

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

YASUYUKI KACHI

Extremal contractions from 4-dimensional manifolds to 3-folds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 24, n° 1 (1997), p. 63-131

http://www.numdam.org/item?id=ASNSP_1997_4_24_1_63_0

© Scuola Normale Superiore, Pisa, 1997, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Extremal Contractions from 4-Dimensional Manifolds to 3-Folds

YASUYUKI KACHI

0. – Introduction

In the classification theory of higher dimensional algebraic varieties, it is believed that every algebraic variety is transformed, via successive birational maps determined by extremal rays, either to a minimal model or to a model which admits a structure of Mori fiber space, *i.e.* a fiber space determined by an extremal ray. This is the so-called Minimal Model Conjecture (see *e.g.* [18]), which was classical in dimension 2, and was solved affirmatively also in dimension 3 by Mori [30], with [28], [29], Reid [38], Kawamata [12], [13], [14], Shokurov [42]. (See also [45], Kollár-Mori [25], Shokurov [43] and Kawamata [17] for further developments.) In the first place, it was Mori [28] who introduced the notion of extremal ray. He proved the existence of the contraction morphism associated to any extremal ray in the case of smooth projective 3-folds, and completely classified those structures [28]. Kawamata [12] then generalized the existence of the contraction morphism to the case of singular 3-folds which was necessary for the Minimal Model Conjecture. After then Kawamata [13] and Shokurov [42] furthermore generalized it to an arbitrary dimension, even though the Minimal Model Conjecture itself is still unsolved in dimension greater than or equal to 4.

Thus, it is worth trying to investigate the structures of those contractions also for dimension greater than or equal to 4. There are several results known so far in this direction (Ando [1], Beltrametti [5], Fujita [8], Kawamata [15], and Andreatta-Wisniewski [3]). Especially, Kawamata [15] proved the existence of flips from smooth 4-folds, which, together with the Termination theorem [18] after [42], should be considered as the first step toward generalizing the Minimal Model Conjecture to dimension 4.

Our interest is in the extremal contractions from smooth projective 4-folds to 3-folds. More generally, let $g : X \rightarrow Y$ be the contraction of an extremal ray of smooth projective n -folds X such that $\dim Y = n - 1$. For $n = 3$, Mori [28] proved that g is a conic bundle (see Beauville [4], cf. Sarkisov [40], Ando [1]).

Moreover for an arbitrary $n \geq 4$, if we assume that g is equi-dimensional, then Ando [1] proved that g is a conic bundle also in this case. If $n \geq 4$, however, the situation is more complicated, namely, it is no longer true in general that g is equi-dimensional. Actually, even in the case $n = 4$, g may admit a finite number of 2-dimensional fibers (Beltrametti [5] Example 3.6, Mukai, and Reid, we shall give essentially the same example as theirs in Example 11.1). On the other hand, it can be shown that g is still a conic bundle elsewhere, by modifying the argument of [28], [1] (see also Proposition 2.2 below).

The purpose of this paper is to describe the structure of 2-dimensional fibers of $g : X \rightarrow Y$, when $n = 4$. The classification results will be given in Theorems 0.6, 0.7 and 0.8 below. In the course of determining those local structures, we necessarily need the results on other types of contractions from 4-folds, such as flips [15], divisorial contractions [1], [3], and flops.

We put no assumptions on the singularities of Y , although we assume that X is smooth. The general theories (Kawamata-Matsuda-Matsuki [18] and Kollár [19], [20]) tell us that Y has only isolated $\mathbb{Q}$ -factorial rational singularities.

Throughout this paper, we fix the following notation unless otherwise stated:

NOTATION 0.1. Let X be a 4-dimensional smooth projective variety, R an extremal ray of X , and $g : X \rightarrow Y$ the contraction morphism associated to R . Assume that $\dim Y = 3$. Let E be any 2-dimensional fiber of g , $E = \bigcup_{i=1}^n E_i$ the irreducible decomposition of E , and $v_i : \tilde{E}_i \rightarrow E_i$ the normalization of E_i . Let V be a sufficiently small analytic neighborhood of $P := g(E)$ in Y , let $U := g^{-1}(V)$, and $g_U := g|_U : U \rightarrow V$.

To state our main result, we shall prepare some terminologies and notations.

DEFINITION 0.2. Let

$$l_E(R) := \min \{ (-K_X \cdot C) \mid C \text{ is an irreducible rational curve contained in } E \},$$

and call the length of R at E . It is easily seen that $l_E(R) = 1$ or 2 .

Also, for any curve (or 1-cycle) C of X , we call $(-K_X \cdot C)$ the length of C .

DEFINITION 0.3. Under the Notation 0.1, $g_U|_{U-E} : U - E \rightarrow V - P$ is a proper flat morphism whose fiber is 1-dimensional. Furthermore, this is actually a conic bundle by the results of Ando [1]. (See also Proposition 2.2 below). A 1-dimensional closed subscheme C of X is called a limit conic if there is a curve A in (V, P) passing through P such that C is the fiber at P of the 1-parameter family $\{X_t\}_{t \in A}$ induced from $g_U|_{g^{-1}(A)-E} : g^{-1}(A) - E \rightarrow A - P$ by taking the closure (see also Notation 2.1.)

We note that $(-K_X \cdot C) = 2$, since a general fiber ($\simeq \mathbb{P}^1$) of g has length 2. In particular, C satisfies either one of the following:

- $$\left\{ \begin{array}{l} (0.3.1) \ C \text{ is irreducible, generically reduced, and } (-K_X \cdot C) = 2. \\ (0.3.2) \ C \text{ is generically reduced and } C_{\text{red}} = l \cup l' \text{ for irreducible rational} \\ \quad \text{curves } l, l' \subset E \text{ with } (-K_X \cdot l) = (-K_X \cdot l') = 1. \\ (0.3.3) \ C \text{ is not generically reduced, } (-K_X \cdot C) = 2, \text{ and } (-K_X \cdot C_{\text{red}}) = 1. \end{array} \right.$$

DEFINITION 0.4. We define the analytically-local relative cone of curves $\overline{NE}^{\text{an}}(X \supset E / Y \ni P)$ and the analytically-local relative Picard number $\rho^{\text{an}}(X \supset E / Y \ni P)$ of g along E as $\overline{NE}(U/V)$ and $\rho(U/V)$ for a sufficiently small $V \ni P$ and $U := g^{-1}(V)$, respectively ([39], [13], [14], [34]).

Now we come to stating our main results. They consist of four parts: Theorem 0.5 (general properties), Theorems 0.6, 0.7 and 0.8 (classifications).

THEOREM 0.5. Let X be a 4-dimensional smooth projective variety, R an extremal ray of X , and $g : X \rightarrow Y$ the contraction morphism associated to R . Assume that $\dim Y = 3$. Then the following holds:

(1) $|-K_X|$ is g -free.

(2) Let E be any 2-dimensional fiber of g , and let $P := g(E)$. Then

$$\rho^{\text{an}}(X \supset E / Y \ni P) \leq 2.$$

(3) Either one of the following holds;

(CL) (Connected by limit conics) For any two points x, y of E , there exists a limit conic in E which passes through both x and y , or

(MSW) (Mukai-Shepherd-Barron-Wisniewski type) $E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, $E_1 \cap E_2$ is a point, and $N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. $g_U|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle. In this case, for any points $x \in E_1$ and $y \in E_2$, there exists a limit conic in E which passes through both x and y .

In particular, E is rationally chain connected, in either case. (See Definition 0.10).

According to Definition 0.2 and Theorem 0.5 (2), there are exactly three possible types for 2-dimensional fibers E of g :

$$(0.5.1) \quad \begin{cases} \text{Type (A)} : l_E(R) = 2, \\ \text{Type (B)} : l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 2, \text{ or} \\ \text{Type (C)} : l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1. \end{cases}$$

Let us give a classification result for each type of E in the following:

THEOREM 0.6 (For Type (A)). Assume $l_E(R) = 2$. Then

(A) E is irreducible and is isomorphic to $\mathbb{P}^2$, $N_{E/X} \simeq \Omega_{\mathbb{P}^2}^1(1)$, and Y is smooth at P . In particular, $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$, and $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.

Furthermore, the following holds: (local elementary transformation) let U, V , and $g_U : U \rightarrow V$ be as in Notation 0.1. Let $x \in E$ be an arbitrary point.

Then there exists a smooth surface $S_x \subset U$ proper over V such that $g|_{S_x-x} : S_x - x \rightarrow g(S_x) - P$ is an isomorphism, and that $S_x \cap E = \{x\}$ intersecting transversally. Let $\varphi : \overline{U} \rightarrow U$ be the blow-up with center S_x . Then $-K_{\overline{U}}$ is $(g|_U) \circ \varphi$ -ample, and $\varphi^{-1}(E) \simeq \Sigma_1$. Let $\varphi^+ : \overline{U} \rightarrow U^+$ be the contraction

associated to the extremal ray of $\overline{NE}(\overline{U}/V)$ other than φ . Then $U^+ \simeq V \times \mathbb{P}^1$. Let $g^+ : U^+ \rightarrow V$ be the first projection.

Then

$$(g|_U) \circ \varphi = g^+ \circ \varphi^+,$$

and φ^+ is the blow-up with center S^+ which is a smooth surface proper over V such that $S^+ \supset g^{+-1}(P) \simeq \mathbb{P}^1$ with the normal bundle $N_{g^{+-1}(P)/S^+} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$.

THEOREM 0.7 (For Type (B)). Assume that

$$l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 2.$$

Then E is one of the following:

(B-0) (=MSW in Theorem 0.5) (Mukai-Shepherd-Barron-Wisniewski type):
 $E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, $E_1 \cap E_2$ is a point, and $N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

In particular

$$\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \quad (i = 1, 2).$$

(B-1) E is irreducible and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1).$$

(B-2) E is irreducible and is isomorphic to Σ_1 (see Notation 0.14)

$$\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\Sigma_1}(M + 2l)$$

(B-3) $E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $E_1 \cap E_2$ is a line of E_1 and is a ruling of E_2 .

$$\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1), \text{ and } \mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1).$$

(B-4) $E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq \Sigma_1$. $E_1 \cap E_2$ is a line of E_1 and is the negative section of E_2 .

$$\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1), \text{ and } \mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{\Sigma_1}(M + 2l).$$

(B-5) $E = E_1 \cup E_2 \cup E_3$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$ and $E_3 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $E_i \cap E_3$ ($i = 1, 2$) is a line in E_i and a ruling in E_3 . Moreover $E_1 \cap E_2$ is a point.

$$\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \quad (i = 1, 2), \text{ and } \mathcal{O}_{E_3}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1).$$

In (B-0), P is an ordinary double singular point of Y , and $g_U|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle; in (B-1) $\sim$ (B-5), P is a smooth point of Y , and $g_U|_{U-E} : U - E \rightarrow V - P$ is a conic bundle with an irreducible discriminant divisor.

THEOREM 0.8 (For Type (C)). *Assume that*

$$l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

Then E is one of the following:

(C-1) *E is irreducible, $E \simeq \mathbb{P}^2$, and $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$.*

(C-2) *$E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, and $E_1 \cap E_2$ is a line of both E_i .*

$$\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \quad (i = 1, 2).$$

(C-3) *E is irreducible. Let $v : \tilde{E} \rightarrow E$ be the normalization. Then $\tilde{E} \simeq S_m$ and (see Notation 0.14)*

$$v^* \mathcal{O}_E(-K_X) \simeq \mathcal{O}_{S_m}(1).$$

(C-4) *E has two or three irreducible components: $E = E_1 \cup E_2 (\cup E_3)$. Let $v_i : \tilde{E}_i \rightarrow E_i$ be the normalization of E_i ($i = 1, 2, 3$). Then $\tilde{E}_i \simeq S_{m_i}$ for some $m_i \geq 2$, and*

$$v_i^* \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{S_{m_i}}(1).$$

Moreover, let v_i be the vertex of $\tilde{E}_i$, then $v_1(v_1) = v_2(v_2) (= v_3(v_3)) =: Q$ in E . $E_i \cap E_j$ is a ruling of E_i, E_j ($i \neq j$).

(C-5) *$E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq S_2$. $E_1 \cap E_2$ is a line of E_1 and a ruling of E_2 .*

$$\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1), \text{ and } \mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{S_2}(1).$$

In each case (C-1) $\sim$ (C-5), $g_U|_{U-E} : U - E \rightarrow V - P$ is a conic bundle with a non-empty discriminant divisor.

We do not know at present any example for (C-3) with $m \geq 4$, or (C-4) above. Note that if E is of type (C-3) with $m \geq 4$, then E is necessarily non-normal, and similarly for E_i 's of type (C-4).

By the way, as for the property of fibers of extremal contractions in a more general setting, Kawamata showed the following as an application of his adjunction ([16] lemma) and Miyaoka-Mori's theorem ([26]):

REMARK 0.9 (Kawamata [16], Theorem 2). Let X be a projective variety of an arbitrary dimension, and $g : X \rightarrow Y$ the contraction of an extremal ray of X . Then any non-trivial fiber of g is covered by rational curves.

In the context of this direction, we obtain a stronger result (Corollary 0.11 below) when we concentrate on $\dim X \leq 4$, by virtue of [28], [1], [5], [8], [15], [22], [6], [3], together with our Theorem 0.5 (3). To state the result, we recall the notion of rational chain-connectedness and rational connectedness (Kollár-Miyaoka-Mori [22], [23], [24], Campana [6]):

DEFINITION 0.10. A scheme S proper over $\mathbb{C}$ is rationally chain connected if for two arbitrary closed points P, Q of S , there exists finitely many rational curves $C_1, \dots, C_r$ on S such that $\bigcup_{i=1}^r C_i$ is connected, $P \in C_1$, and $Q \in C_r$. S is rationally connected if for two general closed points P, Q of S , there exists an irreducible rational curve on S which passes through both P and Q .

COROLLARY 0.11. Let X be a smooth projective variety with $\dim X \leq 4$, and $g : X \rightarrow Y$ the contraction of an extremal ray of X . Then any non-trivial fiber of g is rationally chain connected.

QUESTION 0.12. Does the same hold when $\dim X$ is arbitrary?

REMARK 0.13. Under the same assumption as in Corollary 0.11, it is not true in general that any irreducible component of any fiber of g is rationally connected, even in the case $\dim X = 4$ (Example 11.9, inspired by Hidaka-Oguiso).

This paper is organized as follows: First in Section 1, we recall the construction of the universal family of extremal rational curves of length 1, following Ionescu [10], which is a generalization of Mori [27]. We shall prove Proposition 1.4, which roughly gives the structure of those $\tilde{E}_i$'s which are covered by rational curves of length 1. The idea is mostly indebted to Kawamata [15].

In Section 2, we recall the construction of the universal family containing a general fiber of g , due to Mori [28]. Especially in Proposition 2.2, we prove that g is a conic bundle outside any 2-dimensional fiber of g , by the argument of Ando [1] after [28]. As a consequence of this construction, we obtain several informations about E . More precisely, we shall prove that E contains no 1-dimensional irreducible component (Proposition 2.4), and that each E_i is a rational surface (Proposition 2.6). These, together with Proposition 1.4, give us a coarse classification of $\tilde{E}_i$'s (Theorem 2.8).

In Section 3, we shall prove Theorem 3.1, which asserts that either (CL) any two points of E are joined by a limit conic, or (MSW) E is of Mukai-Shepherd-Barron-Wisniewski type, as described in Theorem 0.5 (3). We are inspired about this by Campana [6], Kollár-Miyaoka-Mori [22]. The proof will be divided into three parts;

$$\left\{ \begin{array}{l} \rho^{\text{an}}(X \supset E / Y \ni P) = 1 \implies (\text{CL}), \\ \rho^{\text{an}}(X \supset E / Y \ni P) \geq 2, g_U|_{U-E} : U-E \rightarrow V-P \text{ is a } \mathbb{P}^1\text{-bundle} \implies (\text{MSW}), \text{ and} \\ \rho^{\text{an}}(X \supset E / Y \ni P) \geq 2, g_U|_{U-E} : U-E \rightarrow V-P \text{ is not a } \mathbb{P}^1\text{-bundle} \implies (\text{CL}). \end{array} \right.$$

The first part is easier, and the proof will be given in Corollary 3.4. We prove the second part in Theorem 3.5. In this case $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$, and both the extremal rays define flipping contractions (Kawamata [15]). Thus, by virtue of Kawamata's characterization [15], we determine the structure of E as described in (MSW). For the last part, we will give in Theorem 3.7 the structure theorem of the cone $\overline{NE}^{\text{an}}(X \supset E / Y \ni P)$, which gives a one-to-one correspondence between the set of extremal rays of $\overline{NE}^{\text{an}}(X \supset E / Y \ni P)$ and the set of rational

curves of length 1 in E , modulo numerical equivalence in U . Then the last part follows directly from Theorem 3.7 (Corollary 3.8).

In Section 4, we shall prove the relative freeness of $|-K_X|$ (Theorem 4.1). This proof heavily relies on Kawamata's base-point-free technique (*e.g.* [15]). In the course of the proof, we need a sort of property which the direct image sheaf $g_*\mathcal{O}_U(-K_U)$ should satisfy (Lemma 4.3).

So far we prepared most of the tools for classifying E 's, and we then go to the classification from Section 5 on. Firstly in Section 5, we deal with the case $l_{E_i}(R) = 2$ for some i , especially with the Type (A) in (0.5.1). For this case the answer is simple:

E is irreducible and is isomorphic to $\mathbb{P}^2$. The normal bundle $N_{E/X}$ is isomorphic to $\Omega_{\mathbb{P}^2}^1(1)$. P is a smooth point of Y (Theorem 5.1). Firstly by the results of the previous sections, E is irreducible and $\tilde{E} \simeq \mathbb{P}^2$ (Lemma 5.4). To prove the smoothness of E , we consider a blow-up $\bar{U} \supset \bar{E}$ of $U \supset E$. Then $-K_{\bar{U}}$ is still ample over V (Lemma 5.5), and another contraction $\bar{U} \rightarrow U^+$ is found to be a divisorial contraction with $U^+ \simeq V \times \mathbb{P}^1$ (Ando [1]). Consequently $\bar{E}$ is smooth [1], and so is E (Proposition 5.6). The latter half of this section is devoted to determining $N_{E/X}$. To do this we consider $|- \frac{1}{2}K_U|$, and cut out $N_{E/X}$ by its member. Then we find from Mori [28] and Andreatta-Wiśniewski [2] that $N_{E/X}$ is a uniform bundle, and then by Van de Ven's theorem [46], we conclude $N_{E/X} \simeq \Omega_{\mathbb{P}^2}^1(1)$ (Theorem 5.8).

Section 6 is a preparation of later sections. In this section, we recall Kollár-Miyaoka-Mori's technique of glueing chain of rational curves [23], [24] (see Theorem 6.2). It was Andreatta-Wiśniewski [3] who first applied this method to classify 4-dimensional divisorial contractions, and they are very much useful also for our case. We are much indebted to their idea.

In Section 7, we prove $\rho^{\text{an}}(X \supset E/Y \ni P) \leq 2$ (Theorem 7.1). To do this, we consider a pair, say $\{l_1, l_2\}$, of rational curves of length 1 in E which intersects with each other, and belong to distinct extremal rays in $\overline{NE}^{\text{an}}(X \supset E/Y \ni P)$ (Lemma 7.3). Then it can be shown in Lemma 7.7 that l_1, l_2 are two intersecting rulings of the same E_i which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, unless $l_1 + l_2$ deforms outside E . This part essentially needs Kollár-Miyaoka-Mori's glueing technique given in the previous section. So if we assume $\rho^{\text{an}}(X \supset E/Y \ni P) \geq 3$, then E admits another irreducible component which is also isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This contradicts the condition (CL) of Theorem 3.1.

In Section 8, we classify E with $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E/Y \ni P) = 2$, namely, of Type (B) in (0.5.1). The result is given in Theorem 8.1. In this case, our g is factored locally by a birational contraction. Moreover, if $g_U|_{U-E} : U-E \rightarrow V-P$ is a $\mathbb{P}^1$ -bundle, then E is of Mukai-Shepherd-Barron-Wiśniewski type, as in Theorem 3.5. If not, then all the local contractions are of divisorial type. So in this case we can apply the classifications of 4-dimensional divisorial contractions due to Andreatta-Wiśniewski [3] (non-equi-dimensional case) and Ando [1] (equi-dimensional case) to get our classification.

The remaining sections are devoted to the study of E of $l_E(R) = 1$ and

$\rho^{\text{an}}(X \supset E / Y \ni P) = 1$, namely, of Type (C) in (0.5.1). This case is a little more complicated.

Before going to the classification, we first prove in Section 9 that the deformation locus of any rational curve of length 1 is purely codimension 1 (Theorem 9.1) also for the case $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$, as well as Theorem 3.7 for the case $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$ and not of Mukai-Shepherd-Barron-Wiśniewski type. The proof is, however, quite different from that of Theorem 3.7. To be more precise, we follow the argument of Kawamata [15], as follows: we have to exclude the case that E is a union of $\mathbb{P}^2$'s, and $g_U|_{U-E} : U-E \rightarrow V-P$ is a $\mathbb{P}^1$ -bundle (Assumption 9.2). If such case happened, then, by the argument of ([15] (2.4)), together with our Lemma 4.3, the normal bundle of those $\mathbb{P}^2$'s in X would all be isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ (Proposition 9.6). From this we easily obtain a contradiction to the assumption $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$. Kawamata's method used here consists of the theory of vector bundles on $\mathbb{P}^2$ ([46], [9], [35]), and the formal function theory.

In Section 10 we classify E 's of Type (C). The hardest part, in the course of the classification, is to exclude the possibility that some $\tilde{E}_i$ is a geometrically ruled surface (Proposition 10.3). Then we find that E is either an irreducible $\mathbb{P}^2$, a union of two $\mathbb{P}^2$'s, a union of $\mathbb{P}^2$ and S_2 , or a union of some rational cones (Proposition 10.6). In the last case, it will be proved furthermore that the number of irreducible components is at most 3, by considering the flopping contraction from a certain blow-up of U (Proposition 10.7). This is a relative and 4-dimensional analogue of (double) projections developed by Fano-Iskovskikh [11]-Takeuchi [44]. Thus we get the classification as given in Theorem 10.1.

Finally in Section 11, we give some examples.

NOTATION 0.14. Throughout this paper, we shall use the symbols Σ_m , S_m and $\mathcal{O}_{S_m}(1)$ in the following sense: we denote by $\Sigma_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)) \rightarrow \mathbb{P}^1$ the m -th Hirzebruch surface. Its any fiber l is called a *ruling* of Σ_m . If $m \geq 1$, then there is a unique section M with $(M^2) < 0$, called the *minimal section*, or the *negative section*. We denote by S_m the image of $\Phi_{|M+ml|} : \Sigma_m \rightarrow \mathbb{P}^{m+1}$. A *ruling* on S_m is the image of a ruling of Σ_m by Φ . Furthermore, the symbol $\mathcal{O}_{S_m}(1)$ always means the line bundle $\mathcal{O}_{\mathbb{P}^{m+1}}(1) \otimes \mathcal{O}_{S_m}$ on S_m .

Acknowledgements. The author would like to express his sincere gratitude for Professor Y. Kawamata, who encouraged him for a long time and gave him much instruction. He is grateful to Professor S. Mori who suggested this problem to him with various meaningful advices, especially told him Beltrametti, Mukai, Reid's example (Example 11.1), which was the starting point of this work. He is also grateful to Professor S. Mukai who not only invited him to Nagoya University in August, 1993 giving him a special hospitality and entertainment, but also pointed out to him a mistake of the first version of this paper, including

the lack of Mukai-Shepherd-Barron-Wisniewski type, and told him a systematic way of constructing many examples.

He would like to express his hearty gratitude for Professors M. Andreatta and J. Wiśniewski who kindly communicated to him their work of 4-dimensional divisorial contractions. Especially, Professor J. Wiśniewski first suggested to him a possibility of the existence of Mukai-Shepherd-Barron-Wisniewski type, with much encouragement. He is grateful to Professor N.I. Shepherd-Barron who first taught him an example which is neither $\mathbb{P}^2$ nor a quadric surface (Example 11.6), and this suggestion was very much helpful to completing the final step of this paper. He is grateful to Professors F. Hidaka and K. Oguiso who helped him constructing Example 11.9, which gives a counter-example for irreducible special fibers of extremal contractions to be rationally connected.

He is grateful to Professor E. Sato who kindly invited him to Kyushu University in March, 1995 and gave him an opportunity of talking on this work. He is grateful to Professor J. Kollár who gave him helpful suggestions during his stay at University of Utah in April, 1995. He also would like to thank Professors Y. Miyaoka, T. Fujita, T. Ando, A. Corti, K. Matsuki, V. Alexeev and M.Reid for their lots of helpful comments and encouragements.

1. – The universal family of extremal rational curves of length 1 (in the case $l_E(R) = 1$)

ASSUMPTION 1.0. Throughout this section, we assume

$$(1.0.1) \quad l_E(R) = 1.$$

In this section, we recall the construction of the universal family of extremal rational curves of length 1 following Ionescu [10] after Mori [27], and to prove Proposition 1.4 below, which is simple but contains the key idea of this paper. We are inspired these arguments by Kawamata [15].

CONSTRUCTION 1.1 (Ionescu [10] (0.4)). Let l_0 be an irreducible rational curve of length 1 which is contained in E . Let

$$\alpha : \mathbb{P}^1 \rightarrow l_0 \subset X$$

be the composition of the normalization of l_0 and the closed immersion. We consider the deformation of α .

(1.1.0) Let H be any irreducible component containing the point $[\alpha]$ of the Hilbert scheme $\text{Hom}(\mathbb{P}^1, X)$, and $\mathcal{U}_H \rightarrow H$ the corresponding universal family. There is a natural inclusion $\mathcal{U}_H \subset \mathbb{P}^1 \times X \times H$. Let $\mathbb{P}^1 \times X \times H \rightarrow X \times H$ be the projection, and $\text{Im}(\mathcal{U}_H) \subset X \times H$ the image of $\mathcal{U}_H$ in $X \times H$. Consider the projection $\rho : \text{Im}(\mathcal{U}_H) \rightarrow H$. We take the maximal Zariski open set H_0 of

H over which ρ is flat, and let $\eta : H_0 \rightarrow \text{Hilb}_X$ be the induced morphism. Let $T_H \subset \text{Hilb}_X$ be the closure of the image of η , and

$$p_H : S_H \subset X \times T_H \rightarrow T_H$$

the induced universal family over T_H . Note that T_H is irreducible by construction. Let $q_H : S_H \rightarrow X$ be the natural projection. Then by construction,

$$(1.1.1) \quad (-K_X \cdot q_H(p_H^{-1}(t))) = 1$$

for all $t \in T_H$. Let

$$\bar{\eta} : H \dashrightarrow T_H \subset \text{Hilb}_X$$

be the rational map induced from η . Then

(1.1.2) The original l_0 is parametrized by points of T_H which correspond to $[\alpha] \in H$ by this $\bar{\eta}$. Denote such set of points by $T_H(l_0)$.

DEFINITION 1.2. *In the above, let V be any analytic open subset of Y passing through $P := g(E)$, and let $U := g^{-1}(V)$. Obviously $p_H(q_H^{-1}(U)) \supset T_H(l_0)$ (1.1.2). Let $T_H(U \supset l_0)$ be the union of all connected components of $p_H(q_H^{-1}(U))$ which contain at least one point of $T_H(l_0)$. Then we define the whole deformation locus $L_U(l_0)$ of l_0 in U as*

$$(1.2.1) \quad L_U(l_0) := \bigcup_H q_H(p_H^{-1}(T_H(U \supset l_0))),$$

where the union is taken over all irreducible components H of $\text{Hom}(\mathbb{P}^1, X)$ containing $[\alpha]$, as in (1.1.0). Remark that

$$L_X(l_0) = \bigcup_H q_H(S_H), \text{ and } L_U(l_0) \subset L_X(l_0) \cap U.$$

The following inequality is due to Ionescu, which is essentially based on Mori's estimation of the dimension of Hilbert schemes ([27], Proposition 3).

LEMMA 1.3 (Ionescu [10] (0.4) (2),(3)). *If H is chosen to be of maximal dimension among those in (1.1.0), then $\dim T_H \geq 2$. $\square$*

The following is inspired by Kawamata [15]. Actually, it is a slight generalization of ([15], (2.2)).

PROPOSITION 1.4. *Let E_1 be a 2-dimensional irreducible component of E , let $\nu : \tilde{E}_1 \rightarrow E_1$ be the normalization of E_1 , and $\mu : E'_1 \rightarrow \tilde{E}_1$ the minimal resolution of $\tilde{E}_1$. Assume that E_1 is covered by rational curves of length 1 in E_1 . Then E'_1 is isomorphic either to $\mathbb{P}^2$ or a geometrically ruled surface.*

PROOF. Let $p_H : S_H \rightarrow T_H$, and $q_H : S_H \rightarrow X$ be as in Construction 1.1. Then by assumption, for a suitable H , there is a 1-dimensional irreducible closed subset T_1 of T_H such that

$$q_H(p_H^{-1}(T_1)) = E_1.$$

Let us denote for simplicity $T_H = T$, $S_H = S$, $p_H = p$, and $q_H = q$. Let $S_1 := p^{-1}(T_1) \subset S$:

$$q(S_1) = E_1.$$

Moreover, let $\tilde{S}_1, \tilde{T}_1$ be the normalization of S_1, T_1 , respectively, and $\gamma : \tilde{S}_1 \rightarrow \tilde{T}_1$ the induced morphism from $p|_{S_1} : S_1 \rightarrow T_1$. (See the diagram (1.4.3) below). Then we claim that

(1.4.1) γ is a $\mathbb{P}^1$ -bundle. In particular $\tilde{S}_1$ is smooth (cf. [10] or [15]).

In fact, a general fiber of γ is isomorphic to $\mathbb{P}^1$, since F has at most isolated singularities. On the other hand, any fiber of γ is irreducible and generically reduced, since so is p . Furthermore it has no embedded points, since $\tilde{S}_1$ is Cohen-Macaulay and $\tilde{T}_1$ is smooth. Hence any fiber of γ is isomorphic to $\mathbb{P}^1$, namely, γ is a $\mathbb{P}^1$ -bundle, and we have (1.4.1).

Since $q|_{S_1} : S_1 \rightarrow E_1$ is surjective (1.4.0), we have the induced surjective morphism $\beta : \tilde{S}_1 \rightarrow \tilde{E}_1$. Then there is a suitable birational morphism $\delta : S'_1 \rightarrow \tilde{S}_1$ such that S'_1 dominates E'_1 : $\beta' : S'_1 \rightarrow E'_1$. Since a general fiber $l \simeq \mathbb{P}^1$ of $\gamma \circ \delta : S'_1 \rightarrow \tilde{T}_1$ satisfies $(-K_{S'_1} \cdot l) = 2$ (1.4.1), and is sent birationally to $l' := \beta'(l)$, we have

$$(1.4.2) \quad (-K_{E'_1} \cdot l') \geq 2.$$

$$(1.4.3) \quad \begin{array}{ccccc} l' \subset E'_1 & \xrightarrow{\mu} & \tilde{E}_1 & \xrightarrow{\nu} & E_1 \subset X \\ \uparrow & \uparrow \beta' & \uparrow \beta & q|_{S_1} \uparrow & \uparrow q \\ l \subset S'_1 & \xrightarrow[\delta]{} & \tilde{S}_1 & \longrightarrow & S_1 \subset S \\ & & \downarrow \gamma & & \downarrow p \\ & & \tilde{T}_1 & \longrightarrow & T_1 \subset T \end{array}$$

Now assume that E'_1 is isomorphic neither to $\mathbb{P}^2$ nor a geometrically ruled surface, to get a contradiction. Then each extremal ray of $\overline{NE}(E'_1)$ is spanned by a (-1) -curve ([28] Chapter 2). Thus by (1.4.2), there are two (-1) -curves C_1, C_2 (maybe equal) and a pseudo-effective 1-cycle C' such that

$$(1.4.4) \quad l' \equiv C_1 + C_2 + C'.$$

Moreover, $\mu(C_i) \neq 0$ ($i = 1, 2$), since μ is a minimal resolution. Thus by the ampleness of $-\nu^*K_X$ on $\tilde{E}_1$, we have

$$(1.4.5) \quad \begin{cases} (-\mu^*\nu^*K_X \cdot C_i) = (-\nu^*K_X \cdot \mu(C_i)) \geq 1 & (i = 1, 2), \text{ and} \\ (-\mu^*\nu^*K_X \cdot C') = (-\nu^*K_X \cdot \mu(C')) \geq 0. \end{cases}$$

By intersecting $-\mu^*\nu^*K_X$ with the numerical equivalence (1.4.4),

$$(1.4.6) \quad (-\mu^*\nu^*K_X \cdot l') \geq 2.$$

On the other hand, by construction (1.4.3), $\nu \circ \mu(l')$ is parametrized by T_1 ;

$$\nu \circ \mu(l') = q(p^{-1}(t)) \quad (\exists t \in T_1).$$

Thus

$$(1.4.7) \quad (-\mu^*\nu^*K_X \cdot l') = (-K_X \cdot \nu \circ \mu(l')) = 1$$

(1.1.1). This contradicts (1.4.6), and hence the proposition. $\square$

DEFINITION 1.5. Let $p_H : S_H \rightarrow T_H$ and $q_H : S_H \rightarrow X$ be as in Construction 1.1. Then for any $x \in X$, we define

$$T_{H,x} := p_H(q_H^{-1}(x)).$$

Note that $p_H : q_H^{-1}(x) \xrightarrow{\sim} T_{H,x}$.

This gives a subfamily of p_H parametrizing extremal rational curves of length 1 and passing through the point x .

LEMMA 1.6 (Wiśniewski [48], Claim in (1.1)).

$\dim T_{H,x} \leq 1$ for any $x \in E$. $\square$

The following proposition is for the case $\dim T_{H,x} = 1$.

PROPOSITION 1.7. Assume $l_{E_1}(R) = 1$ and $\dim T_{H,x} = 1$ for some H and for some $x \in \text{Reg } E_1 - \bigcup_{j \neq 1} E_j$. (See Definition 1.5). Then $\tilde{E}_1 \simeq \mathbb{P}^2$ and $\nu^*\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$.

PROOF. Choose a 1-dimensional irreducible component T_1 of $T_{H,x}$, and let $\tilde{T}_1$ be its normalization. Then, as in the proof of Proposition 1.4, we have a $\mathbb{P}^1$ -bundle $\gamma : \tilde{S}_1 \rightarrow \tilde{T}_1$ over $\tilde{T}_1$ and a surjective morphism $\beta : \tilde{S}_1 \rightarrow \tilde{E}_1$. (See also the diagram (1.4.3)). Let $\tilde{x} \in \tilde{E}_1$ be such that $\nu(\tilde{x}) = x$. Then $\beta^{-1}(\tilde{x})$ gives a section of γ . Since $\tilde{x}$ is a smooth point of $\tilde{E}_1$, we necessarily have $\tilde{E}_1 \simeq \mathbb{P}^2$. Moreover, if we write

$$\nu^*\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(d) \quad (d \in \mathbb{Z}_{>0}),$$

then $d = 1$ by $l_{E_1}(R) = 1$. $\square$

2. – The universal family of extremal rational curves of length 2

In this section, we shall first recall the construction of the universal family of extremal rational curves of length 2 on X containing a general fiber of g ([28]). We shall prove that the total space of such a family is a birational modification of X (Proposition 2.2). This is a 4-dimensional analogue of [28], (3.24).

As an application, it will be proved that a 2-dimensional fiber E of g never admits a 1-dimensional irreducible component (Proposition 2.4), and that each irreducible component of E is a rational surface (Proposition 2.6). Moreover, in Theorem 2.8, we coarsely classify the normalization of each irreducible component of E .

NOTATION 2.1. Let $f \simeq \mathbb{P}^1$ be a general fiber of g , let $\text{Hilb}_{X,[f]}$ be the unique irreducible component of the Hilbert scheme Hilb_X of X containing the point $[f]$, and let $W := (\text{Hilb}_{X,[f]})_{\text{red}}$. Let $\phi : Z \rightarrow W$ be the induced universal family over W , and $\pi : Z \rightarrow X$ the natural projection. Note that π is surjective, since ϕ parametrizes a general fiber of g .

Let B be the whole set of points of Y which give 2-dimensional fibers of g , and let $Y^\circ := Y - B$, $X^\circ := g^{-1}(Y^\circ)$. Let $E = X_b$ be any 2-dimensional fiber. Then a *limit conic* in E as in Definition 0.3 is nothing but the curve parametrized by a point of $\phi(\pi^{-1}(E))$.

PROPOSITION 2.2. *Under Notation 2.1 above, the following holds:*

- (1) g is a conic bundle on Y° . In particular, Y° is smooth, and $g|_{X^\circ} : X^\circ \rightarrow Y^\circ$ is flat.
- (2) $\dim Z = 4$ and $\pi : Z \rightarrow X$ is a birational morphism onto X , which induces an isomorphism $\pi^{-1}(X^\circ) \xrightarrow{\sim} X^\circ$.
- (3) $\phi|_{\pi^{-1}(X^\circ)} = g|_{X^\circ}$ under the identification $\pi|_{\pi^{-1}(X^\circ)} : \pi^{-1}(X^\circ) \xrightarrow{\sim} X^\circ$.

$$(2.2.0) \quad \begin{array}{ccc} Z & \xrightarrow{\pi} & X \supset X^\circ \\ \phi \downarrow & & \downarrow g \quad \downarrow g|_{X^\circ} \\ W & \dashrightarrow & Y \supset Y^\circ \end{array}$$

PROOF. By the similar argument to Ando ([1], Theorem 3.1), which is a generalization of Mori ([28], (3.25)), g is a conic bundle on Y° . Thus (1) holds.

For (2), (3), we shall follow the argument of ([28], (3.24)). Let f be as in Notation 2.1. First by $N_{f/X} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$, $H^1(N_{f/X}) = 0$. So Hilb_X is smooth at $[f]$ and $\dim W = \dim_{[f]} \text{Hilb}_X = h^0(N_{f/X}) = 3$. Since each fiber of ϕ is of dimension 1, we have

$$(2.2.1) \quad \dim Z = 4.$$

By (1), we regard $g|_{X^\circ} : X^\circ \rightarrow Y^\circ$ as a flat family of closed subschemes of X in a trivial way. Then by the universal property of Hilb_X , we have morphisms $j : Y^\circ \rightarrow W$ and $i : X^\circ \rightarrow Z$ such that the following is a fibered product diagram:

$$(2.2.2) \quad \begin{array}{ccc} X^\circ & \xrightarrow{i} & Z \\ g|_{X^\circ} \downarrow & & \downarrow \phi \\ Y^\circ & \xrightarrow{j} & W \end{array}$$

Then the composite $\pi \circ i : X^\circ \rightarrow X$ obviously coincides with the natural inclusion $X^\circ \subset X$. Hence by (2.2.1), i is an open immersion and π is a birational morphism, and we get (2). Since ϕ is flat, j is also an open immersion (the diagram (2.2.2)), and hence (3). $\square$

REMARK 2.3.

- (1) Let $C_w := \pi(\phi^{-1}(w))$ for each $w \in W$. Then $[C_w] \in R$ and $(-K_X \cdot C_w) = 2$.
- (2) π has connected fibers, and $\text{Exc } \pi$ is purely codimension 1 in Z , by a form of Zariski Main theorem.
- (3) $F := \phi(\pi^{-1}(E))$ is connected. In particular, any two limit conics in E deform inside E to each other. $\square$

PROPOSITION 2.4. *Under the Notation 2.1, let E be a 2-dimensional fiber of g . Then E is purely 2-dimensional.*

PROOF. Assume that E has a 1-dimensional irreducible component, to get a contradiction. Then by the connectedness of E , E has irreducible components l_0 and E_1 such that

$$(2.4.1) \quad \dim l_0 = 1, \quad \dim E_1 = 2, \quad \text{and} \quad l_0 \cap E_1 \neq \emptyset.$$

Let $F := \phi(\pi^{-1}(E))$. First by Proposition 2.2, l_0 is contained in a limit conic (Definition 0.3 or Notation 2.1):

$$(2.4.2) \quad l_0 \subset \pi(\phi^{-1}(w_0)) \quad (\exists w_0 \in F).$$

By Remark 2.3 (3),

(2.4.3) Any two limit conics deform inside E to each other.

Since l_0 forms a whole irreducible component of E , it follows from (2.4.2) and (2.4.3) that any limit conic C contains l_0 , and is of type (0.3.2) in Definition 0.3. That is:

$$(2.4.4) \quad \pi(\phi^{-1}(w)) = l_0 \cup l_w \quad (\exists l_w : \text{rational curve of length 1, } \neq l_0).$$

In particular,

(2.4.5) l_0 is a rational curve of length 1.

Then by Lemma 1.3, l_0 deforms outside E . Let V be a sufficiently small analytic neighborhood of $P = g(E)$ in Y and let $U := g^{-1}(V)$. Then $g|_{U-E} : U - E \rightarrow V - P$ is a conic bundle (Proposition 2.2), and any deformation of l_0 in U , other than l_0 itself, is an irreducible component of a degenerate fiber of $g|_{U-E}$. Thus by the result of Beauville [4] (see also Sarkisov [40] or Ando [1]),

(2.4.6) The whole deformation locus $L := L_U(l_0)$ of l_0 inside U (Definition 1.2) is purely codimension 1.

Take any irreducible component L_1 of L . Then L_1 is a prime divisor with $L_1 \cap E = l_0$, and in particular

(2.4.7) $L_1 \cap E_1 = l_0 \cap E_1$, which is a non-empty finite set.

Since U is a smooth 4-fold, and since $\dim L_1 = 3$, $\dim E_1 = 2$, (2.4.7) is impossible. Hence the result. $\square$

LEMMA 2.5. *Under the Notation 2.1, $\pi(\text{Exc } \pi)$ is exactly the union of all 2-dimensional fibers of g : $\pi(\text{Exc } \pi) = g^{-1}(B)$.*

PROOF. By Proposition 2.2, $\pi(\text{Exc } \pi) \subset g^{-1}(B)$. Assume that $\pi(\text{Exc } \pi) \subsetneq g^{-1}(B)$, to get a contradiction. Then there exist a 2-dimensional fiber E of g and an irreducible component E_i of E such that

$$\dim(\pi(\text{Exc } \pi) \cap E_i) \leq 1.$$

Note that

$$(2.5.0) \quad \dim E_i = 2$$

by Proposition 2.4.

(2.5.1) Let $x \in E_i - \pi(\text{Exc } \pi)$ be a general point.

Then $\pi^{-1}(x)$ is a point, and

(2.5.2) $C^x := \pi \phi^{-1} \phi \pi^{-1}(x)$ is the unique limit conic which passes through x .

Let S be the intersection of two general very ample divisors in X both of which pass through x such that

$$(2.5.3) \quad \dim(S \cap E) = 0 \quad \text{and} \quad (S - x) \cap C^x = \emptyset.$$

Let $\bar{S}$ be the proper transform of S in Z , and let

$$(2.5.4) \quad D := \pi \phi^{-1} \phi(\bar{S}).$$

Since a general fiber of $\phi|_{\bar{S}} : \bar{S} \rightarrow \phi(\bar{S})$ is isomorphic to $\mathbb{P}^1$, $\phi^{-1}\phi(\bar{S})$ is a prime divisor in Z , and hence D is a prime divisor in X . Moreover, since D is disjoint from a general fiber of g (2.5.4), and since g is the contraction of an extremal ray of X , it follows that

$$(2.5.5) \quad D \equiv_0.$$

Note that $D \cap E \neq \emptyset$, since $D \cap E \ni x$. Thus if $D \not\supset E$, then we can find an irreducible curve C in E such that $D \not\supset C$ and $D \cap C \neq \emptyset$, in particular, $(D \cdot C) > 0$, which contradicts (2.5.5). Hence we must have

$$(2.5.6) \quad D \supset E.$$

From this and the definition of D (2.5.4),

(2.5.7) E is covered by some limit conics all of which pass through at least one point of $S \cap E$.

In particular, $S \cap E \supsetneq \{x\}$ by (2.5.2). Since $S \cap E$ is a finite set of points (2.5.3), it follows from (2.5.7), together with (2.5.0), that there exist $y \in S \cap E$ and a 1-parameter family $\{C_t\}_{t \in T}$ of limit conics such that

$$(2.5.8) \quad E_i \subset \bigcup_{t \in T} C_t \quad \text{and} \quad C_t \ni y \quad (\forall t \in T).$$

Since $x \in E_i$ (2.5.1), this implies that

(2.5.9) There is a limit conic which passes through both x and y .

On the other hand, $y \notin C^x$ (2.5.3). This and (2.5.9) contradict (2.5.2). Hence $\pi(\text{Exc } \pi) \supset E$. $\square$

PROPOSITION 2.6. *Let E be any 2-dimensional fiber of g . Then:*

- (1) *Each irreducible component of E is a rational surface.*
- (2) *There exists a finite set of points $\{y_1, \dots, y_r\} \subset E$ such that E is covered by some limit conics all of which pass through at least one of $y_1, \dots, y_r$.*

PROOF. Let $\psi : Z' \rightarrow Z$ be a resolution of the normalization of Z , and let $\Phi := \pi \circ \psi : Z' \rightarrow X$. Since $\pi(\text{Exc } \pi) = E$ (Lemma 2.5 (1)), we have

$$(2.6.1) \quad \Phi(\text{Exc } \Phi) = E.$$

Moreover, by the same reason as in Lemma 2.5 (2),

$$(2.6.2) \quad \text{Exc } \Phi \text{ is purely codimension 1 in } Z'.$$

In particular, for any irreducible component E_i of E and for a general point x of E_i , $\Phi^{-1}(x)$ is purely 1-dimensional. Hence if we take the intersection S of two general very ample divisors in X , then we may assume that

(2.6.3) $\dim(S \cap E) = 0$, and $\Phi^{-1}(S \cap E)$ is purely 1-dimensional.

Since $R^1\Phi_*\mathcal{O}_{Z'} = 0$,

(2.6.4) $\Phi^{-1}(S \cap E)$ is a union of finitely many $\mathbb{P}^1$'s.

(cf. [7]). Moreover, by Bertini's theorem,

(2.6.5) $\Phi^{-1}(S)$ is an irreducible surface in Z' .

By considering the image by $\psi : Z' \rightarrow Z$, it follows from (2.6.4) and (2.6.5) that

(2.6.6) $\pi^{-1}(S \cap E)$ is a union of finitely many rational curves, and

(2.6.7) $\pi^{-1}(S)$ is an irreducible surface in Z containing $\pi^{-1}(S \cap E)$.

Consider $\phi|_{\phi^{-1}\phi\pi^{-1}(S \cap E)} : \phi^{-1}\phi\pi^{-1}(S \cap E) \rightarrow \phi\pi^{-1}(S \cap E)$. Since any fiber of ϕ is a union of rational curves, it follows from (2.6.6) that

(2.6.8) $\phi^{-1}\phi\pi^{-1}(S \cap E)$ is a union of rational surfaces, and so is $\pi\phi^{-1}\phi\pi^{-1}(S \cap E)$.

On the other hand, by (2.6.7), $D := \pi\phi^{-1}\phi\pi^{-1}(S)$ is a prime divisor of X . Since D is disjoint from a general fiber of g ,

(2.6.9) $D \supset E$,

by the same reason as in (2.5.6). Hence

(2.6.10) $E = D \cap E = \pi\phi^{-1}\phi\pi^{-1}(S \cap E)$.

(2.6.8) and (2.6.10) prove (1). (2.6.3) and (2.6.10) prove (2). $\square$

LEMMA 2.7. *Let E be a 2-dimensional fiber of g , let $E = \bigcup E_i$ be the irreducible decomposition, and $v_i : \tilde{E}_i \rightarrow E_i$ the normalization of E_i . Let l be a rational curve of length 1 in E .*

- (1) *If $l_{E_i}(R) = 2$, then $(\tilde{E}_i, v_i^*\mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*
- (2) *If $L_X(l) \subset E$, then $L_X(l)$ is a union of some E_i 's each of which satisfies $(\tilde{E}_i, v_i^*\mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.*
- (3) *If $L_X(l) \not\subset E$, then $L_X(l)$ is purely codimension 1. Each irreducible component of $L_X(l)$ contains E , whenever it meets E .*
- (4) *If $L_X(l) \not\subset E$ for at least one rational curve l of length 1, then the same holds for any such l .*

PROOF. (1) Let $\pi : Z \rightarrow X$ and $\phi : Z \rightarrow W$ be as in Notation 2.1. If we choose a smooth point x of E_i sufficiently general so that $x \notin \bigcup_{j \neq i} E_j$, we have $\dim \pi^{-1}(x) = 1$ by Lemma 2.5 (2). Take an irreducible component W_1 of $\phi(\pi^{-1}(x))$, and let $Z_1 := \phi^{-1}(W_1)$. Moreover let $\tilde{Z}_1, \tilde{W}_1$ be the normalization of Z_1, W_1 , respectively, and $\gamma : \tilde{Z}_1 \rightarrow \tilde{W}_1$ the induced morphism. Then by $l_{E_i}(R) = 2$, each fiber of γ is irreducible and generically reduced. Thus γ is a $\mathbb{P}^1$ -bundle, by the same argument as in (1.4.1). Since x is chosen to be a smooth point of E_i , exactly the same argument as in the proof of Proposition 1.7 proves $\tilde{E}_i \simeq \mathbb{P}^2$. Then $v^* \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ by $l_{E_i}(R) = 2$.

(2) Let $H, T = T_H, S = S_H, p = p_H$ and $q = q_H$ be as in Construction 1.1 such that $\dim H$ is maximal among those in (1.1.0). $\dim T \geq 2$ by Lemma 1.3. By assumption, $q(S) \subset L_X(l) \subset E$. Then for any point $x \in q(S)$, $\dim p(q^{-1}(x)) = \dim q^{-1}(x) \geq 1$, since $\dim S = \dim T + 1 \geq 3$ and $\dim q(S) \leq 2$. Thus the assumption of Proposition 1.7 is satisfied, and it follows that

$$(\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$$

for any E_i such that $E_i \subset L_X(l)$.

(3) By assumption, there is an irreducible component l' of a 1-dimensional fiber X_b of g such that l and l' are parametrized by T_H for a same H (Construction 1.1). Let $L_1 := q_H(S_H)$ for such an H . Since g is a conic bundle near X_b (Proposition 2.2), it follows that L_1 is a prime divisor [4]. In particular, L_1 forms an irreducible component of $L_X(l)$ (1.2.2). Then by the same reason as in (2.5.6),

$$(2.7.1) \quad L_1 \supset E.$$

Next assume that $L_X(l)$ has another irreducible component L_2 with $L_2 \cap E \neq \emptyset$. By (2.7.1), $L_2 \not\subset E$. Thus L_2 contains at least one irreducible curve, say l'' , which is contained in a 1-dimensional fiber of g . Then again by [4], $\dim L_2 = 3$. Combining this and (2.7.1), we get (3).

(4) Let l and l' be two rational curves of length 1 in E and assume that $L_X(l) \not\subset E$. We would like to deduce

$$L_X(l') \not\subset E.$$

In fact, if $L_X(l') \subset E$, then by (2), we have

$$(2.7.2) \quad \begin{cases} l' \subset E_i, \\ (\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), \text{ and} \\ v_i^* l' \text{ is a line in } \tilde{E}_i \end{cases}$$

for some i . On the other hand, $L_X(l) \supset E \supset E_i$ by (3). Since E_i is an irreducible component of a fiber of the contraction g associated to the extremal

ray R and $[l] \in R$, it follows that E_i is actually an union of some rational curves which are all deformations of l inside X . Since $(\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ (2.7.2) and $(-K_X \cdot l) = 1$, $v_i^* l$ is also a line in $\tilde{E}_i$. Again by (2.7.2), we conclude that l and l' deform inside X to each other, in particular, $L_X(l) = L_X(l')$. Since $L_X(l) \not\subset E$ and $L_X(l') \subset E$, we get a contradiction, and (4) is proved. $\square$

THEOREM 2.8. *For each E_i , $(\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X))$ is isomorphic either to one of the following:*

- (a) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$,
- (b) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$,
- (c) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, m))$ ($m \geq 1$),
- (d) $(\Sigma_m, \text{an ample section})$ ($m \geq 1$), or
- (e) $(S_m, \mathcal{O}_{S_m}(1))$ ($m \geq 2$).

In particular, $v_i^ \mathcal{O}_{E_i}(-K_X)$ is globally generated for each i .*

PROOF. We shall divide our proof into cases.

Case (1) Either $l_E(R) = 2$, or $l_E(R) = 1$ and $L_X(l) \subset E$ for each rational curve l of length 1 in E . In this case, if $l_{E_i}(R) = 2$, then by Lemma 2.7 (1), $\tilde{E}_i$ is of type (b). If $l_{E_i}(R) = 1$, then by Lemma 2.7 (2), $\tilde{E}_i$ is of type (a).

Case (2) There is at least one rational curve l of length 1 in E such that $L_X(l) \not\subset E$. In this case, by Lemma 2.7 (3), $L_X(l) \supset E$, and in particular each E_i is covered by rational curves of length 1 in E_i . Hence by Proposition 1.4, $\tilde{E}_i$ is isomorphic either to $\mathbb{P}^2$, a geometrically ruled surface, or a cone obtained by contracting the negative section of a geometrically ruled surface. Moreover, each E_i is a rational surface by Proposition 2.6. Hence $\tilde{E}_i$ is of type either (a), (c), (d), or (e). $\square$

3. – Rational 2-chain connectedness of E

The aim of this section is to prove the following theorem. We are inspired this by Campana [6] and Kollár-Miyaoka-Mori [22].

THEOREM 3.1. *Either one of the following holds;*

- (CL) *(Connected by limit conics) For any two points x, y of E , there exists a limit conic in E which passes through both x and y , or*
- (MSW) *(Mukai-Shepherd-Barron-Wiśniewski type) $E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, $\dim(E_1 \cap E_2) = 0$, and $N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. $g_U|_{U-E} :$*

$U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle. In particular $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$. In this case, for any points $x \in E_1$ and $y \in E_2$, there exists a limit conic in E which passes through both x and y .

(In the case (MSW), we will prove later in Proposition 8.2 that $E_1 \cap E_2$ is actually a single point).

NOTATION 3.2. Let E_i be any irreducible component of E , and x a sufficiently general point of E_i . Let $S_x \ni x$ be the connected component of the intersection of two general g_U -very ample divisors in U both of which pass through x , so that

$$(3.2.1) \quad \begin{cases} S_x \text{ is a smooth surface proper over } V, \text{ and} \\ S_x \cap E = \{x\} \text{ intersecting transversally.} \end{cases}$$

Let $\pi : Z \rightarrow X$ and $\phi : Z \rightarrow W$ be as in Notation 2.1, and let

$$(3.2.2) \quad \begin{cases} Z_U := \pi^{-1}(U), \quad W_U := \phi(\pi^{-1}(U)), \\ \pi_U := \pi|_{Z_U} : Z_U \rightarrow U, \quad \text{and} \quad \phi_U := \phi|_{Z_U} : Z_U \rightarrow W_U. \end{cases}$$

Moreover let

$$(3.2.3) \quad D_x := \pi_U \phi_U^{-1} \phi_U \pi_U^{-1}(S_x).$$

This is disjoint from a general fiber of $g_U : U \rightarrow V$.

We remark that $\pi_U^{-1}(S_x)$ is an irreducible surface in Z_U by Bertini's theorem, as in (2.6.5). Hence (3.2.4) D_x (3.2.3) is a prime divisor in U .

PROPOSITION 3.3. *Under the Notation 3.2, assume that*

$$D_x \equiv 0 \text{ in } U.$$

Then x and any point of E are contained in a same limit conic.

PROOF. If $D_x \not\supset E$, then there is an irreducible curve C in E such that $D_x \not\supset C$ and $D_x \cap C \neq \emptyset$, in particular $(D_x \cdot C) > 0$, which contradicts the assumption $D_x \equiv 0$. Hence

$$(3.3.1) \quad D_x \supset E.$$

By the definition of D_x (3.2.3), D_x is a union of curves which are parametrized by ϕ and intersect with S_x . Since $S_x \cap E = \{x\}$ (3.2.1), $D_x \cap E$ is a union of limit conics which pass through x . Thus (3.3.1) implies that x and any point of E are contained in a same limit conic. $\square$

COROLLARY 3.4. *Assume*

$$\rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

Then any two points of E are contained in a same limit conic in E .

PROOF. Let x be as in Notation 3.2. Then the assumption $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$ implies $D_x \equiv 0$, since D_x is disjoint from a general fiber of $g_U : U \rightarrow V$. Hence by Proposition 3.3,

(3.4.1) x and any point of E are joined by a limit conic.

Since x is chosen to be a general point of an arbitrary irreducible component E_i of E , and since the property of being a limit conic is preserved under degenerations, the result follows. $\square$

Next we shall prove Theorem 3.1 also in the case $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$.

THEOREM 3.5. *Assume that*

$$\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2 \text{ and } g_U|_{U-E} : U - E \rightarrow V - \mathbb{P} \text{ is a } \mathbb{P}^1\text{-bundle.}$$

Then E satisfies the following:

- (1) $E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, $\dim(E_1 \cap E_2) = 0$, and $N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.
- (2) For any point x of E_1 and any point y of E_2 , there exists a limit conic which passes through both x and y .
- (3) (Structure of the cone $\overline{NE}^{\text{an}}(X \supset E / Y \ni P) - I$):
 $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$.

Let $\{R_1, R_2\}$ be the set of extremal rays of $\overline{NE}(U/V)$. Then R_i defines the flipping contraction [15] which contracts E_i to a point ($i = 1, 2$).

PROOF. First by the assumption, any rational curve l of length 1 in E (if exists) satisfies $L_X(l) \subset E$. Hence by Lemma 2.7 (1) and (2),

$$(3.5.0) \quad \tilde{E}_j \simeq \mathbb{P}^2 \quad (\forall j),$$

regardless of $l_{E_j}(R) = 1$ or 2. Consider the cone of curves $\overline{NE}(U/V)$. This is spanned by finitely many extremal rays, since $-K_U$ is g_U -ample. In particular

(3.5.1) $\overline{NE}(U/V)$ has at least two extremal rays.

Choose any extremal ray of $\overline{NE}(U/V)$ and let $h : U \rightarrow U_h$ be the associated contraction. Then we claim that

(3.5.2) h never contracts a general fiber of $g_U : U \rightarrow V$.

In fact, if h contracts a general fiber of g_U , then obviously h contracts all limit conics in E . Hence by Proposition 2.6 (2), $h(E)$ must be a point, and h coincides with g_U , a contradiction to our assumption $\rho(U/V) \geq 2$. Hence (3.5.2). Since $g_U|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle by the assumption, (3.5.2) implies that

(3.5.3) h is a flipping contraction with $\text{Exc } h \subset E$.

Then by Kawamata [15],

(3.5.4) h contracts an unique irreducible component E_i of E such that

$$E_i \simeq \mathbb{P}^2 \text{ and } N_{E_i/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}.$$

From (3.5.1) and (3.5.4), there are at least two irreducible components, say E_1 and E_2 , of E such that

$$(3.5.5) \quad E_1 \simeq E_2 \simeq \mathbb{P}^2, \text{ and } N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}.$$

Let h_i ($i = 1, 2$) be the flipping contraction which contracts E_i to a point.

Assume that E has at least three irreducible components, to get a contradiction. Let E_3 be any other one. Since $\tilde{E}_3 \simeq \mathbb{P}^2$ (3.5.0), h_i never contracts any curve in E_j ($i \in \{1, 2\}, j \in \{1, 2, 3\} - \{i\}$). Thus

$$(3.5.6) \quad \dim(E_i \cap E_j) \leq 0 \quad (\forall i, j \in \{1, 2, 3\}, i \neq j).$$

This contradicts the fact that any two limit conics in E deform inside E to each other (Remark 2.3 (3)). Hence

$$(3.5.7) \quad E = E_1 \cup E_2.$$

Moreover by the same reason as in (3.5.6), we have

$$(3.5.8) \quad \dim(E_1 \cap E_2) = 0.$$

(3.5.5), (3.5.7) and (3.5.8) prove the theorem. $\square$

Next we investigate the structure of the cone $\overline{NE}^{\text{an}}(X \supset E / Y \ni P)$ in the rest case, i.e. the case $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$ and $g_U|_{U-E} : U - E \rightarrow V - P$ is not a $\mathbb{P}^1$ -bundle.

NOTATION 3.6. Till the end of this section, for any curve l which is contained in a fiber of g_U , the symbol $[l]$ means the numerical equivalence class of l in U , not in X : $[l] \in \overline{NE}(U/V)$.

THEOREM 3.7 (Structure of the cone $\overline{NE}^{\text{an}}(X \supset E / Y \ni P) - II$). Assume that

$$\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2 \text{ and } g_U|_{U-E} : U - E \rightarrow V - P \text{ is not a } \mathbb{P}^1\text{-bundle}.$$

Then the following holds:

(1) For each extremal ray R_k of $\overline{NE}(U/V)$, there exists a rational curve l_k of length 1 in E such that $R_k = \mathbb{R}_{\geq 0}[l_k]$. Moreover, R_k defines a divisorial contraction $\varphi_k : U \rightarrow U'_k$ which contracts an irreducible divisor D_k to a surface, and $(D_k \cdot l_k) = -1$. Set theoretically, $D_k = L_U(l_k)$.

(2) Conversely, for each rational curve l of length 1 in E , $\mathbb{R}_{\geq 0}[l]$ forms an extremal ray of $\overline{NE}(U/V)$. If we regard $L_U(l)$ as a reduced divisor, then $-K_U + L_U(l)$ is a supporting divisor of $\mathbb{R}_{\geq 0}[l]$.

PROOF. First by the assumption and Lemma 2.7 (4),

(3.7.0) $L_U(l) \not\subset E$ for any rational curve l of length 1 in E .

(1) Take an arbitrary extremal ray R_k of $\overline{NE}(U/V)$ and let φ_k be the associated contraction. First we claim:

(3.7.1) φ_k is a divisorial contraction.

In fact, if φ_k is of fiber type, then by Proposition 2.6 (2), it must coincide with g_U , which is absurd. Thus φ_k is a birational contraction. Assume that φ_k is a flipping type contraction. Then by [15], φ_k contracts an irreducible component $E_i \simeq \mathbb{P}^2$ of E to a point, with a line l of E_i being of length 1. Since φ_k is an isomorphism outside E_i , l never deforms outside E , which contradicts (3.7.0). Hence (3.7.1).

Moreover, since any divisor of U is sent by g_U either to V itself or to a divisor on V , it follows that

(3.7.2) φ_k contracts an irreducible divisor D_k to a surface.

Next, consider $\varphi_k|_{D_k - D_k \cap E} : D_k - D_k \cap E \rightarrow \varphi_k(D_k - D_k \cap E)$. Since $g|_{U-E} : U - E \rightarrow V - P$ is a conic bundle (Proposition 2.2) and φ_k never contracts a whole fiber of g_U as above,

(3.7.3) $\varphi_k|_{D_k - D_k \cap E}$ is a $\mathbb{P}^1$ -bundle whose fiber l_k satisfies

$$N_{l_k/U} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \text{ and } (-K_U \cdot l_k) = 1.$$

Moreover for such l_k , consider the exact sequence:

$$0 \longrightarrow N_{l_k/D_k} \longrightarrow N_{l_k/U} \longrightarrow \mathcal{O}_{D_k}(D_k) \otimes \mathcal{O}_{l_k} \longrightarrow 0.$$

Since $\deg N_{l_k/D_k} = 0$ and $\deg N_{l_k/U} = -1$ (3.7.3),

$$(3.7.4) \quad (D_k \cdot l_k) = -1.$$

By considering a degeneration of l_k 's into E , it follows that R_k is spanned by a rational curve of length 1 which is actually contained in E . Hence we get (1).

(2) Assume that there exists a rational curve l of length 1 in E such that $\mathbb{R}_{\geq 0}[l]$ is not an extremal ray of $\overline{NE}(U/V)$, and shall derive a contradiction. By (3.7.0), $L_U(l)$ is purely codimension 1. So we regard this as a reduced divisor on U and denote by L ;

$$L := L_U(l).$$

Then we claim:

(3.7.5) $-K_U + L$ is g_U -ample.

In fact by (1), any extremal ray of $\overline{NE}(U/V)$ is written as $R_k = \mathbb{R}_{\geq 0}[l_k]$ for a certain rational curve l_k of length 1 in E . Then again by (3.7.0), l_k deforms to a rational curve l'_k which is an irreducible component of a degenerate fiber of the conic bundle $g|_{U-E} : U - E \rightarrow V - P$. If $L \supset l'_k$, then obviously $l \equiv_{g_U} l'_k$, which contradicts the assumption that $\mathbb{R}_{\geq 0}[l]$ is not an extremal ray of $\overline{NE}(U/V)$. Thus $L \not\supset l'_k$, in particular $(L \cdot l_k) \geq 0$. On the other hand, $(-K_U \cdot l_k) > 0$ for each k , since $-K_U$ is g_U -ample. Thus $(-K_U + L \cdot l_k) > 0$ for each k . This means that $-K_U + L$ is g_U -ample, since $\overline{NE}(U/V) = \sum \mathbb{R}_{\geq 0}[l_k]$, and (3.7.5) is proved.

On the other hand,

$$(3.7.6) \quad (L \cdot l) = -1.$$

In fact, let l' be a deformation of l which is not contained in E (3.7.0), and is contained in a unique irreducible component, say L_1 , of L . Then l' is a fiber of the $\mathbb{P}^1$ -bundle $g|_{L_1-L_1 \cap E} : L_1 - L_1 \cap E \rightarrow g(L_1 - L_1 \cap E)$. Thus from

$$0 \longrightarrow N_{l'/L_1} \longrightarrow N_{l'/U} \longrightarrow \mathcal{O}_{L_1}(L_1) \otimes \mathcal{O}_{l'} \longrightarrow 0$$

we get (3.7.6).

Since $(-K_U \cdot l') = 1$, (3.7.6) says that $(-K_U + L \cdot l') = 0$, which contradicts the above claim, and hence (2). $\square$

Now we come to the proof of Theorem 3.1 in the case $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$ and $g_U|_{U-E} : U - E \rightarrow V - P$ is not a $\mathbb{P}^1$ -bundle.

COROLLARY 3.8. *Assume that*

$$\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2 \text{ and } g_U|_{U-E} : U - E \rightarrow V - P \text{ is not a } \mathbb{P}^1\text{-bundle.}$$

Then any two points of E are contained in a same limit conic.

PROOF. Let E_i be any irreducible component of E , and x a general smooth point of E_i . Let $S_x \ni x$ be as in Notation 3.2, in particular be satisfying the condition (3.2.1). Let $\Delta^\circ \subset V - P$ be the discriminant locus of the conic bundle $g|_{U-E} : U - E \rightarrow V - P$ (Proposition 2.2), $\Delta := \Delta^\circ \cup \{P\}$, and $\Delta = \bigcup_j \Delta_j$ the irreducible decomposition. Note that

$$(3.8.0) \quad \Delta^\circ \neq \emptyset$$

by the assumption. Then

$$(3.8.1) \quad \dim \Delta_j = 2 \quad (\forall j)$$

[4]. Hence by the definition of S_x ,

$$(3.8.2) \quad g(S_x) \not\supset \Delta_j \quad (\forall j).$$

Let D_x be as in (3.2.3). Recall that

(3.8.3) D_x is a prime divisor in U (3.2.4).

Take any extremal ray R_k of $\overline{NE}(U/V)$. This is spanned by a rational curve l of length 1 in U (Theorem 3.7). Consider the whole deformation locus $L_U(l)$ of l in U . Then by Theorem 3.7, $g(L_U(l))$ coincides with some Δ_j . Thus by (3.8.2), we can choose a deformation of l which is disjoint from D_x , in particular $(D_x \cdot l) = 0$. Since $\overline{NE}(U/V)$ is spanned by extremal rays, it follows that

$$(3.8.4) \quad D_x \equiv 0 \quad \text{in } U.$$

Thus the assumption of Proposition 3.3 is satisfied, and hence

(3.8.5) x and any point of E are joined by a limit conic.

Since x is chosen to be a general point of an arbitrary irreducible component E_i of E , and since the property of being a limit conic is preserved under degenerations, the result follows. $\square$

3.9. PROOF OF THEOREM 3.1. Corollary 3.4, Theorem 3.5 and Corollary 3.8 immediately imply Theorem 3.1. $\square$

4. – Relative base-point-freeness of $|-K_X|$

In this section, we shall prove the following theorem, whose proof heavily relies on Kawamata's base-point-free technique, especially ([15], (2.3)):

THEOREM 4.1. *The linear system $|-K_X|$ is g -free. Namely, the natural homomorphism*

$$\rho : g^* g_* \mathcal{O}_X(-K_X) \rightarrow \mathcal{O}_X(-K_X)$$

is surjective.

DEFINITION 4.2. *Let $B \subset Y$ be the whole set of points which give 2-dimensional fibers of g , as in the Notation 2.1. Since g is a conic bundle over $Y - B$ (Proposition 2.2), the above ρ is surjective on $g^{-1}(Y - B)$. Thus we concentrate on a neighborhood $U = g^{-1}(V)$ of any 2-dimensional fiber E of g . We shall prove that $\rho|_U$ is surjective.*

For any torsion-free $\mathcal{O}_V$ -module $\mathcal{G}$, and for any section $v \in \mathcal{G}$, we define $Z_{\mathcal{G}}(v)$, the zero locus of v in $\mathcal{G}$, as the closed analytic subspace of V defined by the image ideal of the $\mathcal{O}_V$ -homomorphism:

$$^a v : \mathcal{G}^\vee \rightarrow \mathcal{O}_V, \quad \mathcal{G}^\vee \ni \varphi \mapsto \varphi(v) \in \mathcal{O}_V.$$

Moreover let

$$B_G := \{Q \in V \mid \mathcal{G}_Q \text{ is not a free } \mathcal{O}_{V,Q}\text{-module}\}.$$

Note that

$$(4.2.1) \quad \dim B_G \leq 1, \text{ and } \{Q \in V \mid \mathrm{Tor}_1^{\mathcal{O}_{V,Q}}(\mathcal{G}_Q, \mathbb{C}(Q)) \neq 0\} \subset B,$$

where $\mathbb{C}(Q) = \mathcal{O}_{V,Q}/\mathfrak{m}_Q$ is the residue field of $\mathcal{O}_{V,Q}$. If $Q \in V - B_G$, then $Q \in Z_G(v)$ if and only if $v(Q) = 0$ in $\mathbb{C}(Q)$.

LEMMA 4.3 (communicated by Y. Kawamata and N. Takahashi). For any irreducible member $D \in |-K_U|$, there exists another $D' \in |-K_U|$ such that

- (1) D and D' give the same element of $g_{U*}\mathcal{O}_U(-K_U) \otimes \mathbb{C}(P)$, and
- (2) D' never contains a whole fiber of $g_U|_{U-E} : U-E \rightarrow V-P$, as analytic subspaces of U .

PROOF. Let $D \in |-K_U|$ be any irreducible member. If the condition (2) of this lemma fails for $D' = D$, then from D we shall find another $D' \in |-K_U|$ which satisfies both (1) and (2).

Let $\mathcal{F}_3 := (g_U)_*\mathcal{O}_U(-K_U)$. Since $g_U|_{U-E} : U-E \rightarrow V-P$ is a conic bundle (Proposition 2.2), it follows that $\dim H^0(U_Q, \mathcal{O}_U(-K_U) \otimes \mathcal{O}_{U_Q}) = 3$ for any $Q \in V-P$, and thus $\mathcal{F}_3$ is a reflexive sheaf of rank 3 on V which is locally free outside P . Let $s \in \mathcal{F}_3$ be the section corresponding to D . Then

(4.3.0) For any $Q \in V-P$, $Q \in Z_{\mathcal{F}_3}(s)$ is equivalent to $D \supset U_Q$ as analytic subspaces of U .

In particular, $Z_{\mathcal{F}_3}(s)$ is at most 1-dimensional, since $g_U|_D : D \rightarrow V$ is generically finite, and since D is irreducible.

If $\dim Z_{\mathcal{F}_3}(s) = 0$, then by (4.3.0) there is nothing to prove, so assume $\dim Z_{\mathcal{F}_3}(s) = 1$. By (4.3.0) again,

(4.3.1) For the lemma, it is enough to show the existence of $t \in \mathfrak{m}_P \mathcal{F}_3$ such that $Z_{\mathcal{F}_3}(s+t) \subset \{P\}$.

Take first of all a saturated filtration

$$(4.3.2) \quad \begin{cases} 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 & \text{with } \mathrm{rk} \mathcal{F}_i = i \text{ such that} \\ s \notin \mathcal{F}_2, \text{ and} \\ Z_{\mathcal{F}_3}(s) \cap B \subset \{P\}, \end{cases}$$

where $B := B_{\mathcal{F}_3/\mathcal{F}_1}$, as defined in Definition 4.2. Note that

$$(4.3.3) \quad \dim B \leq 1 \text{ and } \{Q \in V \mid \mathrm{Tor}_1^{\mathcal{O}_{V,Q}}((\mathcal{F}_3/\mathcal{F}_1)_Q, \mathbb{C}(Q)) \neq 0\} \subset B$$

(4.2.1). Then we claim:

CLAIM 1. *There exists $t \in \mathfrak{m}_P(\mathcal{F}_2/\mathcal{F}_1)$ such that for any $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| < 1$,*

$$\dim Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \leq 1.$$

PROOF. If $\dim Z_{\mathcal{F}_3/\mathcal{F}_1}(s) \leq 1$, then the claim trivially holds by letting $t = 0$. Assume $\dim Z_{\mathcal{F}_3/\mathcal{F}_1}(s) = 2$. First we take $t \in \mathfrak{m}_P(\mathcal{F}_2/\mathcal{F}_1)$ so that

$$(4.3.4) \quad \dim(Z_{\mathcal{F}_2/\mathcal{F}_1}(t) \cap Z_{\mathcal{F}_3/\mathcal{F}_1}(s)) = 1.$$

Let $B' := B_{\mathcal{F}_3/\mathcal{F}_2}$. Then

$$\dim B' \leq 1 \text{ and } \{Q \in V \mid \text{Tor}_1^{\mathcal{O}_V, \mathcal{Q}}((\mathcal{F}_3/\mathcal{F}_2)_Q, \mathbb{C}(Q)) \neq 0\} \subset B'$$

(4.2.1). In particular,

$$Z_{\mathcal{F}_2/\mathcal{F}_1}(t) \subset Z_{\mathcal{F}_3/\mathcal{F}_1}(t) \subset Z_{\mathcal{F}_2/\mathcal{F}_1}(t) \cup B'.$$

From this and (4.3.4),

$$(4.3.5) \quad \dim(Z_{\mathcal{F}_3/\mathcal{F}_1}(t) \cap Z_{\mathcal{F}_3/\mathcal{F}_1}(s)) = 1.$$

Second, for any $\varepsilon \in \mathbb{C}$,

$$(4.3.6) \quad Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \subset Z_{\mathcal{F}_3/\mathcal{F}_2}(s + \varepsilon t) = Z_{\mathcal{F}_3/\mathcal{F}_2}(s)$$

by $t \equiv 0$ in $\mathcal{F}_3/\mathcal{F}_2$. Let $\{V_k\}_k$ be the set of all 2-dimensional irreducible subspaces V_k of V passing through P such that

$$V_k \not\subset Z_{\mathcal{F}_3/\mathcal{F}_1}(s) \text{ and } V_k \subset Z_{\mathcal{F}_3/\mathcal{F}_2}(s).$$

In particular,

$$(4.3.7) \quad Z_{\mathcal{F}_3/\mathcal{F}_2}(s) = \bigcup_k V_k \cup Z_{\mathcal{F}_3/\mathcal{F}_1}(s) \cup (\text{one dimensional subspaces of } V).$$

Let

$$r_k := \inf\{r \in \mathbb{R}_{>0} \mid V_k \subset Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \text{ for some } \varepsilon \in \mathbb{C} - \{0\} \text{ with } |\varepsilon| = r\}.$$

Then obviously $r_k > 0$ for each k , and for any $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| < \min_k r_k$, we have

$$(4.3.8) \quad Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \not\supset V_k \text{ for each } k.$$

Thus

$$\begin{aligned} Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) &= Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \cap Z_{\mathcal{F}_3/\mathcal{F}_2}(s) && \text{(by (4.3.6))} \\ &= \bigcup_k (Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \cap V_k) \cup (Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \cap Z_{\mathcal{F}_3/\mathcal{F}_1}(s)) \\ &\quad \cup (Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \cap (\text{one dimensional subspace of } V)) && \text{(by (4.3.7))} \\ &= \bigcup_k (Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) \cap V_k) \cup ((Z_{\mathcal{F}_3/\mathcal{F}_1}(t) \cap Z_{\mathcal{F}_3/\mathcal{F}_1}(s)) \\ &\quad \cup (\text{one dimensional subspace of } V)), \end{aligned}$$

which is of dimension at most 1 (4.3.5), (4.3.8). Hence we get Claim 1. $\square$

Next we prove the following:

CLAIM 2. *Let $t \in \mathfrak{m}_P(\mathcal{F}_2/\mathcal{F}_1)$ be as given in Claim 1 above. Let $t' \in \mathfrak{m}_P\mathcal{F}_2$ be an arbitrary lifting of t . Then for any $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| < 1$, there is a suitable $u_\varepsilon \in \mathfrak{m}_P\mathcal{F}_1$ such that for any $\eta \in \mathbb{C}$ with $0 < |\eta| < 1$,*

$$Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u_\varepsilon) \subset B,$$

where $B = B_{\mathcal{F}_3/\mathcal{F}_1}$ as in (4.3.2).

PROOF. Since

$$Z_{\mathcal{F}_3}(s + \varepsilon t') \subset Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t),$$

we have $\dim Z_{\mathcal{F}_3}(s + \varepsilon t') \leq 1$ by Claim 1. If $Z_{\mathcal{F}_3}(s + \varepsilon t') \subset B$, then there is nothing to prove. Assume $Z_{\mathcal{F}_3}(s + \varepsilon t') \not\subset B$. First we can take $u = u_\varepsilon \in \mathfrak{m}_P\mathcal{F}_1$, depending on the value ε , so that

$$(4.3.9) \quad Z_{\mathcal{F}_1}(u) \cap Z_{\mathcal{F}_3}(s + \varepsilon t') = \{P\}.$$

(Recall that $\mathcal{F}_1$ is a reflexive sheaf of rank 1 on V). Since $Z_{\mathcal{F}_3}(u) \subset Z_{\mathcal{F}_1}(u) \cup B$ (4.3.3), we have

$$(4.3.10) \quad Z_{\mathcal{F}_3}(u) \cap Z_{\mathcal{F}_3}(s + \varepsilon t') \subset B$$

by (4.3.9). Second, for any $\eta \in \mathbb{C}$,

$$(4.3.11) \quad Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \subset Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t' + \eta u) = Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t)$$

by $u \equiv 0$ in $\mathcal{F}_3/\mathcal{F}_1$. Let $\{W_k\}_k$ be the set of all 1-dimensional irreducible subspaces of V passing through P such that

$$W_k \not\subset Z_{\mathcal{F}_3}(s + \varepsilon t') \quad \text{and} \quad W_k \subset Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t),$$

and in particular

$$(4.3.12) \quad Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) = \bigcup_k W_k \cup Z_{\mathcal{F}_3}(s + \varepsilon t').$$

Let

$$q_k := \inf\{q \in \mathbb{R}_{>0} \mid W_k \subset Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \text{ for some } \eta \in \mathbb{C} - \{0\} \text{ with } |\eta| = q\}.$$

Then obviously $q_k > 0$ for each k , and for any $\eta \in \mathbb{C}$ with $0 < \eta < \min_k q_k$, we have

$$(4.3.13) \quad Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \cap W_k = \{P\} \quad \text{for each } k.$$

Thus

$$\begin{aligned}
 Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) &= Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \cap Z_{\mathcal{F}_3/\mathcal{F}_1}(s + \varepsilon t) && \text{(by (4.3.11))} \\
 &= \bigcup_k (Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \cap W_k) \\
 &\quad \cup (Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \cap Z_{\mathcal{F}_3}(s + \varepsilon t')) && \text{(by (4.3.12))} \\
 &= \bigcup_k (Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u) \cap W_k) \\
 &\quad \cup ((Z_{\mathcal{F}_3}(s + \varepsilon t') \cap Z_{\mathcal{F}_3}(u)) \subset B, && \text{(by (4.3.10), (4.3.13))}
 \end{aligned}$$

and the Claim 2 is proved. $\square$

Finally, if we choose a sufficiently small $\varepsilon > 0$, and then choose a sufficiently small $\eta > 0$, $Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u_\varepsilon) \cap B \subset \{P\}$ by (4.3.2). From this together with Claim 2 above, we actually have $Z_{\mathcal{F}_3}(s + \varepsilon t' + \eta u_\varepsilon) \subset \{P\}$. Hence we get (4.3.1), and the proof of Lemma 4.3 is completed. $\square$

From now on, assume that the cokernel $\text{Bs } |-K_U|$ of $\rho|_U$ is nonempty so as to derive a contradiction.

LEMMA 4.4. *A general member D of $|-K_U|$ has at most canonical singularities.*

PROOF. We will follow the argument of the proof of the base-point-free theorem originated by Kawamata.

We take $\varphi : U' \rightarrow U$, a resolution of the base locus $\text{Bs } |-K_U|$ of $|-K_U|$ such that

$$(4.4.1) \quad \begin{cases} \varphi^*|-K_U| = |D'| + \sum r_i G_i, \\ K_{U'} = \varphi^* K_U + \sum a_i G_i, \\ -\varphi^* K_U - \sum \delta_i G_i \text{ is } g \circ \varphi\text{-ample,} \end{cases}$$

where $\sum_{i=1}^N G_i$ is a simple normal crossing divisor on U' , D' is the proper transform of D such that

$$(4.4.2) \quad \text{Bs } |D'| = \emptyset,$$

$r_i, a_i \in \mathbb{Z}_{\geq 0}$ with $(r_i, a_i) \neq (0, 0)$, and $\delta_i \in \mathbb{Q}_{>0}$ with $0 < \delta_i < 1$. Note that

$$(4.4.3) \quad \varphi\left(\bigcup_i G_i\right) = \text{Bs } |-K_U|.$$

Let

$$c := \min \frac{a_i + 1 - \delta_i}{r_i}.$$

By openness of the condition of ampleness, we may shrink δ_i 's if necessary so that the minimum c is attained exactly for a single i , say $i = 1$. Let

$$\begin{cases} A : &= \sum_{i \geq 2} (-cr_i + a_i - \delta_i) G_i, \quad \text{and} \\ B : &= G_1. \end{cases}$$

Here we note $-cr_i + a_i - \delta_i \geq -1$, with equality if and only if $i = 1$. In particular,

$$(4.4.4) \quad \begin{cases} \lceil A \rceil \geq 0, \\ \text{Supp} \lceil A \rceil \subset \text{Exc } \varphi, \quad \text{and} \\ A - B = \sum_{i \geq 1} (-cr_i + a_i - \delta_i) G_i. \end{cases}$$

Then

CLAIM. $a_i \geq r_i$ for all i .

PROOF. If $a_i + 1 \leq r_i$ for some i , then $c < \frac{a_i+1}{r_i} \leq 1$ i.e. $2 - c > 1$. Moreover since D' is nef (4.4.2) and $c > 0$,

$$C := -\varphi^* K_U - K_{U'} + (A - B) = cD' - (2 - c)\varphi^* K_U - \sum_{i \geq 1} \delta_i G_i$$

is $g_U \circ \varphi$ -ample. Thus

$$R^1(g_U \circ \varphi)_* \mathcal{O}_{U'} (-\varphi^* K_U + \lceil A \rceil - B) = 0$$

by the Kawamata-Viehweg vanishing theorem (e.g. [18]), and hence the following restriction homomorphism is surjective:

$$s : (g_U \circ \varphi)_* \mathcal{O}_{U'} (-\varphi^* K_U + \lceil A \rceil) \longrightarrow H^0(B, \mathcal{O}_B(-\varphi^* K_U + \lceil A \rceil)).$$

The left-hand side is naturally equal to $g_{U,*} \mathcal{O}_U(-K_U)$ by $\varphi_* \mathcal{O}_{U'}(\lceil A \rceil) \simeq \mathcal{O}_U$ (4.4.4). On the other hand, since $v_i^* \mathcal{O}_{E_i}(-K_U)$ is globally generated for each i (Theorem 2.8), and since $\varphi(B) \subset \text{Bs } |-K_U| \subset E$ (4.4.3), the right-hand side of s does not vanish. Hence $\varphi(B) \not\subset \text{Bs } |-K_U|$, which contradicts (4.4.3), and the claim is proved. $\square$

PROOF OF LEMMA 4.4 (continued). Now we have $a_i - r_i \geq 0$. Let D' be a general member of $|D'|$ by abuse of notation. D' is smooth by (4.4.2) and Bertini's theorem. Then by (4.4.1) together with the adjunction, we have

$$K_{D'} = \varphi^* K_D + \sum_{i \geq 1} (a_i - r_i)(G_i|_{D'}),$$

which implies that D has at most canonical singularities. $\square$

Let D be a general member of $|-K_U|$ as in Lemma 4.4, and $h : D \rightarrow \tilde{V}$ the Stein factorization of $g_U|_D : D \rightarrow V$. Then h is a projective bimeromorphic morphism. Let F be its exceptional locus.

LEMMA 4.5. *Let $F = \sum_i F_{1,i} + \sum_j F_{2,j} + \sum_k F'_{2,k}$ be the irreducible decomposition of F , where*

$$\begin{cases} \dim F_{1,i} = 1, \\ (\dim F_{2,j}, \dim h(F_{2,j})) = (2, 0), \quad \text{and} \\ (\dim F'_{2,k}, \dim h(F'_{2,k})) = (2, 1). \end{cases}$$

Let $F_1 := \sum_i F_{1,i}$, $F_2 := \sum_j F_{2,j}$, and $F'_2 := \sum_k F'_{2,k}$. Then

$$\text{Bs } |-K_U| = F_2.$$

(Note that each $F_{2,j}$ coincides with an irreducible component of E .)

PROOF. Let $L := D|_D$, then by the exact sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U(D) \longrightarrow \mathcal{O}_D(L) \longrightarrow 0$$

together with $R^1 g_{U*} \mathcal{O}_U = 0$ [18], we have

$$(4.5.0) \quad \text{Bs } |D| = \text{Bs } |L|.$$

Let us choose $x \in (F_1 \cup F'_2) \cap E - F_2$ arbitrarily. By Lemma 4.3, if two 1-dimensional irreducible components, say l_1 and l_2 , of a fiber of h meets at x , then $l_1 \cup l_2$ never deforms outside E in D . So we can find an effective Cartier divisor M of D with $M \ni x$ such that

$$\begin{cases} (4.5.1) & M \cap (\text{the fiber of } h \text{ passing through } x) = \{x\}, \quad \text{and} \\ (4.5.2) & (M \cdot l) = 1 \text{ for any irreducible component } l \text{ of any} \\ & \text{one-dimensional fiber of } h \text{ with } M \cap l \neq \emptyset. \end{cases}$$

Hence $L - M$ is h -nef, so again by [18], $R^1 h_* \mathcal{O}_D(L - M) = 0$. In particular

$$h_* \mathcal{O}_D(L) \rightarrow h_* \mathcal{O}_M(L)$$

is surjective. Thus $x \notin \text{Bs } |L| = \text{Bs } |-K_U|$, and we have $\text{Bs } |-K_U| \subset F_2$.

On the other hand, F_2 is a union of some irreducible components of E (as remarked just before this proof), in particular, F_2 does not depend on the choice of a general $D \in |-K_U|$. Thus we necessarily have $F_2 \subset \text{Bs } |-K_U|$. Hence $\text{Bs } |-K_U| = F_2$. $\square$

4.6. PROOF OF THEOREM 4.1.

For a general h -very ample divisor H on D , we have $H = H_1 + H_2$, $H_1 \cap (F_1 \cup F'_2) = H_2 \cap F_2 = \emptyset$ by shrinking $\tilde{V}$ if necessary. Then $|H_2|$ gives a projective bimeromorphic morphism $h_1 : D \rightarrow \tilde{V}'$ whose exceptional locus is just F_2 . Let F° be a connected component of F_2 , D° a neighborhood of F° in D , $V^\circ := h_1(D^\circ)$, and consider $h^\circ := h_1|_{D^\circ} : D^\circ \rightarrow V^\circ$.

(4.6.0) h° is a projective bimeromorphic morphism which contracts a connected exceptional divisor F° to a point, and each irreducible component of F° coincides with some E_i . Moreover, let us denote $L|_{D^\circ}$ again by L for simplicity. Then by (4.5.0) and Bertini's theorem,

$$(4.6.1) \quad \text{Sing } D^\circ \subset \text{Bs } |L|.$$

Let $h^\circ(F^\circ) =: P^\circ \in V^\circ$. Let $L_0 \in |L|$ be a general member, $r \in \mathbb{Z}$ with $r \gg 0$, and L' a general hyperplane section of (V°, P°) . Let

$$(4.6.2) \quad L_r := L_0 + r h^{\circ*} L' \in |L|.$$

We take a resolution $\psi : D' \rightarrow D^\circ$ of singularities of D° such that

$$(4.6.3) \quad \begin{cases} \psi^* L_r = \sum r'_i G'_i, \\ K_{D'} = \psi^* K_{D^\circ} + \sum a'_i G'_i \sim \sum a'_i G'_i, \text{ and} \\ \psi^* L - \sum \delta'_i G'_i \text{ is } h^\circ \circ \psi\text{-ample,} \end{cases}$$

where $\sum G'_i$ is a simple normal crossing divisor on D' , $r'_i, a'_i \in \mathbb{Z}$ with $r'_i \geq 0$, and $\delta'_i \in \mathbb{Q}_{>0}$ with $0 < \delta'_i < 1$. Note that

$$(4.6.4) \quad \psi \left(\bigcup_i G'_i \right) \subset \text{Bs } |L| \cup \text{Sing } D^\circ = F^\circ$$

by (4.6.1) and Lemma 4.5. Moreover by Lemma 4.4,

$$(4.6.5) \quad a'_i \geq 0$$

for each i . Then, for any G'_i , we have $r'_i \geq r$ by virtue of the definition of L_r (4.6.2), and $\psi(G'_i) \subset F^\circ$ (4.6.4). So if we define

$$c' := \min \frac{a'_i + 1 - \delta'_i}{r'_i},$$

then we have

$$(4.6.6) \quad 0 < c' \ll 1,$$

since a'_i does not depend on the choice of the number r . We may assume that the minimum c' is attained exactly for a single i , say $i = 1$, by shrinking δ'_i 's if necessary. Let

$$\begin{cases} A' : &= \sum_{i \geq 2} (-c'r'_i + a'_i - \delta'_i)G'_i, \quad \text{and} \\ B' : &= G'_1. \end{cases}$$

Then

$$\lceil A' \rceil \geq 0, \quad A' - B' = \sum_{i \geq 1} (-c'r'_i + a'_i - \delta'_i)G'_i$$

and

$$\begin{aligned} C' &:= \psi^*L - K_{D'} + (A' - B') = (1 - c')\psi^*L - \sum \delta'_i G'_i \\ &= (1 - c') \left(\psi^*L - \sum \frac{\delta'_i}{1 - c'} G'_i \right) \end{aligned}$$

is $h^\circ \circ \psi$ -ample (4.6.3), (4.6.6). Thus $R^1(h^\circ \circ \psi)_* \mathcal{O}_{D'}(\psi^*L + \lceil A' \rceil - B') = 0$ [18], and the restriction homomorphism

$$(h^\circ \circ \psi)_* \mathcal{O}_{D'}(\psi^*L + \lceil A' \rceil) = h^\circ_* \mathcal{O}_{D^\circ}(L) \longrightarrow H^0(B', \mathcal{O}_{B'}(\psi^*L + \lceil A' \rceil))$$

is surjective. Moreover, since $\psi(B') \subset E$ (4.6.4), and since each $v_i^* \mathcal{O}_{E_i}(L)$ is globally generated (Theorem 2.8),

$$H^0(B', \mathcal{O}_{B'}(\psi^*L + \lceil A' \rceil)) \neq 0,$$

as in the proof of Lemma 4.4. Thus we obtain

$$\psi(B') \not\subset \text{Bs } |L|.$$

This contradicts $\psi(B') \subset \text{Bs } |L|$ (4.6.4), and the proof of Theorem 4.1 is complete. $\square$

5. – The case $l_{E_i}(R) = 2$ for some i

ASSUMPTION 5.0. In this section, we assume that

(5.0.1) E contains an irreducible component, say E_1 , with $l_{E_1}(R) = 2$.

The following is the main result of this section:

THEOREM 5.1. *Under Assumption 5.0, the following holds:*

- (I) *E is irreducible; $E = E_1$, and is isomorphic to $\mathbb{P}^2$. $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$. Y is smooth at P . In particular, $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$.*
- (II) *The normal bundle $N_{E/X}$ of E in X is isomorphic to $\Omega_{\mathbb{P}^2}^1(1)$.*
- (III) *(local elementary transformation) Let $x \in E$ be an arbitrary point. Then there exists a smooth surface $S_x \subset U$ proper over V such that $g|_{S_x-x} : S_x-x \rightarrow g(S_x)-P$ is an isomorphism, and that $S_x \cap E = \{x\}$ intersecting transversally. Let $\varphi : \bar{U} \rightarrow U$ be the blow-up with center S_x . Then $-K_{\bar{U}}$ is $g_U \circ \varphi$ -ample, and $\varphi^{-1}(E) \simeq \Sigma_1$. Let $\varphi^+ : \bar{U} \rightarrow U^+$ be the contraction associated to the extremal ray of $\overline{NE}(\bar{U}/V)$ other than φ . Then $U^+ \simeq V \times \mathbb{P}^1$. Let $g^+ : U^+ \rightarrow V$ be the first projection. Then $g_U \circ \varphi = g^+ \circ \varphi^+$, and φ^+ is the blow-up with center S^+ which is a smooth surface proper over V such that $S^+ \supset g^{+-1}(P) \simeq \mathbb{P}^1$ with the normal bundle $N_{g^{+-1}(P)/S^+} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$.*

First we shall prove:

LEMMA 5.2. *$g|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle.*

PROOF. Assume to the contrary. Then there is a rational curve l of length 1 which deforms outside E . Then by Lemma 2.7 (3), $L_X(l) \supset E$, which contradicts the assumption (5.0.1). $\square$

LEMMA 5.3. *$\rho^{\text{an}}(X \supset E / Y \ni P) = 1$. (See Definition 0.4).*

PROOF. If $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$, then by Lemma 5.2 and Theorem 3.5, E is a union of two $\mathbb{P}^2$'s whose lines are of length 1. This contradicts our assumption (5.0.1). $\square$

LEMMA 5.4. *Under the Assumption 5.0, E is irreducible: $E = E_1$.*

PROOF. By Lemma 5.3 and Theorem 3.1, any two points of E are joined by a limit conic. Since $l_{E_1}(R) = 2$, E is necessarily irreducible. $\square$

LEMMA 5.5. *Let x be a general smooth point of E , and let S_x be as in Notation 3.2. Let $\varphi : \bar{U} \rightarrow U$ be the blow-up of U with center S_x . Then $-K_{\bar{U}}$ is $g_U \circ \varphi$ -ample.*

PROOF. First by (3.2.1),

(5.5.0) For every one-dimensional fiber C of φ , $C \simeq \mathbb{P}^1$ and $(-K_{\bar{U}} \cdot C) = 1$.

Let $\bar{E} := \varphi^{-1}(E)$, $\varphi_{\bar{E}} := \varphi|_{\bar{E}} : \bar{E} \rightarrow E$, and $L := \text{Exc } \varphi$. Moreover let $\nu : \tilde{E} \rightarrow E$, $\bar{\nu} : (\bar{E})^\sim \rightarrow \bar{E}$ be the normalization of E , $\bar{E}$, respectively, and

$(\varphi_{\bar{E}})^{\sim} : (\bar{E})^{\sim} \rightarrow \tilde{E}$ the induced morphism.

$$(5.5.1) \quad \begin{array}{ccc} \Sigma_1 \simeq (\bar{E})^{\sim} & \xrightarrow{\bar{v}} & \bar{E} \subset \bar{U} \\ (\varphi_{\bar{E}})^{\sim} \downarrow & & \varphi_{\bar{E}} \downarrow \quad \downarrow \varphi \\ \mathbb{P}^2 \simeq \tilde{E} & \xrightarrow[\nu]{} & E \subset U \\ & & \downarrow \quad \downarrow g_U \\ & & P \in V \end{array}$$

By Lemma 2.7 (1) and Lemma 5.4,

$$(5.5.2) \quad \tilde{E} \simeq \mathbb{P}^2, \text{ and } \nu^* \mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(2).$$

By (3.2.1) again, $\varphi_{\bar{E}}$ is just the blow-up of E with center x . Since x is a smooth point of E , $(\varphi_{\bar{E}})^{\sim}$ coincides with $\varphi_{\bar{E}}$ in an analytic neighborhood of x , and it follows that $(\varphi_{\bar{E}})^{\sim}$ is the blow-up of $\tilde{E} \simeq \mathbb{P}^2$ with center x . In particular

$$(5.5.3) \quad (\bar{E})^{\sim} \simeq \Sigma_1.$$

(5.5.4) Let $\tilde{M}, \tilde{l}$ be the minimal section and a ruling of $(\bar{E})^{\sim} \simeq \Sigma_1$, respectively, and let $M := \bar{v}(\tilde{M})$, $l := \bar{v}(\tilde{l})$. Note that $\varphi(M) = \{x\}$. We claim that

$$(5.5.5) \quad \bar{v}^* \mathcal{O}_{\bar{E}}(-K_{\bar{U}}) \text{ is ample.}$$

In fact, since φ contracts M (5.5.4),

$$(5.5.6) \quad (\bar{v}^*(-K_{\bar{U}}) \cdot \tilde{M}) = (-K_{\bar{U}} \cdot M) = 1 > 0$$

(5.5.0). On the other hand, since $(\varphi_{\bar{E}})^{\sim}(\tilde{l})$ is a line in $\tilde{E} \simeq \mathbb{P}^2$, and since $\nu^* \mathcal{O}_E(-K_U) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ (5.5.2), we have

$$(-K_U \cdot \varphi(l)) = (-K_U \cdot (\nu \circ (\varphi_{\bar{E}})^{\sim})(\tilde{l})) = 2.$$

Moreover, since $\varphi(l)$ intersects with the center S_x of φ transversally at the single point $\{x\}$, we have $(L \cdot l) = 1$. Hence

$$(5.5.7) \quad \begin{aligned} (\bar{v}^*(-K_{\bar{U}}) \cdot \tilde{l}) &= (-K_{\bar{U}} \cdot l) = (\varphi^*(-K_U) - L \cdot l) \\ &= (-K_U \cdot \varphi(l)) - (L \cdot l) = 2 - 1 > 0. \end{aligned}$$

(5.5.6) and (5.5.7) prove (5.5.5).

Finally, recall that $g_U|_{U-E} : U-E \rightarrow V-P$ is a $\mathbb{P}^1$ -bundle (Lemma 5.2). Moreover since $l_E(R) = 2$, any limit conic in E is generically reduced, and thus $S_x - \{x\}$ is a subsection of $g_U|_{U-E}$, namely, $g|_{S_x - \{x\}} : S_x - \{x\} \rightarrow g(S_x) - P$ is an isomorphism. Hence

$$(5.5.8) \quad g_U \circ \varphi \text{ is a conic bundle over } V - P.$$

In particular, $-K_{\bar{U}}$ is $g_U \circ \varphi$ -ample on $\bar{U} - \bar{E}$. By combining this and (5.5.5), we conclude that $-K_{\bar{U}}$ is $g_U \circ \varphi$ -ample. $\square$

PROPOSITION 5.6. *Under the Assumption 5.0, E is normal: $E \simeq \mathbb{P}^2$, and V is smooth. Moreover, (III) of Theorem 5.1 holds.*

PROOF. We will follow the notations in the proof of Lemma 5.5. Since $\rho(\overline{U}/V) = 2$, it follows from Lemma 5.5 that

(5.6.1) $\overline{NE}(\overline{U}/V)$ admits exactly two extremal rays, one of which defines φ .

Let $\varphi^+ : \overline{U} \rightarrow U^+$ be the contraction associated to the other extremal ray, and $g^+ : U^+ \rightarrow V$ the structure morphism. By construction, φ^+ is a divisorial contraction such that $g^+(\text{Exc } (\varphi^+)) = g(S_x)$ (5.5.8).

Let $E^+ := \varphi^+(\overline{E})$ and $\varphi_E^+ := \varphi^+|_{\overline{E}} : \overline{E} \rightarrow E^+$. Let $\nu^+ : (E^+)^{\sim} \rightarrow E^+$ be the normalization of E^+ , and $(\varphi_E^+)^{\sim} : (\overline{E})^{\sim} \rightarrow (E^+)^{\sim}$ the induced morphism.

$$(5.6.2) \quad \begin{array}{ccc} \Sigma_1 \simeq (\overline{E})^{\sim} & \xrightarrow{\quad \bar{\nu} \quad} & \overline{E} \subset \overline{U} \\ (\varphi_E^+)^{\sim} \downarrow & & \varphi_E^+ \downarrow \quad \downarrow \varphi^+ \\ (E^+)^{\sim} & \xrightarrow[\nu^+]{} & E^+ \subset U^+ \\ & & \downarrow \quad \downarrow g^+ \\ & & P \in V \end{array}$$

By the upper-semi-continuity of the fiber dimension of φ^+ , φ^+ contracts at least one curve in $\overline{E}$. Let M, l be as in (5.5.4), then φ contracts M . Thus φ^+ never contracts M , and φ^+ necessarily contracts l 's. In particular,

(5.6.3) E^+ is an irreducible rational curve, and φ^+ has no 2-dimensional fibers. Thus by the result of Ando ([1] Theorem 2.3),

(5.6.4) Both U^+ and $S^+ := \varphi^+(\text{Exc } (\varphi^+))$ are smooth, and φ^+ is the blow-up with center S^+ . In particular, all fibers of $\varphi_E^+ : \overline{E} \rightarrow E^+$ are isomorphic to $\mathbb{P}^1$.

Let $\text{Sing } E^+ = \{q_1, \dots, q_r\}$. Then by (5.6.4)

$$(5.6.5) \quad \text{Sing } \overline{E} = \coprod_{k=1}^r \varphi^{+-1}(q_k).$$

Since M dominates E^+ through φ^+ , and since $\overline{E}$ is smooth along M , it follows that $\text{Sing } E^+ = \emptyset$. Hence $\overline{E}$ is also smooth, by (5.6.5). Namely, we have

$$(5.6.6) \quad E^+ \simeq \mathbb{P}^1, \quad \text{and} \quad \overline{E} \simeq \Sigma_1$$

(5.5.3). Note in particular that every fiber of $g^+ : U^+ \rightarrow V$ is isomorphic to $\mathbb{P}^1$. Since U^+ is smooth (5.6.4),

(5.6.7) (V, P) is a smooth germ, and g^+ is a trivial $\mathbb{P}^1$ -bundle: $U^+ \simeq V \times \mathbb{P}^1$

([1], Theorem 3.1). Finally, since $\varphi_{\bar{E}}$ is an isomorphism outside $M = \varphi^{-1}(x)$, (5.6.6) implies that E is smooth outside x . Moreover E is smooth also at x , by the assumption of Lemma 5.5. Hence we get the smoothness of the whole E , i.e.

$$(5.6.8) \quad E \simeq \mathbb{P}^2.$$

At the same time, the above proof just gives the elementary transformation as described in (III) of Theorem 5.1. $\square$

Determination of the normal bundle $N_{E/X}$ (cf. Andreatta-Wisńiewski [2])

From now on we shall prove (II) of Theorem 5.1. First we shall prove:

LEMMA 5.7. *Under the Assumption 5.0 and Proposition 5.6,*

- (1) *There is a divisor L in U such that $2L \sim -K_U$. Moreover $Bs |L| = \emptyset$.*
- (2) *Take a member L of $|L|$ such that $L \not\supset E$. Then L is a smooth 3-fold, and $g|_L : L \rightarrow V$ is a divisorial contraction which contracts a divisor to a curve.*

PROOF. Since $-K_U$ is g_U -ample, $H^i(U, \mathcal{O}_U) = 0$ for $i \geq 1$ [18]. On the other hand, since $E \simeq \mathbb{P}^2$ (Proposition 5.6), $H^i(E, \mathcal{O}_E) = 0$ for $i \geq 1$. Furthermore since U can be retracted to E , $H^2(U, \mathbb{Z}) \simeq H^2(E, \mathbb{Z})$. Thus

(5.7.1) The natural $\text{Pic } U \rightarrow \text{Pic } E$ is an isomorphism.

(5.7.2) Let L be a divisor on U which corresponds to $\mathcal{O}_{\mathbb{P}^2}(1)$ on E under (5.7.1).

Since $-K_U$ corresponds to $\mathcal{O}_{\mathbb{P}^2}(2)$ (Lemma 2.7 (1) with Proposition 5.6), we have

$$(5.7.3) \quad 2L \sim -K_U.$$

Then by a version of the non-vanishing theorem ([2] Theorem 3.1),

$$(5.7.4) \quad Bs |L| \not\supset E.$$

Choose any $L \in |L|$ such that $L \not\supset E$. Then $C := L \cap E$ is a line in $\mathbb{P}^2$ (5.7.2). Then

(5.7.5) L is smooth.

Indeed, let x be any point of C and B_x an analytic neighborhood of x in U . Then in B_x , E is the intersection of two smooth hypersurfaces. In order to obtain C , it is enough to cut out by one more hypersurface L . Since C is still smooth, L is necessarily smooth in B_x . Thus L is smooth along C . By shrinking V if necessary, we can make L everywhere smooth, and we have (5.7.5).

Consider $g|_L : L \rightarrow V$. This has connected fibers by $(L \cdot f) = 1$ (5.7.3), and thus is a birational morphism onto V . Moreover since

$$(5.7.6) \quad -2K_L \sim -2K_U - 2L|_L \sim -K_U|_L$$

(5.7.3), $-K_L$ is $g|_L$ -ample. Thus by [28], $g|_L$ is a divisorial contraction, and $|-K_L|$ is $g|_L$ -free. Since $-K_L \sim L|_L$,

$$(5.7.7) \quad \text{Bs } |L|_L = \emptyset.$$

Finally, by the exact sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U(L) \longrightarrow \mathcal{O}_L(L) \longrightarrow 0$$

with $R^1(g_U)_*\mathcal{O}_U = 0$, we conclude that

$$\text{Bs } |L| = \text{Bs } |L|_L = \emptyset$$

(5.7.7). □

THEOREM 5.8. *Under the Assumption 5.0, let $N_{E/X}$ be the normal bundle of $E \simeq \mathbb{P}^2$ (Proposition 5.6) in X . Then*

$$N_{E/X} \simeq \Omega_{\mathbb{P}^2}^1(1).$$

PROOF. Consider the linear system

$$\Lambda := \{L|_E \in |\mathcal{O}_{\mathbb{P}^2}(1)| \mid L \in |L|\}.$$

Then by Lemma 5.7 Λ is free, and thus it must coincide with the whole $|\mathcal{O}_{\mathbb{P}^2}(1)|$. (Recall that any proper sub-linear system of $|\mathcal{O}_{\mathbb{P}^2}(1)|$ has a base point). Namely,

(5.8.1) For any line C in $E \simeq \mathbb{P}^2$, there exists $L_C \in |L|$ such that $L_C|_E = C$.

In particular, L_C is smooth, and $g|_{L_C} : L_C \rightarrow V$ is a divisorial contraction which contracts a divisor to a curve (Lemma 5.7 (2)). Note that C is a fiber of $g|_{L_C}$. Then

$$N_{E/X} \otimes \mathcal{O}_C \simeq N_{C/L_C} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

[28]. Since the line C is arbitrarily chosen, it follows that $N_{E/X}$ is a uniform bundle. Since $c_1(N_{E/X}) = -1$, by Van de Ven's theorem [46], $N_{E/X}$ is isomorphic either to

$$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \quad \text{or} \quad \Omega_{\mathbb{P}^2}^1(1).$$

In the former case, we have $H^0(N_{E/X}) \neq 0$ and $H^1(N_{E/X}) = 0$, which implies that E deforms inside X , a contradiction. Hence $N_{E/X} \simeq \Omega_{\mathbb{P}^2}^1(1)$. □

6. – Glueing chain of rational curves

DEFINITION 6.0. *Let B be a 1-dimensional reduced proper scheme, and $B = \bigcup_{i \in I} B_i$ the irreducible decomposition. Then B is called a tree of $\mathbb{P}^1$'s if the following conditions are satisfied:*

- $$\left\{ \begin{array}{l} (6.0.1) \quad B_i \simeq \mathbb{P}^1 \text{ for each } i, \\ (6.0.2) \quad \text{for any } i, j (i \neq j), B_i \cap B_j \text{ is either } \emptyset \text{ or one point intersecting transversally, and} \\ (6.0.3) \quad \text{there is no subset } \{i_1, \dots, i_r\} \subset I \ (r \geq 2) \text{ such that for each} \\ \quad k = 1, \dots, r-1, B_{i_k} \cap B_{i_{k+1}} \neq \emptyset \text{ and } B_{i_r} \cap B_{i_1} \neq \emptyset. \end{array} \right.$$

Let X be an algebraic variety. A 1-dimensional closed subset C of X is called a chain of rational curves if there is a tree $B = \bigcup B_i$ of $\mathbb{P}^1$'s and a morphism $\varphi_0 : B \rightarrow X$ whose image is equal to C such that $\varphi_0|_{B_i}$ is birational onto its image for each i .

In this section, we shall recall Kollár-Miyaoka-Mori's method of glueing chain of rational curves [23], [24]. To be more precise, we shall give a sufficient condition when we can glue a given chain C of rational curves on X to an irreducible curve C' by a deformation inside X . We will give it only for the simplest case, namely, the case that C' is rational, and that there is no prescribed fixing point under deformations, which is suitable for our aim. Andreatta-Wisniewski [3] essentially used this method to classify 4-dimensional divisorial contractions which contracts divisors to surfaces. We are much inspired by their idea.

NOTATION 6.1. Let X be a smooth projective variety and $C \subset X$ a chain of rational curves on X . By Definition 6.0, there is a tree B of $\mathbb{P}^1$'s and a morphism $\varphi_0 : B \rightarrow X$ which is birational onto C .

Let $S \rightarrow \mathbb{A}^1$ be a proper surjective morphism from a smooth surface S whose fiber at 0 is isomorphic to B , and another fiber all isomorphic to $\mathbb{P}^1$. (Such a morphism can easily be constructed by a successive blowing-up from $\mathbb{P}^1 \times \mathbb{A}^1$).

We consider the connected component of the relative Hilbert scheme $\text{Hom}_{\mathbb{A}^1}(S, X \times \mathbb{A}^1)$ over $\mathbb{A}^1$ containing $[\varphi_0]$, with the structure morphism ([23], (1.1)):

$$\lambda : \text{Hom}_{\mathbb{A}^1}(S, X \times \mathbb{A}^1)_{[\varphi_0]} \rightarrow \mathbb{A}^1.$$

THEOREM 6.2 (Kollár-Miyaoka-Mori. See [23] (1.2)). *Under the Notation 6.1, the following holds:*

- (1) $\dim \text{Hom}_{\mathbb{A}^1}(S, X \times \mathbb{A}^1)_{[\varphi_0]} \geq \dim X + (-K_X \cdot C) + 1$.
- (2) $\dim \text{Hom}(B, X)_{[\varphi_0]} \geq \dim X + (-K_X \cdot C)$.
- (3) *If the equality holds in (2), then C , as a 1-cycle of X , deforms to an irreducible rational curve inside X . Namely, there is an analytic open subset $A \ni 0$ of $\mathbb{A}^1$, and a morphism $\varphi : S \times_{\mathbb{A}^1} A \rightarrow X$ such that $\varphi|_{S_0} = \varphi_0$.*

PROOF. (1) is a direct consequence of [23] (1.2). Since $\text{Hom}(B, X)_{[\varphi_0]}$ is the fiber of λ at 0, (2) follows from (1). Finally, the assumption of (3) implies that $\text{Hom}_{\mathbb{A}^1}(S, X \times \mathbb{A}^1)_{[\varphi_0]}$ dominates $\mathbb{A}^1$ through λ . Thus (3) holds. $\square$

7. – Proof of $\rho^{\text{an}}(X \supset E / Y \ni P) \leq 2$

The main result of this section is:

THEOREM 7.1. $\rho^{\text{an}}(X \supset E / Y \ni P) \leq 2$.

ASSUMPTION 7.2. To prove Theorem 7.1, we may assume

$$(7.2.1) \quad l_E(R) = 1,$$

according to Theorem 5.1. From now on we assume

$$(7.2.2) \quad \rho(U/V) \geq 3$$

to get a contradiction, till the end of this section. In particular, we may assume that

(7.2.3) $g_U|_{U-E} : U - E \rightarrow V - P$ is not a $\mathbb{P}^1$ -bundle (Theorem 3.5), and the assumption of Theorem 3.7 is satisfied.

In this section, for any curve l which is contained in a fiber of g_U , the symbol $[l]$ means the numerical equivalence class of l in U , not in X : $[l] \in \overline{NE}(U/V)$, as in the Notation 3.6.

LEMMA 7.3. Assume $\rho(U/V) \geq 3$, and let f be a general fiber of g_U . Then there exist two rational curves l, l' of length 1 in E which satisfy

$$l \cap l' \neq \emptyset, \quad \mathbb{R}_{\geq 0}[l] \neq \mathbb{R}_{\geq 0}[l'], \quad \text{and} \quad [f] \notin \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[l'].$$

PROOF. By assumption $l_E(R) = 1$, there exists a couple $\{l_1, l_2\}$ of rational curves in E with $l_1 \cap l_2 \neq \emptyset$ and

$$(7.3.1) \quad f \equiv l_1 + l_2.$$

Since $\rho(U/V) \geq 3$ by assumption, $\overline{NE}(U/V)$ has at least three extremal rays. Thus by Theorem 3.7 and the connectedness of E , there is another rational curve l_3 of length 1 in E such that $l_i \cap l_3 \neq \emptyset$ for $i = 1$ or 2 , and

$$(7.3.2) \quad \mathbb{R}_{\geq 0}[l_3] \neq \mathbb{R}_{\geq 0}[l_1], \quad \mathbb{R}_{\geq 0}[l_2].$$

Without loss of generality, we may assume

$$(7.3.3) \quad l_1 \cap l_3 \neq \emptyset.$$

Then

$$[l_1] + [l_3] = [f] - [l_2] + [l_3] \quad (\text{by (7.3.1)})$$

$$\neq [f]. \quad (\text{by (7.3.2)})$$

This, together with (7.3.2) and (7.3.3), proves our lemma. $\square$

LEMMA 7.4. *Let $\{l, l'\}$ be as in Lemma 7.3. Then these are contained in the same irreducible component, say E_i , of E .*

PROOF. Let $B = B_1 \cup B_2$ be the reduced scheme which is isomorphic to the union of two distinct lines in $\mathbb{P}^2$. Let $C := l \cup l'$, and consider any morphism $\alpha : B \rightarrow C \subset X$ such that $\alpha|_{B_1}, \alpha|_{B_2}$ gives the normalization of l, l' , respectively.

We shall divide our proof into cases:

Case (1) $\dim \text{Hom}_{[\alpha]}(B, X) \geq 7$. In this case, we shall derive a contradiction by assuming that l and l' are not contained in the same irreducible component of E . By $\dim \text{Aut } B = 4$, it follows that C deforms, as a 1-cycle of U , with an at least 3-dimensional parameter space. This is possible only when

$$\begin{cases} l \subset E_i, l' \subset E_j \quad (\exists i \neq \exists j), \\ \tilde{E}_i \simeq \tilde{E}_j \simeq \mathbb{P}^2, \text{ and} \\ \dim(E_i \cap E_j) = 1, \end{cases}$$

in view of Theorem 2.8, where $\tilde{E}_i$ is the normalization of E_i , etc. Let C be any irreducible curve in $E_i \cap E_j$. Then $[l], [l'] \in \mathbb{R}_{\geq 0}[C]$, which contradicts $\mathbb{R}_{\geq 0}[l] \neq \mathbb{R}_{\geq 0}[l']$.

Case (2) $\dim \text{Hom}_{[\alpha]}(B, X) \leq 6$. In this case, Theorem 6.2 (3) says that $l \cup l'$ deforms to an irreducible rational curve of length 2 inside X . If such a deformation goes outside E , then it must coincide with the whole fiber of g_U , a contradiction to $f \neq l + l'$. Thus $l \cup l'$ deforms inside E to an irreducible rational curve of length 2, in particular, l and l' must be contained in the same irreducible component of E . $\square$

LEMMA 7.5. *Let $\{l, l'\}$ and E_i be as in Lemma 7.3 and 7.4, and let φ, φ' be the contraction associated to the extremal rays $\mathbb{R}_{\geq 0}[l]$ and $\mathbb{R}_{\geq 0}[l']$, respectively. Then*

$$\dim \varphi(E_i) = \dim \varphi'(E_i) = 1.$$

PROOF. By assumption,

$$(7.5.1) \quad \begin{cases} \dim \varphi(l) = \dim \varphi'(l') = 0, \text{ and} \\ \dim \varphi(l') = \dim \varphi'(l) = 1. \end{cases}$$

In particular $\dim \varphi(E_i) \geq 1, \dim \varphi'(E_i) \geq 1$. First assume $\dim \varphi(E_i) = \dim \varphi'(E_i) = 2$, and let $\varphi(E_i)^\sim, \varphi'(E_i)^\sim$ be the normalization of $\varphi(E_i), \varphi'(E_i)$, respectively. Then $\tilde{E}_i \rightarrow \varphi(E_i)^\sim$ and $\tilde{E}_i \rightarrow \varphi'(E_i)^\sim$ are birational morphisms which are both not isomorphisms and contracts distinct curves (7.5.1). This is impossible, by Theorem 2.8.

Thus let us assume $\dim \varphi(E_i) = 2$ and $\dim \varphi'(E_i) = 1$, say, and consider $D := \text{Exc } \varphi = L_U(l)$, which is a prime divisor (Theorem 3.7 (1)). In this case, obviously $D \not\supset E_i$. In particular $(D \cdot l') > 0$, since $l \cap l' \neq \emptyset$. On the other hand, take a deformation $\bar{l}'$ of l' which is not contained in E (Theorem 3.7). Then since $f \neq l + \bar{l}' \equiv l + l'$ by assumption, $D \cap \bar{l}' = \emptyset$ and thus $(D \cdot \bar{l}') = 0$, a contradiction. Hence Lemma 7.5. $\square$

LEMMA 7.6. *Under Lemma 7.3, 7.4 and 7.5, φ and φ' have no 2-dimensional fibers which intersect with E_i .*

PROOF. Assume that φ , say, has a 2-dimensional fiber F with $F \cap E_i \neq \emptyset$. Such an F is classified by Andreatta-Wiśniewski [3]:

(7.6.1) F is irreducible and is isomorphic either to $\mathbb{P}^2$ or to a singular quadric surface S_2 in $\mathbb{P}^3$, and $\mathcal{O}_F(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{S_2}(1)$, respectively.

Let

$$(7.6.2) \quad l := F \cap E_i.$$

Then l is the fiber of $\varphi|_{E_i}$ at the point $\varphi(F)$. By Lemma 7.5, l is purely 1-dimensional in F . From this and (7.6.1), l is connected. In particular, $\varphi|_{E_i}$ has connected fibers near $\varphi(F)$. Thus l is an irreducible rational curve of length 1, since so is a general fiber of $\varphi|_{E_i}$ [1]. Again by (7.6.1),

(7.6.3) l is a line (respectively a ruling) of F when $F \simeq \mathbb{P}^2$ (respectively $F \simeq S_2$).

Let l'' be a general line (respectively a general ruling) of F , and let l' be the unique fiber of $\varphi'|_{E_i}$ passing through the point $Q := l \cap l''$. In particular $l' \cap l'' \ni Q$. Let $C := l' \cup l''$. Then C deforms, as a 1-cycle of U , with a 2-dimensional (respectively 1-dimensional) parameter space. So denote by $B = B_1 \cup B_2$ the reduced scheme isomorphic to two intersecting lines in $\mathbb{P}^2$, and by $\alpha : B \rightarrow C \subset X$ a morphism such that $\alpha|_{B_1}$, $\alpha|_{B_2}$ are isomorphisms with $\alpha(B_1 \cap B_2) = Q$. Then

$$\dim \text{Hom}_{[\alpha]}(B, X) \leq 6 \text{ (respectively } \leq 5 \text{)}.$$

Thus by Theorem 6.2,

(7.6.4) The case $F \simeq S_2$ cannot happen,

and for the case $F \simeq \mathbb{P}^2$, C deforms to an irreducible rational curve inside X . Moreover, such a deformation never goes outside E , since $l' + l'' \equiv l' + l \neq f$. On the other hand, $l' \subset E_i$, $\not\subset F$, and $l'' \not\subset E_i$, $\subset F$, a contradiction. Hence Lemma 7.6. $\square$

LEMMA 7.7. *Under Lemma 7.3 and 7.4, $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$. l and l' are two intersecting rulings.*

PROOF. Let $C := \varphi(E_i)$ and $C' := \varphi'(E_i)$ in view of Lemma 7.5:

$$\varphi|_{E_i} : E_i \rightarrow C, \quad \varphi'|_{E_i} : E_i \rightarrow C'.$$

Then C and C' are both rational curves, since $\varphi(l') = C$ and $\varphi'(l) = C'$. By [1] and Lemma 7.6,

(7.7.1) $\varphi|_{\text{Exc } \varphi}$ and $\varphi'|_{\text{Exc } \varphi'}$ are both $\mathbb{P}^1$ -bundles near E_i .

We shall prove that $C \simeq C' \simeq \mathbb{P}^1$. Assume C , say, is singular: $\text{Sing } C = \{P_1, \dots, P_r\}$. Then by (7.7.1)

$$(7.7.2) \quad \text{Sing } E_i = \prod_{k=1}^r \varphi^{-1}(P_k).$$

Since φ and φ' are the contractions of two distinct extremal rays, $\varphi'(\varphi^{-1}(P_k)) = C'$ for each k . On the other hand, again by (7.7.1), $\varphi'^{-1}(C' - \text{Sing } C')$ is smooth, a contradiction. Hence C must be smooth, and similarly for C' .

Again by (7.7.1), E_i is smooth. Since E_i has two distinct $\mathbb{P}^1$ -bundle structures:

$$\varphi|_{E_i} : E_i \rightarrow C \simeq \mathbb{P}^1 \quad \text{and} \quad \varphi'|_{E_i} : E_i \rightarrow C' \simeq \mathbb{P}^1,$$

it follows that $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$. □

LEMMA 7.8. *Let $\{l, l'\}$ and E_i be as in Lemma 7.7. Then there exists another irreducible component $E_j \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of E , together with a couple of intersecting rulings $\{m, m'\}$, such that $f \equiv l + m$.*

PROOF. Let $[f]$ be the numerical class of a general fiber of g_U , as in Lemma 7.3.

First, even if we replace the original $\{l, l'\}$ by another couple of intersecting rulings of $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$, the assumption of Lemma 7.3 is preserved. So we shall make the following assumption:

(7.8.0) By abuse of notation, we shall also denote by l (respectively l') an arbitrary ruling of $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ which is linearly equivalent to the original l (respectively l') given in Lemma 7.3. Recall that $\mathbb{R}_{\geq 0}[l] \neq \mathbb{R}_{\geq 0}[l']$.

Since each l as in (7.8.0) is isomorphic to $\mathbb{P}^1$, l deforms, as a 1-cycle of U , with an at least 2-dimensional parameter space (Lemma 1.3), while in E_i , l deforms only with 1-dimensional parameters. Thus those l 's which are not contained in any other E_j are actually contained in some limit conics:

(7.8.1) There is a connected 1-cycle $f_l \subset E$ of length 2 which contains l as its irreducible component, and $[f_l] = [f]$.

Let us denote the other irreducible component of f_l by m :

$$(7.8.2) \quad f_l =: l + m.$$

Note that since $[f_l] \notin \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[l']$ by assumption,

$$(7.8.3) \quad [m] \notin \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[l'].$$

Since every irreducible curve in E_i is spanned by $[l]$ and $[l']$ (Lemma 7.7),

$$(7.8.4) \quad E_i \not\supset m.$$

Let $Q \in l \cap m$, and let $l' \subset E_i$ be as in (7.8.0) which passes through Q . Then

$$\begin{aligned} [m] + [l'] &= [l] + [m] - [l] + [l'] \\ &= [f] - [l] + [l'] \quad (7.8.2) \\ &\neq [f]. \quad (7.8.0) \end{aligned}$$

Hence the pair $\{m, l'\}$ satisfies the assumption of Lemma 7.3, and it follows from Lemma 7.7 that there exists an irreducible component $E_j \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of E such that m, l' are intersecting rulings of E_j . By (7.8.4), $E_i \neq E_j$, and Lemma 7.8 is proved. $\square$

Now we come to the proof of Theorem 7.1, namely, we assume $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 3$ to get a contradiction.

PROOF OF THEOREM 7.1.

Assume $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 3$. Then (CL) of Theorem 3.1 holds.

Let $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $E_j \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be as in Lemma 7.8, and take a general $x \in E_i$, and a general $y \in E_j$. Since $\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{E_j}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$, (CL) of Theorem 3.1 is impossible. Hence $\rho^{\text{an}}(X \supset E / Y \ni P) \leq 2$. $\square$

8. – The classification of E in the case $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$

THEOREM 8.1. Assume that $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$. Then E is one of the following holds:

- (0) (Mukai-Shepherd-Barron-Wiśniewski type) $E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, $E_1 \cap E_2$ is a point, and $N_{E_1/X} \simeq N_{E_2/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. In particular $\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ ($i = 1, 2$).
- (1) E is irreducible and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.
- (2) E is irreducible and is isomorphic to Σ_1 , and $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\Sigma_1}(M + 2l)$ (Notation 0.14).
- (3) $E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $E_1 \cap E_2$ is a line of E_1 and is a ruling of E_2 . $\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, and $\mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.
- (4) $E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq \Sigma_1$. $E_1 \cap E_2$ is a line of E_1 and is the negative section of E_2 . $\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, and $\mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{\Sigma_1}(M + 2l)$.
- (5) $E = E_1 \cup E_2 \cup E_3$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$ and $E_3 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $E_i \cap E_3$ ($i = 1, 2$) is a line in E_i and is a ruling in E_3 . Moreover $E_1 \cap E_2$ is a point. $\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ ($i = 1, 2$), and $\mathcal{O}_{E_3}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.

In (0), P is an ordinary double singular point of Y , and $g_U|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle.

In (1) $\sim$ (5), P is a smooth point of Y , and $g_U|_{U-E} : U - E \rightarrow V - P$ is a conic bundle with an irreducible discriminant divisor.

First we shall prove:

PROPOSITION 8.2. Assume that $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$ and $g_U|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle. Then (0) of Theorem 8.1 holds.

PROOF. By Theorem 3.5, the remaining properties we have to check are

$$\begin{cases} (8.2.1) & \#(E_1 \cap E_2) = 1, \text{ and} \\ (8.2.2) & (V, P) \text{ is an ordinary double point.} \end{cases}$$

For (8.2.1): Assume $\#(E_1 \cap E_2) \geq 2$, to get a contradiction. Let l be any line of $E_1 \simeq \mathbb{P}^2$ which passes through at least two points $\{Q_1, Q_2\}$ of $E_1 \cap E_2$. Then by g -freeness of $|-K_X|$, there exists a member $D \in |-K_U|$ such that $D|_{E_1} = l$. Such a D is necessarily smooth, by exactly the same argument as in (5.7.5). Hence $R^1(g_U|_D)_* \mathcal{O}_D = 0$ [18], and $H^1(\mathcal{O}_{D \cap E}) = 0$. On the other hand, $D \cap E$ has two irreducible components $D \cap E_1$ and $D \cap E_2$, which intersect at two distinct points Q_1, Q_2 with each other, a contradiction. Thus we have (8.2.1).

For (8.2.2): Consider a general member $D \in |-K_U|$. Then $D \cap E = l_1 \cup l_2$ such that l_i is a line in E_i ($i = 1, 2$) and $l_1 \cap l_2 = \emptyset$. Moreover

$$(8.2.3) \quad N_{l_i/D} \simeq N_{E_i/X} \otimes \mathcal{O}_{l_i} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.$$

Take the Stein factorization $h : D \rightarrow \tilde{V}$ of $g_U|_D : D \rightarrow V$. We note that since the fiber of g_U at P has two connected components, the double cover $\tilde{V} \rightarrow V$ is étale. Moreover by Lemma 4.3, h is a small contraction, and $\text{Exc } h = l_1 \coprod l_2$. Thus by (8.2.3), it follows that $h(l_i)$ is an ordinary double point of $\tilde{V}$ ($i = 1, 2$). Since $(V, P) \simeq (\tilde{V}, h(l_i))$ through the étale double cover $\tilde{V} \rightarrow V$, we are done. $\square$

ASSUMPTION 8.3. In the rest of this section, we assume that

$$(8.3.1) \quad \rho^{\text{an}}(X \supset E / Y \ni P) = 2 \text{ and } g_U|_{U-E} : U - E \rightarrow V - P \text{ is not a } \mathbb{P}^1\text{-bundle.}$$

In particular, the assumption of Theorem 3.7 is satisfied, and (CL) of Theorem 3.1 holds.

LEMMA 8.4. Under Assumption 8.3, let E_1 be any irreducible component of E , and let φ, φ' be the contraction morphisms associated to the two extremal rays of $\overline{NE}(U/V)$. Then, up to the permutation of $\{\varphi, \varphi'\}$, one of the following holds:

(a) $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1)$, $\tilde{E}_1 \simeq \Sigma_1$, and

$$\nu^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\Sigma_1}(\tilde{M} + 2\tilde{l}),$$

where $\tilde{M}$ and $\tilde{l}$ are the minimal section and a ruling of Σ_1 , respectively,

(b) $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (1, 1)$, $\tilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and

$$\nu^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1), \text{ or}$$

(c) $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 0)$, E_1 is normal, is isomorphic to $\mathbb{P}^2$ or S_2 , and

$$\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{S_2}(1),$$

respectively.

PROOF. First of all, we shall prove

$$(8.4.1) \quad (\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1), (1, 1), \text{ or } (2, 0).$$

In fact, $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 2)$ is impossible by Theorem 2.8, via exactly the same argument as in Lemma 7.5. Next if $\dim \varphi'(E_1) = 0$, then φ never contracts any curves, in particular $\dim \varphi(E_1) = 2$. Hence (8.4.1) holds.

Case (a), (b) $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1), (1, 1)$.

In these cases $\rho(E_1) \geq 2$, and it follows from Theorem 2.8 that $\tilde{E}_1$ is isomorphic to a Hirzebruch surface: $\tilde{E}_1 \simeq \Sigma_m$. Moreover since

$$(8.4.2) \quad \nu^* \mathcal{O}_{E_1}(-K_X) \text{ is ample on } \tilde{E}_1,$$

any section of $\tilde{E}_1 \simeq \Sigma_m \rightarrow \mathbb{P}^1$, except for the minimal one, has length at least $m + 1$. Thus by Theorem 3.1, it necessarily follows that $m \leq 1$, i.e.

$$(8.4.3) \quad \tilde{E}_1 \simeq \Sigma_1 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1.$$

Assume $\tilde{E}_1 \simeq \Sigma_1$.

(8.4.4) Let $\tilde{M}$, $\tilde{l}$ be the minimal section and a ruling of $\tilde{E}_1 \simeq \Sigma_1$, respectively, and let $M := \nu(\tilde{M})$, $l := \nu(\tilde{l})$.

Then again by Theorem 3.1 $(-K_X \cdot M) = (-K_X \cdot l) = 1$, in particular, $-\nu^* K_X \sim \tilde{M} + 2\tilde{l}$. Obviously $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1)$.

Next, assume $\tilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(8.4.5) Let $\{\tilde{l}, \tilde{l}'\}$ be a couple of intersecting rulings in $\tilde{E}_1$, and let $l := \nu(\tilde{l})$, $l' := \nu(\tilde{l}')$, such that φ' contracts $\nu(l')$.

Then by Theorem 3.1, $(-K_X \cdot l) = (-K_X \cdot l') = 1$, in particular $-\nu^* K_X \sim \tilde{l} + \tilde{l}'$. We have two possibilities:

$$(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1), \text{ or } (1, 1).$$

Since by Theorem 3.7, $\mathbb{R}_{\geq 0}[l]$ is the extremal ray of $\overline{NE}(U/V)$ defining φ , the former one cannot happen, and we have $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (1, 1)$. Hence (a) and (b).

Case (c) $(\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 0)$.

In this case, E_1 is normal and is isomorphic either to $\mathbb{P}^2$ or S_2 , as a direct consequence of Andreatta-Wisniewski [3]. $\square$

PROPOSITION 8.5. *In Lemma 8.4 (a), E_1 is normal: $E_1 \simeq \Sigma_1$. Y is smooth at P . Moreover if E is reducible, then E has exactly two irreducible components. Let E_2 be another one. Then $E_2 \simeq \mathbb{P}^2$, and $E_1 \cap E_2$ is the minimal section of E_1 and is a line of E_2 .*

PROOF. Let $\varphi : U \rightarrow U_\varphi$, $\varphi' : U \rightarrow U_{\varphi'}$ be as in Lemma 8.4 such that

$$(8.5.0) \quad (\dim \varphi(E_1), \dim \varphi'(E_1)) = (2, 1).$$

Let $\nu : \tilde{E}_1 \rightarrow E_1$ be the normalization of E_1 , and let $\tilde{M}, \tilde{l} \subset \tilde{E}_1$, $M, l \subset E_1$ be as in (8.4.4). Recall that

$$(8.5.1) \quad \tilde{E}_1 \simeq \Sigma_1, \quad \text{and} \quad \nu^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\Sigma_1}(\tilde{M} + 2\tilde{l}).$$

We first claim that

$$(8.5.2) \quad \varphi' : U \rightarrow U_{\varphi'} \text{ has no 2-dimensional fibers which intersect with } E_1.$$

Actually, assume that φ' has a fiber F with $\dim F = 2$ and $m := F \cap E_1 \neq \emptyset$. Then m is a fiber of $\varphi'|_{E_1} : E_1 \rightarrow \varphi'(E_1)$. In particular, a general fiber of $\varphi'|_{E_1}$ is disjoint from m . Then by (8.5.1), a general point of E_1 and that of F never can be joined by a limit conic, which contradicts Theorem 3.1. Hence φ' has no 2-dimensional fibers, and (8.5.2) is proved.

Let $h' : U_{\varphi'} \rightarrow V$ be the structure morphism. By (8.5.2) and [1],

$$(8.5.3) \quad U_{\varphi'} \text{ is smooth, } \varphi' \text{ is the blow-up of a smooth codimension 2 center, and } \varphi'|_{\text{Exc } \varphi'} : \text{Exc } \varphi' \rightarrow \varphi'(\text{Exc } \varphi') \text{ is a } \mathbb{P}^1\text{-bundle.}$$

We divide our proof into cases.

Case (1) E is irreducible: $E = E_1$. In this case, since $\dim \varphi'(E) = 1$ (8.5.0), h' has no 2-dimensional fibers. Since $U_{\varphi'}$ is smooth (8.5.3), h' is a $\mathbb{P}^1$ -bundle [1], in particular V is smooth, and $\varphi'(E) \simeq \mathbb{P}^1$. Moreover since $\varphi'|_E : E \rightarrow \varphi'(E)$ is a $\mathbb{P}^1$ -bundle (8.5.3), E is smooth: $E \simeq \Sigma_1$. Thus we get the proposition in the Case (1).

Case (2) E is reducible. First $\varphi(M)$ and $\varphi'(l)$ are points. Let E_2 be any other irreducible components of E . Then by Theorem 3.1 and Lemma 8.4,

$$(8.5.4) \quad E_2 \simeq \mathbb{P}^2.$$

Since φ' does not contract E_2 (8.5.2), φ contracts E_i , and hence

$$(8.5.5) \quad E_1 \cap E_2 \supset M.$$

Since M is a rational curve of length 1, M is a line in E_2 in $\mathbb{P}^2$ and in particular $M \simeq \mathbb{P}^1$. Hence by (8.5.3), E_1 is smooth:

$$(8.5.6) \quad E_1 \simeq \Sigma_1.$$

Next we claim that E_2 is unique:

$$(8.5.7) \quad E = E_1 \cup E_2.$$

In fact, if there are at least two irreducible components E_2, E_3 of E which both meet E_1 , then by (8.5.4) and (8.5.5) $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1} \subset N_{M/X}$ as in [AW2], which is absurd since $(-K_X \cdot M) = 1$. Hence (8.5.7) is proved.

Consider $\varphi' : U \rightarrow U_{\varphi'}$. Let

$$(8.5.8) \quad \overline{E} := \varphi'(E) = \varphi'(E_2).$$

Then

(8.5.9) , $\varphi'|_{E_2} : \mathbb{P}^2 \simeq E_2 \rightarrow \overline{E}$ is a finite birational morphism, namely, is the normalization morphism.

Let $L := \text{Exc } \varphi'$;

$$(8.5.10) \quad -K_U = -\varphi'^* K_{U_{\varphi'}} - L.$$

Then for a general line l in $E_2 \simeq \mathbb{P}^2$, $(L \cdot l) > 0$ and hence

$$(8.5.11) \quad (-K_{U_{\varphi'}} \cdot \varphi'(l)) = (-K_U \cdot l) + (L \cdot l) \geq 2$$

(8.5.10). In particular, $h' : U_{\varphi'} \rightarrow V$ satisfies the following:

$$(8.5.12) \quad \begin{cases} h'^{-1}(P) = \overline{E}, \text{ which is irreducible and whose normalization is } \mathbb{P}^2, \\ -K_{U_{\varphi'}} \text{ is } h'\text{-ample, and} \\ \varphi'|_{U_{\varphi'} - \overline{E}} : U_{\varphi'} - \overline{E} \rightarrow V - P \text{ is a } \mathbb{P}^1\text{-bundle.} \end{cases}$$

Based on the facts (8.5.3), (8.5.11) and (8.5.12), we can proceed exactly the same argument as in the proof of Lemma 5.5 and Proposition 5.6 to deduce that

$$(8.5.13) \quad \overline{E} \text{ is smooth: } \overline{E} \simeq \mathbb{P}^2, \text{ and } V \text{ is smooth.}$$

Then by (8.5.9), $\varphi'|_{E_2}$ is an isomorphism. In particular

$$(8.5.14) \quad E_1 \cap E_2 = M.$$

(8.5.4), (8.5.6), (8.5.7), (8.5.13) and (8.5.14) prove our proposition also in Case (2). $\square$

PROPOSITION 8.6. *In Lemma 8.4 (b), E_1 is normal: $E_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Y is smooth at P . Moreover if E is reducible, then E has two or three irreducible components. Let E_2 be another one. Then $E_2 \simeq \mathbb{P}^2$, and $E_1 \cap E_2$ is a ruling of E_1 and is a line of E_2 .*

If E has three irreducible components; $E = E_1 \cup E_2 \cup E_3$, then $E_2 \cap E_3$ is one point lying on E_1 .

PROOF. Let $\varphi : U \rightarrow U_\varphi$, $\varphi' : U \rightarrow U_{\varphi'}$ be as in Lemma 8.4 such that

$$(8.6.0) \quad (\dim \varphi(E_1), \dim \varphi'(E_1)) = (1, 1).$$

Recall that

$$(8.6.1) \quad \widetilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1, \text{ and } v^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1).$$

We divide our proof into cases.

Case (1) E is irreducible: $E = E_1$. In this case, consider $\varphi|_E : E \rightarrow \varphi(E)$ and $\varphi'|_E : E \rightarrow \varphi'(E)$. $\varphi(E)$, $\varphi'(E)$ are both irreducible curves (8.6.0). Since φ and φ' has no 2-dimensional fibers, the smoothness of E follows exactly from the same argument as in Lemma 7.7. Moreover U_φ is smooth [1], and hence the structure morphism $h : U_\varphi \rightarrow V$ is a $\mathbb{P}^1$ -bundle [1], in particular V is smooth. Thus we are done in the Case (1).

Case (2) E is reducible. In this case, as in (8.5.4),

(8.6.2) *Any other irreducible component of E is isomorphic to $\mathbb{P}^2$, and is a 2-dimensional fiber of φ or φ' .*

We claim that

(8.6.3) *If both φ and φ' have 2-dimensional fibers, then $E = E_1 \cup E_2 \cup E_3$, $E_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $E_2 \simeq E_3 \simeq \mathbb{P}^2$, $E_1 \cap E_2$ and $E_1 \cap E_3$ are rulings intersecting to each other. Let E_2 (respectively E_3) be an arbitrary 2-dimensional fiber of φ (respectively φ'). Note that*

$$(8.6.4) \quad E_2 \simeq E_3 \simeq \mathbb{P}^2$$

(8.6.2), and

(8.6.5) *$E_1 \cap E_2$ (respectively $E_1 \cap E_3$) is the fiber of $\varphi|_{E_1}$ (respectively $\varphi'|_{E_1}$) at the point $\varphi(E_2)$ (respectively $\varphi(E_3)$).*

As in (7.6.3),

(8.6.6) *$l := E_1 \cap E_2$ (respectively $l' := E_1 \cap E_3$) is a line in E_2 (respectively E_3).*

Choose $Q \in E_1 \cap E_2 \cap E_3 = l \cap l'$ arbitrarily, and consider the linear system

$$\Lambda_Q := \{D \in |-K_U| \mid D \ni Q\}.$$

We claim that

(8.6.7) There exists a point $Q' \in E_1$ such that $\text{Bs } \Lambda_Q = \{Q, Q'\}$.

In fact, since $\text{Bs } |-K_U| = \emptyset$ (Theorem 4.1), for any line m in E_2 (respectively E_3), there exists $D \in |-K_U|$ such that $D|_{E_2} = m$ (respectively $D|_{E_3} = m$), as in (5.8.1). Hence

$$(8.6.8) \quad \text{Bs } \Lambda_Q \subset E_1.$$

In particular, for a general $D \in \Lambda_Q$, $D \not\supset l$, $D \not\supset l'$. Hence if we let

$$\nu^* \Lambda_Q := \{ \nu^*(D|_{E_1}) \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)| \mid D \in \Lambda_Q \},$$

then

(8.6.9) $\nu^* \Lambda_Q$ contains an irreducible member.

Since

$$\nu^* |-K_U| := \{ \nu^*(D|_{E_1}) \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)| \mid D \in |-K_U| \}$$

is a free linear sub-system of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|$ (Theorem 4.1), we have $\dim \nu^* |-K_U| \geq 2$, and hence

$$(8.6.10) \quad \dim \nu^* \Lambda_Q \geq 1.$$

From (8.6.9) and (8.6.10), $\#\text{Bs } \nu^* \Lambda_Q = (\nu^*(D|_{E_1}))_{E_1}^2 = 2$. This, together with (8.6.8), proves (8.6.7). Next we claim that

(8.6.11) A general member D of Λ_Q is smooth.

By (8.6.7) and Bertini's theorem, it is sufficient to prove that a general $D \in \Lambda_Q$ is smooth at Q and Q' . Since $D|_{E_2}$ is a line in $E_2 \simeq \mathbb{P}^2$, D is smooth at Q , as the same reason as in (5.7.5). Similarly if Q' is contained in a 2-dimensional fiber of φ or φ' , then D is smooth at Q' . So assume that Q' is not contained in any 2-dimensional fiber of φ and φ' . Then

$$\varphi|_{E_1} : E_1 \rightarrow \varphi(E_1) \quad \text{and} \quad \varphi'|_{E_1} : E_1 \rightarrow \varphi'(E_1)$$

are both $\mathbb{P}^1$ -bundles in a neighborhood of $\varphi(Q')$ and $\varphi'(Q')$, respectively [1]. Hence E_1 is smooth at Q' , and the normalization morphism $\nu : \tilde{E}_1 \rightarrow E_1$ is an isomorphism at Q' . Then by (8.6.9), $D|_{E_1}$ is smooth at Q' . Then by the same reason as above, D is smooth at Q' . Hence (8.6.11).

(8.6.12) Next we shall prove (8.6.3).

By (8.6.11), $H^1(\mathcal{O}_{D \cap E}) = 0$ [18]. If E has more than three irreducible components, $D \cap E$ contains at least four distinct irreducible rational curves $D \cap E_i$ meeting at Q , a contradiction. Thus (8.6.3) is proved.

So from now on, we assume that φ' , say, has no 2-dimensional fibers. Then $\varphi'|_{E_1} : E_1 \rightarrow \varphi'(E_1)$ is a $\mathbb{P}^1$ -bundle [1], and $l \simeq \mathbb{P}^1$ (8.6.6) is a section. Hence E_1 is smooth:

$$(8.6.13) \quad E_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

By combining the above,

$$(8.6.14) \quad \begin{cases} E = \bigcup_{i=1}^n E_i, \\ E_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1, \quad E_2 \simeq \dots \simeq E_n \simeq \mathbb{P}^2, \\ E_1 \cap E_i \text{ is a ruling in } E_1 \text{ and a line in } E_i \ (i = 2, \dots, n), \text{ and} \\ E_1 \cap E_i \sim E_1 \cap E_j \text{ in } E_1. \end{cases}$$

The rest we have to do is to prove $E = E_1 \cup E_2$, and the smoothness of V .

Assume that E has more than two irreducible components, and let $E_2 \simeq E_3 \simeq \mathbb{P}^2$ be as in (8.6.14). First $E_1 \cap E_2 = E_1 \cap E_3$ is impossible, exactly by the same reason as in (8.5.7). So $E_1 \cap E_2$ and $E_1 \cap E_3$ are two disjoint rulings in E_1 . Then $E_2 \cap E_3 \neq \emptyset$ by Theorem 3.1. Hence

$$(8.6.15) \quad E_2 \cap E_3 \neq \emptyset \text{ and } E_1 \cap E_2 \cap E_3 = \emptyset.$$

Since $E_2 \simeq E_3 \simeq \mathbb{P}^2$,

$$(8.6.16) \quad \dim(E_2 \cap E_3) = 0.$$

Choose $Q \in E_2 \cap E_3$, and general lines $l_Q \subset E_2$, $l'_Q \subset E_3$ which both pass through Q . Let $B := l_Q \cup l'_Q$ regarding as a reduced scheme, and consider the closed immersion $i : B \hookrightarrow X$. Since $l_Q \equiv l'_Q$ (8.6.14), $l_Q + l'_Q \neq f$ (a general fiber of g_U), and hence $l_Q \cup l'_Q$ never deform outside E (Theorem 3.7). Thus

$$\dim \operatorname{Hom}_{[1]}(B, X) = 6.$$

By Theorem 6.2, $l_Q \cup l'_Q$ deforms to an irreducible curve. This is absurd, since $l_Q \subset E_2$, $\not\subset E_3$, and $l'_Q \not\subset E_2$, $\subset E_3$.

Hence E has exactly two irreducible components, namely, $E = E_1 \cup E_2$ in (8.6.14). Assume φ' , say, satisfies $\dim \varphi'(E_2) = 2$. Then $\bar{E} := \varphi'(E)$ is an irreducible surface, and hence the structure morphism $h' : U_{\varphi'} \rightarrow V$ satisfies the same condition as in (8.5.12). Thus as in (8.5.13), we conclude that V is smooth. $\square$

PROPOSITION 8.7. *Under the same assumption as in Lemma 8.4, assume that E has at least one irreducible component E_1 which is isomorphic either to $\mathbb{P}^2$ or S_2 (Lemma 8.4 (c)). Then E has another irreducible component which is isomorphic either to Σ_1 or $\mathbb{P}^1 \times \mathbb{P}^1$.*

PROOF. Since $\rho(U/V) = 2$ while $\rho(E_1) = 1$, E necessarily admits another irreducible component. Now we assume furthermore that all the irreducible components of E are isomorphic either to $\mathbb{P}^2$ or S_2 , to get a contradiction.

We put the equivalence relation into $\{E_1, \dots, E_n\}$, the whole set of irreducible components of E , as follows:

(8.7.1) $E_i \sim E_j$ if and only if there exists a subset $\{E_{i,0} = E_i, E_{i,1}, \dots, E_{i,r} = E_j\}$ of $\{E_1, \dots, E_n\}$ such that for each $k = 0, \dots, r-1$, $E_{i,k} \cap E_{i,k+1}$ contains a rational curve of length 1.

Then by the same reason as above, this relation is disconnected, *i.e.*

(8.7.2) There is a partition $\{1, \dots, n\} = I \coprod J$ ($I, J \neq \emptyset$) such that $E_i \sim E_{i'}$ for each $i, i' \in I$, and $E_i \not\sim E_j$ for each $i \in I$ and $j \in J$.

Choose $i \in I$ arbitrarily, and let l be any line (*respectively* any ruling) of E_i when $E_i \simeq \mathbb{P}^2$ (*respectively* $E_i \simeq S_2$). Consider the deformation locus $L_U(l)$. By Theorem 3.7, this is a prime divisor in U . Moreover

$$(8.7.3) \quad \begin{cases} L_U(l) \supset \bigcup_{i \in I} E_i, \text{ and} \\ L_U(l) \not\supset E_j \ (\forall j \in J) \end{cases}$$

by the above construction. By the connectedness of E , we find some $j \in J$ such that $L_U(l) \cap E_j \neq \emptyset$. Since U is a smooth 4-fold and $\dim L_U(l) = 3$, $\dim E_j = 2$,

$$(8.7.4) \quad \dim(L_U(l) \cap E_j) = 1$$

for such j . Let C be any irreducible component of $L_U(l) \cap E_j$, and let l_j be a general line or a general ruling in E_j . Then $[l], [l_j] \in \mathbb{R}_{\geq 0}[C]$, and hence

$$(8.7.5) \quad l \equiv l_j \quad \text{in } U.$$

Since $l_j \not\subset L_U(l)$,

$$(8.7.6) \quad (L_U(l) \cdot l_j) \geq 0.$$

On the other hand,

$$(8.7.7) \quad (L_U(l) \cdot l) = -1$$

(Theorem 3.7). (8.7.6) and (8.7.7) contradict (8.7.5).

Hence E admits at least one irreducible component which is isomorphic neither to $\mathbb{P}^2$ nor S_2 . By Lemma 8.4, Proposition 8.5 and 8.6, such a component must be isomorphic either to Σ_1 or $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

PROOF OF THEOREM 8.1. Now our Theorem 8.1 is an easy consequence of Lemma 8.4, Proposition 8.5, 8.6, and 8.7. $\square$

9. – Deformation loci of extremal rational curves of length 1
(the case $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$)

ASSUMPTION 9.0. Throughout this and the next sections, we assume

$$(9.0.1) \quad l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

In this and the next sections, we shall investigate the structure of E under the above assumption. In this section, we shall prove the following:

THEOREM 9.1. *Let $g : X \rightarrow Y$, R , E and P be as above. Assume*

$$l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

Then for any rational curve l of length 1 in E , the whole deformation locus $L_X(l)$ of l in X (Definition 1.2) is purely codimension 1 in X . Each irreducible component of $L_X(l)$ contains E , whenever it meets E .

ASSUMPTION 9.2. According to Lemma 2.7, it is sufficient to do the following in order to prove Theorem 9.1:

(9.2.1) We assume that $g|_{U-E} : U - E \rightarrow V - P$ is a $\mathbb{P}^1$ -bundle, and shall derive a contradiction.

LEMMA 9.3. *Under the Assumption 9.0 and 9.2,*

$$(E_i, \mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ for any irreducible component } E_i \text{ of } E.$$

PROOF. For any irreducible component E_i of E , we have $l_{E_i}(R) = 1$ by Theorem 5.1. Then

$$(\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ for each } i,$$

by Assumption 9.2 and Lemma 2.7 (2).

We shall prove that each E_i is normal, i.e. E_i itself is isomorphic to $\mathbb{P}^2$. First of all, consider the rational map associated to $|-K_X|_{E_i}|$:

$$\Phi : E_i \rightarrow \mathbb{P}^N.$$

Note that since $|-K_X|_{E_i}|$ is free (Theorem 4.1), the above Φ is actually a morphism.

We claim that $N = \dim |-K_X|_{E_i}| = 2$. In fact, there is a natural injective homomorphism

$$H^0(E_i, \mathcal{O}_{E_i}(-K_X)) \hookrightarrow H^0(\tilde{E}_i, v_i^* \mathcal{O}_{E_i}(-K_X)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)).$$

The last term has dimension 3, whereas for the first term,

$$\dim H^0(E_i, \mathcal{O}_{E_i}(-K_X)) \geq 3,$$

since the linear system $| -K_X|_{E_i}|$ on a surface E_i is free and ample. Thus the above injection must be an isomorphism and we deduce $N = 2$. Then we consider the composition;

$$\mathbb{P}^2 \simeq \tilde{E}_i \xrightarrow{\nu_i} E_i \xrightarrow{\Phi} \mathbb{P}^2$$

which is exactly the rational map associated to the linear system $|\nu_i^*(-K_X|_{E_i})| = |\mathcal{O}_{\mathbb{P}^2}(1)|$, namely, an isomorphism $\mathbb{P}^2 \rightarrow \mathbb{P}^2$. Thus the morphism $\Phi : E_i \rightarrow \mathbb{P}^2$ is set theoretically a bijection. By Zariski Main theorem, we conclude that Φ is actually an isomorphism: $E_i \simeq \mathbb{P}^2$. $\square$

From now on, we shall show that the normal bundle $N_{E_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ for each E_i (Proposition 9.6), following the argument of Kawamata ([15] (2.4)).

NOTATION 9.4. Let E_i be an arbitrary irreducible component of E , and l_0 any line in E_i which is not contained in any other E_j . Then we can take a member D of $| -K_U|$ such that

$$(9.4.1) \quad \begin{cases} (1) & D \cap E_i = l_0, \text{ and} \\ (2) & D \text{ is smooth along } l_0. \end{cases}$$

In fact, we can choose first of all $D \in | -K_U|$ which satisfies (1) by the g -freeness of $| -K_U|$ and $-K_U|_{E_i} = \mathcal{O}_{\mathbb{P}^2}(1)$. Then the same argument as in (5.7.5) shows that D is necessarily smooth along l_0 , namely, D satisfies also (2).

Let $h_D : D \rightarrow \tilde{V}$ be the Stein factorization of $g|_D : D \rightarrow V$. This is a bimeromorphic morphism and $h_D(l_0)$ is a point. Let $\text{Exc}(h_D)$ be its exceptional locus. Each irreducible component of $\text{Exc}(h_D)$ has dimension either 1 or 2.

LEMMA 9.5. *Under the Notation 9.4, if we choose a sufficiently general $D \in | -K_U|$ which satisfies (9.4.1), then $g|_{D-(D \cap E)} : D - (D \cap E) \rightarrow Y - P$ is a finite morphism. In particular, l_0 forms a whole irreducible component of $\text{Exc}(h_D)$.*

PROOF. Since each fiber of $g|_{U-E} : U - E \rightarrow V - P$ is isomorphic to the reduced $\mathbb{P}^1$ (Assumption 9.2), this is an immediate consequence of Lemma 4.3. $\square$

PROPOSITION 9.6 (Kawamata [15] (2.4)). *Under Assumption 9.2, Lemma 9.3 and Notation 9.4, the normal bundle $N_{E_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ for each E_i .*

PROOF. Let $l_0 \subset E_i$ be any line with $l_0 \not\subset \bigcup_{j \neq i} E_j$. Then by Lemma 9.5, there is a member $D_0 \in | -K_U|$ which satisfies (9.4.1) such that $g|_{D_0-D_0 \cap E} : D_0 - D_0 \cap E \rightarrow Y - P$ is a finite morphism. By deforming D_0 inside $| -K_U|$, we obtain a one-parameter family $\mathcal{D} \rightarrow \Delta$ (where Δ is the unit disc) of members $D_t := (\mathcal{D})_t$ of $| -K_U|$ which satisfies the following:

$$(9.6.0) \quad \begin{cases} l_t := D_t \cap E_i \text{ is a line in } E_i, \\ D_t \text{ is smooth along } l_t, \text{ and} \\ g|_{D_t-(D_t \cap E)} : D_t - (D_t \cap E) \rightarrow Y - P \text{ is a finite morphism.} \end{cases}$$

Let $l'_t := D_t \cap (\bigcup_{j \neq i} E_j)$. Then $g|_U$ induces a surjective morphism ψ defined as:

$$\begin{array}{ccc} \psi : \mathcal{D} & \longrightarrow & V \times \Delta \\ \bigcup & & \bigcup \\ g|_{D_t} : D_t & \longrightarrow & V \times \{t\} \end{array}$$

We take the Stein factorization $\psi' : \mathcal{D} \rightarrow (V \times \Delta)^\sim$ of ψ . Then

(9.6.1) l_t is an irreducible component of the exceptional locus of $(\psi')_t : D_t \rightarrow (V \times \Delta)_t^\sim$ for each $t \in \Delta$

by (9.6.0). A general ψ' -very ample divisor H on $\mathcal{D}$ is decomposed as

$$H = H_1 + H_2, \quad H_1 \cap l_t = \emptyset, \quad H_2 \cap l'_t = \emptyset \quad (\forall t \in \Delta),$$

by shrinking $(V \times \Delta)^\sim$ if necessary. Then $|H_1|$ gives a projective surjective morphism $\varphi : \mathcal{D} \rightarrow \mathcal{D}'$ which is, on each fiber over Δ , a flopping contraction of the smooth 3-fold D_t whose exceptional locus is just $l_t \simeq \mathbb{P}^1$ (9.6.0). Thus there are only three possibilities:

$$(9.6.2) \quad N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ or } \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$$

Reid [38], whereas the left-hand side is naturally isomorphic to $N_{E_i/X} \otimes \mathcal{O}_{l_t}$.

From now on, we shall show that the type of N_{l_t/D_t} in (9.6.2) is locally constant on t . Let L be an arbitrary divisor on $\mathcal{D}$ such that $(L \cdot l_t) < 0$ for all $t \in \Delta$, and let $L' := \varphi(L)$.

CLAIM 1. *$\dim R^1 \varphi_{t*} \mathcal{O}_{D_t}(L)$ is constant on $t \in \Delta$, by shrinking Δ if necessary.*

PROOF. By ([36] Theorem 3), we have another projective bimeromorphic morphism (the so-called *symultaneous flop* of φ) $\varphi^+ : \mathcal{D}^+ \rightarrow \mathcal{D}'$ such that the proper transform L^+ of L is φ^+ -ample. Then by [18], $R^1 \varphi^+_* \mathcal{O}_{\mathcal{D}^+}(L^+ - D_t^+) = 0$, since $D_t^+ \equiv 0$. Thus by the exact sequence;

$$0 \longrightarrow \mathcal{O}_{\mathcal{D}^+}(L^+ - D_t^+) \longrightarrow \mathcal{O}_{\mathcal{D}^+}(L^+) \longrightarrow \mathcal{O}_{D_t^+}(L^+) \longrightarrow 0$$

we have a surjection

$$\varphi^+_* \mathcal{O}_{\mathcal{D}^+}(L^+) \twoheadrightarrow \varphi^+_* \mathcal{O}_{D_t^+}(L^+).$$

Let $D'_t := (D')_t$, then this surjection is nothing but

$$(9.6.3) \quad \mathcal{O}_{\mathcal{D}'}(L') \twoheadrightarrow \mathcal{O}_{D'_t}(L').$$

On the other hand, by

$$0 \longrightarrow \mathcal{O}_{\mathcal{D}}(L - D_t) \longrightarrow \mathcal{O}_{\mathcal{D}}(L) \longrightarrow \mathcal{O}_{D_t}(L) \longrightarrow 0,$$

we have the induced exact sequence:

$$\begin{aligned} \varphi_* \mathcal{O}_{\mathcal{D}}(L) &\rightarrow \varphi_{t*} \mathcal{O}_{D_t}(L) \rightarrow R^1 \varphi_* \mathcal{O}_{\mathcal{D}}(L - D_t) \\ &\rightarrow R^1 \varphi_* \mathcal{O}_{\mathcal{D}}(L) \rightarrow R^1 \varphi_{t*} \mathcal{O}_{D_t}(L) \rightarrow 0 \end{aligned}$$

Here the first arrow is equal to $\mathcal{O}_{\mathcal{D}'}(L') \rightarrow \mathcal{O}_{D'_t}(L')$ and this is surjective (9.6.3). Thus the following is exact:

$$0 \longrightarrow R^1 \varphi_* \mathcal{O}_{\mathcal{D}}(L - D_t) \longrightarrow R^1 \varphi_* \mathcal{O}_{\mathcal{D}}(L) \longrightarrow R^1 \varphi_{t*} \mathcal{O}_{D_t}(L) \longrightarrow 0.$$

Hence $\dim R^1 \varphi_{t*} \mathcal{O}_{D_t}(L)$ is independent on t . $\square$

PROOF OF PROPOSITION 9.6 (continued). Let I_t be the defining ideal of l_t (with reduced structure) in D_t . By the formal function theorem,

$$(R^1\varphi_{t*}\mathcal{O}_{D_t}(L))^{\wedge}_{\varphi_t(l_t)} = \varprojlim H^1(l_t, \mathcal{O}_{D_t}/I_t^m \otimes \mathcal{O}_{D_t}(L)).$$

Here the inverse limit above is taken with respect to the sequence of homomorphisms $\{\rho_m\}_{m \geq 2}$ given by:

$$(9.6.4) \quad \left\{ \begin{array}{ll} H^1(I_t^{m-1}/I_t^m \otimes \mathcal{O}_{D_t}(L)) & \longrightarrow H^1(\mathcal{O}_{D_t}/I_t^m \otimes \mathcal{O}_{D_t}(L)) \\ & \xrightarrow{\rho_m} H^1(\mathcal{O}_{D_t}/I_t^{m-1} \otimes \mathcal{O}_{D_t}(L)) \\ & \longrightarrow 0 \quad (m \geq 3) \text{ and} \\ 0 \longrightarrow H^1(I_t/I_t^2 \otimes \mathcal{O}_{D_t}(L)) & \longrightarrow H^1(\mathcal{O}_{D_t}/I_t^2 \otimes \mathcal{O}_{D_t}(L)) \\ & \xrightarrow{\rho_2} H^1(\mathcal{O}_{D_t}(L)) \longrightarrow 0 \end{array} \right.$$

Note that each ρ_m ($m \geq 2$) is surjective.

CLAIM 2. *Let L be chosen so that $(L \cdot l_t) = -1$. Then, $R^1\varphi_{t*}\mathcal{O}_{D_t}(L) \neq 0$ if and only if $N_{l_t/D_t} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. (See (9.6.2)).*

PROOF. If $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ i.e. $I_t/I_t^2 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, then

$$H^1(I_t/I_t^2 \otimes \mathcal{O}_{D_t}(L)) = H^1(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \neq 0$$

and ρ_2 in (9.6.4) is not an isomorphism. Thus $R^1\varphi_{t*}\mathcal{O}_{D_t}(L) \neq 0$.

If $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ i.e. $I_t/I_t^2 \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ respectively, then $H^1(\mathcal{O}_{D_t}(L)) = H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and

$$H^1(I_t^{m-1}/I_t^m \otimes \mathcal{O}_{D_t}(L)) = H^1(S^{m-1}(I_t/I_t^2) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

Thus $H^1(\mathcal{O}_{D_t}/I_t^m \otimes \mathcal{O}_{D_t}(L)) = 0$ for all m by (9.6.4), and hence

$$R^1\varphi_{t*}\mathcal{O}_{D_t}(L) = 0. \quad \square$$

CLAIM 3. *Let L be chosen so that $(L \cdot l_t) = -2$. Then*

(i) $\dim R^1\varphi_{t*}\mathcal{O}_{D_t}(L) = 1$ if $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

(ii) $\dim R^1\varphi_{t*}\mathcal{O}_{D_t}(L) \geq 2$ if $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

PROOF. First we note $\dim H^1(\mathcal{O}_{l_t}(L)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2)) = 1$. In particular, $R^1\varphi_{t*}\mathcal{O}_{D_t}(L) \neq 0$ by (9.6.4). If $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ i.e. $I_t/I_t^2 \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, then

$$H^1(I_t^{m-1}/I_t^m \otimes \mathcal{O}_{D_t}(L)) = H^1(S^{m-1}(I_t/I_t^2) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0,$$

and again by (9.6.4), all ρ_m ($m \geq 2$) is an isomorphism. Thus (i) holds.

If $N_{l_t/D_t} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ i.e. $I_t/I_t^2 \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, then

$$\dim H^1(I_t/I_t^2 \otimes \mathcal{O}_{D_t}(L)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) = 1,$$

and $\dim H^1(\mathcal{O}_{D_t}/I_t^2 \otimes \mathcal{O}_{D_t}(L)) = 2$ in (9.6.4). Thus (ii) holds. $\square$

PROOF OF PROPOSITION 9.6 (continued). Claim 1, Claim 2, and Claim 3 implies that the isomorphism type of $N_{l_t/D_t} \simeq N_{E_i/X} \otimes \mathcal{O}_{l_t}$ is constant on $t \in \Delta$. Since l_0 was arbitrarily chosen so that $l_0 \not\subset \bigcup_{j \neq i} E_j$, we deduce that there are at most finitely many lines at which $N_{E_i/X}$ jumps. We again put $N_{E_i/X} \otimes \mathcal{O}_l = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $(a, b) = (-1, -1)$, $(0, -2)$ or $(1, -3)$, for a general line l of E_i . If $(a, b) = (0, -2)$ or $(1, -3)$, then by ([35] p. 205, Theorem 2.1.4) which is a generalization of Grauert-Mülich's theorem [9], $N_{E_i/X}$ in fact splits into line bundles:

$$N_{E_i/X} \simeq \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b).$$

In particular $H^0(N_{E_i/X}) \neq 0$ and $H^1(N_{E_i/X}) = 0$, which implies that E_i deforms inside X , a contradiction. Thus (a, b) must be $(-1, -1)$ and, in turn, by Van de Ven's theorem [46], we have $N_{E_i/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, and the proof of Proposition 9.6 is now completed. $\square$

9.7. PROOF OF THEOREM 9.1. We assume (9.2.1), and the results Lemma 9.3, Proposition 9.6, and will derive a contradiction. Since $E_i \simeq \mathbb{P}^2$ and $N_{E_i/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ for each E_i (Lemma 9.3 and Proposition 9.6),

(9.7.1) There is a birational morphism $\varphi_i : X \rightarrow X'_i$ over Y to some normal 4-fold X'_i such that $\varphi_i(E_i)$ is a point, say Q_i , and φ_i is an isomorphism elsewhere.

Let $\psi_i : X'_i \rightarrow Y$ be the induced morphism. $g|_{X-E_i} = \psi_i|_{X'_i-Q_i}$ by construction.

First we claim that

(9.7.2) E is reducible.

In fact, if E is irreducible, then the fiber of $\psi := \psi_i : X' \rightarrow Y$ at $P = g(E)$ is the point $Q := Q_i$, while another fiber of ψ is a curve since $g|_{X-E} = \psi|_{X'-Q}$, which contradicts the upper semi-continuity of the fiber dimension. Thus E has at least two irreducible components.

Next,

(9.7.3) For any E_i, E_j ($i \neq j$) with $E_i \cap E_j \neq \emptyset$, $E_i \cap E_j$ is a finite set of points.

In fact, if $E_i \cap E_j$ contains a curve, say C , then $\varphi_i(C) = \{Q_i\}$, and $\varphi_i|_{E_j} : E_j \rightarrow \varphi_i(E_j)$ is an isomorphism outside C . This is impossible, since $E_j \simeq \mathbb{P}^2$. Hence (9.7.3) holds.

(9.7.2) and (9.7.3) imply $\rho^{\text{an}}(X \supset E / Y \ni P) \geq 2$, which contradicts the assumption. Hence the theorem. $\square$

10. – The classification of E in the case $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$

In this section, we shall classify E in the case $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$. This case is difficult, and we have at present a partial answer to the classification:

THEOREM 10.1. *Assume that*

$$l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

Then one of the following holds:

- (1) *E is irreducible, $E \simeq \mathbb{P}^2$, and $\mathcal{O}_E(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$.*
- (2) *$E = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$, and $E_1 \cap E_2$ is a line of both E_i .*

$$\mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \quad (i = 1, 2).$$

- (3) *E is irreducible. Let $v : \tilde{E} \rightarrow E$ be the normalization. Then $\tilde{E} \simeq S_m$, and*

$$v^* \mathcal{O}_E(-K_X) \simeq \mathcal{O}_{S_m}(1).$$

- (4) *E has two or three irreducible components: $E = E_1 \cup E_2 (\cup E_3)$. Let $v_i : \tilde{E}_i \rightarrow E_i$ be the normalization of E_i ($i = 1, 2, 3$). Then $\tilde{E}_i \simeq S_{m_i}$ for some $m_i \geq 2$, and*

$$v_i^* \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{S_{m_i}}(1).$$

Moreover, let v_i be the vertex of $\tilde{E}_i$, then $v_1(v_1) = v_2(v_2) (= v_3(v_3)) =: Q$ in E . $E_i \cap E_j$ is a ruling of E_i, E_j ($i \neq j$).

- (5) *$E = E_1 \cup E_2$, $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq S_2$. $E_1 \cap E_2$ is a line of E_1 and a ruling of E_2 .*

$$\mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1), \text{ and } \mathcal{O}_{E_2}(-K_X) \simeq \mathcal{O}_{S_2}(1).$$

First of all we shall prove the following:

PROPOSITION 10.2. *Under Assumption 9.0, the following holds:*

- (1) *Let l be any rational curve of length 1 in E . Then each irreducible component L of $L_U(l)$ (Definition 1.2) contains E and is numerically g_U -trivial: $L \supset E, L \equiv_{g_U} 0$.*
- (2) *Let l and l' be two rational curves of length 1 in E shch that $l \cup l'$ deforms outside E . Then l and l' deform inside E to each other.*

PROOF. (1) Since $L_X(l)$ is purely codimension 1 in X (Theorem 9.1), $L_U(l)$ is also purely codimension 1 in U . Since each irreducible component L of $L_U(l)$ is disjoint from a general fiber of g_U and since $\rho(U/V) = 1$, we have (1).

(2) By the assumption, there is an irreducible component L (respectively L') of $L_U(l)$ (respectively $L_U(l')$) such that $g_U(L) = g_U(L')$. Then by (1), $L = L'$. In particular, l and l' deform inside U , and hence inside E , to each other. $\square$

PROPOSITION 10.3. *Under Assumption 9.0, E never contains any irreducible component whose normalization is a geometrically ruled surface.*

PROOF. Assume that E contains an irreducible component, say E_1 , whose normalization $\tilde{E}_1$ is isomorphic to a geometrically ruled surface. Then by Theorem 3.1,

$$(10.3.0) \quad \tilde{E}_1 \simeq \Sigma_1 \quad \text{or} \quad \mathbb{P}^1 \times \mathbb{P}^1,$$

as in the proof of Lemma 8.4. Let $\{\tilde{M}, \tilde{l}\}$ be the minimal section and a ruling (respectively two intersecting rulings) of $\tilde{E}_1$, when $\tilde{E}_1 \simeq \Sigma_1$ (respectively $\tilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$), and $M := \nu(\tilde{M})$ and $l := \nu(\tilde{l})$. Then

$$(10.3.1) \quad \nu^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\Sigma_1}(\tilde{M} + 2\tilde{l}) \quad (\text{respectively } \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\tilde{M} + \tilde{l})).$$

Recall that $|-K_X|$ is g -free (Theorem 4.1). Thus for a general smooth $D \in |-K_U|$, $C_D := D \cap E_1$ is a smooth irreducible curve with $\nu^* C_D \sim \tilde{M} + 2\tilde{l}$ (respectively $\tilde{M} + \tilde{l}$), which is ample. If E_1 has a singular locus of dimension 1, C_D intersects with $\text{Sing } E_1$ for a general D , a contradiction. Hence

$$(10.3.2) \quad \#\text{Sing } E_1 < +\infty.$$

We shall prove that

CLAIM. *M and l never deform inside E to each other.*

PROOF. Assume to the contrary, and let

$$(10.3.3) \quad E^- := \bigcup_{i \neq 1} E_i.$$

Since M and l never deform inside E_1 to each other (10.3.2),

(10.3.4) There is a suitable deformation l' of l inside E_1 (respectively there are suitable deformations M' of M and l' of l inside E_1) such that M and l' (respectively M' and l') deform inside E^- to each other.

Let us denote l' (respectively M' and l') again by l (respectively M and l), respectively, by abuse of notations. Then (10.3.4) is equivalent to say that there is a family $\{m_t\}_{t \in T}$ of 1-cycles of U with a 1-dimensional connected parameter space T (maybe reducible) such that

$$(10.3.5) \quad (-K_U \cdot m_t) = 1, \quad m_t \subset E^- \quad (\forall t \in T), \quad m_{t_0} = M, \quad \text{and } m_{t_1} = l.$$

By the freeness of $|-K_U|$ (Theorem 4.1), we can take a general smooth member $D \in |-K_U|$ so that

$$\begin{cases} (10.3.6) & \dim(D \cap E) = 1, \quad H^1(\mathcal{O}_{D \cap E}) = 0, \\ (10.3.7) & Q_0 := D \cap M \quad \text{and} \quad Q_1 := D \cap l \\ & \text{are distinct points of } D \cap E_1, \quad \text{and} \\ (10.3.8) & \#\{t \in T \mid m_t \subset D\} < +\infty. \end{cases}$$

Let $C_1 := D \cap E_1$, and $C^T := D \cap \bigcup_{t \in T} m_t$. Obviously

$$(10.3.9) \quad Q_0, Q_1 \in C_1 \cap C^T.$$

Moreover by (10.3.5) and (10.3.8),

$$(10.3.10) \quad C^T \text{ is connected, purely 1-dimensional, and is contained in } E^-.$$

(10.3.7), (10.3.9) and (10.3.10) imply that $D \cap E$ has two connected sub-1-cycles C_1 and C^T which have no common irreducible components, and both of which contains Q_0 and Q_1 . This contradicts $H^1(\mathcal{O}_{D \cap E}) = 0$ (10.3.6), as in the proof of Proposition 8.5 or 8.6. Hence the claim. $\square$

PROOF OF PROPOSITION 10.3 (continued). We shall get a contradiction in the following cases, in view of (10.3.0):

$$\text{Case(1)} \quad \tilde{E}_1 \simeq \Sigma_1.$$

$$\text{Case(2)} \quad \tilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

For Case (1) : Take any irreducible component L of $L_U(M)$ which contains M . Then by the claim above, together with the fact that M is rigid inside E_1 , we have $L \not\supset l$, in particular $(L \cdot l) > 0$. This contradicts $L \equiv 0$ (Proposition 10.2 $_{SU}$).

(1)). Hence $\tilde{E}_1 \simeq \Sigma_1$ is impossible.

For Case (2): Recall that $\nu^* \mathcal{O}_{E_1}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$

(10.3.1). Since any two points of E_1 are joined by a limit conic (Theorem 3.1), and since $\# \text{Sing } E_1 < +\infty$ (10.3.2), it follows that

$$(10.3.11) \quad \text{For some limit conic } C_0 \text{ which is contained in } E_1, \nu^* \mathcal{O}_{E_1}(C_0) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1).$$

Again by (10.3.2),

$$\{\nu^* C \mid C \text{ is a deformation of } C_0 \text{ in } E_1\} = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|.$$

Hence there is a deformation C of a general fiber of g which is contained in E_1 such that C is reducible: $C = M + l$, and

$$\nu^* M \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0)|, \quad \nu^* l \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)|.$$

Then by Proposition 10.2 (2), M and l deform inside E to each other, which contradicts the above claim. Hence $\tilde{E}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is also impossible, and the proof of Proposition 10.3 is finished. $\square$

By combining Theorem 2.8 and Proposition 10.3, we immediately get:

COROLLARY 10.4. *Assume*

$$l_E(R) = 1 \text{ and } \rho^{\text{an}}(X \supset E / Y \ni P) = 1.$$

Then $(\tilde{E}_i, \nu_i^* \mathcal{O}_{E_i}(-K_X))$ is isomorphic to one of the following:

- (1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, or
- (2) $(S_m, \mathcal{O}_{S_m}(1))$ ($m \geq 2$).

□

COROLLARY 10.5. *Under the assumption $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$, Any two rational curves of length 1 deform inside E to each other.*

PROOF. By Corollary 10.4,

(10.5.1) Any two rational curves of length 1 in E_i deform inside E_i to each other.

On the other hand, let l be any rational curve of length 1 in any irreducible component E_i of E . Then

$$(10.5.2) \quad L_U(l) \supset E$$

(Proposition 10.2 (1)). (10.5.1) and (10.5.2) immediately imply that l deforms to any rational curve of length 1 in E . □

PROPOSITION 10.6. *If E has an irreducible component, say E_1 , whose normalization $\tilde{E}_1$ is isomorphic to $\mathbb{P}^2$, then E satisfies one of the following:*

- (1) E is irreducible and is isomorphic to $\mathbb{P}^2$,
- (2) E has two or three irreducible components: $E = E_1 \cup E_2 (\cup E_3)$, all of which are isomorphic to $\mathbb{P}^2$, and $E_1 \cap E_2$ ($E_2 \cup E_3$) is a line in E_i ($i = 1, 2(, 3)$), or
- (3) $E = E_1 \cup E_2$ with $E_1 \simeq \mathbb{P}^2$ and $E_2 \simeq S_2$. $E_1 \cap E_2$ is a line of E_1 and a ruling of E_2 .

PROOF. Assume that E_1 say, is an irreducible component of E whose normalization $\tilde{E}_1$ is isomorphic to $\mathbb{P}^2$. Then since $|-K_X|$ is g -free (Theorem 4.1), exactly the same argument as in Lemma 9.3 shows that $E_1 \simeq \mathbb{P}^2$. Thus, by Corollary 10.4,

For each irreducible component E_i of E , one of the following holds:

$$\begin{cases} (10.6.1) & E_i \simeq \mathbb{P}^2, \text{ or} \\ (10.6.2) & \text{The normalization } \tilde{E}_i \text{ of } E_i \text{ is isomorphic to } S_m \text{ } (m \geq 2). \end{cases}$$

Now assume that E has irreducible components one of which is of type (10.6.1) and another of type (10.6.2).

Since any two rational curves of length 1 deform inside E to each other (Corollary 10.5), we may assume that there is a couple $\{E_i, E_j\}$ of irreducible components of type (10.6.1), (10.6.2), respectively, of E such that

(10.6.3) $E_i \cap E_j$ contains a rational curve l_0 which is a line of E_i and is the image by v_j of a ruling of $\tilde{E}_j$. In particular, l_0 contains the image by v_j of the vertex of $\tilde{E}_j \simeq S_m$.

(10.6.4) Take a general line $l \neq l_0$ of E_i which passes through v_j , but is not contained in E_j .

Since $|-K_X|$ is g -free and since $\mathcal{O}_{E_i}(-K_U) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, we can take $D \in |-K_U|$ such that $D|_{E_i} = l$. Such a D is necessarily smooth, by the same reason as in (5.7.5) or (9.4.1). In particular

$$(10.6.5) \quad H^1(\mathcal{O}_{D \cap E}) = 0.$$

On the other hand, since $D \supset l \ni v_j$ and since $v_j^* \mathcal{O}_{E_j}(-K_X) \simeq \mathcal{O}_{S_m}(1)$, it follows that $D \cap E_j$ consists of m rational curves meeting at v_j ;

$$(10.6.6) \quad D \cap E_j = \bigcup_{k=1}^m l_k, \quad \bigcap_{k=1}^m l_k \ni v_j,$$

($m \geq 2$). Since $D \supset l$ and $E_j \not\supset l$ (10.6.4), it follows that $D \cap E$ contains $m+1$ rational curves meeting at v_j . If $m \geq 3$, then this contradicts (10.6.5). If $m = 2$, then since D is smooth, E_j is Cohen-Macaulay at v_j so that E_j itself is isomorphic to S_2 .

Hence we proved that

(10.6.7) Either all irreducible components of E are $\mathbb{P}^2$, $E = E_1 \cup E_2$ with $E_1 \simeq \mathbb{P}^2$, $E_2 \simeq S_2$, or all irreducible components are of type (10.6.2).

What we are left to prove is the following:

(10.6.8) If any irreducible component of E is $\mathbb{P}^2$, then E has at most three irreducible components;

(10.6.9) If E is reducible and $E_1 \simeq \mathbb{P}^2$, then the intersection with any other component is a line of E_1 .

For (10.6.8): Assume that E has at least four $\mathbb{P}^2$'s, to get a contradiction. By Corollary 10.5 again, E has three irreducible components E_1, E_2 , and E_3 such that $E_1 \cap E_2$ contains a line l_{12} of both E_1 and E_2 , and $E_2 \cap E_3$ contains a line l_{23} of both E_2 and E_3 . Then by considering a general $D \in |-K_U|$ passing through the point $l_{12} \cap l_{23}$, we get a contradiction as exactly the same argument as above. Hence (10.6.8).

For (10.6.9): By (10.6.7), (10.6.8), either $E = E_1 \cup E_2 (\cup E_3)$ with $E_1 \simeq E_2 (\simeq E_3) \simeq \mathbb{P}^2$, or $E = E_1 \cup E_2$ with $E_1 \simeq \mathbb{P}^2$, $E_2 \simeq S_2$.

First if $E_1 \cap E_2$ contains a curve other than a line of E_1 , then for a general $D \in |-K_U|$, $D \cap E$ has two irreducible curves intersecting with at least two distinct points to each other, a contradiction. Next if $E_1 \cap E_2$ has a 0-dimensional connected component, say $\{Q\}$, then consider a general line l in E_1 passing through Q . Then we can take a smooth $D \in |-K_U|$ such that $D|_{E_1} = l$, as in (5.7.5) or (9.4.1). Since $E_1 \cap E_2$ contains a line of E_1 which does not pass through Q , it follows that $D \cap E$ has two irreducible components meeting at two distinct points, again a contradiction. Hence (10.6.9). $\square$

PROPOSITION 10.7. *If E consists only of irreducible components E_i whose normalization $\tilde{E}_i$ are isomorphic to S_{m_i} ($m_i \geq 2$), then E satisfies one of the following:*

(1) E is irreducible: $\tilde{E} \simeq S_m$, or

(2) E has two or three irreducible components: $E = \bigcup_{i=1}^N E_i$ ($N = 2$ or 3), $\tilde{E}_i \simeq S_{m_i}$, and $E_i \cap E_j$ is the image by v_i of a ruling of $\tilde{E}_i$ for $(i, j) = (1, 2), (2, 3)$. The images of the vertices are the same point.

PROOF. This proof is inspired by Iskovskikh [11], Takeuchi [44]. (cf. [41], [37], [31], [32]). We consider a relative and 4-dimensional analogue of the projection $U \dashrightarrow U'$ of Fano-Iskovskikh-Takeuchi in the following:

If E is irreducible, then there is nothing to prove.

Assume that E is reducible. By Corollary 10.4 and Proposition 10.6, each E_i has the normalization $\tilde{E}_i \simeq S_{m_i}$ ($m_i \geq 2$). Let l_i and v_i be the image by v_i of an arbitrary ruling and the vertex of $\tilde{E}_i$, respectively. Since l_i 's are the only rational curves of length 1 in E_i , v_i ($i = 1, \dots, n$) must be the same point of E , by Theorem 3.1. We denote the point $v_1 = \dots = v_n$ simply by $v \in E$.

Let $\Delta^\circ \subset V - P$ be the discriminant divisor [4] of the conic bundle $g_U|_{U-E} : U - E \rightarrow V - P$, and let $\Delta := \Delta^\circ \cup P$. Moreover let $S_v \ni v$ be the connected component of the intersection of two general g_U -very ample divisors both of which pass through v so that

$$(10.7.1) \quad \begin{cases} S_v \text{ is a smooth surface proper over } V, \\ g_U|_{S_v-v} : S_v - v \rightarrow g_U(S_v) - P \text{ is an isomorphism, and} \\ \dim(g_U(S_v) \cap \Delta) \leq 1. \end{cases}$$

Consider the blow-up $\varphi : \bar{U} \rightarrow U$ with center S_v . Let $\bar{E}$, $\bar{E}_i$, and $\bar{l}_i$ be the proper transform of E , E_i , and l_i , respectively, and let $m := \varphi^{-1}(v) \simeq \mathbb{P}^1$. Since $\bar{U}$ is also a smooth 4-fold,

$$(10.7.2) \quad \bar{E}_i \supset m \quad (\forall i).$$

Since $\rho(U/V) = 1$, an easy calculation similar to (5.5.5) shows

$$(10.7.3) \quad \begin{cases} (-K_{\bar{U}} \cdot \bar{l}_i) = 0, & (-K_{\bar{U}} \cdot m) = 1, \text{ and} \\ \overline{NE}(U/V) = \mathbb{R}_{\geq 0}[\bar{l}_i] + \mathbb{R}_{\geq 0}[m]. \end{cases}$$

Consider the contraction morphism $\psi : \overline{U} \rightarrow U'$ associated to the half-line $\mathbb{R}_{\geq 0}[\overline{l}_i]$. By (10.7.3),

$$(10.7.4) \quad \begin{cases} -K_{\overline{U}} \equiv 0, \text{ and} \\ B := \psi(m) \text{ is not a point.} \end{cases}$$

Since l_i 's cover the whole $\overline{E}_i$, and these are all contracted by ψ , it follows that $\dim \psi(\overline{E}_i) \leq 1$. From this, (10.7.2), and (10.7.4),

$$(10.7.5) \quad \psi(\overline{E}_i) = B \quad (\forall i).$$

On the other hand, since $g_U|_{U-E} : U - E \rightarrow V - P$ is a conic bundle and $\varphi|_{\overline{U}-\overline{E}}$ is the blow-up of a codimension 2 subsection (10.7.1),

(10.7.6) Any irreducible curve in $\overline{U} - \overline{E}$ which is contracted by ψ must be lying over a point of $g_U(S_v) \cap \Delta$ via $g_U \circ \varphi$.

Since $\dim(g_U(S_v) \cap \Delta) \leq 1$ (10.7.1), it follows that $\dim \text{Exc}(\psi|_{\overline{U}-\overline{E}}) = 2$, and hence $\dim \text{Exc} \psi = 2$. Namely, ψ is a flopping contraction. From this and (10.7.5),

(10.7.7) $\overline{E}_i$'s are irreducible components of $\text{Exc} \psi$, with $\dim \psi(\overline{E}_i) = 1$.

(10.7.8) Now we shall prove that E has at most three irreducible components:

$$E = E_1 \cup E_2 (\cup E_3).$$

Assume that E has $n \geq 4$ irreducible components, to get a contradiction. Consider the fiber C of ψ over a general point of B (10.7.5). Then

(10.7.9) C has n irreducible components meeting at a point of m .

On the other hand, by (10.7.4) and [18], $R^1\psi_*\mathcal{O}_{\overline{U}} = 0$, and in particular

$$(10.7.10) \quad H^1(\mathcal{O}_C) = 0.$$

These contradict with each other. Hence (10.7.8).

Finally, by Corollary 10.5,

(10.7.11) There exists a rational curve l of length 1 which is contained in $E_1 \cap E_2$.

Assume that $E_1 \cap E_2 \supsetneq l$, to get a contradiction.

If $E_1 \cap E_2$ contains a curve other than l , then for a general smooth $D \in |-K_X|$ (Theorem 4.1), $D \cap E$ has two irreducible components which intersect at two distinct points with each other, a contradiction to $H^1(\mathcal{O}_{D \cap E}) = 0$.

If $E_1 \cap E_2$ contains a point, say Q , as a connected component, then let us go back to $\psi : \overline{U} \rightarrow U'$, and let $\overline{Q} := \varphi^{-1}(Q)$. Then the fiber C^Q of ψ containing $\overline{Q}$ has two irreducible components which meet at both $\overline{Q}$ and $C^Q \cap m$, which contradicts $H^1(\mathcal{O}_{C^Q}) = 0$ as in (10.7.10).

Hence $E_1 \cap E_2$ is exactly the image by v_i of some ruling in $\widetilde{E}_i \simeq S_{m_i}$ ($i = 1, 2$) (10.7.11), and we get (2) of the proposition. $\square$

PROOF OF THEOREM 10.1. Now the Theorem 10.1 is the combination of Proposition 10.6 and 10.7. $\square$

CONCLUSION. Theorem 0.5 is a combination of Theorem 3.1, 4.1, and 7.1.

Theorem 0.6 comes directly from Theorem 5.1. (The assumption of Theorem 5.1 is weaker than that of Theorem 0.6).

Theorem 0.7 and 0.8 are exactly Theorem 8.1 and 10.1 with 9.1, respectively.

11. – Examples

EXAMPLE 11.1. (Beltrametti ([5] Example 3.6), Mukai, Reid) Let $(x_0 : x_1 : x_2)$, $(y_0 : y_1 : y_2 : y_3)$ be the homogeneous coordinate of $\mathbb{P}^2$, $\mathbb{P}^3$, respectively, and let

$$X := \{x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

X is a Fano 4-fold of Picard number 2 (cf. [47], [33]). Let $g : X \rightarrow Y = \mathbb{P}^3$ be the second projection. Then X is a smooth projective 4-fold, and g is the contraction of an extremal ray of X . g has a fiber $E \simeq \mathbb{P}^2$ at $P = (0 : 0 : 0 : 1)$, while the other fibers are all lines in $\mathbb{P}^2$. $l_E(R) = 2$.

EXAMPLE 11.2. Let $g : X \rightarrow \mathbb{P}^3$ be as in Example 11.1. Let D be a general (2,2)-divisor of X , and $\beta : X' \rightarrow X$ the double cover branched over D . Then $g \circ \beta : X' \rightarrow Y$ has a fiber $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $l_{E'}(R) = 1$ and $\rho^{\text{an}}(X' \supset E' / Y \ni P) = 2$.

EXAMPLE 11.3. Let $g : X \rightarrow \mathbb{P}^3$ be as in Example 11.1. Let D be a general (2,2)-divisor of X whose intersection with $E \simeq \mathbb{P}^2$ is a reducible conic in E . Let $\beta : \tilde{X} \rightarrow X$ be the double cover branched over D . Then $g \circ \beta : X' \rightarrow Y$ has a fiber $E' \simeq S_2$. $l_{E'}(R) = 1$ and $\rho^{\text{an}}(X' \supset E' / Y \ni P) = 1$.

EXAMPLE 11.4. Let $g : X \rightarrow \mathbb{P}^3$ be as in Example 11.1. Let D be a general (2,2)-divisor of X whose intersection with $E \simeq \mathbb{P}^2$ is a double line in E . Let $\beta : \tilde{X} \rightarrow X$ be the double cover branched over D . Then $g \circ \beta : X' \rightarrow Y$ has a reducible fiber $E' = E_1 \cup E_2$, $E_1 \simeq E_2 \simeq \mathbb{P}^2$. $l_{E'}(R) = 1$ and $\rho^{\text{an}}(X' \supset E' / Y \ni P) = 1$.

EXAMPLE 11.5. Let $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2 : y_3)$ be as in Example 11.1. Let

$$X := \{x_0^2y_0 + x_1^2y_1 + x_2^2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^3,$$

and $g : X \rightarrow \mathbb{P}^3$ the second projection. Then g has a fiber $E \simeq \mathbb{P}^2$ at $y = (0 : 0 : 0 : 1)$, while the other fibers are all conics in $\mathbb{P}^2$. $l_E(R) = 1$ and $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$.

Shepherd-Barron told us the following:

EXAMPLE 11.6 (Shepherd-Barron). Let $E_0 \simeq S_3 \subset \mathbb{P}^4$ be the cone over a twisted cubic curve in $\mathbb{P}^3$. Let $C \subset E_0$ be a general quadric hypersurface section in $\mathbb{P}^4$, and $\pi : X \rightarrow \mathbb{P}^4$ the blow-up with center C . Let $E \subset X$ be the proper transform of E_0 . Then X is a Fano 4-fold of Picard number 2. Let $g : X \rightarrow Y$ be the contraction of the extremal ray R other than π . Then $Y \simeq \mathbb{P}^3$, and g has the fiber $E \simeq S_3$. $l_E(R) = 1$, and $\rho^{\text{an}}(X \supset E / Y \ni P) = 1$.

Mukai told us the following:

EXAMPLE 11.7 (Mukai). Let $E_0 \simeq \Sigma_1 \subset \mathbb{P}^4$ be a cubic surface in $\mathbb{P}^4$. Let $C \subset E_0$ be a general quadric hypersurface section, $\pi : X \rightarrow \mathbb{P}^4$ the blow-up with center C , and $E \subset X$ the proper transform of E_0 . Then X is a Fano 4-fold of Picard number 2, and the other contraction $g : X \rightarrow Y \simeq \mathbb{P}^3$ has the fiber $E \simeq \Sigma_1$. $l_E(R) = 1$, and $\rho^{\text{an}}(X \supset E / Y \ni P) = 2$.

Although we do not know at present any example of Mukai-Shepherd-Barron-Wisńiewski type, Wisńiewski told us the following, which satisfies all of the condition (MSW) except $\rho(X/Y) = 1$. Mukai also constructed a similar example.

EXAMPLE 11.8 (Wisńiewski). (locally of Mukai-Shepherd-Barron-Wisńiewski type) Let Y be a quadric hypersurface in $\mathbb{P}^4$ with an isolated vertex P . Let $h : W \rightarrow Y$ be the blow-up with the center P , and $F := \text{Exc } h \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then W has a structure of a $\mathbb{P}^1$ -bundle

$$W \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)) \xrightarrow{p} \mathbb{P}^1 \times \mathbb{P}^1 =: B,$$

and F is its section. Let $\mathcal{E} := p^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1))$, and

$$\phi_1 : Z^{(1)} := \mathbb{P}(\mathcal{E}) \rightarrow W$$

be the associated $\mathbb{P}^1$ -bundle. Then the tautological section $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathcal{E}|_F$ has the normal bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ in $Z^{(1)}$. Blow-up $Z^{(1)}$ with the center S :

$$\phi_2 : Z \rightarrow Z^{(1)}.$$

Let $\psi := h \circ \phi_1 \circ \phi_2 : Z \rightarrow Y$, then $-K_Z$ is ψ -ample and $\psi^{-1}(P)$ is an 'alternate union' $D_1 \cup D_2$:

$$D_1 \simeq \mathbb{P}^1 \times \Sigma_1, D_2 \simeq \Sigma_1 \times \mathbb{P}^1, \text{ with } D_1 \cap D_2 =: S' \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Let $\{l_1, l_2\}$ be a couple of intersecting rulings in S' . Then $\mathbb{R}_{\geq 0}[l_1]$ and $\mathbb{R}_{\geq 0}[l_2]$ are extremal rays of $\overline{NE}(Z/Y)$. Let

$$\pi_2 := \text{contr}_{\mathbb{R}_{\geq 0}[l_2]} : Z \rightarrow X^{(1)},$$

say, and let $l' := \pi_2(l_1)$. Then π_2 is a divisorial contraction which contracts D_2 to Σ_1 , and $X^{(1)}$ is smooth [1]. The fiber of $X^{(1)} \rightarrow Y$ at P is $\Sigma_1 \cup (\mathbb{P}^2 \times \mathbb{P}^1)$

intersecting along l' with each other, which is the negative section of Σ_1 and is a section of $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Finally let

$$\pi_1 := \text{contr}_{\mathbb{R}_{\geq 0}[l']} : X^{(1)} \rightarrow X.$$

Then π_1 is a divisorial contraction which contracts $\mathbb{P}^2 \times \mathbb{P}^1$ to $\mathbb{P}^2$, and X is smooth [1]. $g : X \rightarrow Y$ has the fiber $E := g^{-1}(P) \simeq \mathbb{P}^2 \cup \mathbb{P}^2$ intersecting at a single point, and is a $\mathbb{P}^1$ -bundle elsewhere. Namely, g is, locally near P , of Mukai-Shepherd-Barron-Wisńiewski type. Unfortunately, we can easily check that $\rho(X/Y) = 2$.

$$(11.8.1) \quad \begin{array}{ccccc} Z & \xrightarrow{\pi_2} & X^{(1)} & \xrightarrow{\pi_1} & X \\ \phi_2 \downarrow & & & & \downarrow g \\ Z^{(1)} & \xrightarrow{\phi_1} & W & \xrightarrow{h} & Y \end{array}$$

$$(11.8.2) \quad \begin{array}{ccccccc} (\mathbb{P}^1 \times \Sigma_1) \cup (\Sigma_1 \times \mathbb{P}^1) & \longrightarrow & \Sigma_1 \cup (\mathbb{P}^2 \times \mathbb{P}^1) & \longrightarrow & \mathbb{P}^2 \cup \mathbb{P}^2 \\ \downarrow & & & & \downarrow \\ \mathbb{P}^1 \times \Sigma_1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & P \end{array}$$

Finally we shall give an example of extremal contraction of smooth 4-folds which is a Del Pezzo fibration, whose special fiber is irreducible and irrational, in connection with Corollary 0.11 (cf. Fujita [8]).

EXAMPLE 11.9 (inspired by Hidaka-Oguiso, after Shepherd-Barron's 11.6 above). Let $E_0 \subset \mathbb{P}^4$ be the cone over an elliptic curve which is of (2,2)-complete intersection in $\mathbb{P}^3$. Let $C \subset E_0$ be a general quadric hypersurface section in $\mathbb{P}^4$, and $\pi : X \rightarrow \mathbb{P}^4$ the blow-up with center C . Let $E \subset X$ be the proper transform of E_0 . Then X is a Fano 4-fold of Picard number 2, and the other extremal ray defines $g : X \rightarrow \mathbb{P}^2$. g is a Del Pezzo fibration of degree 4, while it has a fiber E , which is an irrational surface.

REFERENCES

- [1] T. ANDO, *On extremal rays of the higher dimensional varieties*, Invent. Math. **81** (1985), 347-357.
- [2] M. ANDREATTA – J. WIŚNIEWSKI, *A note on nonvanishing and applications*, Duke Math. J. **72** (1993), 739-755.
- [3] M. ANDREATTA – J. WIŚNIEWSKI, *On good contractions of smooth varieties*, preprint (1996).

- [4] A. BEAUVILLE, *Variétés de Prym et Jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. **10** (1977), 309-391.
- [5] M. BELTRAMETTI, *On d-folds whose canonical bundle is not numerically effective, according to Mori and Kawamata*, Ann. Mat. Pura. Appl. **147** (1987), 151-172.
- [6] F. CAMPANA, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. **25** (1992), 539-545.
- [7] V. I. DANILOV, *Decomposition of certain birational morphisms*, Math. USSR.-Izv. **16** (1981), 419-429.
- [8] T. FUJITA, *On Del Pezzo fibrations over curves*, Osaka Math. J. **27** (1990), 229-245.
- [9] H. GRAUERT – G. MÜLICH, *Vektorbündel vom rang 2 über dem n-dimensionalen komplex-projektiven raum*, Manuscripta. Math. **16** (1975), 75-100.
- [10] P. IONESCU, *Generalized adjunction and applications*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 457-472.
- [11] V. A. ISKOVSKIKH, *Double projection from a line on Fano threefolds of the first kind*, Math. USSR.-Sb. **66** (1990), 265-284.
- [12] Y. KAWAMATA, *Elementary contractions of algebraic 3-folds*, Ann. of Math. **119** (1984), 95-110.
- [13] Y. KAWAMATA, *The cone of curves of algebraic varieties*, Ann. of Math. **119** (1984), 603-633.
- [14] Y. KAWAMATA, *Crepan blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. **127** (1988), 93-163.
- [15] Y. KAWAMATA, *Small contractions of four dimensional algebraic manifolds*, Math. Ann. **284** (1989), 595-600.
- [16] Y. KAWAMATA, *On the length of an extremal rational curve*, Invent. Math. **105** (1991), 609-611.
- [17] Y. KAWAMATA, *Semistable minimal models of threefolds in positive or mixed characteristic*, J. Alg. Geom. **3** (1994), 463-491.
- [18] Y. KAWAMATA – K. MATSUDA – K. MATSUKI, *Introduction to the minimal model problem*, in Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. vol. 10 (T. Oda ed.), Kinokuniya, 1987, pp. 283-360.
- [19] J. KOLLÁR, *Higher direct images of dualizing sheaves I*, Ann. of Math. **123** (1986), 11-42.
- [20] J. KOLLÁR, *Higher direct images of dualizing sheaves II*, Ann. of Math. **124** (1986), 171-202.
- [21] J. KOLLÁR, *Flops*, Nagoya Math. J. **113** (1989), 15-36.
- [22] J. KOLLÁR – Y. MIYAOKA – S. MORI, *Rational curves on Fano varieties*, preprint (1991).
- [23] J. KOLLÁR – Y. MIYAOKA – S. MORI, *Rationally connected varieties*, J. Alg. Geom. **1** (1992), 429-448.
- [24] J. KOLLÁR – Y. MIYAOKA – S. MORI, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), 765-779.
- [25] J. KOLLÁR – S. MORI, *Classification of three dimensional flips*, J. Amer. Math. Soc. **5** (1992), 533-703.
- [26] Y. MIYAOKA – S. MORI, *A numerical criterion for uniruledness*, Ann. of Math. **124** (1986), 65-69.
- [27] S. MORI, *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979), 593-606.

- [28] S. MORI, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. **116** (1982), 133-176.
- [29] S. MORI, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43-66.
- [30] S. MORI, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), 117-253.
- [31] S. MORI – S. MUKAI, *On Fano 3-folds with $B_2 \geq 2$* , in Algebraic and Analytic Varieties, Adv. Stud. in Pure Math. vol. 1 (S. Iitaka ed.), Kinokuniys, 1983, pp. 101-129.
- [32] S. MORI – S. MUKAI, *Classification of Fano 3-folds with $B_2 \geq 2$, I*, Alg. and Top. Theories - to the memory of Dr. T. Miyata, 1985, pp. 496-545.
- [33] S. MUKAI, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. USA **86** (1989), 3000-3002.
- [34] N. NAKAYAMA, *The lower semi-continuity of the plurigenera of complex varieties*, in Algebraic Geometry, Sendai 1985, Adv. St. Pure Math. vol. 10 (T. Oda ed.), Kinokuniya, 1987, pp. 551-590.
- [35] C. OKONEK – M. SCHNEIDER – H. SPINDLER, *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser, Boston, 1980.
- [36] H. PINKHAM, *Factorization of birational maps in dimension 3*, Proc. Sympos. Pure Math. **40** (1983), 343-371.
- [37] M. REID, *Lines on Fano 3-folds according to Shokurov*, preprint (1980).
- [38] M. REID, *Minimal models of canonical 3-folds*, in Algebraic and Analytic Varieties, Adv. Stud. in Pure Math. vol. 1 (S. Iitaka ed.), Kinokuniya, 1983, pp. 131-180.
- [39] M. REID, *Projective morphisms according to Kawamata*, preprint (1983).
- [40] V. G. SARKISOV, *On conic bundle structures*, Math. USSR. Izv. **20** (1983), 355-390.
- [41] V. V. SHOKUROV, *The existence of a straight line on Fano 3-folds*, Math. USSR.-Izv. **15** (1980), 173-209.
- [42] V. V. SHOKUROV, *The nonvanishing theorem*, Math. USSR.-Izv. **26** (1986), 591-604.
- [43] V. V. SHOKUROV, *3-fold log flips*, Math. USSR.-Izv. **40** (1993), 95-202.
- [44] K. TAKEUCHI, *Some birational maps of Fano 3-folds*, Compositio Math. **71** (1989), 265-283.
- [45] J. KOLLÁR *et al.*, *Flips and abundance for algebraic threefolds*, Astérisque vol. 211, Soc. Math. de France, 1992.
- [46] A. VAN DE VEN, *On uniform vector bundles*, Math. Ann. **195** (1972), 245-248.
- [47] P. M. H. WILSON, *Fano fourfolds of index greater than one*, J. Reine. Angew. Math. **379** (1987), 172-181.
- [48] J. WIŚNIEWSKI, *On contraction of extremal rays of Fano manifolds*, J. Reine. Angew. Math. **417** (1991), 141-157.

Department of Mathematical Sciences
 University of Tokyo
 Komaba, Tokyo 153, Japan
 E-mail: kaki@ms.u-tokyo.ac.jp