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On Rational de Rham Cohomology Associated with the Generalized Airy Function

HIRONOBU KIMURA

Dedicated to Professor Toshihusa Kimura
for his 65 th birthday

1. – Introduction

The purpose of this paper is to prove a vanishing theorem on the rational de Rham cohomology associated with the generalized Airy function, which is the function introduced by I.M. Gel'fand *et al.* [GRS].

Let us explain our motivation. The Airy function of a single variable is defined by the integral

$$Ai(x) = \int_{\Delta} e^{tx - t^3/3} dt$$

where Δ is a suitable path in the complex t-plain on which the point comes from infinity along some half line and goes back to infinity along another half line. The directions of the half lines are chosen so that the integrand tends to zero exponentially as t goes to infinity along Δ. Then $Ai(x)$ is an entire function of x and satisfies the ordinary differential equation

$$\left(\frac{d^2}{dx^2} - x \right) Ai(x) = 0$$

having an irregular singular point at $x =$ infinity. The Airy function is important from several viewpoints. From the viewpoint of the theory of differential equations in the complex domain, it is important because it provided the example by which the existence of the Stokes phenomenon at an irregular singular point was recognized for the first time (the phenomenon that the asymptotic behavior of $Ai(x)$ $(x \to \infty)$ changes discontinuously as one varies continuously the asymptotic direction). From the viewpoint of theory of special functions, it is

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important because it forms a class of special functions with Gauss hypergeometric function, Kummer's confluent hypergeometric function, Bessel function and Hermite function such that the members are related each other by a certain limit process so called the confluence, see [KHT2]. It is also to be mentioned that the integral $Ai(x)$ is regarded as a simple case of complex oscillatory integral in which the phase function $xt - x^3/3$ is a deformation of the simple singularity of type A_2 ([AVG]). A generalization of $Ai(x)$ to a function of two variables is given by the integral known as Pearcy integral, which is a 1-dimensional complex oscillatory integral whose phase function is a deformation of the simple singularity of type A_3. These functions are also related to nonlinear integrable Hamiltonian systems, see [O], [OK]. The generalized Airy function discussed in this paper is a generalization of $Ai(x)$ and Pearcy integral to several variables case and it is characterized as solutions of holonomic system on an affine space [GRS]. Outside the singularity of the system, this holonomic system is equivalent to a de Rham system, or in other term, to a completely integrable linear Pfaffian system. To write down explicitly this integrable Pfaffian system, it is necessary to compute the cohomology associated with the generalized Airy function.

We recall the definition of the generalized Airy function. Let r and $n(> r)$ be positive integers and Z be the set of $(r+1) \times (n+1)$ complex matrices $z = (z_0, \dots, z_n)$ of the form

$$z = \begin{pmatrix} 1 & z_{01} & \dots & z_{0n} \\ 0 & z_{11} & \dots & z_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & z_{r1} & \dots & z_{rn} \end{pmatrix}$$

whose first $(r+1)$-minor is not zero. Let H be the Jordan group of size $n+1$, namely

$$H := \left\{ h = \sum_{0 \le i \le n} h_i \Lambda^i \ ; \quad h_i \in \mathbb{C}, \quad h_0 \neq 0 \right\}$$

with the shift matrix $\Lambda = (\delta_{i+1,j})_{0 \le i,j \le n}$. Sometimes we denote an element $h \in H$ by $[h_0, h_1, \dots, h_n]$. Define the biholomorphic map

$$\iota : H \to \mathbb{C}^\times \times \mathbb{C}^n$$

by

$$\iota([h_0, h_1, \dots, h_n]) = (h_0, h_1, \dots, h_n) \ .$$

It is extended to the biholomorphic map between the universal covering manifolds $\tilde{H}$ and $\tilde{\mathbb{C}}^\times \times \mathbb{C}^n$, which we denote by the same letter ι.

Let χ be a character of the universal covering group $\tilde{H}$ of the Jordan group H, see § 3 as for the explicit form of the characters. Since the character of $\tilde{H}$ depends on parameters $\alpha \in \mathbb{C}^n$, as is explained in § 3, we denote it as $\chi(\cdot; \alpha)$. We assume

(A.0) *the parameter* $\alpha \in \mathbb{C}^{n+1}$ *satisfies* $\alpha_0 = -r-1$.
Let $t = (t_0, t_1, \dots, t_r)$ be the coordinates of $\mathbb{C}^{r+1}$. Put

$$\begin{aligned}\tau &= i_v(dt_0 \wedge \cdots \wedge dt_r) \\ &= \sum_{0 \le i \le r} (-1)^i t_i dt_0 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_r \ ,\end{aligned}$$

where $\widehat{dt_i}$ denotes the deletion of dt_i and i_v denotes the interior product defined by the Euler vector field

$$v = t_0 \partial/\partial t_0 + \cdots + t_r \partial/\partial t_r \ .$$

DEFINITION 1.2. *The generalized Airy function is the function on* Z *defined by*

$$\Phi(z;\alpha) = \int_\Delta \chi(\iota^{-1}(tz);\alpha) \cdot \tau \ , \tag{1.1}$$

where Δ *is a cycle.*

Since we are interested in the cohomology group associated with the Airy function which is determined by the integrand and is irrelevant to the cycle, we will not explain the homology group or the cycles in the definition of the Airy function.

The assumption (A.0) says that the r-form $\chi(\iota^{-1}(tz);\alpha)\cdot\tau$ in the integral (1.1) is invariant under the homothety $t \mapsto ct$ for any $c \in \mathbb{C}^\times$ as is seen from the explicit form of χ and therefore it is considered as a form on $\mathbb{P}^r$. We take an affine coordinate $t = (1, t_1, \dots, t_r) \in \mathbb{C}^r \subset \mathbb{P}^r$, then the form τ turns into

$$\tau = dt_1 \wedge \cdots \wedge dt_r$$

and the integral takes the form

$$\chi(\iota^{-1}(tz);\alpha) = \exp F(t,z) \tag{1.2}$$

with the polynomial $F(t,z) \in \mathbb{C}[t,z]$, see (3.2) for its explicit form. We denote the affine t-space $\mathbb{C}^r$ by T.

Let us introduce the twisted rational de Rham complex associated with the Airy integral (1.1). Put $M := T \times Z$ and

$$\begin{aligned}\Omega_T^p &:= \{p - \text{forms in } t \text{ with coefficients polynomial in } t\} \\ \Omega_{M/Z}^p &:= \Omega_T^p \otimes_{\mathbb{C}} S(Z) \ ,\end{aligned}$$

where $S(Z) := \mathbb{C}[z][(\det z')^{-1}]$ is the ring of regular functions on Z, namely, the localization of $\mathbb{C}[z]$ by $\det z'$. Let F be that given in (1.2). Define

$$\nabla_F : \Omega_{M/Z}^p \to \Omega_{M/Z}^{p+1}$$

by

$$\nabla_F \omega = d\omega + dF \wedge \omega \ ,$$

where d is the exterior differentiation with respect to t regarding z as parameters.

DEFINITION 1.3. *The twisted rational de Rham complex associated with the Airy integral is*

$$(\Omega^{\bullet}_{M/Z}, \nabla_F) : 0 \to \Omega^0_{M/Z} \xrightarrow{\nabla_F} \Omega^1_{M/Z} \xrightarrow{\nabla_F} \Omega^2_{M/Z} \xrightarrow{\nabla_F} \cdots \xrightarrow{\nabla_F} \Omega^r_{M/Z} \longrightarrow 0 .$$

Denote by $H^*(\Omega^{\bullet}_{M/Z}, \nabla_F)$ the cohomology group of the complex $(\Omega^{\bullet}_{M/Z}, \nabla_F)$ and call it *the twisted rational de Rham cohomology* associated with Airy integral.

The results of this paper are as follows.

THEOREM 1.4. *Let the parameter* α *satisfy* $\alpha_n \neq 0$. *Then*

(1) $H^p(\Omega^{\bullet}_{M/Z}, \nabla_F) = 0$ *for* $p \neq r$.

(2) $H^r(\Omega^{\bullet}_{M/Z}, \nabla_F)$ *is a free* $S(Z)$*-module such that*

$$rank\ H^r(\Omega^{\bullet}_{M/Z}, \nabla_F) = \binom{n-1}{r} .$$

As to a choice of the basis of the r-th cohomology $H^r(\Omega^{\bullet}_{M/Z}, \nabla_F)$, we have Theorem 3.3 and Proposition 4.1. We state in Section 5 a conjecture concerning a $S(Z)$-basis of the cohomology.

2. – Preliminary on Koszul complex

In this section we recall known results on the cohomology of Koszul complex related to an isolated critical point of a polynomial function.

Let $T := \mathbb{C}^r$ with the coordinates $t = (t_1, \dots, t_r)$ and let $g \in \mathbb{C}[t]$ such that $g(0) = 0$. We assume

(A.1) $g(t)$ *has 0 as a unique isolated critical point*:

$$(dg)(0) = 0 \text{ and } (dg)(t) \neq 0 \text{ if } t \neq 0 .$$

For the above $g(t)$ we consider the Koszul complex

$$(\Omega^{\bullet}_T, dg) : 0 \to \Omega^0_T \overset{dg\wedge}{\to} \Omega^1_T \overset{dg\wedge}{\to} \Omega^2_T \overset{dg\wedge}{\to} \cdots \overset{dg\wedge}{\to} \Omega^r_T \to 0$$

and denote the cohomology by $H^*(\Omega^{\bullet}_T, dg)$. The following result is known.

PROPOSITION 2.1. *Assume that* $g(t) \in \mathbb{C}[t]$ *satisfies (A.1).*

(1) *The ring* $\mathbb{C}[t]/(\partial_t g)$ *is a finite dimensional vector space over* $\mathbb{C}$, *where* $(\partial_t g)$ *is the Jacobian ideal of* $\mathbb{C}[t]$ *generated by the derivatives* $\partial_{t_1} g, \dots, \partial_{t_r} g$.

(2) *We have*

$$H^p(\Omega^{\bullet}_T, dg) = 0 \text{ for } p \neq r$$
$$H^r(\Omega^{\bullet}_T, dg) \simeq \mathbb{C}[t]/(\partial_t g) .$$

The ring $\mathbb{C}[t]/(\partial_t g)$ is called the *Jacobi ring* of g and $\mu(g) := \dim_{\mathbb{C}} \mathbb{C}[t]/(\partial_t g)$ is called the *Milnor number* of g.

Next we consider the special case where g is a quasihomogeneous polynomial.

DEFINITION 2.2. *Let $\rho = (\rho_1, \dots, \rho_r)$, $\rho_i \in \mathbb{Q}_{>0}$. A polynomial $f(t) \in \mathbb{C}[t]$ is said to be quasihomogeneous of ρ-degree $q \in \mathbb{Q}$ if $f(t)$ has the form*

$$f(t) = \sum_{\langle \rho, \nu \rangle = q} a_\nu t^\nu , \quad a_\nu \in \mathbb{C} ,$$

where $t^\nu = t_1^{\nu_1} \dots t_r^{\nu_r}$ and $\langle \rho, \nu \rangle = \rho_1 \nu_1 + \dots + \rho_r \nu_r$. This condition is equivalent to $L_E f(t) = q f(t)$, where L_E is the Lie derivation defined by the Euler vector field

$$E = \rho_1 t_1 \partial/\partial t_1 + \dots + \rho_r t_r \partial/\partial t_r .$$

We write ρ-deg $f = q$.

DEFINITION 2.3. *A polynomial $f(t)$ is said to be of ρ-degree $\leq q$ if $f(t)$ is expressed as a sum of quasihomogeneous polynomials of ρ-degree less than q. We denote this fact as ρ-deg $f \leq q$.*

DEFINITION 2.4. *A p-form $\eta \in \Omega_T^p$ is quasihomogeneous of ρ-degree q if $L_E \eta = q\eta$. When a p-form η is a sum of quasihomogeneous p-forms of ρ-degree less than q then we write ρ-deg $\eta \leq q$.*

Now we consider for a polynomial $g(t)$ the following condition.
(A.2) *$g(t)$ is a quasihomogeneous polynomial of ρ-degree $= n$.*
For $\rho = (\rho_1, \dots, \rho_n)$ we decompose the modules Ω_T^p as

$$\Omega_T^p = \bigoplus_{q \in \mathbb{Q}} \Omega_{T,q}^p , \quad \Omega_{T,q}^p = \{\eta \in \Omega_T^p \ ; \ L_E \eta = q\eta\}$$

Assume the conditions (A.1) and (A.2) for the function g. Then the Koszul complex decomposes as $(\Omega_T^\bullet, dg) = \oplus_{q \in \mathbb{Q}} (\Omega_T^\bullet(q), dg)$ with

$$(\Omega_T^\bullet(q), dg) : 0 \to \Omega_{T,q-nr}^0 \xrightarrow{dg\wedge} \Omega_{T,q-n(r-1)}^1 \xrightarrow{dg\wedge} \dots \xrightarrow{dg\wedge} \Omega_{T,q}^r \to 0$$

and we get

$$H^p(\Omega_T^\bullet, dg) = \bigoplus_{q \in \mathbb{Q}} H^p(\Omega_T^\bullet(q), dg) .$$

In particular

$$H^r(\Omega_T^\bullet, dg) = \bigoplus_{q \in \mathbb{Q}} H^r(\Omega_T^\bullet(q), dg) \simeq \mathbb{C}[t]/(\partial_t g) ,$$

where the isomorphism is constructed so that the representative $[h(t) dt_1 \wedge \dots \wedge dt_r]$ of $H^r(\Omega_T^\bullet(q), dg)$ corresponds to $[h(t)] \in \mathbb{C}[t]/(\partial_t g)$ and $h(t)$ can be given by some quasihomogeneous polynomial.

Next we consider a deformation $G = G(t, x) \in \mathbb{C}[t, x]$ of $g(t)$, where x is the coordinates of an affine space X. Put $M = T \times X$ and consider the cochain complex

$$(\Omega_{M/X}^\bullet, dG) : 0 \to \Omega_{M/X}^0 \xrightarrow{dG\wedge} \Omega_{M/X}^1 \xrightarrow{dG\wedge} \Omega_{M/X}^2 \xrightarrow{dG\wedge} \dots \xrightarrow{dG\wedge} \Omega_{M/X}^r \longrightarrow 0 ,$$

where $\Omega_{M/X}^p = \Omega_T^p \otimes \mathbb{C}[x]$. We assume for $G(t, x)$
(A.3) *The ρ-degree of $G(t, x) - g(t)$ is strictly less that n.*

PROPOSITION 2.5. [N]. *Under the conditions (A.1)–(A.3), we have*
(1) $H^p(\Omega^\bullet_{M/X}, dG) = 0$ *if* $p \neq r$.
(2) $H^r(\Omega_{M/X}, dG)$ *is a free* $\mathbb{C}[x]$*-module of rank* $\mu(g)$.
(3) *Let* $\omega_1, \dots, \omega_\mu \in \Omega^r_T$ *be quasihomogeneous forms which form a* $\mathbb{C}$*-basis of* $H^r(\Omega^\bullet_T, dg)$. *Then* $[\omega_1], \dots, [\omega_\mu]$ *give a* $\mathbb{C}[x]$*-basis of* $H^r(\Omega^\bullet_{M/X}, dG)$.

3. – Koszul complex associated with the Airy function

In this section we consider the cohomology of Koszul complex associated with the integral (1.1). Since we need in the following the explicit form of the character χ of the group $\tilde{H}$, we begin by recalling it.

Define a sequence of polynomials $\theta_m(v)$ of $v = (v_1, v_2, \dots)$ by

$$\log(1 + v_1 T + v_2 T^2 + \dots) = \sum_{m \geq 1} \theta_m(v) T^m \ , \tag{3.1}$$

where T is an indeterminate. Explicitly,

$$\begin{aligned} \theta_1 &= v_1 \ , \\ \theta_2 &= v_2 - \frac{1}{2} v_1^2 \ , \\ \theta_3 &= v_3 - v_1 v_2 + \frac{1}{3} v_1^3 \ , \\ \theta_4 &= v_4 - v_1 v_3 - \frac{1}{2} v_2^2 + v_1^2 v_2 - \frac{1}{4} v_1^4 \end{aligned}$$

and so on. Note that if we define the weight of v_i to be i, then $\theta_m(v)$ is quasihomogeneous of weight m. Using these polynomials the character $\chi : \tilde{H} \to \mathbb{C}^\times$ is given by

$$\chi(h; \alpha) = h_0^{\alpha_0} \exp\left(\sum_{1 \leq i \leq n} \alpha_i \theta_i(h/h_0) \right)$$

with parameters $\alpha = (\alpha_0, \dots, \alpha_n)$. We assume the condition $\alpha_0 = -r-1$ in the following, see the assumption (A.0). Let Z be the set of $(r+1) \times (n+1)$ complex matrices whose first $(r+1)$-minor does not vanish and whose first column vector is ${}^t(1, 0, \dots, 0)$ as is defined in Introduction. Take $z = (z_0, \dots, z_n) \in Z$ and put

$$l(t) = (1, l_1(t), \dots, l_n(t)) = tz \ , \quad l_i(t) = tz_i = z_{0i} + t_1 z_{1i} + \dots + t_r z_{ri} \ .$$

Then

$$\chi(\iota^{-1}(tz);\alpha) = \exp F(t,z)\ , \quad F(t,z) = \sum_{1\le i\le n} \alpha_i\theta_i(l(t))\ . \tag{3.2}$$

Note that $F(t,z) \in \mathbb{C}[t,z]$. We want to study the Koszul complex of dF. As a first step, we treat a simple case. Put

$$X := \{x = (1_{r+1}, x'')\ ; \quad x'' \in \text{ Mat } (r+1, n-r)\}\ .$$

We can regard X as a subvariety of Z isomorphic to an affine space $\mathbb{C}^{(r+1)(n-r)}$. Let $S(X) := \mathbb{C}[x'']$ be the ring of regular functions on X. Take $x = (1_{r+1}, x'') \in X$ and put

$$g(t) = \alpha_n\theta_n(t_1, \dots, t_r, 0, \dots, 0) \in \mathbb{C}[t]$$
$$G(t,x) = \log \chi(\iota^{-1}(tx);\alpha) \in S(X)[t]\ .$$

Consider the Koszul complex $(\Omega^\bullet_{M_0/X}, dG)$, where $M_0 := T \times X$. Note here that if we let $\rho = (1, 2, \dots, r)$ be the weight of $t = (t_1, \dots, t_r)$, then $g(t)$ is a quasihomogeneous polynomial such that ρ-deg $g = n$.

PROPOSITION 3.1. *Assume that* $\alpha_n \neq 0$.

(1) *The polynomial* $g(t)$ *has the isolated critical point 0 and the Milnor number is* $\mu(g) = \binom{n-1}{r}$.

(2) $H^p(\Omega^\bullet_{M_0/X}, dG) = 0$ *for* $p \neq r$.

(3) $H^r(\Omega^\bullet_{M_0/X}, dG)$ *is a free* $S(X)$*-module of rank* $\mu(g)$.

(4) *Let* $\omega_1, \dots, \omega_\mu$ *be a basis of* $H^r(\Omega^\bullet_T, dg)$, *then* $[\omega_1], \dots, [\omega_\mu]$ *form a* $S(X)$*-basis of* $H^r(\Omega^\bullet_{M_0/X}, dG)$.

PROOF OF PROPOSITION 3.1 (1). Assume that $\alpha_n = 1$ without loss of generality. We put

$$V = \{t \in \mathbb{C}^r\ ; \quad \partial_{t_1} g = \dots = \partial_{t_r} g = 0\}$$

and we show $V = \{0\}$. Differentiating the both sides of (3.1) with respect to v_i, we get

$$\frac{T^i}{1 + v_1T + v_2T^2 + \dots} = \sum_{m\ge 1} \frac{\partial\theta_m}{\partial v_i} T^m\ .$$

Define a sequence of polynomials $\phi_m(v)$ by

$$\frac{1}{1 + v_1T + v_2T^2 + \dots} = \sum_{m\ge 0} \phi_m(v)T^m\ . \tag{3.3}$$

Then

$$\frac{\partial\theta_n}{\partial v_i} = \phi_{n-i}(v)\ , \quad i \ge 1\ . \tag{3.4}$$

Put $t = (t_1, t_2, \dots, t_r, 0, \dots)$ into (3.3) in place of v and denote the resulting polynomials by $\phi_m(t)$. Then the assertion can be restated as

$$V = \{t \in \mathbb{C}^r \ ; \quad \phi_{n-r}(t) = \cdots = \phi_{n-1}(t) = 0\} = \{0\} \ .$$

From (3.3) we see that $\phi_m(t)$ satisfy

$$(1 + t_1T + t_2T^2 + \cdots + t_rT^r)(1 + \phi_1(t)T + \phi_2(t)T^2 + \dots) = 1$$

and therefore by equating the coefficients of T^m of both sides we get the recurrence formula

$$\phi_m(t) + t_1\phi_{m-1}(t) + \cdots + t_r\phi_{m-r}(t) = \begin{cases} 1 & \text{if } m = 0 \ , \\ 0 & \text{if } m > 0 \ , \end{cases} \tag{3.5}$$

where we put $\phi_m(t) = 0$ for $m < 0$ by convention. Suppose that there is a point $0 \neq a \in V$ and take an index $1 \le p \le r$ such that

$$a_p \neq 0 \quad \text{and} \quad a_{p+1} = \cdots = a_r = 0 \ .$$

First we assert that $\phi_i(a) = 0$ $(1 \le i \le n-1)$. In fact, considering (3.5) for $m = n - r - 1 + p$ and putting $t = a$ in it, we have

$$\phi_{n-r-1+p}(a) + a_1\phi_{n-r-2+p}(a) + \cdots + a_p\phi_{n-r-1}(a) = 0 \ .$$

Since $a_p \neq 0$, we have $\phi_{n-r-1}(a) = 0$. Next we take $m = n-r-2+p$ in (3.5) and get $\phi_{n-r-2}(a) = 0$. Inductively we can see that $\phi_1(a) = \cdots = \phi_{n-1}(a) = 0$. Again using the formula (3.5) for $m = 1$, $\phi_1(a) + a_1 = 0$, we have $a_1 = 0$. The formula (3.5) for $m = 2$, $\phi_2(a) + a_1\phi_1(a) + a_2\phi_0(a) = 0$, yields $a_2 = 0$. Proceeding in the same way we get $a_1 = a_2 = \cdots = a_p = 0$, which contradicts the assumption $a_p \neq 0$ for the point $a \in V$. Once the isolation of the critical point $t = 0$ for g is established, its Milnor number is computed by the formula

$$\mu(g) = \prod_{1 \le i \le r} \left(\frac{n}{\rho_i} - 1 \right) = \binom{n-1}{r} \ .$$

See [AVG] as to this type of formula for the Milnor number. □

In view of Propositions 2.1 and 2.2, to prove (2), (3) and (4) of Proposition 3.1, it is sufficient to show:

LEMMA 3.2. *Let $\rho = (1, 2, \dots, r)$ be the weight of $t = (t_1, \dots, t_r)$. Then $g(t)$ is a quasihomogeneous polynomial of ρ-degree $= n$ and $G(t, x) - g(t)$ is of ρ-degree $< n$.*

PROOF OF LEMMA 3.2. Put $l(t) = (1, l_1(t), \ldots, l_n(t)) = tx$. Then we have $l_1(t) = t_1, \ldots, l_r(t) = t_r$. By the definition (3.1) of the polynomials $\theta_m(v)$, we have

$$\theta_m(v) = \sum_{1 \le k \le m} \frac{(-1)^{k+1}}{k} \sum_{i_1 + \cdots + i_k = m, i_j \ge 1} v_{i_1} \ldots v_{i_k} \ .$$

First we show that ρ-deg $\theta_m(l(t)) \le m$. In fact, for an index $(i_1, \ldots, i_k)$ such that $i_1 + \ldots i_k = m$, we have

$$\rho\text{-deg} l_{i_1}(t) \ldots l_{i_k}(t) \le i_1 + \cdots + i_k = m$$

because $l_i(t) = t_i$ for $1 \le i \le r$ and ρ-deg $l_i(t) < r$ for $i \ge r + 1$. The first assertion of the lemma is obvious since $\theta_n(t_1, \ldots, t_r, 0, \ldots, 0)$ is a quasihomogeneous polynomial in t which is a sum of terms $l_{i_1}(t) \ldots l_{i_k}(t)$ satisfying $i_1 + \cdots + i_k = n$ and $1 \le i_1, \ldots, i_k \le r$. To prove the second assertion, noting that ρ-deg $\theta_m(l(t)) < n$ for $m < n$, it is sufficient to show

$$\rho\text{-deg}[\theta_n(l(t)) - \theta_n(t_1, \ldots, t_r, 0, \ldots, 0)] < n \ . \tag{3.6}$$

Since $\theta_n(l(t)) - \theta_n(t_1, \ldots, t_r, 0, \ldots, 0)$ is a linear combination of the terms $l_{i_1}(t) \ldots l_{i_k}(t)$ with an index $(i_1, \ldots, i_k)$ satisfying $i_1 + \cdots + i_k = n$, with some $i_p \ge r + 1$, we get

$$\begin{aligned} &\rho\text{-deg}[\theta_n(l(t)) - \theta_n(t_1, \ldots, t_r, 0, \ldots, 0)] \\ &\qquad \le \max_{i_1 + \cdots + i_k = n} \rho\text{-deg} l_{i_1}(t) \ldots l_{i_k}(t) \\ &\qquad < i_1 + \cdots + i_k = n \ . \end{aligned}$$

This proves (3.6) and the second assertion of the lemma. Thus the proof of Proposition 3.1 is also completed. □

We turn to the study of the cohomology group of the complex $(\Omega^\bullet_{M/Z}, dF)$, where $M = T \times Z$. Take $z \in Z$ and put

$$f(t) = \alpha_n \theta_n(tz_1, \ldots, tz_r, 0, \ldots, 0) \ .$$

Let $p : Z \to X$ be the smooth map given by $p(z) = (1_{r+1}, (z')^{-1}z'')$. Put $\pi : M = T \times Z \to M_0 = T \times X : (t, z) \mapsto (t, p(z))$. Define the map $\beta : M \to M$ by $\beta(t, z) = (tz', z)$. Note that we can consider $S(Z)$ as a $S(X)$-module by the ring homomorphism $p^* : S(X) \to S(Z)$. The map π induces the $S(X)$-morphism

$$\pi^* : \Omega^\bullet_{M_0/X} \to \Omega^\bullet_{M/Z} , \quad \pi^*\left(\sum a_I(t, x) dt_I\right) = \sum (p^* a_I)(t, z) dt_I \tag{3.7}$$

and β induces the isomorphism of $S(Z)$-modules:

$$\beta^* : \Omega^\bullet_{M/Z} \to \Omega^\bullet_{M/Z} \ .$$

It is clear that π^* is an injective $S(X)$-homomorphism. We shall show:

THEOREM 3.3. *Assume that* $\alpha_n \neq 0$.

(1) *For any* $z \in Z$, $f(t)$ *has a unique isolated critical point and the Milnor number is* $\mu(f) = \binom{n-1}{r}$.

(2) $H^p(\Omega^\bullet_{M/Z}, dF) = 0$ *for* $p \neq r$,

(3) $H^r(\Omega^\bullet_{M/Z}, dF)$ *is a free* $S(Z)$*-module of rank* $\mu(f)$ *such that*

$$\begin{aligned} H^r(\Omega^\bullet_{M/Z}, dF) &\simeq H^r(\Omega^\bullet_{M_0/X}, dG) \otimes_{S(X)} S(Z) \\ &\simeq H^r(\Omega^\bullet_T, dg) \otimes_{\mathbf{C}} S(Z) \,. \end{aligned}$$

(4) *Let* $[\omega_1], \ldots, [\omega_\mu]$ *be the* $S(X)$*-basis of* $H^r(\Omega^\bullet_{M_0/X}, dG)$ *given in Proposition 3.1. Then* $[\beta^*\pi^*\omega_1], \ldots, [\beta^*\pi^*\omega_\mu]$ *provide a* $S(Z)$*-basis of* $H^r(\Omega^\bullet_{M/Z}, dF)$.

To prove the theorem, we relate $(\Omega^\bullet_{M_0/X}, dG)$ and $(\Omega^\bullet_{M/Z}, dF)$.

LEMMA 3.4. *The map* $\beta : M \to M$ *induces an isomorphism*

$$\beta^* : H^*(\Omega^\bullet_{M/Z}, \pi^*dG) \to H^*(\Omega^\bullet_{M/Z}, dF)$$

which sends $[\omega]$ *to* $[\beta^*\omega]$.

PROOF. It is sufficient to notice that the following diagram commutes.

$$\begin{array}{ccc} \Omega^p_{M/Z} & \xrightarrow{\pi^*dG\wedge} & \Omega^{p+1}_{M/Z} \\ \beta^* \downarrow & & \downarrow \beta^* \\ \Omega^p_{M/Z} & \xrightarrow{dF\wedge} & \Omega^{p+1}_{M/Z} \end{array} \tag{3.8}$$

where the horizontal arrows denote homomorphisms of $S(Z)$-module. To see this, put $\beta(t,z) = (tz', z) = (s,z)$ and take $\omega \in \Omega^p_{M/Z}$, $\omega = \sum a_I ds_I$, where $I = \{i_1, \ldots, i_p\}$ and $ds_I = ds_{i_1} \wedge \cdots \wedge ds_{i_p}$. Then

$$\begin{aligned} \beta^*(\pi^*dG \wedge \omega) &= \beta^*(d\log\chi(\iota^{-1}(s(1_{r+1}, (z')^{-1}z'')); \alpha) \wedge \omega) \\ &= d\log\chi(\iota^{-1}(tz'(1_{r+1}, (z')^{-1}z'')); \alpha) \wedge \beta^*\omega \\ &= dF \wedge \beta^*\omega \,. \end{aligned}$$

□

Next we compare the cohomologies $H^*(\Omega^\bullet_{M/Z}, \pi^*dG)$ and $H^*(\Omega^\bullet_{M_0/X}, dG)$. We assert

LEMMA 3.5. *We have the isomorphism*

$$\pi^* : H^*(\Omega^\bullet_{M_0/X}, dG) \otimes_{S(X)} S(Z) \to H^*(\Omega^\bullet_{M/Z}, \pi^*dG) \,.$$

PROOF. We have a cochain map

$$
\begin{array}{ccc}
\Omega^p_{M/Z} & \xrightarrow{\pi^* dG\wedge} & \Omega^{p+1}_{M/Z} \\
\pi^*\uparrow & & \uparrow\pi^* \\
\Omega^p_{M_0/X} & \xrightarrow{dG\wedge} & \Omega^{p+1}_{M_0/X}
\end{array}
\tag{3.9}
$$

where the map π^* denoted by the vertical arrow is defined by (3.7). Making a tensor product with $S(X)$-module $S(Z)$, we get from the diagram (3.8)

$$
\begin{array}{ccc}
\Omega^p_{M/Z} & \xrightarrow{\pi^* dG\wedge} & \Omega^{p+1}_{M/Z} \\
\pi^*\otimes id\uparrow & & \uparrow\pi^*\otimes id \\
\Omega^p_{M_0/X}\otimes S(Z) & \xrightarrow{(dG\wedge)\otimes id} & \Omega^{p+1}_{M_0/X}\otimes S(Z)
\end{array}
\tag{3.10}
$$

Here the vertical homomorphism $\pi^*\otimes id$ is an isomorphism of $S(Z)$-modules. In fact, the surjectivity is clear from the definition of π^*. To see the injectivity, notice that π^* in the diagram (3.8) is injective and that $S(Z)$ is a flat $S(X)$-module. Thus the cochain isomorphism leads to

$$
\begin{aligned}
H^*(\Omega^\bullet_{M/Z}, \pi^* dG) &\simeq H^*(\Omega^\bullet_{M_0/X}\otimes S(Z), dG\otimes id) \\
&\simeq H^*(\Omega^\bullet_{M_0/X}, dG)\otimes_{S(X)} S(Z) \ .
\end{aligned}
$$

□

PROOF OF THEOREM 3.3. By Lemma 3.4 and 3.5, we have

$$
\begin{aligned}
H^*(\Omega^\bullet_{M/Z}, dF) &\overset{(\beta^*)^{-1}}{\simeq} H^*(\Omega^\bullet_{M/Z}, \pi^* dG) \\
&\overset{(\pi^*)^{-1}}{\simeq} H^*(\Omega^\bullet_{M_0/X}, dG)\otimes_{S(X)} S(Z) \\
&\simeq \begin{cases} 0 & p\neq r \\ H^r(\Omega^\bullet_T, dg)\otimes_{\mathbb{C}} S(Z) & p=r \ . \end{cases}
\end{aligned}
$$

The last isomorphism is assured by Proposition 3.1. Since $H^r(\Omega^\bullet_T, dg)$ is a $\mathbb{C}$-vector space of dimension μ, we have Theorem 3.3. □

REMARK 3.6. *Lemmas 3.4 and 3.5 still hold if one consider the de Rham complexes* $(\Omega^\bullet_{M/Z}, \nabla_F)$, $(\Omega^\bullet_{M/Z}, \nabla_{\pi^*G})$ *and* $(\Omega^\bullet_{M_0/X}, \nabla_G)$ *in place of the Koszul complex* $(\Omega^\bullet_{M/Z}, dF)$, $(\Omega^\bullet_{M/Z}, \pi^* dG)$ *and* $(\Omega^\bullet_{M_0/X}, dG)$. *This fact will be used in the next section.*

4. – Cohomology of Koszul and de Rham complex

In this section we relate the cohomology of the Koszul complex $(\Omega^\bullet_{M/Z}, dF)$ and that of the de Rham complex $(\Omega^\bullet_{M/Z}, \nabla_F)$, and as a consequence we complete the proof of Theorem 1.4. We adopt the notations in Section 3.

PROPOSITION 4.1. *We have an isomorphism*

$$H^*(\Omega^\bullet_{M/Z}, \nabla_F) \simeq H^*(\Omega^\bullet_{M/Z}, dF) .$$

Let $[\eta_1], \dots, [\eta_\mu]$ *be the basis of* $H^*(\Omega^\bullet_{M/Z}, dF)$ *given in Theorem 3.3 (4), then the classes* $[\eta_1], \dots, [\eta_\mu]$ *considered as elements in* $H^*(\Omega^\bullet_{M/Z}, \nabla_F)$ *give its* $S(Z)$*-basis.*

Note that we have isomorphisms

$$\begin{aligned} H^*(\Omega^\bullet_{M/Z}, dF) &\simeq H^*(\Omega^\bullet_{M_0/X}, dG) \otimes_{S(X)} S(Z) \ , \\ H^*(\Omega^\bullet_{M/Z}, \nabla_F) &\simeq H^*(\Omega^\bullet_{M_0/X}, \nabla_G) \otimes_{S(X)} S(Z) \end{aligned} \tag{4.1}$$

by virtue of Theorem 3.3 and Remark 3.6. Therefore it will suffice to show the isomorphism of $S(X)$-modules

$$H^*(\Omega^\bullet_{M_0/X}, dG) \simeq H^*(\Omega^\bullet_{M_0/X}, \nabla_G) .$$

For the sake of simplicity of notations, we write $\Omega^\bullet$ instead of $\Omega^\bullet_{M_0/X}$ and ∇ instead of ∇_G, and put

$$C^\bullet_G := (\Omega^\bullet, dG), \quad C^\bullet_\nabla := (\Omega^\bullet, \nabla) .$$

Let $\rho = (1, \dots, n)$ be the weight of the variables $t_1, \dots, t_r$. Recall that a p-form $\eta \in \Omega^p$ is said to be of ρ-degree less than q if η is a sum of quasihomogeneous p-forms of ρ-degree $\le q$, namely, if $\eta = \sum_I a_I dt_I$ with $I = \{i_1, \dots, i_p\} \subset \{1, 2, \dots, r\}$, then there holds for any I:

$$\rho\text{-deg} a_I + \sum_{i_k \in I} i_k \le q .$$

In order to compare the complexes $C^\bullet_G$ and $C^\bullet_\nabla$, we introduce an increasing filtration to these complexes. Put

$$\Omega^p_{\le q} := \{\omega \in \Omega^p ;\ \rho\text{-deg}\omega \le q\} , \quad \text{for } q \in \mathbb{Z} .$$

Let

$$Gr_q \Omega^p = \Omega^p_{\le q} / \Omega^p_{\le q-1}$$

be its associated module. Noting that G is a polynomial of ρ-deg $\le n$ with the nontrivial quasihomogeneous part of ρ-degree n, we introduce subcomplexes of $C^\bullet_G$ and $C^\bullet_\nabla$ by

$$\begin{aligned} \mathcal{F}_q C^\bullet_G &: \ \dots \xrightarrow{dG\wedge} \Omega^p_{\le q+(p-r)n} \xrightarrow{dG\wedge} \Omega^{p+1}_{\le q+(p-r+1)n} \xrightarrow{dG\wedge} \dots \xrightarrow{dG\wedge} \Omega^r_{\le q} \longrightarrow 0 \\ \mathcal{F}_q C^\bullet_\nabla &: \ \dots \xrightarrow{\nabla} \Omega^p_{\le q+(p-r)n} \xrightarrow{\nabla} \Omega^{p+1}_{\le q+(p-r+1)n} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^r_{\le q} \longrightarrow 0 . \end{aligned}$$

These subcomplexes make $C_G^\bullet$ and $C_\nabla^\bullet$ filtered complexes with increasing filtrations $\{\mathcal{F}_q C_G^\bullet\}_{q\in\mathbb{Z}}$ and $\{\mathcal{F}_q C_\nabla^\bullet\}_{q\in\mathbb{Z}}$, respectively.

Put

$$Gr_q C_G^\bullet := \mathcal{F}_q C_G^\bullet / \mathcal{F}_{q-1} C_G^\bullet \ , \quad Gr_q C_\nabla^\bullet := \mathcal{F}_q C_\nabla^\bullet / \mathcal{F}_{q-1} C_\nabla^\bullet \ , \quad q \in \mathbb{Z} \ .$$

The filtrations $\{\mathcal{F}_g C_G^\bullet\}$ and $\{\mathcal{F}_g C_V^\bullet\}$ induce the filtrations for the cohomologies $H^*(C_G^\bullet)$ and $H^*(C_\nabla^\bullet)$. The associated modules are denoted by $Gr_q H^*(C_G^\bullet)$ and $Gr_q H^*(C_\nabla^\bullet)$, respectively. We show

LEMMA 4.2. *For $q \in \mathbb{Z}$, we have*

(1) $Gr_q C_G^\bullet = Gr_q C_\nabla^\bullet$ *for* $q \in \mathbb{Z}$,

(2) $H^p(Gr_q C_G^\bullet) = 0$ *for* $p \neq r$,

(3) $H^p(Gr_q C_\nabla^\bullet) = 0$ *for* $p \neq r$,

(4) $H^p(C_\nabla^\bullet) = 0$ *for* $p \neq r$,

(5) $Gr_q H^r(C_G^\bullet) \simeq H^r(Gr_q C_G^\bullet) \simeq H^r(Gr_q C_\nabla^\bullet) \simeq Gr_q H^r(C_\nabla^\bullet)$.

PROOF. The assertion (1) follows immediately by noticing that, in the expression

$$\nabla\eta = d\eta + dG \wedge \eta \ , \quad \eta \in \Omega^p \ ,$$

the exterior differentiation d preserves the ρ-degree of the form η and the multiplication $dG\wedge$ augment the ρ-degree by n. To show (2), notice that

$$Gr_q C_G^\bullet \simeq (C_g^\bullet(q) \otimes S(X), dg) \ ,$$

where g is the quasihomogeneous part of G of ρ-degree $= n$, and therefore

$$H^p(Gr_q C_G^\bullet) \simeq H^p(C_g^\bullet(q), dg\wedge) \otimes S(X) \ . \tag{4.2}$$

Since g is a quasihomogeneous polynomial in t with the isolated critical point 0 by Proposition 3.1 (1), it follows from Proposition 2.1 (1) and (4.2), we have $H^p(Gr_q C_G^\bullet) = 0$ for $p \neq r$. The assertion (3) is derived from (1) and (2) because

$$H^p(Gr_q C_G^\bullet) \simeq H^p(Gr_q C_\nabla^\bullet) \quad \text{for any } p, q \in \mathbb{Z} \ . \tag{4.3}$$

Next we prove (4) and (5). Let $\{E_m\}$ be the spectral sequence defined by the filtered complex $C_\nabla^\bullet$ with respect to the filtration $\mathcal{F}$. Then the assertion (3) says that

$$E_1^{s,t} = 0 \quad \text{for} \quad s + t \neq r \ , \tag{4.4}$$

therefore the spectral sequence degenerates at E_1-term and we have (5):

$$Gr_q H^p(C_\nabla^\bullet) = E_\infty^{q,p-q} = \cdots = E_1^{q,p-q} = H^p(Gr_q C_\nabla^\bullet) \ . \tag{4.5}$$

In particular if $p \neq r$ we have $Gr_q H^p(C_\nabla^\bullet) = 0$, $q \in Z$ from (3). This proves

$$H^p(C_\nabla^\bullet) = \cdots = \mathcal{F}_0 H^p(C_\nabla^\bullet) = \mathcal{F}_{-1} H^p(C_\nabla^\bullet) = 0 \ .$$

This proves the assertion (4). □

PROOF OF PROPOSITION 4.1. Since the filtrations of $C_G^\bullet$ and $C_\nabla^\bullet$ are regular, those of cohomologies $H^r(C_G^\bullet)$ and $H^r(C_\nabla^\bullet)$ are also regular. Therefore it follows from Lemma 4.2 (5) and Lemma 4.1.1 of [G] that the map $H^r(C_G^\bullet) \ni [\omega] \mapsto [\omega] \in H^r(C_\nabla^\bullet)$ gives the isomorphism of $S(X)$-modules. □

PROOF OF THEOREM 1.4. The vanishing of the cohomology $H^p(\Omega^\bullet_{M/Z}, \nabla_F)$ for $p \neq r$ follows from (4.1) and Lemma 4.2 (4). Moreover we have

$$\begin{aligned} H^r(\Omega^\bullet_{M/Z}, \nabla_F) &\simeq H^r(\Omega^\bullet_{M/Z}, dF) \\ &\simeq H^r(\Omega^\bullet_T, dg) \otimes_{\mathbb{C}} S(Z) \end{aligned}$$

from Theorem 3.3 (3) and Proposition 4.1. □

5. – Discussion

In this section we state a conjecture about a choice of $\mathbb{C}$-basis of the Jacobi ring $\mathbb{C}[t]/(\partial_t g)$ for the quasihomogeneous polynomials

$$g(t) = \theta_n(t_1, \ldots, t_r, 0, \ldots, 0)$$

with the isolated singular point $t = 0$ (see Proposition 3.1 (1)). To state the conjecture we prepare the terminology and notation. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition, namely, a sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0 \ .$$

Put $|\lambda| := \lambda_1 + \cdots + \lambda_p$ and call it the *weight* of λ. Usually we visualize the partition λ as in Figure 1 in the following way. Consider the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq i \leq p$, $1 \leq j \leq \lambda_i$ in the plane. In marking such points we adopt the convention that the first coordinate i increases as one goes downwards, and the second coordinate j increases as one goes from left to right. Then we place a square at each point of the set and we get the diagram as in Figure 1 which is called the *Young diagram* of λ. Put $l(\lambda) := p$ and call it the *length* of λ. By λ' we denote the conjugate of λ, namely the Young diagram obtained by reflection in the main diagonal, see Figure 1. Let $\mathbf{Y}$ denote the set of Young diagrams and put

$$\mathbf{Y}_{r,n-r-1} := \{\lambda \in \mathbf{Y} \ ; \quad l(\lambda) \leq r, \ l(\lambda') \leq n - r - 1\} \ .$$

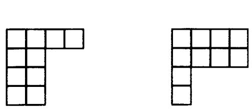

Figure 1. $\lambda = (4, 2, 2, 2)$ and $\lambda' = (4, 4, 1, 1)$

DEFINITION 5.1. *For $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbf{Y}$ such that $l(\lambda) = p \leq r$, the Schur polynomial $s_\lambda(x_1, \dots, x_r)$ is a symmetric polynomial*

$$s_\lambda(x_1, \dots, x_r) = \frac{\begin{vmatrix} x_1^{\lambda_1+r-1} & \dots & x_r^{\lambda_1+r-1} \\ \vdots & & \vdots \\ x_1^{\lambda_{r-1}+1} & \dots & x_r^{\lambda_{r-1}+1} \\ x_1^{\lambda_r} & \dots & x_r^{\lambda_r} \end{vmatrix}}{\begin{vmatrix} x^{r-1} & \dots & x_r^{r-1} \\ \vdots & & \vdots \\ x_1 & \dots & x_r \\ 1 & \dots & 1 \end{vmatrix}}$$

of degree

$$\deg\ s_\lambda(x_1, \dots, x_r) = |\lambda| \ .$$

Here we used the convention $\lambda_{p+1} = \dots = \lambda_r = 0$.

Let $t_i, \dots, t_r$ be the i-th elementary symmetric function of $x_1, \dots, x_r$. Then $s_\lambda(x)$ is represented as a polynomial of t which we denote by $S_\lambda(t)$. If we define the weight of $t = (t_1, \dots, t_r)$ by $\rho = (1, \dots, r)$, then $S_\lambda(t)$ is a quasi-homogeneous polynomial of ρ-deg $S_\lambda(t) = |\lambda|$. Our conjecture is stated as follows.

CONJECTURE. Let $g = \theta_n(t_1, \dots, t_r, 0, \dots, 0)$, then the classes $\{[S_\lambda(t)]\}_{\lambda \in \mathbf{Y}_{r,n-r-1}}$ in $J := \mathbb{C}[t]/(\partial_t g)$ form a $\mathbb{C}$-basis of J.

We shall give an evidence of this conjecture. Let ρ be the weight of $t = (t_1, \dots, t_r)$ as in the preceeding sections. It is known that the Jacobi ring $J = \mathbb{C}[t]/(\partial_t g)$ is a $\mathbb{C}$-vector space of dimension $\mu(g) = \binom{r-1}{n}$ and admits $\mathbb{C}$-basis consisting of monomials. Define the Poincaré polynomial of J by

$$P_g(T) := \sum_i \mu_i T^i, \ \mu_i := \sharp\{\text{monomials of } \rho\text{-deg} = i \text{ which form a basis of } J\}.$$

Then we know [AVG] that

$$P_g(T) = \prod_{i=1}^{r} \frac{T^{n-i} - 1}{T^i - 1} \ .$$

On the other hand, we compute the Poincaré polynomial $P_Y(T)$ of $\mathbf{Y}_{r,n-r-1}$ defined by

$$P_Y(T) := \sum_i \nu_i T^i \ , \ \ \nu_i := \sharp\{\lambda \in \mathbf{Y}_{r,n-r-1} \ ; \ \ |\lambda| = i\} \ .$$

It is known [Mac] that $P_Y(T)$ coincides with $P_g(T)$. Another evidence of the conjecture is given in [K3].