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New CR invariants and their application to the CR equivalence problem


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1. – Introduction

Let $M$ be a strictly pseudoconvex CR manifold (of hypersurface type) of CR dimension $n' - 1$. Let $K(M) = \Omega^{n,0}(M)$ be its canonical bundle and $K^0(M) = K(M) - \{ \text{zero section} \}$. Let $C(M) = K^0(M)/\mathbb{R}_+$. Then $C(M)$ is a principal circle bundle over $M$ and, by work of C.L. Fefferman [4], with each fixed pseudohermitian structure $\theta$ on $M$ one may associate a Lorentz metric $g$ on $C(M)$. This is the Fefferman metric of $(M, \theta)$. Its properties are closely tied to those of the base CR manifold. For instance, if $M$ is a real hypersurface in $\mathbb{C}^n$ then the null geodesics of the Fefferman metric project on biholomorphic invariant curves (known as the chains of $M$, cf. S.S. Chern & J. Moser [1]). Although not fully understood as yet, the Fefferman metric proved useful in a number of situations, e.g. provided a simpler proof (cf. L.K. Koch [9]) of the striking result of H. Jacobowitz (cf. [6]) that two nearby points of a strictly pseudoconvex CR manifold are joined by a chain. See also C.R. Graham [5], for a characterization of Fefferman metrics among all Lorentz metrics on $C(M)$.

By classical work of S.S. Chern & J. Simons [2], the Pontrjagin forms of a riemannian manifold are conformal invariants. On the other hand, the restricted conformal class of the Fefferman metric is known (cf. J.M. Lee [10]) to be a CR invariant. This led us to investigate whether the result by S.S. Chern & J. Simons may carry over to Lorentz geometry. We find (cf. Theorem 2) that the Pontrjagin forms $P(\Omega^\ell)$ of the Fefferman metric are CR invariants of $M$. Also, whenever $P(\Omega^\ell) = 0$, the De Rham cohomology class of the corresponding transgression form is a CR invariant, as well. As an application, we show that a necessary condition for $M$ to be globally CR equivalent to a sphere $S^{2n-1}$ is that $P_1(\Omega^2) = 0$ (i.e. the first Pontrjagin form of $(C(M), g)$ must vanish) and the corresponding transgression form gives an integral cohomology class (cf. Theorem 3).

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2. – The Fefferman metric

Let \((M, T_{1,0}(M))\) be an orientable CR manifold (of hypersurface type) of CR dimension \(n - 1\), where \(T_{1,0}(M) \subset T(M) \otimes \mathbb{C}\) denotes its CR structure. Its Levi distribution \(H(M) = \text{Re}(T_{1,0}(M) \oplus T_{0,1}(M))\) carries the complex structure \(J : H(M) \to H(M)\) given by \(J(Z + \overline{Z}) = i(Z - \overline{Z})\) for any \(Z \in T_{1,0}(M)\). Here \(T_{0,1}(M) = 
\overline{T_{1,0}(M)}\). Overbars denote complex conjugation and \(i = \sqrt{-1}\).

The annihilator \(E \subset T^*(M)\) of \(H(M)\) is a trivial line bundle, hence it admits global nowhere vanishing cross sections \(\theta \in \Gamma^\infty(E)\), each of which is referred to as a pseudohermitian structure. The Levi form \(L_\theta\) is given by \(L_\theta(Z, W) = -i(d\theta)(Z, W)\) for any \(Z, W \in T_{1,0}(M)\). Two pseudohermitian structures \(\theta, \hat{\theta}\) are related by \(\hat{\theta} = e^{2u}\theta\) for some \(C^\infty\) function \(u : M \to \mathbb{R}\) and the corresponding Levi forms satisfy \(L_{\hat{\theta}} = e^{2u}L_\theta\). This accounts for the (already highly exploited, cf. e.g. D. Jerison & J.M. Lee [7], and references therein) analogy between CR and conformal geometry. If \(L_\theta\) is nondegenerate for some choice of \(\theta\) (and thus for all) then \((M, T_{1,0}(M))\) is a nondegenerate CR manifold. Any nondegenerate CR manifold, on which a pseudohermitian structure \(\theta\) has been fixed, admits a unique linear connection \(\nabla\) (the Tanaka-Webster connection) parallelizing both the Levi form and the complex structure (in the Levi distribution). Cf. also [3] for an axiomatic description of the Tanaka-Webster connection.

A complex valued \(p\)-form \(\omega\) on \(M\) is a \((p, 0)\)-form if \(T_{0,1}(M) \not\subset \omega = 0\). Let \(\Omega^{\rho, \sigma}(M)\) be the bundle of all \((p, 0)\)-forms on \(M\). Set \(K(M) = \Omega^{n,0}(M)\). There is a natural action of \(\mathbb{R}^+ = (0, \infty)\) on \(K^0(M) = K(M) - \{0\}\) and the quotient space \(C(M) = K^0(M)/\mathbb{R}^+\) is a principle \(S^1\)-bundle over \(M\). Let \(\pi : C(M) \to M\) be the projection. A local frame \([\theta^a]\) of \(T_{1,0}(M)^*\) on \(U \subseteq M\) induces the trivialization chart:

\[\pi^{-1}(U) \to U \times S^1, \quad [\omega] \mapsto \left( x, \frac{\lambda}{|\lambda|} \right)\]

where \(\omega \in K^0(M), \pi([\omega]) = x\) and \(\omega = \lambda (\theta \wedge \theta^1 \wedge \cdots \wedge \theta^{n-1})\) \(x\) with \(\lambda \in \mathbb{R}, \lambda \neq 0\). Define \(\gamma : \pi^{-1}(U) \to [0, 2\pi]\) by \(\gamma([\omega]) = \arg(\lambda)\). Moreover, consider the (globally defined) 1-form \(\sigma\) on \(C(M)\) given by:

\[\sigma = \frac{1}{n+1} \left( d\gamma + \pi^* \left( i\omega^a - \frac{i}{2} h^{a\bar{b}} dh_{a\bar{b}} - \frac{R}{2n} \right) \right)\]

Here \(h_a^{\alpha\bar{\beta}}, \omega^a_\beta\) and \(R = h^{a\bar{b}} R_{a\bar{b}}\) are respectively the (local) components of the Levi form, the connection 1-forms (of the Tanaka-Webster connection) and the pseudohermitian scalar curvature (cf. e.g. (2.17) in [12], p. 34).

Let us extend the Hermitian form \(\langle Z, W \rangle_\theta = L_\theta(Z, \overline{W})\) to the whole of \(T(M) \otimes \mathbb{C}\) by requesting that \(\langle Z, \overline{W} \rangle_\theta = 0, \langle \overline{Z}, W \rangle_\theta = \langle Z, W \rangle_\theta\) and \(\langle T, V \rangle_\theta = 0\) for any \(Z, W \in T_{1,0}(M), V \in T(M) \otimes \mathbb{C}\). Then:

\[(1) \quad g = \pi^*(\cdot, \cdot)_\theta + 2(\pi^*\theta) \otimes \sigma\]
is a semi-riemannian metric on \( C(M) \). Assume from now on that \( M \) is strictly pseudoconvex and choose \( \theta \) so that \( L_\theta \) is positive definite. Then \( g \) is a Lorentz metric on \( C(M) \), known as the Fefferman metric of \( (M, \theta) \). By a result of J.M. Lee (cf. [10], p. 418) if \( \hat{\theta} = e^{2u} \theta \) is another pseudohermitian structure and \( \hat{g} \) the corresponding Fefferman metric, then \( \hat{g} = e^{2(u \circ \pi)} g \).

3. Pontrjagin forms

Let \( I^\ell(GL(2n)) \) be the space of all invariant polynomials of degree \( \ell \), i.e. symmetric multilinear maps \( P : \mathfrak{gl}(2n)^\ell \to \mathbb{R} \) which are \( ad(GL(2n)) \)-invariant. Here \( \mathfrak{gl}(2n) \) is the Lie algebra of \( GL(2n) = GL(2n, \mathbb{R}) \). Also, if \( G \) is a linear space then \( G^\ell = G \otimes \cdots \otimes G \) (\( \ell \) terms). Let \( Q_\ell \in I^\ell(GL(2n)) \), \( 1 \leq \ell \leq 2n \), be the natural generators of the ring of invariant polynomials on \( \mathfrak{gl}(2n) \) (cf. [2], p. 57, for the explicit expressions of the \( Q_\ell \)). Let \( (M, T_{1,0}(M)) \) be a strictly pseudoconvex CR manifold of CR dimension \( n - 1 \) and \( \theta \) a pseudohermitian structure on \( M \) so that \( L_\theta \) is positive definite. Let \( g \) be the Fefferman metric of \( (M, \theta) \). Let \( F(C(M)) \to C(M) \) be the principal \( GL(2n) \)-bundle of all linear frames on \( C(M) \) and \( \omega \in \Gamma^\infty(T^*(F(C(M))) \otimes \mathfrak{gl}(2n)) \) the connection 1-form (of the Levi-Civita connection) of the Lorentz manifold \( (C(M), g) \). Then:

**Theorem 1.** The characteristic forms \( Q_{2\ell+1}(\Omega^{2\ell+1}) \) vanish for any \( 0 \leq \ell \leq n - 1 \).

Here \( \Omega = D\omega \) is the curvature 2-form of \( \omega \). Also, for any \( P \in I^\ell(GL(2n)) \) we set \( P(\Omega^\ell) = P \circ \Omega^\ell \) where \( \Omega^\ell = \Omega \wedge \cdots \wedge \Omega \) (\( \ell \) terms). Let us prove Theorem 1. To this end, let \( \mathcal{L}(C(M)) \to C(M) \) be the principal \( O(2n - 1, 1) \)-bundle of all Lorentz frames, i.e. \( u = (c, \{X_i\}) \in \mathcal{L}(C(M)) \) if \( \epsilon_{\alpha} = 1 \), \( 1 \leq \alpha \leq 2n - 1 \) and \( \epsilon_{2n} = -1 \), \( c \in C(M) \). Here \( O(2n - 1, 1) \) is the Lorentz group. Let \( \mathfrak{o}(2n - 1, 1) \) be its Lie algebra. By hypothesis:

\[
\omega_u(T_u(\mathcal{L}(C(M)))) \subseteq \mathfrak{o}(2n - 1, 1)
\]

i.e. \( \epsilon \omega_u(X) + \omega_u(X) \epsilon = 0 \) for any \( X \in T_u(\mathcal{L}(C(M))) \), \( u \in \mathcal{L}(C(M)) \). Here \( \epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_{2n}) \). Let \( \{E^i_j\} \) be the canonical basis of \( \mathfrak{gl}(2n) \) and set \( \omega = \omega^i_j \otimes E^i_j \), \( \Omega = \Omega^i_j \otimes E^i_j \). We claim that:

\[
(2) \quad \epsilon^i \Omega^j_j + \epsilon^j \Omega^i_i = 0
\]

at all points of \( \mathcal{L}(C(M)) \), as a form \( F(C(M)) \). Here \( \epsilon^i = \epsilon_i \). As \( \Omega \) is horizontal, it suffices to check (2) on horizontal vectors (hence tangent to \( \mathcal{L}(C(M)) \)). We have:

\[
\epsilon^i \Omega^j_j = \epsilon^i (d\omega^j_j + \omega^i_j \wedge \omega^j_j) = d(-\epsilon^j \omega^i_j) + \sum_k (-\epsilon^k \omega^i_k) \wedge \omega^j_j = -\epsilon^j \Omega^j_j
\]
on \( T_u(\mathcal{L}(C(M))) \) for any \( u \in \mathcal{L}(C(M)) \), etc. Next, note that for any \( A \in \mathfrak{o}(2n-1,1) \) one has i) \( \text{tr}(A) = 0 \), ii) \( \text{tr}(AB) = 0 \), for any \( B \in M_{2n}(\mathbb{R}) \) satisfying \( B = \epsilon B^t \epsilon \), and iii) \( \text{tr}(A^{2^\ell+1}) = 0 \). Then:

\[
\text{tr}(A_1 \cdots A_{2^\ell+1}) = 0
\]

for any \( A_1, \ldots, A_{2^\ell+1} \in \mathfrak{o}(2n-1,1) \) (the proof is by induction over \( \ell \)). Since \( Q_{2^\ell+1}(\Omega^{2^\ell+1}) \) is invariant, we need only show that it vanishes at the points of \( \mathcal{L}(C(M)) \). But at these points the range of \( \Omega^{2^\ell+1} \) lies (by (2)-(3)) in the kernel of \( Q_{2^\ell+1} \). Our Theorem 1 is proved.

Let \( P \in I^\ell(GL(2n)) \). The transgression form \( TP(\omega) \) is given by:

\[
TP(\omega) = \ell \int_0^1 P(\omega \wedge \Omega_t^{\ell-1}) dt
\]

where \( \Omega_t = \tau \Omega + (1/2)\tau(t-1)[\omega,\omega], 0 \leq t \leq 1 \). By Chern-Weil theory (cf. e.g. [8], vol. II, p. 297) one has \( P(\Omega^\ell) = dTP(\omega) \). By Theorem 1, the transgression forms \( TQ_{2^\ell+1}(\omega) \) are closed, hence we get the cohomology classes \( [TQ_{2^\ell+1}(\omega)] \in H^{4\ell+1}(F(C(M)), \mathbb{R}) \). Note that:

\[
[TQ_{2^\ell+1}(\omega)] \in \ker(j^*)
\]

where \( j^* : H^{4\ell+1}(F(C(M)), \mathbb{R}) \rightarrow H^{4\ell+1}(\mathcal{L}(C(M)), \mathbb{R}) \) is induced by \( j : \mathcal{L}(C(M)) \subset F(C(M)) \). Indeed \( TQ_{2^\ell+1}(\omega) \) may be written as:

\[
TQ_{2^\ell+1}(\omega) = \sum_{i=0}^{2^\ell} B_i Q_{2^\ell+1}(\omega \wedge [\omega,\omega]^i \wedge \Omega^{2^\ell-i})
\]

for some constants \( B_i > 0 \). As \( j^* \omega \) is \( \mathfrak{o}(2n-1,1) \)-valued, the same argument as in the proof of Theorem 1 shows that \( j^* TQ_{2^\ell+1}(\omega) = 0 \), q.e.d. One has to work with \( j^* \omega \) (rather than \( \omega \) at a point of \( \mathcal{L}(C(M)) \)) because \( \omega \) (unlike its curvature form) is not horizontal.

If \( g_0 \) is a riemannian metric on \( C(M) \) with connection 1-form \( \omega_0 \) and \( O(C(M)) \rightarrow C(M) \) is the principal \( O(2n) \)-bundle of orthonormal (with respect to \( g_0 \)) frames on \( C(M) \), then orthonormalization of frames gives a deformation retract \( F(C(M)) \rightarrow O(C(M)) \) and hence (cf. Proposition 4.3 in [2], p. 58) the corresponding transgression forms \( TQ_{2^\ell+1}(\omega_0) \) are exact. As to the Lorentz case, in general (4) need not imply exactness of \( TQ_{2^\ell+1}(\omega) \). For instance \( \mathbb{R}^1_1 \) is a Lorentz manifold for which the homomorphism \( j^* : H^1(F(\mathbb{R}^2_1), \mathbb{R}) \rightarrow H^1(\mathcal{L}(\mathbb{R}^2_1), \mathbb{R}) \) (induced by \( j : \mathcal{L}(\mathbb{R}^2_1) \subset F(\mathbb{R}^2_1) \)) has a nontrivial kernel. Here \( \mathbb{R}^N_0 = (\mathbb{R}^N, \langle \cdot, \cdot \rangle_{N-v,v}) \) and \( \langle \cdot, \cdot \rangle_{N-v,v} = \sum_{i=1}^N x_i y_i - \sum_{i=N-v+1}^N x_i y_i \). Indeed, as both \( F(\mathbb{R}^2_1) \) and \( \mathcal{L}(\mathbb{R}^2_1) \) are trivial bundles \( j^* \) may be identified with the homomorphism \( j^* : H^1(GL(2), \mathbb{R}) \rightarrow H^1(O(1,1), \mathbb{R}) \) (induced by \( j : O(1,1) \subset GL(2) \)). The Lorentz group \( O(1,1) \) has four components, each diffeomorphic to \( \mathbb{R} \). Hence \( H^1(O(1,1)) = 0 \). Moreover \( O(2) \subset GL(2) \) is a homotopy equivalence, hence \( \ker(j^*) = H^1(GL(2), \mathbb{R}) = H^1(O(2), \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \) (as \( O(2) \) has two components, each diffeomorphic to \( S^1 \)).

At this point, we may state the following:
THEOREM 2. Let $M$ be a strictly pseudoconvex CR manifold of CR dimension $n - 1$ and $P \in \Omega^\ell(GL(2n))$. Then $P(\Omega^\ell)$ is a CR invariant of $M$. Moreover, if $P(\Omega^\ell) = 0$, then the cohomology class $[TP(\omega)] \in H^{2\ell-1}(F(C(M)), \mathbb{R})$ is a CR invariant of $M$. In particular $[\mathcal{T}Q_{2\ell+1}(\omega)] \in H^{4\ell+1}(F(C(M)), \mathbb{R})$ is a CR invariant.

4. - Applications

Let $M$ be a strictly pseudoconvex CR manifold. Assume that $M$ is realizable as a real hypersurface in $\mathbb{C}^n$. If $\varphi : M \to \mathbb{C}^n$ is the given immersion, then $\eta = \varphi^*dz_1 \wedge \cdots \wedge dz^n$ is a nowhere zero global $(n, 0)$-form on $M$, hence $C(M)$ is a trivial bundle. By work of C.L. Fefferman [4], there is a smooth defining function $\psi$ of $M$ satisfying the complex Monge-Ampère equation:

$$J(\psi) \equiv \det \begin{pmatrix} \eta^2/\partial z^j & \eta \partial^2/\partial z^j \partial z^k \\ \partial \psi/\partial z^j & \partial^2 \psi/\partial z^j \partial z^k \end{pmatrix} = 1$$

to second order along $M$, so that $F^*h$ is the Fefferman metric of $(M, \hat{\theta})$, $\hat{\theta} = \frac{i}{2} \varphi^*(\bar{\partial} - \partial)\psi$, where $h$ is the Lorentz metric given by:

$$h = -\frac{i}{n+1} \left( \partial - \bar{\partial} \right) \psi \otimes d\gamma + j^* \left\{ \frac{\partial^2 \psi}{\partial z^j \partial \bar{z}^k} d\gamma \otimes d\bar{\zeta} \right\}$$

and $F : C(M) \approx M \times S^1$ the diffeomorphism induced by $\eta$. Also $\gamma$ is a local coordinate on $S^1$ and $j : M \times S^1 \subset \mathbb{C}^{n+1}$. Let $\theta$ be any pseudohermitian structure on $M$ (so that $L_\theta$ is positive definite). Then $\hat{\theta} = e^{2u}\theta$ for some smooth function $u$ on $M$, and an inspection of (1) shows that $F^*h$ and $g$ are conformally equivalent Lorentz metrics. On the other hand $h = j^*G$ where $G$ is the semi-riemannian metric on $\mathbb{C}^n \times \mathbb{C}_*$ given by:

$$G = |\xi|^{2/(n+1)} \left\{ \begin{array}{c} \psi \\ (n+1)^2 |\xi|^{-2} d\xi \otimes d\bar{\xi} + \frac{\partial^2 \psi}{\partial z^j \partial \bar{z}^k} d\gamma \otimes d\bar{\zeta} \\ + \frac{1}{n+1} \left( (\partial \psi) \otimes \frac{d\bar{\xi}}{\xi} + \frac{d\xi}{\xi} \otimes (\bar{\partial} \psi) \right) \end{array} \right\}$$

where $(z, \zeta) = (z^1, \cdots, z^n, \zeta)$ are complex coordinates. Summing up, if $M$ is realizable then $(C(M), g)$ admits a global conformal immersion in $(\mathbb{C}^n \times \mathbb{C}_*, G)$, hence (in view of Theorem 5.14 in [2], p. 64) it is reasonable to expect that some of the CR invariants furnished by Theorem 2 are obstructions towards the global embeddability of a given, abstract, CR manifold $M$. While we leave this as an open problem, we address the following simpler situation. Assume $M$ to be equivalent to $S^{2n-1}$. Then $C(M)$ is diffeomorphic to the Hopf manifold $H^n = S^{2n-1} \times S^1$. On the other hand, note that $I_{n+1} = \{ \xi \in \mathbb{C} : \xi^{n+1} = 1 \}$ acts freely on $\mathbb{C}^n \times \mathbb{C}_*$ as a properly discontinuous group of complex analytic
transformations. Hence the quotient space $V_{n+1} = (\mathbb{C}^n \times \mathbb{C}_*)/I_{n+1}$ is a complex $(n+1)$-dimensional manifold. Consider the biholomorphism $p : V_{n+1} \to \mathbb{C}^n \times \mathbb{C}_*$ given by $p((z, \xi)) = (z/\xi, \xi^{n+1})$ for any $[z, \xi] \in V_{n+1}$ and set $\phi_0 = p^{-1} \circ j \circ F$. Next:

$$G_0 = \sum_{j=1}^{n} dz^j \otimes d\bar{z}^j - d\xi \otimes d\bar{\xi}$$

is $I_{n+1}$-invariant, hence gives rise to a globally defined semi-riemannian metric of index 2 on $V_{n+1}$. Note that $(V_{n+1}, G_0)$ is locally isometric to $\mathbb{R}^{2n+2}$.

**Lemma 1.** $\phi_0 : (C(M), g) \to (V_{n+1}, G_0)$ is a conformal immersion.

Indeed, let $\psi(z) = |z|^2 - 1$. A calculation then shows that $G_0 = p^* G$. Finally, it may be seen that $F : (C(M), g) \to (H^n, h)$ is a conformal diffeomorphism.

Let $P_i \in I^{2i}(GL(2n))$ be given by:

$$\det \left( \lambda I_{2n} - \frac{1}{2\pi} A \right) = \sum_{i=0}^{n} P_i(A \otimes \cdots \otimes A)\lambda^{2n-2i} + O(\lambda^{2n-\text{odd}})$$

i.e. the invariant polynomials obtained by ignoring the powers $\lambda^{2n-\text{odd}}$. We obtain the following:

**Theorem 3.** Let $M$ be a strictly pseudoconvex CR manifold of CR dimension $n - 1$ and $\theta$ a pseudohermitian structure on $M$ so that $L_\theta$ is positive definite. Let $g$ be the Fefferman metric of $(M, \theta)$. Let $\omega$ be the connection 1-form of $g$ and $\Omega$ its curvature 2-form. If $M$ is CR equivalent to $S^{2n-1}$ then $P_1(\Omega^2) = 0$ and $[TP_1(\omega)] \in H^3(F(C(M)), \mathbb{Z})$, provided $n \geq 3$.

To prove Theorem 3, we study the geometry of the second fundamental form of the immersion $\phi = p^{-1} \circ j : H^n \to (\mathbb{C}^n \times \mathbb{C}_*, G)$. Set $C_n = \sqrt{n + 1}/\sqrt{2(n + 1)}$. The tangent vector fields $\xi_a$ given by:

$$\xi_1 = C_n \left( z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} + \xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right)$$

$$\xi_2 = C_n \left( z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} - (n + 2) \left( \xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right) \right)$$

are such that $G(\xi_1, \xi_2) = 0$, $G(\xi_1, \xi_1) = 1$ and $G(\xi_2, \xi_2) = -1$, and form a frame of the normal bundle of $\phi$. Since $p$ is a biholomorphism (with the inverse $p^{-1}(z, \xi) = [z\xi^{1/(n+1)}, \xi^{1/(n+1)}]$) we have:

$$p^* \frac{\partial}{\partial z^j} = \xi^{-1/(n+1)} \frac{\partial}{\partial z^j}$$

$$p^* \frac{\partial}{\partial \xi} = \xi^{-1/(n+1)} \left( -z^j \frac{\partial}{\partial z^j} + (n + 1) \xi \frac{\partial}{\partial \xi} \right).$$
By (5) the Christoffel symbols of the Levi-Civita connection $\nabla^0$ of $(V_{n+1}, G_0)$ vanish. The Levi-Civita connection $\nabla$ of $(\mathbb{C}^n \times \mathbb{C}_*, G)$ is related to $\nabla^0$ by:

$$p_* \left( \nabla^0_X Y \right) = \nabla_{p_* X} p_* Y$$

for any $X, Y \in T(V_{n+1})$. A calculation shows that:

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^k} = 0; \quad \nabla_{\frac{\partial}{\partial \zeta}} \frac{\partial}{\partial \zeta} = -\frac{n}{n+1} \frac{\partial}{\partial \zeta}$$

$$\nabla_{\frac{\partial}{\partial \zeta}} \frac{\partial}{\partial z^j} = \frac{1}{n+1} \frac{\partial}{\partial z^j}.$$

Tangent vector fields on $H^n$ are of the form $X + Y$ with $X = A^j \frac{\partial}{\partial z^j} + \overline{A^j} \frac{\partial}{\partial \zeta}$ and $Y = B \frac{\partial}{\partial \zeta} + \overline{B} \frac{\partial}{\partial \zeta}$ satisfying $A^j \overline{z}_j + \overline{A^j} z_j = 0$, respectively $B \overline{z} + \overline{B} \zeta = 0$. Here $z^j = z_j$. It follows that:

$$(6) \quad \nabla_X \xi_1 = C_n \frac{n+2}{n+1} X, \quad \nabla_X \xi_2 = -\frac{C_n}{n+1} X$$

$$(7) \quad \nabla_Y \xi_1 = \frac{C_n}{n+1} \left\{ Y + B \overline{z} \frac{\partial}{\partial z^j} + \overline{B} \zeta \frac{\partial}{\partial z^j} \right\}$$

$$(8) \quad \nabla_Y \xi_2 = \frac{C_n}{n+1} \left\{ -(n+2)Y + B \overline{z} \frac{\partial}{\partial z^j} + \overline{B} \zeta \frac{\partial}{\partial z^j} \right\}.$$

Let $A_a = A_{\xi_a}$ be the Weingarten operator corresponding to the normal section $\xi_a$. We shall need the following:

**Lemma 2.** The first Pontrjagin form of $(H^n, h)$ is:

$$\frac{1}{4\pi^2} \Psi_{12} \wedge \Psi_{12}$$

where (with respect to a local coordinate system $(x^i)$ on $H^n$):

$$\Psi_{12} = h \left( \frac{\partial}{\partial x^i}, A_1 A_2 \frac{\partial}{\partial x^j} \right) dx^i \wedge dx^j.$$

We shall prove Lemma 2 later on. Recall the Ricci equation (of the given immersion $\phi$, cf. e.g. (2.7) in [13], p. 22):

$$G(R(X, Y)\xi, \xi') = G(R^\perp(X, Y)\xi, \xi') + h([A_\xi, A_{\xi'}]X, Y)$$

where $R, R^\perp$ denote respectively the curvature tensor fields of $(\mathbb{C}^n \times \mathbb{C}_*, G)$ and of the normal connection. As a consequence of (6)-(8) $\xi_a$ are parallel in the normal bundle, hence the immersion $\phi$ has a flat normal connection ($R^\perp = 0$). On the other hand $R = 0$ (because $(V_{n+1}, G_0)$ is flat) and the Ricci equation...
shows that the Weingarten operators $A_a$ commute. Then $\Psi_{12} = 0$ and our Lemmas 1 and 2 together with Theorem 2 yield $P_1(\Omega^2) = 0$.

Let $q : H^3(F(C(M)), \mathbb{R}) \to H^3(F(C(M)), \mathbb{R}/\mathbb{Z})$ be the natural homomorphism. By Theorem 3.16 in [2], p. 56, since $P_1(\Omega^2) = 0$, there is a cohomology class $\alpha \in H^3(C(M), \mathbb{R}/\mathbb{Z})$ so that $p^*_F \alpha = q([TP_1(\omega)])$, where $p_F : F(C(M)) \to C(M)$ is the projection. Yet, for the Hopf manifold $H^3(H^n, \mathbb{R}/\mathbb{Z}) = 0$ provided $n \geq 3$, hence $[TP_1(\omega)] \in \ker(q)$ and then by the exactness of the Bockstein sequence:

$$\cdots \to H^3(F(C(M)), \mathbb{Z}) \to H^3(F(C(M)), \mathbb{R}) \to H^3(F(C(M)), \mathbb{R}/\mathbb{Z}) \to H^4(F(C(M)), \mathbb{R}) \to \cdots$$

it follows that $[TP_1(\omega)]$ is an integral class.

5. Proof of Theorem 2

Let $\varphi \in \Gamma^\infty(T^*(F(C(M))) \otimes \mathbb{R}^{2n})$ be the canonical 1-form and set $\varphi = \varphi^i \otimes e_i$, where $\{e_i\}$ is the canonical basis in $\mathbb{R}^{2n}$. Moreover, let $E_i = B(e_i)$ be the corresponding standard horizontal vector fields (cf. e.g. [8], vol. I, p. 119). Let $u : M \to \mathbb{R}$ be a $C^\infty$ function and let $g$ be the Fefferman metric of $(M, e^{2u})$. Let $\hat{\omega}$ be the corresponding connection 1-form. Then:

(9) \[ \hat{\omega}^i_j = \omega^i_j + d(u \circ \rho)\delta^i_j + E_j(u \circ \rho)\varphi^i - \varepsilon_i E_i(u \circ \rho)\varepsilon_j \varphi^i \]

at all points of $\mathcal{L}(C(M))$, as forms on $F(C(M))$. Here $\rho = \pi \circ p_F$. The proof is to relate the Levi-Civita connections of the conformally equivalent Fefferman metrics $g$ and $\hat{g}$, followed by a translation of the result in principal bundle terminology. We omit the details. Consider the 1-parameter family of Lorentz metrics $g(s) = e^{2u(s)}g$, $0 \leq s \leq 1$, on $C(M)$. Let $\omega(s)$ be the corresponding connection 1-form and set $\omega' = \frac{d}{ds}\{\omega(s)\}_{s=0}$. By (9) (applied to $s(u \circ \rho)$ instead of $u \circ \rho$) we obtain:

(10) \[ \omega'^i_j = d(u \circ \rho)\delta^i_j + E_j(u \circ \rho)\varphi^i - \varepsilon_i E_i(u \circ \rho)\varepsilon_j \varphi^i \]

at all points of $\mathcal{L}(C(M))$, as forms on $F(C(M))$. Let $P \in I^\ell(GL(2n))$. We wish to show that $P(\Omega^\ell)$ is invariant under any transformation $\hat{\theta} = e^{2u}\theta$. Note that a relation of the form:

(11) \[ TP(\hat{\omega}) = TP(\omega) + \text{exact} \]

yields $P(\hat{\Omega}^\ell) = P(\Omega^\ell)$, hence we only need to prove (11). Since the $Q_\ell$ generate $I(GL(2n))$ we may assume that $P$ is a monomial in the $Q_\ell$. Using
Proposition 3.7 in [2], p. 53, an inductive argument shows that it is sufficient to prove (11) for $P = Q\ell$. It is enough to prove that:

$$(12) \quad \frac{d}{ds}\{T Q\ell(\omega(s))\} = exact.$$ 

Since each point on the curve $s \mapsto g(s)$ is the initial point of another such curve, it suffices to prove (12) at $s = 0$. By Proposition 3.8 in [2], p. 53, we know that:

$$\frac{d}{ds}\{T Q\ell(\omega(s))\}_{s=0} = \ell Q\ell(\omega' \wedge \Omega^{\ell-1}) + exact$$

hence it is enough to show that $Q\ell(\omega' \wedge \Omega^{\ell-1}) = exact$. Using (10) and the identity:

$$Q\ell(\psi \wedge \Omega^{\ell-1}) = \sum_{i_1, \ldots, i_\ell} \psi_{i_2}^{i_1} \wedge \Omega_{i_3}^{i_2} \wedge \cdots \wedge \Omega_{i_1}^{i_\ell}$$

(cf. (4.2) in [2], p. 57) for any $gl(2n)$-valued form $\psi$ on $F(C(M))$, we may conduct the following calculation:

$$Q\ell(\omega' \wedge \Omega^{\ell-1}) = \sum \omega_{i_2}^{i_1} \wedge \Omega_{i_3}^{i_2} \wedge \cdots \wedge \Omega_{i_1}^{i_\ell}$$

$$= \sum d(u \circ \rho) \wedge \Omega_{i_3}^{i_2} \wedge \cdots \wedge \Omega_{i_1}^{i_\ell}$$

$$+ \sum \left( E_{i_2} (u \circ \rho) \varphi_{i_1}^{i_2} - \epsilon_{i_1} E_{i_1} (u \circ \rho) \epsilon_{i_2} \varphi_{i_2}^{j_2} \right) \wedge \Omega_{i_3}^{i_2} \wedge \cdots \wedge \Omega_{i_1}^{i_\ell}$$

Recall the structure equations, cf. e.g. [8], vol. I, p. 121. As $g$ is Lorentz, $\omega$ is torsion free. Hence $\varphi_{i_1}^{i_2} \wedge \Omega_{i_1}^{i_\ell} = 0$. This and (2) also yield $\epsilon_{i_2} \varphi_{i_2}^{j_2} \wedge \Omega_{i_3}^{i_2} = 0$. Hence:

$$Q\ell(\omega' \wedge \Omega^{\ell-1}) = d(u \circ \rho) \wedge Q\ell^{-1}(\Omega^{\ell-1}) = exact$$

(because $d Q\ell^{-1}(\Omega^{\ell-1}) = 0$) at all points of $\mathcal{L}(C(M))$, as a form on $F(C(M))$. This suffices because both $Q\ell(\omega' \wedge \Omega^{\ell-1})$ and $(u \circ \rho)Q\ell^{-1}(\Omega^{\ell-1})$ are invariant forms.

6. - Proof of Lemma 2

Recall (cf. e.g. [8], vol. II, p. 313) that:

$$P\ell(\Omega^{2\ell}) = c_\ell \sum \delta_{i_1 \ldots i_{2\ell}}^{j_1 \ldots j_{2\ell}} \Omega_{j_1}^{i_1} \wedge \cdots \wedge \Omega_{j_{2\ell}}^{i_{2\ell}}$$

where $c_\ell = 1/((2\pi)^{2\ell}(2\ell)!)$ and the summation runs over all ordered subsets $(i_1, \ldots, i_{2\ell})$ of $\{1, \ldots, 2n\}$ and all permutations $(j_1, \ldots, j_{2\ell})$ of $(i_1, \ldots, i_{2\ell})$ and
\( \delta_{i_1 \cdots i_2 t}^{j_1 \cdots j_2 t} \) is the sign of the permutation. We need the Gauss equation (cf. e.g. (2.4) in [13], p. 21):

\[
R^\ell_{kij} = B^a_{jk} A^\ell_{ai} - B^a_{ik} A^\ell_{aj}
\]

where \( R^\ell_{kij}, B^a_{jk} \) are respectively the curvature tensor field of \((H^n, h)\) and the second fundamental form of \( \phi \) (with respect to a local coordinate system \((U, x^i)\) on \( H^n \)). Also \( A_a \partial_i = A^j_{ai} \partial_j \) where \( \partial_i \) is short for \( \partial/\partial x^i \). The Gauss equation and the identity:

\[
R(X, Y)Z = u \left( 2\Omega(X^*, Y^*)_u (u^{-1}Z) \right)
\]

(cf. [8], vol. I, p. 133) for any \( X, Y, Z \in T_x(H^n) \) and some \( u \in F(H^n)_x \), furnish:

\[
2\Omega^\ell_* = Y^p_* X^k_* (B^a_{jk} A^\ell_{ai} - B^a_{ik} A^\ell_{aj}) \, dx^i \wedge dx^j
\]

(where \( X^j : p_F^{-1}(U) \to \mathbb{R} \) are fibre coordinates on \( F(H^n) \) and \((Y^j)_* = (X^j)_*^{-1}\)). Using:

\[
B^a_{jk} = A^r_{aj} h_{rk}
\]

a calculation leads to:

\[
2P_1(\Omega^2) = -c_1 \left( B^{a1}_{j1k1} A^{k2}_{a1p1} B^{a2}_{j2k2} A^{k1}_{a2p2} - B^{a1}_{p1k1} A^{k2}_{a1j1} B^{a2}_{j2k2} A^{k1}_{a2p2} \right) \, dx^{p1} \wedge dx^{j1} \wedge dx^{p2} \wedge dx^{j2}
\]

hence:

\[
P_1(\Omega^2) = c_1 \sum_{a, b} \Psi_{ab} \wedge \Psi_{ab}
\]

where \( \Psi_{ab} \) is the 2-form on \( F(H^n) \) given by:

\[
\Psi_{ab} = h(A_a \partial_i, A_b \partial_j) \, dx^i \wedge dx^j.
\]

Finally, note that \( \Psi_{11} = \Psi_{22} = 0 \) and \( \Psi_{21} = -\Psi_{12} \) and Lemma 2 is proved. Note that the proof works for any codimension two submanifold of a flat riemannian manifold.

**REFERENCES**


