 - \varepsilon) \times \Omega$, we have:

a) If $r_0 \le 1 + \frac{N}{2}$ then u belongs to $B_{r_0}^{T_1+\varepsilon,T_2-\varepsilon}$. Moreover, there exists a constant C depending on ε and $T_2 - T_1$ such that

$$\|u\|_{B_{r_0}^{T_1+\varepsilon,T_2-\varepsilon}} \le C\|f\|_{L^{r_0}(\Sigma)}\,,$$

b) If $r_0 > 1 + \frac{N}{2}$ then the solution u belongs to $C^{0,\alpha}(\Sigma_\varepsilon)$, for some $\alpha > 0$. Moreover there is a constant C depending on ε and $T_2 - T_1$ such that

$$\|u\|_{C^{0,\alpha}(\overline{\Sigma_\varepsilon})} \le C\|f\|_{L^{r_0}(\Sigma)}\,,$$

c) If, in addition, $f \in C^{0,\alpha}(\overline{\Sigma})$ for some $\alpha > 0$ then $u \in C^{(1,\alpha)(2,\alpha)}(\overline{\Sigma_\varepsilon})$.

REMARK 5.1. In case $f \in C^{0,\alpha}(\overline{\Sigma})$ and u satisfies the equation above then u is a classical solution in $\overline{\Sigma}$.

The next regularity result involves the boundary conditions.

LEMMA A.3. (Global Regularity) *Assume that Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $T_1 < T_2$ and set Σ as before. If $u \in B_r$ (See Section 3) satisfies*

$$\frac{\partial u}{\partial t} - \Delta u = f \quad in \quad \Sigma$$

where $f \in L^{r_0}(\Sigma)$ with $r \le r_0$, then we have:

a) *If $r_0 \le 1 + \frac{N}{2}$ then $u \in B_{r_0}$,*

b) *If $r_0 > 1 + \frac{N}{2}$ then $u \in C^{0,\alpha}(\overline{\Sigma})$ for some $\alpha > 0$ and u satisfies the boundary conditions*

$$u(t,x) = 0 \quad for\ all\ t \in [-T, T]\ and\ x \in \partial\Omega$$

and

$$u(-T,x) - u(T,x) = 0 \quad for\ all\ x \in \Omega\,,$$

c) *If $f \in C^{0,\alpha}(\overline{\Sigma})$ for some $\alpha > 0$ then $u \in C^{(1,\alpha)(2,\alpha)}(\overline{\Sigma})$.*

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