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Plane Curves with Hyperbolic and C-hyperbolic Complements

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We find sufficient conditions for the complement $\mathbb{P}^2 \backslash C$ of a plane curve C to be C-hyperbolic. The latter means that some covering over $\mathbb{P}^2 \backslash C$ is Carathéodory hyperbolic. This implies that this complement $\mathbb{P}^2 \backslash C$ is Kobayashi hyperbolic, and (due to Lin's theorem) the fundamental group $\pi_1(\mathbb{P}^2 \backslash C)$ does not contain a nilpotent subgroup of finite index. We also give explicit examples of irreducible such curves of any even degree $d \geq 6$.

1. – Introduction

A complex space X is said to be *C-hyperbolic* if there exists a non-ramified covering $Y \to X$ such that Y is Carathéodory hyperbolic, i.e. the points in Y are separated by bounded holomorphic functions (see Kobayashi [21]). If there exists a covering Y of X such that for any point $p \in Y$ there exist only finitely many points $q \in Y$ which cannot be separated from p by bounded holomorphic functions on Y, then we say that X is *almost C-hyperbolic*. There is a general problem: *Which quasiprojective varieties are uniformized by bounded domains in* $\mathbb{C}^n$? In particular, such a variety must be C-hyperbolic. Here we study plane projective curves whose complements are C-hyperbolic. We prove the following

THEOREM 1.1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of geometric genus g. Assume that its dual curve C^* is an immersed curve of degree n.*

a) *If $g \geq 1$, then $\mathbb{P}^2 \backslash C$ is C-hyperbolic.*

b) *If $g = 0$, $n \geq 5$ and C^* is a generic rational nodal curve, then $\mathbb{P}^2 \backslash C$ is almost C-hyperbolic.*

c) *In both cases $\mathbb{P}^2 \backslash C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.*

Consider, for instance, an elliptic sextic with 9 cusps (see (6.7)). Such a sextic can be given explicitly by Schläfli's equation (see Gelfand, Kapranov and Zelevinsky [14]). It is dual to a smooth cubic and hence, due to (a),

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its complement is C-hyperbolic. Actually, 6 is the least possible degree of an irreducible plane curve with C-hyperbolic complement (see (7.5)).

Note that C-hyperbolicity implies Kobayashi hyperbolicity. S. Kobayashi [21] proposed the following:

CONJECTURE. *Let $\mathcal{H}(d)$ be the set of all hypersurfaces D of degree d in $\mathbb{P}^n$ such that $\mathbb{P}^n \backslash D$ is complete hyperbolic and hyperbolically embedded into $\mathbb{P}^n$. Then for any $d \geq 2n+1$ the set $\mathcal{H}(d)$ contains a Zariski open subset of $\mathbb{P}^{N(d)}$, where $\mathbb{P}^{N(d)}$ is the complete linear system of effective divisors of degree d in $\mathbb{P}^n$.*

For $n > 2$ the problem is still open. For $n = 2$ Y.-T. Siu and S.-K. Yeung [32] gave a proof of the above conjecture in the particular case of plane curves of sufficiently large degree ($d > 10^6$). However, even for $n = 2$ and for small d it is not so easy to construct explicit examples of irreducible plane curves in $\mathcal{H}(d)$ (see Zaidenberg [37] and literature therein). For the case of reducible curves see e.g. Dethloff, Schumacher and Wong [8,9].

The first examples of smooth curves in $\mathcal{H}(d)$ of any even degree $d \geq 30$ were constructed by K. Azukawa and M. Suzuki [3]. A. Nadel [24] mentioned such examples for any $d \geq 18$ which is divisible by 6. K. Masuda and J. Noguchi [23] obtained smooth curves in $\mathcal{H}(d)$ for any $d \geq 21$.

In Zaidenberg [36] the existence of smooth curves in $\mathcal{H}(d)$ is proven (by deformation arguments) for arbitrary $d \geq 5$; however, their equations are not quite explicit. For instance, the equation of a smooth quintic in $\mathcal{H}(5)$ includes five parameters which should be chosen successively small enough, with unexplicit upper bounds.

In a series of papers by M. Green [16], J. Carlson and M. Green [4] and H. Grauert and U. Peternell [15] sufficient conditions were found for irreducible plane curves of genus $g \geq 2$ to be in $\mathcal{H}(d)$. This leads to examples of irreducible (but singular) curves in $\mathcal{H}(d)$ with $d \geq 9$ (see the remark after (6.5)).

Generalizing the method of Green [16] (see the proof of Theorem 3.1, a)), we obtain for any even $d \geq 6$ families of irreducible curves in $\mathcal{H}(d)$ described in terms of genus and singularities. While in all the examples known before the curves were of genus at least two, now we obtain such examples of elliptic or rational curves. They are all singular, and the method used is not available to get such examples of smooth curves, even of higher genus. On the other hand, it is clear that an elliptic or a rational curve with hyperbolic complement must be singular.

We have presented above a family of elliptic sextics with C-hyperbolic and hyperbolically embedded complements. Another example of curves with hyperbolically embedded complements in degree 6, is the family of rational sextics with four nodes and six cusps, where the cusps are on a conic (6.11). Such a curve is dual to a generic rational nodal quartic, and therefore, one can easily write down its explicit equation.

In fact, C-hyperbolicity is a much stronger property than Kobayashi hyperbolicity. To show this, recall that a *Liouville* complex space is a space Y such that all the bounded holomorphic functions on Y are constant. This property is just opposite of being Carathéodory hyperbolic. By Lin's theorem (see Lin [22], Theorem B) a Galois covering Y of a quasiprojective variety X is Liouville if its group of deck transformations is *almost nilpotent*, i.e. it contains a nilpotent subgroup of finite index. If follows that, as soon as the fundamental group $\pi_1(\mathbb{P}^2\backslash C)$ is almost nilpotent, any covering over $\mathbb{P}^2\backslash C$ is a Liouville one. In particular, this is so for a nodal (not necessarily irreducible) plane curve C(1). Indeed, due to the Deligne-Fulton theorem (see Deligne [7] and Fulton [13]), in the latter case the group $\pi_1(\mathbb{P}^2\backslash C)$ is abelian. As a corollary we obtain that for curves mentioned in Theorem 1.1 the group $\pi_1(\mathbb{P}^2\backslash C)$ is not almost nilpotent (see Proposition 7.1). For $n = \deg C^* \geq 2g - 1$ this group even contains a free subgroup with two generators (see Section 7.a).

This shows that C-hyperbolicity of $\mathbb{P}^2\backslash C$ can be easily destroyed under small deformations of C, by passing to a smooth or nodal approximating curve C'. Observe that hyperbolicity of projective complements is often stable and, in particular, smooth curves with hyperbolic complements form an open subset (see Zaidenberg [36]). Whereas the locus of curves with C-hyperbolic complements is contained in the locus of curves with singularities worse than ordinary double points.

The paper is organized as follows. In Section 2 we summarize necessary background on plane algebraic curves, hyperbolic complex analysis and on $\mathrm{PGL}(2, \mathbb{C})$-actions on $\mathbb{P}^n$. In Section 3 we formulate Theorem 3.1 which is a generalization of Theorem 1.1. Its proof is given in Sections 3-5. In this theorem we give sufficient conditions of C-hyperbolicity of a complement of a plane curve together with its *artifacts*, i.e. certain of its inflectional and cuspidal tangent lines. A curve has no artifacts exactly when its dual is an immersed curve.

Section 6 is devoted to examples of curves of low degrees with hyperbolic and C-hyperbolic complements. In Section 7.a we discuss the fundamental groups of the complements of curves with immersed dual. In Proposition 7.5 we prove that 6 is the minimal degree of irreducible curves with C-hyperbolic complement. We also establish genericity of inflectional tangents (i.e. artifacts) of a generic plane curve (Proposition 7.6).

A part of the results of this paper was reported at the Hayama Conference on Geometric Complex Analysis (Japan, March 1995; see Dethloff and Zaidenberg [10]). In the course of its preparation we had useful discussion on different related topics with D. Akhiezer, F. Bogomolov, M. Brion, R. O. Buchweitz, F. Catanese, H. Kraft, F. Kutzschebauch, S. Orevkov and V. Sergiescu. Their advice, references and information were very helpful. We are grateful to all of them. The first named author would like to thank the Institut Fourier in Grenoble and the second named author would like to thank the SFB 170 "Geometry and Analysis" in Göttingen for their hospitality.

(1) I.e. a curve with only normal crossing singularities.

2. – Preliminaries

a) Background on plane algebraic curves

One says that a reduced curve C in $\mathbb{P}^2$ has *classical singularities* if all its singular points are nodes and ordinary cusps. It is called *a Plücker curve* if both C and the dual curve C^* have only classical singularities and no flecnode, i.e. no flex at a node(2). We say that C is *an immersed curve* if the normalization mapping $\nu : C_{\text{norm}} \to C \hookrightarrow \mathbb{P}^2$ is an immersion, or, which is equivalent, if all the irreducible local analytic branches of C are smooth (in particular, this is so if C has only ordinary singularities(3)).

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$ and of geometric genus g. Then $d^* = \deg C^*$ (i.e. the class of C) is defined by the class formula (see Namba [25], (1.5.4))

$$d^* = 2(d + g - 1) - \sum_{p \in \operatorname{sing} C} (m_p - r_p)\,, \tag{1}$$

where $m_p = \operatorname{mult}_p C$ and r_p is the number of irreducible analytic branches of C at p. Thus, $d^* \geq 2(d + g - 1)$, where the equality holds if and only if C is an immersed curve.

We will need the following corollary of the genus formula (see Namba [25], (2.1.10)):

$$2g \leq (d-1)(d-2) - \sum_{p \in \operatorname{sing} C} m_p(m_p - 1)$$

and $2g = (d-1)(d-2) - 2\delta$ for a nodal curve with δ nodes. For the reader's convenience we recall also the usual Plücker formulas:

$$g = 1/2(d-1)(d-2) - \delta - \kappa = 1/2(d^*-1)(d^*-2) - b - f$$
$$d^* = d(d-1) - 2\delta - 3\kappa \quad \text{and} \quad d = d^*(d^*-1) - 2b - 3f$$

for a Plücker curve C with δ nodes, κ cusps, b bitangent lines and f flexes.

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$ and let $\nu : C^*_{\text{norm}} \to C^*$ be the normalization of the dual curve. Following Zariski [38], p. 307, p. 326 and M. Green [16] (see also Dolgachev and Libgober [11]), consider the mapping $\rho_C : \mathbb{P}^2 \to S^n C^*_{\text{norm}}$ of $\mathbb{P}^2$ into the n-th symmetric power of C^*_{norm}, where $n = \deg C^*$ and where $\rho_C(z) = \nu^*(l_z) \subset S^n C^*_{\text{norm}}$ (here $z \in \mathbb{P}^2$ and $l_z \subset \mathbb{P}^{2*}$ is the dual line). It is easy to check that $\rho_C : \mathbb{P}^2 \to S^n C^*_{\text{norm}}$ is a holomorphic embedding, which we call in the sequel *the Zariski embedding*. We denote by $\mathbb{P}^2_C$ the image $\rho_C(\mathbb{P}^2)$ in $S^n C^*_{\text{norm}}$, by D_n the union of the diagonal divisors in $(C^*_{\text{norm}})^n$ and by $\Delta_n = s_n(D_n) \subset S^n C^*_{\text{norm}}$ the discriminant divisor, i.e. the

(2) Observe that the Plücker formulas are still valid if the latter condition is omitted, but in this case one must count separately the flexes and nodes which are coming from flecnodes or biflecnodes.
(3) I.e. singularities where all the local branches are smooth and pairwise transversal.

ramification locus of the branched covering $s_n : (C^*_{\text{norm}})^n \to S^n C^*_{\text{norm}}$. Thus, we have the diagram

$$\begin{array}{ccccccc} & & & (C^*_{\text{norm}})^n & \supset & D_n \\ (2) & & & {\scriptstyle s_n}\downarrow & & \downarrow \\ & C \subset \mathbb{P}^2 & \overset{\rho_C}{\hookrightarrow} & \mathbb{P}^2_C \subset S^n(C^*_{\text{norm}}) & \supset & \Delta_n\,. \end{array}$$

It is easily seen that $C \subset \rho_C^{-1}(\Delta_n)$. Besides C, this preimage may also contain some lines which we call *artifacts*.

To be more precise, denote by L_C the union of the dual lines in $\mathbb{P}^2$ of the cusps of C^* (by *a cusp* we mean an irreducible singular local branch). Clearly, L_C consists of the inflectional tangents of C and the cuspidal tangents at those cusps of C which are not simple, i.e. which can not be resolved by just one blow-up. Due to an analogy in tomography, we call L_C *the artifacts* of C. These artifacts arise naturally as soon as C^* is not immersed, namely we have

$$\rho_C^{-1}(\mathbb{P}^2_C \cap \Delta_n) = C \cup L_C\,.$$

Indeed, a point $z \in \mathbb{P}^2 \backslash C$ is contained in $\rho_C^{-1}(\Delta_n)$ if and only if its dual line l_z passes through a cusp of C^*.

If $C \subset \mathbb{P}^2$ is a rational curve of degree $d > 1$, then $C^*_{\text{norm}} \cong \mathbb{P}^1$, $S^n\mathbb{P}^1 \cong \mathbb{P}^n$, and hence the Zariski embedding ρ_C embeds $\mathbb{P}^2$ into $\mathbb{P}^n \cong S^n\mathbb{P}^1$, where $n = \deg C^*$. The normalization map $\nu : \mathbb{P}^1 \to C^* \subset \mathbb{P}^2$ can be given as $\nu = (g_0 : g_1 : g_2)$, where $g_i(z_0, z_1) = \sum_{j=0}^n b_j^{(i)} z_0^{n-j} z_1^j$, $i = 0, 1, 2$, are homogeneous polynomials of degree n without common factor.

If $x = (x_0 : x_1 : x_2) \in \mathbb{P}^2$ and $l_x \subset \mathbb{P}^{2*}$ is the dual line, then $\rho_C(x) = \nu^*(l_x) \in S^n\mathbb{P}^1 = \mathbb{P}^n$ is defined by the equation $\sum_{i=0}^2 x_i g_i(z_0 : z_1) = 0$. Thus, $\rho_C(x) = (a_0(x) : \dots : a_n(x))$, where $a_j(x) = \sum_{i=0}^2 x_i b_j^{(i)}$.

Therefore, in the case of a plane rational curve C the Zariski embedding $\rho_C : \mathbb{P}^2 \to \mathbb{P}^n$ is a linear embedding given by the $3\times(n+1)-$matrix $B_C := (b_j^{(i)})$, $i = 0, 1, 2$, $j = 0, \dots, n$, and its image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$ is a plane in $\mathbb{P}^n$.

b) On the Vieta map and the PGL$(2, \mathbb{C})-$ action on $\mathbb{P}^n$

The symmetric power $S^n\mathbb{P}^1$ is naturally identified with $\mathbb{P}^n$ in such a way that the canonical projection $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1$ coincides with *the Vieta ramified covering* given by

$$\big((u_1 : v_1), \dots, (u_n : v_n)\big) \mapsto$$
$$\mapsto \left(\prod_{i=1}^n v_i\right)\big(1 : \sigma_1(u_1/v_1, \dots, u_n/v_n) : \dots : \sigma_n(u_1/v_1, \dots, u_n/v_n)\big)\,,$$

where $\sigma_i(x_1 \dots, x_n)$, $i = 1, \dots, n$, are the elementary symmetric polynomials. This is a Galois covering with the Galois group being the n-th symmetric group

S_n. With $z_i := (u_i : v_i) \in \mathbb{P}^1$, $i = 1, \ldots, n$, we have $s_n(z_1, \ldots, z_n) = (a_0 : \ldots : a_n)$, where z_i, $i = 1, \ldots, n$, are the roots of the binary form $\sum_{i=0}^n a_i u^{n-i} v^i$ of degree n; see Zariski [38], p. 252.

Note that the Vieta map $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1 = \mathbb{P}^n$ is equivariant with respect to the induced actions of the group $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}\,\mathbb{P}^1$ on $(\mathbb{P}^1)^n$ and on $\mathbb{P}^n$, respectively. The branching divisors $D_n \subset (\mathbb{P}^1)^n$ (the union of the diagonals) respectively $\Delta_n \subset \mathbb{P}^n$ (the discriminant divisor), as well as their complements are invariant under these actions. It is easily seen that for $n \geq 3$ the orbit space of the $\mathrm{PGL}(2, \mathbb{C})$-action on $\mathbb{P}^n \backslash \Delta_n$ is naturally isomorphic to the moduli space $M_{0,n}$ of the Riemann sphere with n punctures. Denote by $\tilde{M}_{0,n}$ the quotient $((\mathbb{P}^1)^n \backslash D_n)/\mathbb{P}\mathrm{GL}(2, \mathbb{C})$. We have the following commutative diagram of equivariant morphisms

$$
\begin{array}{ccc}
(\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde{\pi}_n} & \tilde{M}_{0,n} \\
{\scriptstyle s_n}\big\downarrow & & \big\downarrow \\
\mathbb{P}^n \setminus \Delta_n & \xrightarrow{\pi_n} & M_{0,n}\,.
\end{array}
\tag{3}
$$

The cross-ratios $\delta_i(z) = (z_1, z_2; z_3, z_i)$, where $z = (z_1, \ldots, z_n) \in (\mathbb{P}^1)^n$ and $4 \leq i \leq n$, define a morphism

$$\delta^{(n)} = (\delta_4, \ldots, \delta_n) : (\mathbb{P}^1)^n \backslash D_n \to (\mathbb{C}^{**})^{n-3} \backslash D_{n-3}\,,$$

where $\mathbb{C}^{**} := \mathbb{P}^1 \backslash \{0, 1, \infty\}$. By the invariance of cross-ratio $\delta^{(n)}$ is constant along the orbits of the action of $\mathrm{PGL}(2, \mathbb{C})$ on $(\mathbb{P}^1)^n \backslash D_n$. Therefore, it factorizes through a mapping of the orbit space $\tilde{M}_{0,n} \to (\mathbb{C}^{**})^{n-3} \backslash D_{n-3}$. On the other hand, for each point $z \in (\mathbb{P}^1)^n \backslash D_n$ its $\mathrm{PGL}(2, \mathbb{C})$−orbit O_z contains the unique point z' of the form $z' = (0, 1, \infty, z'_4, \ldots, z'_n)$. This defines a regular section $\tilde{M}_{0,n} \to (\mathbb{P}^1)^n \backslash D_n$, and its image coincides with the image of the biregular embedding

$$(\mathbb{C}^{**})^{n-3} \backslash D_{n-3} \ni u = (u_4, \ldots, u_n) \mapsto (0, 1, \infty, u_4, \ldots, u_n) \in (\mathbb{P}^1)^n \backslash D_n\,.$$

This shows that the above mapping $\tilde{M}_{0,n} \to (\mathbb{C}^{**})^{n-3} \backslash D_{n-3}$ is an isomorphism.

In the sequel we treat $\mathbb{P}^n$ as the projectivized space of the binary forms of degree n in u and v. For instance, $e_k = (0 : \ldots : 0 : 1_k : 0 : \ldots : 0) \in \mathbb{P}^n$ corresponds to the forms $cu^{n-k}v^k$, where $c \in \mathbb{C}^*$. Denote by O_q the $\mathrm{PGL}(2, \mathbb{C})$-orbit of a point $q \in \mathbb{P}^n$; it is a smooth quasiprojective variety. If the form q has the roots $z_1, z_2, \ldots$ of multiplicities $m_1, m_2, \ldots$, then we say that O_q is an orbit of type $O_{m_1, m_2, \ldots}$; furthermore, even in the case when O_q is not the only orbit of this type, without abuse of notation we often write $O_{m_1, m_2, \ldots}$ for the orbit O_q itself. Clearly, $O_{e_i} = O_{e_{n-i}}$, $i = 0, \ldots, n$; $O_{e_0} = O_n$ is the only one-dimensional orbit and, at the same time, the only closed orbit; $O_{e_i} = O_{n-i,i}$, $i = 1, \ldots, [n/2]$, are the only two-dimensional orbits. Any other orbit $O_q = O_{m_1,m_2,m_3,\dots}$ has dimension 3 and its closure $\bar{O}_q$ is the union of the orbits O_q, O_n and $O_{m_i,n-m_i}$, $i = 1, 2, \dots$, which follows from Aluffi and Faber [1], Proposition 2.1. Furthermore, for any point $q \in \mathbb{P}^n \backslash \Delta_n$, i.e. for any binary form q without multiple roots, its orbit $O_q = O_{1,1,\dots,1}$ is closed in $\mathbb{P}^n \backslash \Delta_n$, and its closure in $\mathbb{P}^n$ is $\bar{O}_q = O_q \cup S_1$, where $S_1 := O_n \cup O_{n-1,1} = \bar{O}_q \cap \Delta_n$. Therefore, any Zariski closed subvariety Z of $\mathbb{P}^n$ such that $\dim(O_q \cap Z) > 0$ must meet the surface S_1. These observations yield the following lemma(4).

LEMMA 2.1. *If a linear subspace L in $\mathbb{P}^n$ does not meet the surface $S_1 = \bar{O}_{n-1,1} \subset \Delta_n$, then it has at most finite intersection with any of the orbits O_q, where $q \in \mathbb{P}^n \backslash \Delta_n$. In particular, this is so for a generic linear subspace L in $\mathbb{P}^n$ of codimension at least 3.*

For instance, for a given k-tuple of distinct points $z_1, \dots, z_k \in \mathbb{C}$, where $3 \le k \le n$, consider the projective subspace $H_k = H_k(z_1, \dots, z_k) \subset \mathbb{P}^n$, consisting of the binary forms of degree n which vanish at $z_1, \dots, z_k$. Then, clearly, H_k satisfies the above condition, i.e. it does not meet S_1.

c) Background in hyperbolic complex analysis

The next statement follows from Zaidenberg [35], Theorems 1.3, 2.5.

LEMMA 2.2. *Let $C \subset \mathbb{P}^2$ be a curve such that the Riemann surface* $\operatorname{reg} C := C \backslash \operatorname{sing} C$ *is hyperbolic and $\mathbb{P}^2 \backslash C$ is Brody hyperbolic, i.e. it does not contain any entire curve. Then $\mathbb{P}^2 \backslash C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. The condition "*$\operatorname{reg} C$ *is hyperbolic" is necessary for $\mathbb{P}^2 \backslash C$ being hyperbolically embedded into $\mathbb{P}^2$.*

We say that a complex space X is *almost* respectively *weakly Carathéodory hyperbolic* if for any point $p \in X$ there exist only finitely many respectively countably many points $q \in X$ which cannot be separated from p by bounded holomorphic functions. It will be called *almost* respectively *weakly C-hyperbolic* if X has a covering $Y \to X$, where Y is almost respectively weakly Carathéodory hyperbolic. Note that the universal covering $\tilde{X}$ of a C-hyperbolic complex manifold X need not be Carathéodory hyperbolic(5). At the same time, it is weakly Carathéodory hyperbolic.

The next lemma is evident.

LEMMA 2.3. *Let $f : Y \to X$ be a holomorphic mapping of complex spaces. If f is injective (respectively has finite respectively at most countable fibres) and X is C-hyperbolic (respectively almost respectively weakly C-hyperbolic), then Y is C-hyperbolic (respectively almost respectively weakly C-hyperbolic).*

(4)We are grateful to H. Kraft for pointing out the approach used in the proof, and to M. Brion for mentioning the paper Aluffi and Faber [1].

(5)F. Kutzschebauch has constructed a corresponding example of a non-Stein domain $X \subset\subset \mathbb{C}^2$(letter to the authors from 6.7.1995).

3. – Proof of Theorem 1.1 and a generalization

The next theorem gives sufficient conditions for the complement of an irreducible plane curve C and its artifacts L_C to be C-hyperbolic. In the particular case when the dual curve C^* is immersed (i.e. when $L_C = \emptyset$) this leads to Theorem 1.1 of the Introduction.

THEOREM 3.1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of genus g. Put $n = \deg C^*$ and $X = \mathbb{P}^2 \backslash (C \cup L_C)$.*

a) *If $g \geq 1$, then X is C-hyperbolic.*

b) *If $g = 0$, then X is almost C-hyperbolic if at least one of the following conditions is fulfilled:*

 b′) *$i(T_{p^*}A^*, A^*; p^*) \leq n-2$ for any local analytic branch (A^*, p^*) of C^*;*

 b″) *C^* has a cusp and it is not projectively equivalent to one of the curves $(1 : g(t) : t^n)$, $(t : g(t) : t^n)$, where $g \in \mathbb{C}[t]$ and $\deg g \leq n-2$.*

c) *Under any of the assumptions of* (a), (b′), (b″) *X is complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$ if and only if the curve* $\operatorname{reg}(C \cup L_C) = (C \cup L_C) \backslash \operatorname{sing}(C \cup L_C)$ *is hyperbolic.*

Here $i(.,.;.)$ stands for the local intersection multiplicity.

The last statements (c) easily follows from the previous ones in view of Lemma 2.2 and the subsequent remark. Before passing to the proof of (a) and (b) let us make the following observations.

REMARK. Observe that under the conditions of Theorem 1.1 the curve $\operatorname{reg} C$ is hyperbolic. Indeed, the dual of an immersed curve can not be smooth; therefore, this is true as soon as $g \geq 1$, i.e. under the condition in (a). This is also true if C^* is a generic rational curve of degree $n \geq 5$, as it was supposed in 1.1 (b). More generally, let C be a rational curve of degree d such that the dual C^* is an immersed curve of degree $n > 2$. Then by class formula (1) we have: $d = 2(n-1)$ and

$$\sum_{p \in \operatorname{sing} C} (m_p - r_p) = 2(d-1) - n = 3(n-2) \geq 3 .$$

Thus, C has at least three cusps and therefore, $\operatorname{reg} C$ is hyperbolic.

Notice also that condition (b′) is fulfilled for the dual of a generic rational curve of degree $n \geq 5$. This ensures that, indeed, Theorem 1.1 follows from Theorem 3.1.

Proof of Theorem 3.1.a

Let $\rho_C : \mathbb{P}^2 \to S^n(C^*_{\text{norm}})$ be the Zariski embedding introduced in Section 2.a). The covering $s_n : (C^*_{\text{norm}})^n \backslash D_n \to S^n(C^*_{\text{norm}}) \backslash \Delta_n$ is non-ramified.

Thus, we have the commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "\tilde{\rho}_C"] \arrow[d, "\tilde{s}_n"'] & (C^*_{\text{norm}})^n \backslash D_n \arrow[r, hook] \arrow[d, "s_n"] & (C^*_{\text{norm}})^n \\ X \arrow[r, hook, "\rho_C"] & S^n(C^*_{\text{norm}}) \backslash \Delta_n & \end{tikzcd} \tag{4}$$

where $\tilde{s}_n : Y \to X$ is the induced covering. If genus $g(C^*) \geq 2$, then $(C^*_{\text{norm}})^n$ has the polydisc U^n as the universal covering. Passing to the induced covering $Z \to Y$ we can extend (4) to the diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & U^n \arrow[d] \\ Y \arrow[r, hook, "\tilde{\rho}_C"] \arrow[d, "\tilde{s}_n"'] & (C^*_{\text{norm}})^n \arrow[d, "s_n"] \\ X \arrow[r, hook, "\rho_C"] & S^n(C^*_{\text{norm}}) \,. \end{tikzcd} \tag{5}$$

Being a submanifold of the polydisc Z is Carathéodory hyperbolic, and so X is C-hyperbolic. Therefore, we have proved Theorem 3.1.a in the case $g \geq 2$.

Next we consider the case $g = 1$. Denote $E = C^*_{\text{norm}}$. Note that both $E^n \backslash D_n$ and $S^n E \backslash \Delta_n$ are not C-hyperbolic or even hyperbolic, and so we can not apply the same arguments as above.

Represent E as $E = J(E) = \mathbb{C}/\Lambda_\omega$, where Λ_ω is the lattice generated by 1 and $\omega \in \mathbb{C}_+$ (here $\mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0\}$). By Abel's theorem we may assume this identification of E with its jacobian $J(E)$ being chosen in such a way that the image $\rho_C(\mathbb{P}^2)$ is contained in the hypersurface $s_n(H_0) = \phi_n^{-1}(\bar{0}) \cong \mathbb{P}^{n-1} \subset S^n E$, where

$$H_0 := \left\{ z = (z_1, \ldots, z_n) \in E^n \,\middle|\, \sum_{i=1}^n z_i = 0 \right\}$$

is an abelian subvariety in E^n and ϕ_n: $S^n E \to J(E)$ denotes the n-th Abel-Jacobi map. The universal covering $\tilde{H}_0$ of H_0 can be identified with the hyperplane $\sum_{i=1}^n x_i = 0$ in $\mathbb{C}^n = \tilde{E}^n$.

Consider the countable families $\tilde{D}_{ij}$ of parallel affine hyperplanes in $\mathbb{C}^n$ defined by the conditions $x_i - x_j \in \Lambda_\omega$, $i, j = 1, \ldots, n$, $i < j$.

CLAIM. *The domain $\tilde{H}_0 \backslash \bigcup_{i=1}^{n-1} \tilde{D}_{i,i+1}$ is biholomorphic to $(\mathbb{C} \backslash \Lambda_\omega)^{n-1}$.*

Indeed, put $y_k := (x_k - x_{k+1})|\tilde{H}_0$, $i = 1, \ldots, n-1$. It is easily seen that $(y_1, \ldots, y_{n-1})\colon \tilde{H}_0 \to \mathbb{C}^{n-1}$ is a linear isomorphism whose restriction yields a biholomorphism as in the claim.

The universal covering of $(\mathbb{C}\backslash\Lambda_\omega)^{n-1}$ is the polydisc U^n, and so $(\mathbb{C}\backslash\Lambda_\omega)^{n-1}$ is C-hyperbolic. Put $\tilde{D}_n := \bigcup_{i,j=1,\ldots,n} \tilde{D}_{ij}$. The open subset $\tilde{H}_0\backslash\tilde{D}_n$ of $\tilde{H}_0\backslash\bigcup_{i=1}^{n-1}\tilde{D}_{i,i+1} \cong (\mathbb{C}\backslash\Lambda_\omega)^{n-1}$ is also C-hyperbolic (see (2.3)).

Denote by p the universal covering map $\mathbb{C}^n \to (\mathbb{C}/\Lambda_\omega)^n$. The restriction

$$p|\tilde{H}_0\backslash\tilde{D}_n : \tilde{H}_0\backslash\tilde{D}_n \to H_0\backslash D_n \subset E^n\backslash D_n$$

is also a covering map. Therefore, $H_0\backslash D_n$ is C-hyperbolic, and so $s_n(H_0)\backslash\Delta_n$ is C-hyperbolic, too. Since $\rho_C|X : X \to s_n(H_0)\backslash\Delta_n$ is a holomorphic embedding, by Lemma 2.3 X is C-hyperbolic. □

Proof of Theorem 3.1.b′

It consists of the next two lemmas. We freely use the notation from Sections 2.a, 2.b. Remind, in particular, that for a rational curve $C \subset \mathbb{P}^2$ with the dual C^* of degree n, the plane $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \subset \mathbb{P}^n$ is the image of $\mathbb{P}^2$ under the Zariski embedding. The surface $S_1 \subset \Delta_n \subset \mathbb{P}^n$ is the orbit closure $\bar{O}_{n-1,1}$.

LEMMA 3.2. *The complement $X = \mathbb{P}^2\backslash(C \cup L_C)$, where $C \subset \mathbb{P}^2$ is a rational curve, is almost C-hyperbolic whenever $\mathbb{P}^2_C \cap S_1 = \emptyset$.*

PROOF. Consider the following commutative diagram of morphism:

$$(6)\qquad \begin{array}{ccccccc}
Y & \xrightarrow{\tilde{\rho}_C} & (\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde{\pi}_n} & (\mathbb{C}^{**})^{n-3} \setminus D_{n-3} & \hookrightarrow & (\mathbb{C}^{**})^{n-3} \\
{\scriptstyle \tilde{s}_n}\downarrow & & {\scriptstyle s_n}\downarrow & & \downarrow & & \\
X & \xrightarrow{\rho_C} & \mathbb{P}^n \setminus \Delta_n & \xrightarrow{\pi_n} & M_{0,n} & &
\end{array}$$

where $\tilde{s}_n : Y \to X$ is the induced covering (cf. (4)). From Lemma 2.1 it follows that the mapping $\pi_n \circ \rho_C : X \to M_{0,n}$ has finite fibres. Hence, the same is valid for the mapping $\tilde{\pi}_n \circ \tilde{\rho}_C : Y \to (\mathbb{C}^{**})^{n-3}\backslash D_{n-3}$. By Lemma 2.3 Y, and thus also X, are almost C-hyperbolic. □

LEMMA 3.3. *Let $C \subset \mathbb{P}^2$ be a rational curve. Put $n = \deg C^*$. Then $\mathbb{P}^2_C \cap S_1 = \emptyset$ if and only if the condition* (b′) *is fulfilled, i.e. if and only if $i(T_{p^*}A^*, A^*; p^*) < n-1$ for any local analytic branch (A^*, p^*) of the dual curve C^*.*

PROOF. By definition of the Zariski embedding, $q \in \mathbb{P}^2_C \cap S_1$ if and only if, after passing to normalization $\nu : \mathbb{P}^1 \to C^*$ and identifying $\mathbb{P}^2$ with its image $\mathbb{P}^2_C$ under ρ_C, the dual line $l_q \subset \mathbb{P}^{2*}$ cuts out on C^* a divisor of the form $(n-1)a + b$, where $a, b \in \mathbb{P}^1$. Then $p^* := \nu(a) \in C^*$ is the center of a local branch A^* of C^* which violates the condition in (b′). The converse is evidently true. □

REMARKS. 1. If the dual curve C^* has only ordinary cusps and flexes and $n = \deg C^* \geq 5$, then $\mathbb{P}^2_C \cap S_1 = \emptyset$. Indeed, in this case $i(T_{p^*}A^*, A^*; p^*) \leq 3 < n-1$ for any local analytic branch (A^*, p^*) of C^*, and so the result follows from Lemma 3.3.

2. If C^* has a cusp (A^*, p^*) of multiplicity $n-1$, then $\rho_C(l_{p^*}) \subset \mathbb{P}^2_C \cap S_1$, where $l_{p^*} \subset L_C \subset \mathbb{P}^2$ is the dual line of the point $p^* \in \mathbb{P}^{2*}$. Indeed, for any point $q \in l_{p^*}$ its dual line $l_q \subset \mathbb{P}^{2*}$ passes through p^*, and hence we have, as above, $\rho_C(q) \in \mathbb{P}^2_C \cap S_1$.

Next we give an example where both of the conditions (b′), (b″) of Theorem 3.1 are violated.

EXAMPLE 3.4. Let $C^* = (p(t) : q(t) : 1)$ be a parametrized plane rational curve, where $p, q \in \mathbb{C}[t]$ are generic polynomials of degree n and $n-1$, respectively. Thus, C^* is a nodal curve of degree n which is the projective closure of an affine plane polynomial curve with one place at infinity at the point $(1 : 0 : 0)$ which is a smooth point of C^*. The line at infinity $l_2 = \{x_2 = 0\}$ is an inflectional tangent of C^* (of order $n-2$). By Lemma 3.3, $\mathbb{P}^2_C \cap S \neq \emptyset$. Therefore, Lemma 3.2 is not applicable. We do not know whether the complement $\mathbb{P}^2 \backslash C$ of the dual C of C^* in this example is C-hyperbolic or not.

4. – Projective duality and $\mathbb{C}^*$-actions

The proof of Theorem 3.1.b″ is based on a different idea. It needs certain preparations, which is the subject of this section; the proof is done in the next one.

a) Veronese projection, Zariski embedding and projective duality

Let $C \subset \mathbb{P}^2$ be a rational curve with the dual C^* of degree n, and ρ_C: $\mathbb{P}^2 \hookrightarrow \mathbb{P}^2_C \subset \mathbb{P}^n$ be the Zariski embedding. The dual map ρ_C^*: $\mathbb{P}^{n*} \to \mathbb{P}^{2*}$ given by the transposed matrix ${}^tB_C$ (see Section 2.a) defines a linear projection with center $N_C := \mathrm{Ker}^t B_C \subset \mathbb{P}^{n*}$ of codimension 3. The curve C^* is the image under this projection of the rational normal curve $C_n^* = (z_0^n : z_0^{n-1}z_1 : \ldots : z_1^n) \subset \mathbb{P}^{n*}$ (see Veronese [33], p. 208), i.e. $\rho_C^*(C_n^*) = C^*$. Furthermore, C_n^* is the image of $\mathbb{P}^1 \cong C^*_{\mathrm{norm}}$ under the embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{n*}$ defined by the complete linear system $|H| = |n(\infty)| \cong \mathbb{P}^n$. The composition $\nu = \rho_C^* \circ i : \mathbb{P}^1 \to C^* \subset \mathbb{P}^{2*}$ is the normalization map.

The rational normal curve $C_n^* \subset \mathbb{P}^{n*}$ and the discriminant hypersurface $\Delta_n \subset \mathbb{P}^n$ are dual to each other. This yields the following duality:

$$\begin{array}{ccc} (\mathbb{P}^2, C \cup L_C) & \overset{\rho_C}{\hookrightarrow} & (\mathbb{P}^n, \Delta_n) \\ \updownarrow & & \updownarrow \\ (\mathbb{P}^{2*}, C^*) & \overset{\rho_C^*}{\longleftarrow} & (\mathbb{P}^{n^*}, C_n^*)\,. \end{array}$$

To describe this duality in more details, fix a point $q = (z_0^n : z_0^{n-1} z_1 : \ldots : z_1^n) \in C_n^* \subset \mathbb{P}^{n*}$, and let

$$F_q C_n^* = \left\{ T_q^0 C_n^* \subset T_q^1 C_n^* \subset \ldots \subset T_q^{n-1} C_n^* \subset \mathbb{P}^{n*} \right\}$$

be the flag of osculating subspaces to C_n^* at q, where $\dim T_q^k C_n^* = k$, $T_q^0 C_n^* = \{q\}$ and $T_q^1 C_n^* = T_q C_n^*$ is the tangent line to C_n^* at q (see Namba [25], p. 110). For instance, for $q = q_0 = (1 : 0 : \ldots : 0) \in C_n^*$ we have $T_q^k C_n^* = \{x_{k+1} = \ldots = x_n = 0\} \subset \mathbb{P}^{n*}$.

The dual curve $C_n \subset \mathbb{P}^n$ of C_n^* is in turn projectively equivalent to a rational normal curve; namely,

$$C_n = \{p \in \mathbb{P}^n | p = q^* = (z_1^n : -n\, z_0\, z_1^{n-1} : \ldots : (-1)^k \cdot \binom{n}{k} z_0^k\, z_1^{n-k} : \ldots : (-1)^n z_0^n)\} = O_n \,.$$

Furthermore, the dual flag $F_q^{\perp} = \{\mathbb{P}^n \supset H_q^{n-1} \supset \ldots \supset H_q^0\}$, where $H_q^{n-k} := (T_q^{k-1} C_n^*)^{\perp}$, is the flag of osculating subspaces $F_p C_n = \{T_p^{k-1} C_n\}_{k=1}^n$ of the dual rational normal curve $C_n \subset \mathbb{P}^n$. An easy way to see this is to observe that at the dual points $q_0 = (1 : 0 : \ldots : 0) \in C_n^*$ and $p_0 = q_0^* = (0 : \ldots : 0 : 1) \in C_n$ both flags consist of coordinate subspaces, and then to use Aut $\mathbb{P}^1$-homogeneity.

The points of the osculating subspace $H_q^k = T_p^k C_n$ correspond to the binary forms of degree n for which $(z_0 : z_1) \in \mathbb{P}^1$ is a root of multiplicity at least $n-k$. In particular, $H_q^{n-2} = (T_q C_n^*)^{\perp}$ consists of the binary forms which have $(z_0 : z_1)$ as a multiple root. Therefore, the discriminant hypersurface Δ_n is the union of these linear subspaces $H_q^{n-2} \cong \mathbb{P}^{n-2}$ for all $q \in C_n^*$, and thus it is the dual hypersurface of the rational normal curve C_n^*, i.e. each of its points corresponds to a hyperplane in $\mathbb{P}^{n*}$ which contains a tangent line of C_n^*. At the same time, Δ_n is the developable hypersurface of the $(n-2)$-osculating subspaces $H_q^{n-2} = T_p^{n-2} C_n$ of the dual rational normal curve $C_n \subset \Delta_n$; here $T_p^{n-2} C_n \cap C_n = \{p\}$.

Let $d_{ij} \cong \mathbb{P}^1$ be the diagonal of $\mathbb{P}_i^1 \times \mathbb{P}_j^1$. The decomposition $D_{ij} = d_{ij} \times (\mathbb{P}^1)^{n-2}$ of the diagonal hyperplane $D_{ij} \subset D_n$ may be regarded as the trivial fibre bundle $D_{ij} \to \mathbb{P}^1$ with the fibre $(\mathbb{P}^1)^{n-2}$. The subspaces $H_q^{n-2} \subset \Delta_n$ are just the images of the fibres under the Vieta map $s_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$. Moreover, the restriction of s_n to a fibre yields the Vieta map $s_{n-2} : (\mathbb{P}^1)^{n-2} \to \mathbb{P}^{n-2}$. The dual rational normal curve $C_n \subset \mathbb{P}^n$ is the image $s_n(d_n)$ of the diagonal line $d_n := \bigcap_{i,j} D_{ij} = \{z_1 = \ldots = z_n\} \subset (\mathbb{P}^1)^n$.

By duality we have $N_C = \operatorname{Ker} \rho_C^* = (\operatorname{Im} \rho_C)^{\perp}$, i.e. $N_C = (\mathbb{P}_C^2)^{\perp}$. Therefore,

$$\mathbb{P}_C^2 = N_C^{\perp} = \bigcap_{x^* \in N_C} \operatorname{Ker} x^* = \left\{ x \in \mathbb{P}^n | \langle x, x^* \rangle = 0 \quad \text{for all} \quad x^* \in N_C \right\}.$$

A point q on the rational normal curve $C_n^* \subset \mathbb{P}^{n*}$ corresponds to a cusp of C^* under the projection ρ_C^* if and only if the center N_C of the projection meets the tangent developable $TC_n^* = S_1$, which is a ruled surface in $\mathbb{P}^{n*}$, in some point x_q^* of the tangent line $T_qC_n^*$ (see Piene [28]). In this case it meets $T_qC_n^*$ at the only point x_q^*, because otherwise N_C would contain $T_qC_n^*$ and thus also the point q, which is impossible since $\deg C^* = \deg C_n^* = n$.

Let B be a cusp (i.e. a singular local analytic branch) of C^* centered at the point $q_0 = \rho_C^*(q)$. It corresponds to a local branch of C_n^* at the point $q \in C_n^*$ under the normalizing projection $\rho_C^* : C_n^* \to C^*$. Define $L_{B,q_0} := \operatorname{Ker} x_q^* \subset \mathbb{P}^n$ to be the dual hyperplane of the point $x_q^* \in N_C \cap T_qC_n^*$. Since $x_q^* \in N_C$, this hyperplane L_{B,q_0} contains the image $\mathbb{P}_C^2 = \rho_C(\mathbb{P}^2)$. This yields a correspondence between the cusps of C^* and certain hyperplanes in $\mathbb{P}^n$ containing the plane $\mathbb{P}_C^2$. From the definition it follows that L_{B,q_0} contains also the dual linear space $H_q^{n-2} = (T_qC_n^*)^\perp \subset \Delta_n$ of dimension $n-2$. Since the plane $\mathbb{P}_C^2$ is not contained in Δ_n, we have $L_{B,q_0} = \operatorname{span}(\mathbb{P}_C^2, H_q^{n-2})$. It is easily seen that the intersection $\mathbb{P}_C^2 \cap H_q^{n-2}$ coincides with the tangent line $l_{q_0} \subset L_C$ of C, which is dual to the cusp q_0 of C^*. Thus, the artifacts L_C of C are the sections of $\mathbb{P}_C^2$ by those osculating linear subspaces $H_q^{n-2} \subset \Delta_n$ for which q is a cusp of C^*; any other subspace $H_{q'}^{n-2}$ meets the plane $\mathbb{P}_C^2$ in one point of C only.

In what follows by *a special normalization* of the dual rational curve C^* we mean a normalization $\nu : \mathbb{P}^1 \to C^* \subset \mathbb{P}^{2*}$ given in an affine chart in $\mathbb{P}^1$ as $\nu = (h_0(t) : h_1(t) : h_2(t) + t^n)$, where $h_i \in \mathbb{C}[t]$ and $\deg h_i \le n-2$, $i = 0, 1, 2$. Such a curve C^* has a cusp B at the point $q_0 = (0 : 0 : 1)$ which corresponds to $t = \infty$. We will see below that $L_{B,q_0} = \bar{A}_1$, where

$$\bar{A}_1 := \left\{(a_0 : \ldots : a_n) \in \mathbb{P}^n | a_1 = 0\right\} .$$

Clearly, the preimage $\bar{H}_0 := s_n^{-1}(\bar{A}_1) \subset (\mathbb{P}^1)^n$ is the closure of the linear hyperplane in $\mathbb{C}^n$

$$H_0 := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n | \sum_{i=1}^n z_i = 0 \right\} .$$

Note that the choice of normalization of C^* is defined up to the $\mathrm{PGL}(2, \mathbb{C})$-action on $\mathbb{P}^1$, and the induced $\mathrm{PGL}(2, \mathbb{C})$-action on $\mathbb{P}^n$ affects the Zariski embedding. The next lemma ensures the existence of special normalizations.

LEMMA 4.1. *Let $C^* \subset \mathbb{P}^{2*}$ be a rational curve of degree n with a cusp B centered at the point $q_0 = (0 : 0 : 1) \in C^*$, and let $L_{B,q_0} \subset \mathbb{P}^n$ be the corresponding hyperplane which contains the plane $\mathbb{P}_C^2 = \rho_C(\mathbb{P}^2)$. Then C^* admits a special normalization, and under this normalization we have $L_{B,q_0} = \bar{A}_1$ where $\bar{A}_1$ is as above.*

PROOF. The normalization $\nu : \mathbb{P}^1 \cong C^*_{\text{norm}} \to C^* \hookrightarrow \mathbb{P}^2$ can be chosen in such a way that the cusp B corresponds to the local branch of $\mathbb{P}^1$ at $\infty = (1:0) \in \mathbb{P}^1$, and so $\nu(\infty) = q_0$. If $\nu = (g_0 : g_1 : g_2)$ is given by a triple of homogeneous polynomials $g_i(z_0, z_1) = \sum_{j=0}^n b_j^{(i)} z_0^{n-j} z_1^j$, $i = 0, 1, 2$, of degree n, then since $\nu(\infty) = q_0 = (0:0:1)$ we have $\deg_{z_0} g_0 < n$, $\deg_{z_0} g_1 < n$, $\deg_{z_0} g_2 = n$, i.e. $b_0^{(0)} = b_0^{(1)} = 0$, $b_0^{(2)} \neq 0$. Performing the Tschirnhausen transformation

$$\mathbb{P}^1 \ni (z_0 : z_1) \mapsto \left(z_0 - \frac{b_1^{(2)}}{nb_0^{(2)}} z_1 : z_1 \right) \in \mathbb{P}^1$$

we may assume, furthermore, that $b_1^{(2)} = 0$.

CLAIM 1. *The normalization ν as above is a special one, and the image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$ is contained in the hyperplane $\bar{A}_1$.*

Indeed, since C^* has a cusp at q_0, we have $(g_0/g_2)'_{z_1} = (g_1/g_2)'_{z_1} = 0$ at the point $(1:0) \in \mathbb{P}^1$, i.e. $(g_0)'_{z_1} = (g_1)'_{z_1} = 0$ when $z_1 = 0$. This means that $\deg_{z_0} g_0 < n-1$, $\deg_{z_0} g_1 < n-1$, i.e. $b_1^{(0)} = b_1^{(1)} = 0$. And also $b_1^{(2)} = 0$, as it has been achieved above by making use of the Tschirnhausen transformation.

Since $b_1^{(i)} = 0$, $i = 0, 1, 2$, we have $a_1(x) \equiv 0$. Therefore, $\rho_C(x) \in \bar{A}_1$ for any $x \in \mathbb{P}^2$, which proves the claim. □

CLAIM 2. *The dual space H_q^{n-2} to $T_q C_n^*$ is contained in $\bar{A}_1$.*

Indeed, since $\nu(\infty) = q_0$ and $\nu = \rho_C^* \circ i$ with $i : \mathbb{P}^1 \to C_n^* \subset \mathbb{P}^{n*}$ we get $q = (1 : 0 : \ldots : 0)$. Thus, by the preceding considerations the subspace $H_q^{n-2} = (T_q C_n^*)^\perp$ is given by the equations $\{a_0 = a_1 = 0\}$, and hence it is contained in $\bar{A}_1$. □

As before, we have $L_{B,q_0} = \operatorname{span}(\mathbb{P}^2_C, H_q^{n-2})$. Therefore, from Claims 1 and 2 we obtain $L_{B,q_0} = \bar{A}_1$. □

b) Monomial and quasi-monomial rational plane curves

We will use the following terminology. By *a parametrized rational plane curve* we mean a rational curve C in $\mathbb{P}^2$ with a fixed normalization $\mathbb{P}^1 \to C$ of it. *A parametrized monomial* respectively *a parametrized quasi-monomial plane curve* is a parametrized rational plane curve such that all respectively two of its coordinate functions are monomials; the image curve itself is then called *monomial* respectively *quasi-monomial.*

Recall that if $C = (g_0 : g_1 : g_2)$, where $g_i \in \mathbb{C}[t]$, $i = 0, 1, 2$, is a parametrized rational plane curve, then the dual curve C^* has (up to canceling the common factors) the parametrization $C^* = (M_{12} : M_{02} : M_{01})$, where M_{ij} are the 2×2-minors of the matrix

$$\begin{pmatrix} g_0 & g_1 & g_2 \\ g_0' & g_1' & g_2' \end{pmatrix}.$$

The equation of C can be written as $\frac{1}{x_2^d}\operatorname{Res}(x_0g_2 - x_2g_0, x_1g_2 - x_2g_1) = 0$, where $d = \deg C$ and Res means resultant (see e.g. Aure [2], 3.2).

Note that a linear pencil of monomial curves $C_\mu = \{\alpha x_0^l + \beta x_1^{l-k}x_2^k = 0\}$, where $\mu = (\alpha : \beta) \in \mathbb{P}^1$, is self-dual, i.e. the dual curve of a monomial one is again monomial and belongs to the same pencil. In contrast, the dual curve to a quasi-monomial one is not necessarily projectively equivalent to a quasi-monomial curve (recall that two plane curves C, C' are *projectively equivalent* if $C' = \alpha(C)$ for some $\alpha \in \mathbb{P}\mathrm{GL}(3;\mathbb{C}) \cong \operatorname{Aut}\mathbb{P}^2$). The simplest example is the nodal cubic $C = \{(x_0 : x_1 : x_2) = (t : t^3 : t^2 - 1)\}$. Indeed, its dual curve is a quartic with three cusps; but a quasi-monomial curve may have at most two cusps.

Observe that, while the action of the projective group $\mathrm{PGL}(3,\mathbb{C})$ on $\mathbb{P}^2$ does not affect the image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \subset \mathbb{P}^n = S^n\mathbb{P}^1$, the choice of the normalization $\mathbb{P}^1 \to C^*$, defined up to the action of the group $\mathrm{PGL}(2,\mathbb{C}) = \operatorname{Aut}\mathbb{P}^1$, usually does. This is why in the next lemma we have to fix the normalization of a rational plane curve C. This automatically fixes a normalization of its dual curve C^*, and vice versa.

Clearly, projective equivalence between parametrized curves is a stronger relation than just projective equivalence between underlying projective curves themselves.

LEMMA 4.2. *A parametrized rational plane curve $C^* \subset \mathbb{P}^{2*}$ of degree n is projectively equivalent to a parametrized monomial respectively quasi-monomial curve if and only if $\mathbb{P}^2_C \subset \mathbb{P}^n$ is a coordinate plane respectively contains a coordinate axis. This axis is unique if and only if C^* is projectively equivalent to a parametrized quasi-monomial curve, but not to a monomial one.*

PROOF. Let $\nu : t \mapsto (at^k : bt^m : g(t))$, where $a, b \in \mathbb{C}^*$, $g \in \mathbb{C}[t]$ and $t = z_0/z_1 \in \mathbb{P}^1$, define a parametrized quasi-monomial curve $C^* \subset \mathbb{P}^{2*}$ of degree n. Denote $e_k = (0 : \ldots : 0 : 1_k : 0 : \ldots : 0) \in \mathbb{P}^n$. Then ρ_C is given by the matrix $B_C = (b^{(0)}, b^{(1)}, b^{(2)}) = (ae_{n-k}, be_{n-m}, b^{(2)})$, and therefore $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) = \operatorname{span}(b^{(0)}, b^{(1)}, b^{(2)})$ contains the coordinate axis $l_{n-k,n-m}$, where $l_{i,j} := \operatorname{span}(e_i, e_j) \subset \mathbb{P}^n$.

If C^* is a parametrized monomial curve, i.e. if $g(t) = ct^r$, where $c \in \mathbb{C}^*$, then clearly $\mathbb{P}^2_C$ is the coordinate plane $\mathbb{P}_{n-k,n-m,n-r} := \operatorname{span}(e_{n-k}, e_{n-m}, e_{n-r})$. Actually, up to a permutation there should be $0 = r < m < k = n$ and $\gcd(m,n) = 1$; thus, $\mathbb{P}^2_C = \mathbb{P}_{0,n-m,n}$ is a rather special coordinate plane.

Since the projective equivalence of parametrized plane curves does not affect the $\mathbb{P}^2_C$, this yields the first statement of the lemma in one direction.

Vice versa, suppose that $\mathbb{P}^2_C$ coincides with the coordinate plane $\mathbb{P}_{n-k,n-m,n-r}$. Performing a suitable linear coordinate change in $\mathbb{P}^{2*}$ we may assume that $b^{(0)} = e_{n-k}$, $b^{(1)} = e_{n-m}$, $b^{(2)} = e_{n-r}$, i.e. that $\nu(t) = (t^k : t^m : t^r)$. Therefore, in this case the parametrized curve C^* is projectively equivalent to a monomial curve.

Suppose now that $\mathbb{P}^2_C$ contains the coordinate axis $l_{n-k,n-m}$. Performing as above a suitable linear coordinate change in $\mathbb{P}^{2*}$ we may assume that

$b^{(0)} = e_{n-k}, b^{(1)} = e_{n-m}$, and so $\nu(t) = (t^k : t^m : g(t))$. In this case C^* is projectively equivalent to a parametrized quasi-monomial curve. This proves the first assertion of the lemma.

Let $C^* = (at^{n-k} : bt^{n-m} : g(t))$ be a parametrized quasi-monomial curve which is not projectively equivalent to a monomial one. Then as above $\mathbb{P}^2_C \supset l_{k,m}$, and this is the only coordinate axis contained in $\mathbb{P}^2_C$ (indeed, otherwise $\mathbb{P}^2_C$ would be a coordinate plane, that has been excluded by our assumption). The opposite statement is evidently true. This concludes the proof. □

c) $\mathbb{C}^*$-actions

The natural $\mathbb{C}^*$-action on $\mathbb{P}^1$ induces (via the Aut $\mathbb{P}^1$-representations as in Section 2.b) the following $\mathbb{C}^*$-actions on $(\mathbb{P}^1)^n$ respectively on $\mathbb{P}^n = S^n\mathbb{P}^1$:

$$\tilde{G} : \mathbb{C}^* \times (\mathbb{P}^1)^n \ni (\lambda, ((u_1 : v_1), \ldots, (u_n : v_n))) \mapsto ((\lambda u_1 : v_1), \ldots, (\lambda u_n : v_n)) \in (\mathbb{P}^1)^n$$

respectively

$$G : \mathbb{C}^* \times \mathbb{P}^n \ni (\lambda, (a_0 : a_1 : \ldots : a_n)) \mapsto (a_0 : \lambda a_1 : \lambda^2 a_2 : \ldots : \lambda^n a_n) \in \mathbb{P}^n .$$

The Vieta map $s_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$ (see Section 2.b) is equivariant with respect to these $\mathbb{C}^*$-actions and its branching divisors D_n respectively Δ_n are invariant under $\tilde{G}$ respectively G.

LEMMA 4.3. *A parametrized rational plane curve $C^* \subset \mathbb{P}^{2*}$ is projectively equivalent to a parametrized quasi-monomial curve if and only if $\mathbb{P}^2_C \subset \mathbb{P}^n$ contains a one-dimensional G-orbit. This orbit is unique if and only if C^* is projectively equivalent to a parametrized quasi- monomial curve, but not to a monomial one.*

PROOF. Let $\lambda \mapsto (a_0 : \lambda a_1 : \ldots : \lambda^n a_n)$, where $\lambda \in \mathbb{C}^*$, be a parametrization of the G-orbit O_p through the point $p = (a_0 : \ldots : a_n) \in \mathbb{P}^n$. Since the non-zero coordinates are linearly independent as functions of λ, the orbit $O_p \subset \mathbb{P}^n$ is contained in a projective plane if and only if all but at most three of coordinates of p vanish. If p has exactly three non-zero coordinates, then the only plane that contains $\bar{O}_p$ is a coordinate one. If only two of the coordinates of p are non-zero, then the closure $\bar{O}_p$ is a coordinate axis. Since we consider a one-dimensional orbit, the case of one non-zero coordinate is excluded. Now the lemma easily follows from Lemma 4.2. □

5. – Proof of Theorem 3.1.b″

In the sequel "bar" over a letter denotes a projective object, in contrast with the affine ones.

LEMMA 5.1. *Let $\bar{H}_0$ be the hyperplane in $\mathbb{P}^{n-1}$ given by the equation $\sum_{i=1}^n x_i = 0$, and let $\bar{D}_{n-1} = \bigcup_{1\le i<j\le n} \bar{D}_{ij}$ be the union of the diagonal hyperplanes, where $\bar{D}_{ij} \subset \mathbb{P}^{n-1}$ is given by the equation $x_i - x_j = 0$. Then $\bar{H}_0 \backslash \bar{D}_{n-1}$ is C-hyperbolic.*

PROOF. Put $y_i = x_1 - x_{i+1}$, $i = 1, \dots, n-1$. Then $z_i = y_i/y_{n-1}$, $i = 1, \dots, n-2$, are coordinates in the affine chart $\bar{H}_0 \backslash \bar{D}_{1,n} \cong \mathbb{C}^{n-2}$. In these coordinates $\bar{D}_{1,i+1} \cap \bar{H}_0$ respectively $\bar{D}_{i+1,n} \cap \bar{H}_0$ is given by the equation $z_i = 0$ respectively $z_i = 1$, $i = 1, \dots, n-2$. Thus, $\bar{H}_0 \backslash \bar{D}_{n-1} \hookrightarrow (\mathbb{C}^{**})^{n-2}$, where $\mathbb{C}^{**} := \mathbb{P}^1 \backslash \{3 points\}$. By Lemma 2.3 it follows that $\bar{H}_0 \backslash \bar{D}_{n-1}$ is C-hyperbolic. □

REMARK. Using the criterion in Zaidenberg [35], (3.4) one can easily verify that $\bar{H}_0 \backslash \bar{D}_{n-1}$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\bar{H}_0 \cong \mathbb{P}^{n-2}$. Observe, by the way, that for $n = 4$, $\bar{D}_3 \cap \bar{H}_0$ is a complete quadruple[(6)] in $\bar{H}_0 \cong \mathbb{P}^2$.

The proof of Theorem 3.1.b$''$ will be done in several steps.

Basic construction. Let q_0 be a cusp of C^* and $q_0^* \subset L_C \subset \mathbb{P}^2$ be the dual line. We will assume that $q_0 = (0:0:1)$, so that $q_0^* = l_2 = \{x_2 = 0\}$. Due to Lemma 4.1, in what follows we will fix a special normalization $\nu : \mathbb{P}^1 \to C^* \subset \mathbb{P}^{2*}$. Recall (see Section 4.a) that $\nu(\infty) = q_0$ and $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \subset \bar{A}_1 \subset \mathbb{P}^n = S^n\mathbb{P}^1$, where $\bar{A}_1 = \{(a_0 : \dots : a_n) \in \mathbb{P}^n | a_1 = 0\}$. Set $\mathbb{C}^n_a = \{(a_1, \dots, a_n)\} = \{a \in \mathbb{P}^n | a_0 \ne 0\}$, $\mathbb{C}^n_z = s_n^{-1}(\mathbb{C}^n_a) \subset (\mathbb{P}^1)^n$, $A_1 = \bar{A}_1 \cap \mathbb{C}^n_a \cong \mathbb{C}^{n-1}$ and $H_0 = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n_z | \sum_{i=1}^n z_i = 0\} = s_n^{-1}(A_1) \cong \mathbb{C}^{n-1}$. We have $\rho_C(X) \subset \rho_C(\mathbb{P}^2 \backslash l_2) \subset \mathbb{C}^n_a \subset \mathbb{P}^n$. The restriction of the Vieta map yields the non-ramified covering $s_n : H_0 \backslash D_n \to A_1 \backslash \Delta_n$.

Denote by π the canonical projection $\mathbb{C}^n_z \backslash \{0\} \to \mathbb{P}^{n-1}$. Put $\bar{H}_0 := \pi(H_0) \cong \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ and $\bar{D}_{ij} := \pi(D_{ij})$, $\bar{D}_{n-1} := \pi(D_n) = \bigcup_{1\le i<j\le n} \bar{D}_{ij}$. By Lemma 5.1, $\bar{H}_0 \backslash \bar{D}_{n-1}$ is C-hyperbolic.

Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} Y & \overset{\tilde{\rho}_C}{\hookrightarrow} & H_0 \backslash D_n & \xrightarrow{\pi} & \bar{H}_0 \backslash \bar{D}_{n-1} \hookrightarrow \mathbb{P}^{n-2} \\ {\scriptstyle \tilde{s}_n}\downarrow & & \downarrow{\scriptstyle s_n} & & \\ X & \overset{\rho_C}{\hookrightarrow} & A_1 \backslash \Delta_n & & \end{array} \tag{7}$$

where $\tilde{s}_n$ is the induced non-ramified covering. Note that the Vieta map s_n is equivariant with respect to the $\mathbb{C}^*$-actions $\tilde{G}$ on $H_0 \backslash D_n$ and G on $A_1 \backslash \Delta_n$, respectively, and all the fibres of the projection π are one-dimensional $\tilde{G}$-orbits (see (4.c)).

Since $\bar{H}_0 \backslash \bar{D}_{n-1}$ is C-hyperbolic (see Lemma 5.1), by Lemma 2.3 we get that Y, and therefore also X, is almost C-hyperbolic as soon as all the fibres

(6) I.e. a union of six lines through four points in general position.

of the projection $\pi \circ \tilde{\rho}_C : Y \to \bar{H}_0 \backslash \bar{D}_{n-1}$ are finite. To prove Theorem 3.1.b$''$ it is enough to check that this is the case under the condition (b$''$).

CLAIM 1. *If the dual curve C^* (as a parametrized curve) is not projectively equivalent to a quasi-monomial one, then the mapping $\pi \circ \tilde{\rho}_C : Y \to \bar{H}_0 \backslash \bar{D}_{n-1}$ has finite fibres.*

Indeed, since the fibres of π are $\tilde{G}$-orbits, it is enough to show that any $\tilde{G}$-orbit in $H_0 \subset \mathbb{C}^n_{(z)}$ has a finite intersection with $\tilde{\rho}_C(Y)$. Or, what is equivalent, that any G-orbit in $A_1 \subset \mathbb{C}^n_{(a)}$ has a finite intersection with $\rho_C(X) \subset \mathbb{P}^2_C$. We have shown in Lemma 4.3 above that if the latter fails, i.e. if $\mathbb{P}^2_C$ contains a one-dimensional G-orbit, then C^* (parametrized as above) is projectively equivalent to a (parametrized) quasi-monomial curve, which is assumed not to be the case. This yields Claim 1. □

Thus, we may suppose that C^* (as a parametrized curve) is projectively equivalent to a quasi-monomial one. By Lemmas 4.2, 4.3 this means that $\mathbb{P}^2_C$ contains a coordinate line, which is the closure of a one-dimensional G-orbit O_p. Any monomial curve of degree n with a cusp is projectively equivalent (as a parametrized curve, in appropriate parametrization) to one of the curves $(1 : t^k : t^n)$, where $1 \le k \le n-2$. But this is excluded by the conditions in (b$''$). Hence, C^* (as a parametrized curve, with the special normalization chosen above) can not be projectively equivalent to a monomial curve, i.e. the plane $\mathbb{P}^2_C$ is not a coordinate one (see Lemma 4.2).

Let $l_{n-k,n-m} \subset \mathbb{P}^2_C$, $0 \le k < m \le n$, be the only coordinate axis contained in $\mathbb{P}^2_C$. We will distinguish between two cases:

$$\text{(i)} \qquad l := \rho_C^{-1}(l_{n-k,n-m}) \subset L_C \quad \text{and} \qquad \text{(ii)} \qquad l = \rho_C^{-1}(l_{n-k,n-m}) \not\subset L_C\,.$$

Note that (i) respectively (ii) holds if and only if the dual point $q = l^* \in \mathbb{P}^{2*}$ is, respectively is not, a cusp of C^*.

If (i) holds then, as in Claim 1 above, $\pi \circ \tilde{\rho}_C : Y \to \bar{H}_0 \backslash \bar{D}_{n-1}$ has finite fibres, and hence X is almost C-hyperbolic.

Thus, the following claim finishes the proof of (b$''$).

CLAIM 2. *Assume that the dual curve C^* (as a parametrized curve) is projectively equivalent to a quasi-monomial, but not to a monomial one. Then* (ii) *holds if and only if we have one of the two exceptional cases in* (b$''$).

PROOF. Let $\nu = (h_0 : h_1 : t^n + h_2)$, where $h_i \in \mathbb{C}[t]$ and $\deg h_i \le n-2$, $i = 0, 1, 2$, be the special normalization of C^* fixed in the basic construction above. The inclusion $l_{n-k,n-m} = \operatorname{span}(e_{n-k}, e_{n-m}) \subset \mathbb{P}^2_C$ means that $t^k,\ t^m \in \operatorname{span}(h_0, h_1, t^n + h_2)$. Consider two cases

$$(1) \qquad k < m \le n-1, \quad \text{and} \qquad (2) \qquad k < m = n\,.$$

In the first case we have $t^k,\ t^m \in \operatorname{span}(h_0, h_1)$, so that $k,\ m \le n-2$, and without lost of generality we may suppose that $\nu = (t^k : t^m : t^n + h_2(t))$. In the second case we have t^k, $h_2 \in \operatorname{span}(h_0, h_1)$ (so, in particular, $k \leq n-2$), and we may assume that $\nu = (t^k : h_1(t) : t^n)$. With these conventions we have $B_C(e_0) = e_{n-k}$; $B_C(e_1) = e_{n-m}$ in the case (1), so that $l = \rho_C^{-1}(l_{n-k,n-m}) = l_2$, and $B_C(e_2) = e_0$ in the case (2), so that $l = \rho_C^{-1}(l_{n-k,n-m}) = l_1$.

Thus, in the first case $l^* = l_2^* = q_0 = (0 : 0 : 1)$ is the cusp of C^*, and hence (i) holds. In the second case the dual point $l^* = l_1^* = q_1 = (0 : 1 : 0)$ is a cusp of C^* if and only if $k \geq 2$ (indeed, $q_1 \in C^*$ is a smooth point if $k = 1$, and $q_1 \notin C^*$ if $k = 0$). Thus, (ii) holds if and only if $\nu = (t^k : h_1(t) : t^n)$ with $k \leq 1$ and $\deg h_1 \leq n-2$, which are exactly the two exceptional cases of (b″). This completes the proof of Claim 2 and hence the proof of (b″). □

REMARKS. 1. If C^* is one of the exceptional curves mentioned in (3.1.b″) and is not projectively equivalent to a monomial curve, then $X = \mathbb{P}^2 \backslash (C \cup L_C)$ is C-hyperbolic modulo $l_1 = \rho_C^{-1}(l_{n-k,0})$ (in a natural sense). But this is not true, in general, for a plane curve whose dual is a quasi-monomial curve without cusps. An example is a three-cuspidal plane quartic $C \subset \mathbb{P}^2$. Indeed, then C^* is a nodal cubic, which is projectively equivalent to a quasi-monomial curve $t \mapsto (t : t^3 : t^2 - 1)$. The Kobayashi pseudo-distance of $\mathbb{P}^2 \backslash C$ is degenerate on at least seven lines[(7)], and thus $\mathbb{P}^2 \backslash C$ is not C-hyperbolic modulo a line. Furthermore, $\pi_1(\mathbb{P}^2 \backslash C)$ is a finite non-abelian group of order 12 (see Zariski [38], p. 143). Thus, any covering over $\mathbb{P}^2 \backslash C$ is a Liouville one.

2. Let C^* be a monomial curve, and both C^* and C belong to the linear pencil $C_\mu = \{\alpha x_0^n + \beta x_1^k x_2^{n-k} = 0\}$, where $\mu = (\alpha : \beta) \in \mathbb{P}^1$. Set $X = \mathbb{P}^2 \backslash (C \cup L_C)$. Then the Kobayashi pseudodistance k_X is degenerate along any of the members of this linear pencil. At the same time, the distance between points on two distinct members is always positive. In particular, any entire curve $f : \mathbb{C} \to X$ is contained in one of the curves C_μ.

The above proof gives us an additional information that will be used in the concrete examples of Section 6.a to distinguish the exceptional cases. It can be summarized as follows.

PROPOSITION 5.2. *Let $\nu = (h_0 : h_1 : t^n + h_2) : \mathbb{P}^1 \to C^*$ be a special normalization. Then $X = \mathbb{P}^2 \backslash (C \cup L_C)$ is C-hyperbolic in each of the following cases:*

a) *C^* has at least three cusps.*

b) *C^* has two cusps and* $\operatorname{span}(1, h_2) \not\subset \operatorname{span}(h_0, h_1)$.

c) *C^* has one cusp,* $\operatorname{span}(1, h_2) \not\subset \operatorname{span}(h_0, h_1)$, *and* $\operatorname{span}(t, h_2) \not\subset \operatorname{span}(h_0, h_1)$.

The proof is easy and can be omitted.

[(7)] These lines are: the three cuspidal tangents, the three lines through a pair of cusps and the only bitangent line.

6. – Examples

Hereafter $\mathcal{H}(d)$ denotes the set of all plane curves of degree d with complete hyperbolic and hyperbolically embedded complements. In this section we give explicit examples of plane curves with C-hyperbolic complements. Furthermore, we construct, for every even $d \geq 6$, families of irreducible curves in $\mathcal{H}(d)$, especially of elliptic or rational such curves. They are described by the degree, the genus and the singularities of their members or of the dual curves. Most of them arise from Theorem 1.1 in the special case where C^* is a nodal curve, and only the case of maximal cuspidal rational sextics has to be treated in a different way (see Proposition 6.11 below).

a) Reducible curves

6.1. – Two examples of quintics

1. Perhaps, the simplest example is the arrangement C_5 of five lines with two triple points. It is projectively unique and can be given by the equation $x_0x_1x_2(x_0 - x_1)(x_0 - x_2) = 0$. The complement $X = \mathbb{P}^2 \backslash C_5$ is biholomorphic to $(\mathbb{C}^{**})^2$, and thus its universal covering is the bidisc U^2. Hence, X is C-hyperbolic and also complete hyperbolic. However, by Lemma 2.2 X is not hyperbolically embedded into $\mathbb{P}^2$.

2. Another example is a smooth conic C together with its three distinct tangents $L = l_1 \cup l_2 \cup l_3$. This configuration is also projectively unique. We may identify C with the discriminant $\Delta_2 \subset \mathbb{P}^2$, so that Vieta covering $s_2 : (\mathbb{P}_1)^2 \to \mathbb{P}^2$ is branched along C, which is the image of the diagonal $D_2 \subset (\mathbb{P}^1)^2$. The lines l_1, l_2, l_3 are the images of six generators of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, three horizontal ones and three vertical ones. Thus, $X = \mathbb{P}^2 \backslash (C \cup L)$ is covered by $(\mathbb{C}^{**})^2 \backslash D_2$, being, henceforth, C-hyperbolic.

6.2. – Four examples of sextics

1. Modifying example 6.1.1 consider an arrangement C_6 of six lines with three triple points. This is a complete quadruple (cf. the remark after Lemma 5.1). It is projectively unique and can be given by the equation $x_0x_1x_2(x_0 - x_1)(x_0 - x_2)(x_1 - x_2) = 0$. It is known (see e.g. Kaliman [19]) that the universal covering of the complement $X = \mathbb{P}^2 \backslash C_6$ is biholomorphic to the Teichmüller space $T_{0,5}$ of the Riemann sphere with five punctures. Thus, via the Bers embedding $T_{0,5} \hookrightarrow \mathbb{C}^2$ it is biholomorphic to a bounded Bergman domain of holomorphy in $\mathbb{C}^2$, which is contractible and Kobayashi complete hyperbolic. The automorphism group of $T_{0,5}$ is discrete and isomorphic to the mapping class group, or modular group, Mod(0,5) (see Royden [30]). Clearly, the fundamental group $\pi_1(X)$ is a subgroup of finite index in Mod(0,5).

2. The next three examples serve as illustrations to (b″) of Theorem 3.1. Consider a nodal cubic $C \subset \mathbb{P}^2$ together with its three inflectional tangents

$L_C = l_1 \cup l_2 \cup l_3$. They correspond to the cusps of the dual curve C^*, which is a 3-cuspidal quartic. Both C and C^* are projectively unique. By Proposition 5.2.a, we have that $X = \mathbb{P}^2 \backslash (C \cup L_C)$ is almost C-hyperbolic.

3. Let $C \subset \mathbb{P}^2$ be the rational quintic $t \mapsto (2t^5 - t^2 : -(4t^3+1) : 2t)$ with a cusp at the only singular point $(1:0:0)$. The dual curve C^* is the quasi-monomial quartic $t \mapsto (1 : t^2 : t^4 + t)$ given by the equation $(y_0 y_2 - y_1^2)^2 = y_0^3 y_1$. It has the only singular point $q_0 = (0:0:1)$, which is a ramphoid cusp, i.e. it has the multiplicity sequence[(8)] $(2,2,2,1,\dots)$ and $\delta = \mu/2 = 3$, where μ is the Milnor number. Any rational quadric with a ramphoid cusp is projectively equivalent to C^* (see Namba [25], 2.2.5(a)). The artifacts L_C consist of the only cuspidal tangent line $l_2 = \{x_2 = 0\}$ of C. By Proposition 5.2.c, the complement $X = \mathbb{P}^2 \backslash (C \cup l_2)$ is almost C-hyperbolic.

Note that the smooth affine curve $\Gamma = C \backslash l_2 \subset \mathbb{P}^2 \backslash l_2 \cong \mathbb{C}^2$ is isomorphic to $\mathbb{C}^* := \mathbb{C} \backslash \{0\}$, and its complement $X := \mathbb{C}^2 \backslash \Gamma$ is almost C-hyperbolic.

4. Let $C' \subset \mathbb{P}^2$ be the rational quartic $t \mapsto (t^3(2t+1) : -t(4t+3) : -2)$. It has two singular points, a double cusp at the point $(0:0:1)$ (i.e. a cusp with the multiplicity sequence $(2,2,1,\dots)$ and $\delta = 2$) and another one, which is an ordinary cusp. The dual curve $C'^* \subset \mathbb{P}^{2*}$ is the quasi-monomial quartic $t \mapsto (1 : t^2 : t^4 + t^3)$ given by the equation $(y_0 y_2 - y_1^2)^2 = y_0 y_1^3$. It has the same type of singularities as C', namely a double cusp at the point $q_0 = (0:0:1)$ and an ordinary cusp at the point $(1:0:0)$. Therefore, $L_{C'} = l_0 \cup l_2$, where $l_0 = \{x_0 = 0\}$ and $l_2 = \{x_2 = 0\}$. Put $s = 1/t$ and permute the coordinates to obtain the special normalization $s \mapsto (1 + s : s^2 : s^4)$ of C'^*. Now by Proposition 5.2.b, the complement $X = \mathbb{P}^2 \backslash (C' \cup L_{C'})$ is almost C-hyperbolic.

6.3. – Example of a septic

Let things be as in example 6.1.2. Performing the Cremona transformation σ of $\mathbb{P}^2$ with center at the points of intersections of the lines l_1, l_2, l_3, we obtain a 3-cuspidal quartic $C' := \sigma(C)$ together with three new lines $L' = m_1 \cup m_2 \cup m_3$ passing through pairs of cusps of C'. Put $X' = \mathbb{P}^2 \backslash (C' \cup L')$. Since $X = \mathbb{P}^2 \backslash (C \cup L)$ is C-hyperbolic and $\sigma | X : X \to X'$ is an isomorphism, X' is also C-hyperbolic.

6.4. – Two examples of octics

Next we pass to examples to part *a*) of Theorem 3.1. Let $C^* \subset \mathbb{P}^{2*}$ be an irreducible Plücker curve of genus $g \geq 1$ with κ cusps. Then the dual curve $C \subset \mathbb{P}^2$ has $f = \kappa$ ordinary flexes, and L_C is the union of inflectional tangents of C. By the class formula (1), we have $d = \deg C = 2(n + g - 1) - \kappa$. Since all κ inflectional tangents of C are distinct, it follows that $\deg(C \cup L_C) = 2(n + g - 1) \geq 2n \geq 6$. Assume that $\kappa > 0$ to exclude the case when C^*

(8)Recall that *the multiplicity sequence* of a plane analytic germ A at $p_0 \in A$ is the sequence of multiplicities of A at p_0 and in its infinitesimally near points.

is an immersed curve (cf. (6.5), (6.6) below). Since $g \geq 1$, the case when C is a singular cubic has also been excluded. Thus, we have $n \geq 4$, and hence $\deg(C \cup L_c) \geq 8$.

1. The simplest example is a quartic C^* with an ordinary cusp and a node as the only singularities (see Namba [25], p. 133). The dual curve C is an elliptic septic with the only inflectional tangent line $l = L_C$.

2. Another example is a quartic C^* with two ordinary cusps as the only singular points (see Namba [25], p. 133). Here C is an elliptic sextic and L_C is the union of two inflectional tangents of C.

In both examples the assumptions of Theorem 3.1.a are fulfilled, and so $X = \mathbb{P}^2 \backslash (C \cup L_C)$ is C-hyperbolic.

REMARK. It can be checked that in examples 6.1.2, 6.2.1, 6.2.2, 6.3, 6.4.1 and 6.4.2 the conditions of Lemma 2.2 are fulfilled, and hence the corresponding complements are Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$, whereas in 6.1.1, 6.2.3 and 6.2.4 hyperbolic embeddedness fails.

b) Irreducible curves

6.5. – Examples of irreducible curves of genus $g \geq 2$

Theorem 1.1.a can be applied, for instance, to an irreducible curve $C \subset \mathbb{P}^2$ of genus $g \geq 2$ whose dual C^* is a nodal curve of degree $n \geq 4$ with δ nodes. Such a curve C does exist for any given δ with $0 \leq \delta \leq \binom{n-1}{2} - 2$ (see Severi [31], Section 11, p. 347; Oka [26], (6.7)). By the class formula (1) and the genus formula, C has degree $d = n(n-1) - 2\delta$, which can be any even integer from the interval $[2(n+1), n(n-1)]$. The minimal value of d is $d = 10$, which corresponds to a nodal quartic C^* with one node (see Namba [25], p. 130).

REMARK. It was shown by Green [16], Carlson and Green [4] and Grauert and Peternell [15] that an irreducible plane curve C of genus $g \geq 2$ belongs to $\mathcal{H}(d)$ if the following conditions hold:
(i) each tangent line to C^* intersects with C^* in at least two points, and
(ii) $2n < d$, where as before $d = \deg C$ and $n = \deg C^*$.

These conditions are less restrictive than those above, since here C^* may possess cusps. For such a C^* by the genus formula $2g \leq (n-1)(n-2)$, hence $n \geq 4$ for $g \geq 2$, and by (ii) we have $d \geq 9$. Due to the class formula (1), this lower bound is really achieved for the family of duals of the irreducible quartics C^* with an ordinary cusp as the only singular point (see Namba [25], p. 130). However, we do not know whether in this example the complement of C is also C-hyperbolic.

6.6. – Examples of elliptic curves

If the dual C^* of C is an immersed elliptic curve, then by the class formula (1), $d = \deg C = 2n \geq 6$, where $n = \deg C^* \geq 3$. Let C be a sextic in $\mathbb{P}^2$ with nine cusps. Then C is an elliptic Plücker curve whose dual C^* is a smooth cubic; vice versa, the dual of a smooth cubic is a sextic with nine ordinary cusps. From Theorem 1.1.a we get the following

PROPOSITION 6.7. *Every irreducible plane sextic with nine cusps has C-hyperbolic complement and belongs to $\mathcal{H}(6)$.*

Note that up to projective equivalence this family is one dimensional. We refer to Gelfand, Kapranov and Zelevinsky [14], I.2.E for explicit Schläfli's equations of these elliptic sextics. For instance, the dual of the Fermat cubic $-x_0^3+x_1^3+x_2^3=0$ is the sextic

$$x_0^6+x_1^6+x_2^6-2x_0^3x_1^3-2x_1^3x_2^3-2x_0^3x_2^3=0\,.$$

Another example in degree 8 is the family of elliptic curves dual to the nodal quartics with two nodes (see e.g. Namba [25], p. 133 for the existence). Together with (6.4) and (6.5) this yields the following

PROPOSITION 6.8. *For any even $d \geq 6$ there exists a family of irreducible plane curves of degree d and of genus $g \geq 1$ with C-hyperbolic complements which belong to $\mathcal{H}(d)$. It is the family of dual curves to the nodal Plücker curves of degree $n \geq 3$ with δ nodes, with appropriate n and δ.*

6.9. – Examples of rational curves

They illustrate (b) of Theorem 1.1. A generic rational curve C^* of degree $n \geq 3$ is a nodal Plücker curve (see e.g. Aure [2]). Its dual curve C has an even degree $d = 2(n-1)$ and $\kappa = 3(d-2)/2$ cusps. Vice versa, any rational Plücker curve C of even degree $d = 2(n-1)$ with $\kappa = 3(n-2)$ cusps is dual to a nodal curve C^* of degree n. Here κ is the maximal number of cusps which a rational Plücker curve of degree d can possess, and so these curves are called *rational maximal cuspidal curves* (see Zariski [38], p. 267). Applying Theorem 3.1.b$'$ we obtain the following

PROPOSITION 6.10. *For any even degree $d \geq 8$ a rational maximal cuspidal plane curve of degree d belongs to $\mathcal{H}(d)$ and its complement is almost C-hyperbolic.*

What happens with rational maximal cuspidal curves of lower degrees? For $d = 4$ we have a three cuspidal quartic (which is projectively unique; see Namba [25], p. 146). As we saw in Remark 1 after the proof of Theorem 3.1.b$''$, its complement is not even Kobayashi hyperbolic. It remains the case $d = 6$. In this case we have the following

PROPOSITION 6.11. *A generic rational maximal cuspidal plane sextic belongs to $\mathcal{H}(6)$.*

PROOF. We keep the notation of Section 2.b. From the proof of Theorem 3.1.b$'$ we know that such a sextic C is a generic plane section of the discriminant hypersurface $\Delta_4 \subset \mathbb{P}^4$ (by the plane $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$). Clearly, being generic, $\mathbb{P}^2_C$ does not meet the only one-dimensional PGL$(2,\mathbb{C})$-orbit O_4. From the definition of the Zariski embedding it easily follows that it intersects the orbit closure $S_1 = \bar{O}_{3,1}$ respectively $S_2 := \bar{O}_{2,2}$ in the set $K = \{$the cusps of $C\}$ respectively $N = \{$the nodes of $C\}$. Therefore, it intersects the only 3-dimensional orbit $O_{2,1,1}$ contained in Δ_4 in the curve $C \backslash (K \cup N)$. By the Plücker formulas, $\operatorname{card} K = 6$ and $\operatorname{card} N = 4$ (this agrees with the fact that $\deg S_1 = 6$ and $\deg S_2 = 4$; see Aluffi and Faber [1], Proposition 1.1). Let $C_q = \mathbb{P}^2_C \cap \bar{O}_q$, where $O_q = O_{1,1,1,1}$, i.e. $q \in \mathbb{P}^4 \backslash \Delta_4$. Since $\bar{O}_q = O_q \cup S_1$ (see Section 2.b) it is easily seen that the curve $C_q \subset \mathbb{P}^2_C$ meets C exactly in the cusps of C.

Now we use diagram (6). Let $f : \mathbb{C} \to \mathbb{P}^2_C \backslash C = \mathbb{P}^2_C \backslash \Delta_4$ be an entire curve. Since $s_4 : (\mathbb{P}^1)^4 \backslash D_4 \to \mathbb{P}^4 \backslash \Delta_4$ is an unramified covering, f can be lifted to $(\mathbb{P}^1)^4 \backslash D_4$. The curve $\mathbb{C}^{**}$ being hyperbolic, this lifted entire curve has to be contained in a fiber of $\tilde{\pi}_4$, which is an orbit of the PGL$(2,\mathbb{C})$-action on $(\mathbb{P}^1)^4$. The Vieta map s_4 being equivariant, the entire curve $f : \mathbb{C} \to \mathbb{P}^2_C \setminus C$ is contained in a PGL$(2,\mathbb{C})$-orbit, too.

Thus, to see that $\mathbb{P}^2 \backslash C$ is Brody hyperbolic it is enough to show that the quasiprojective curves $C_q \backslash C = O_q \cap \mathbb{P}^2_C$ are hyperbolic for all $q \in \mathbb{P}^4 \backslash \Delta_4$. Once this is done, Proposition 6.11 follows from Lemma 2.2.

It is well known (see Hilbert [18], p. 58 or Popov and Vinberg [29]) that the 3-dimensional PGL$(2,\mathbb{C})$-orbit closures in $\mathbb{P}^4$ form a linear pencil. This pencil of sextic threefolds is generated by its members $3P$ and $2H$, where the irreducible quadric respectively cubic P and H are defined by the basic invariants $\tau_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2$ respectively $\tau_3 = a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3$ (here we use the coordinates where $q(u,v) = a_0 u^4 + 4a_1 u^3 v + 6a_2 u^2 v^2 + 4a_3 u v^3 + a_4 v^4$). The base point set of this linear pencil is the surface $S_1 = H \cap P$, as it follows from the description of the orbit closures in Aluffi and Faber [1] (see Section 2.b).

The restriction of the above pencil to the plane $\mathbb{P}^2_C$ is the linear pencil of plane sextics $\alpha := (C_q)$ generated by $3p$ and $2h$, where $p := P \cap \mathbb{P}^2_C$ and $h := H \cap \mathbb{P}^2_C$ are respectively irreducible conic and cubic. Its base point set $p \cap h$ is the set K of cusps of C (note that C itself is a member of α). The intersection of p and h at the points of K is transversal, because $\operatorname{card} K = p \cdot h = 6$. Since the ideal generated by two distinct members $C' = C_{q'}$ and $C'' = C_{q''}$ is the same as the one generated by $3p$ and $2h$, we have for the local intersection multiplicities at any point $x \in K$

$$i(C', C''; x) = i(3p, 2h; x) = 6i(p, h; x) = 6\,.$$

Assume now that a member C_q of the pencil α has an irreducible component T which intersects C in at most two points $x', x'' \in K$. Since $i(T, C; x) \leq 6$ for $x = x', x''$, we would have $\deg C \cdot \deg T \leq 2$, and hence $\deg T \leq 12$.

If T would be a projective line, then by Bezout's theorem $T \cdot C = \deg C = 6$, hence $i(T, C; x) = 3$ for $x = x'$, x'', and so T should be a common cuspidal tangent of C at these two cusps x', x'', which is impossible for a Plücker curve C.

If, further, T would be a smooth conic, then by the Bezout theorem we would have $i(T, C; x) = 6$ for $x = x', x''$, again in contradiction with the fact that C is a Plücker curve. Indeed, an ordinary cusp (C, P) can be uniformized by $t \mapsto (t^2, t^3 + O(t^4))$, see e.g. Namba [25], 1.5.8, and therefore the local intersection multiplicity of an ordinary cusp with a smooth curve germ (C', P) can be at most 3. To see this, observe that plugging this parametrization into the power series expansion at P of the defining equation of C', its linear term will contain a non-zero monomial in t of order at most 3, which cannot be cancelled by further higher order terms(9).

Thus, there is no irreducible component T as above, and hence all the non-compact curves $C_q \backslash C$ in $\mathbb{P}^2_C \backslash C$ are hyperbolic. □

REMARK. The dimension of the family of all plane rational nodal curves of degree $n \geq 3$ modulo projective equivalence is $3(n-3) = \frac{3}{2}(d-4)$, where $d = 2(n-1)$ is the degree of the dual curves. In particular, the family of curves in Proposition 6.11 is three-dimensional.

7. – Miscellaneous

a) Plane curves with a big fundamental group of the complement

Due to Lin's theorem mentioned in the Introduction (see Lin [22], Theorem 13), we obtain the following

PROPOSITION 7.1. *If $C \subset \mathbb{P}^2$ is one of the curves mentioned in Theorem 3.1, then the group $\pi_1(\mathbb{P}^2 \backslash (C \cup L_C))$ is not almost nilpotent. In particular, it is so in all the examples of Section 6.*

Note that in certain cases more strong fact holds. Let us say that a group G is *big* if it contains a non-abelian free subgroup. By a theorem of von Neumann, a big group is non-amenable. The converse is not true in general; the corresponding examples are due to A. Ol'shanskiy, S. I. Adian and M. Gromov (see e.g. Ol'shanskiy and Shmel'kin [27]). But the groups G in all these examples are not finitely presented. For finitely presented groups the equivalence of bigness and non-amenability is unknown(10). Being non-amenable, a big group can not be almost nilpotent or even almost solvable. As follows from the Nielsen-Schreier theorem, a subgroup of finite index of a big group is big, as well as a normal subgroup with a solvable quotient. Clearly, a group with a big quotient is big.

The following conjecture seems to be plausible.

CONJECTURE 7.2. *If an algebraic variety X is C-hyperbolic, then $\pi_1(X)$ is a big group.*

(9) See also H. Flenner, M. Zaidenberg [12], (1.4) for a more general fact.

(10) We are thankful to V. Sergiescu and V. Guba for this information.

Note that by another Lin's theorem (Lin [22], Theorem B(b)), $\pi_1(X)$ as above can not be an amenable group with a non-trivial center, at least if the universal covering space $\tilde{X}$ is Carathéodory hyperbolic. Observe also that the conjecture is obviously true for $\dim X = 1$.

As far as the complements of plane curves are concerned, we have the following fact[(11)].

THEOREM 7.3. *Let $C \subset \mathbb{P}^2$ be an irreducible curve whose dual C^* is an immersed curve which is neither a line nor a conic nor a nodal cubic. Then the group $\pi_1(\mathbb{P}^2\backslash C)$ is big.*

REMARK. A presentation of the group $\pi_1(\mathbb{P}^2\backslash C)$ for a generic maximal cuspidal curve $C \subset \mathbb{P}^2$ of genus $g = 0$ or 1 was found by Zariski [38], p. 307; cf. also Kaneko [20] for the case $n \geq 2g+1$, where $n = \deg C^*$. However, even if such a presentation is given it might be not so easy to deduce Theorem 7.3.

b) Minimal degree of an irreducible curve with C-hyperbolic complement

Here we show that the examples in Section 6.b are, indeed, at the borderline, as far as the C-hyperbolicity is concerned. Observe that for curves of degree ≤ 4 the complement is not even hyperbolic, since there always exist projective lines which intersect C at most in two points, see e.g. Green [17]. The same remains true for irreducible quintics which are not Plücker.

LEMMA 7.4. *Let $C \subset \mathbb{P}^2$ be an irreducible quintic which is not a Plücker curve. Then $\mathbb{P}^2\backslash C$ is not Brody hyperbolic. Moreover, there exists a line $l_0 \subset \mathbb{P}^2$ which intersects with C in at most two points.*

PROOF. Assume that C has a non-classical singular point p_0 (see Section 2.a). Let l_0 be the tangent line to a local analytic branch of C at p_0. If $\text{mult}_{p_0}C \geq 3$, then $i(C, l_0; p_0) \geq 4$, and so l_0 intersects with C in at most one more point. If $\text{mult}_{p_0}C = 2$, then either $p_0 \in C$ is a tacnode, i.e. C has two smooth branches at p_0 with the same tangent l_0, or C is locally irreducible in p_0 and has the multiplicity sequence $(2, 2, \dots)$ at p_0. In both cases we still have $i(C, l_0; p_0) \geq 4$, and the same conclusion as before holds. It holds also in the case when l_0 is the inflectional tangent to C at a point where C has a flex of order at least 2 (see Namba [25], (1.5)).

Therefore, we may suppose that C has only classical singularities and ordinary flexes. Let q_0 be a singular point of C^* which is not classical. It can not be locally irreducible, since C has only ordinary flexes. If one of the local branches of C^* at q_0 is singular, then the dual line l_0 of q_0 is an inflectional tangent at some flex of C, tangent also at some other point. By Bezout's theorem l_0 is a bitangent line with intersection indices 2 and 3.

It remains to consider the case when C^* has only smooth local branches at q_0. If two of them, say, A_0^* and A_1^*, are tangent to each other, then by duality

(11)For the proof, which was done jointly with S. Orevkov, see G. Dethloff, S. Orevkov, M. Zaidenberg, *Plane curves with a big fundamental group of the complement*, Prépublication de l'Institut Fourier, Grenoble, **354** (1996), 1-26.

the corresponding local branches A_0 and A_1 of C should have common center and also be tangent to each other. This is impossible since C is supposed to have only classical singularities. Thus, $q_0 \in C^*$ should be an ordinary singular point with at least three distinct branches. But then the dual line l_0 of q_0 is tangent in at least three different points of C, which contradicts to Bezout's theorem. □

PROPOSITION 7.5. *The minimal possible degree of an irreducible plane curve with almost C-hyperbolic complement is six, and it has singularities worse than ordinary double points.*

PROOF. By Lemma 7.4 and the preceding remarks, to prove the inequality $d \geq 6$ it is enough to exclude the Plücker quintics. Due to Degtyarev's list [5,6], the fundamental group of the complement of an irreducible Plücker quintic is abelian, and so it is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Thus, the only non-trivial covering Y over $\mathbb{P}^2 \backslash C$ is a finite cyclic one. Being quasiprojective, such an Y is a Liouville variety, and hence $\mathbb{P}^2 \backslash C$ is not almost C-hyperbolic.

The second statement follows from the theorems of Deligne-Fulton and Lin, as it was explained in the introduction. □

c) Genericity of the inflectional tangent lines

In Theorem 3.1 we gave sufficient conditions of (almost) C-hyperbolicity of the complement of a plane curve together with its artifacts. Since the complement of the curve itself is only rarely C-hyperbolic (in particular, this never happens for a nodal curve, see the discussion in the introduction), in order to guarantee C-hyperbolicity we need to add the artifacts, or at least some of them. But then the question arises whether the complement to artifacts themselves is C-hyperbolic. This is the case, for instance, when the artifacts contain the configuration of five lines as in example 6.1. Our aim here is to show that this is not the case for a generic plane curve. Observe that being generic such a curve is smooth, and hence its artifacts are just the inflection tangents.

PROPOSITION 7.6. *If C is a generic plane curve, then the artifacts L_C are in general position. In particular, $\mathbb{P}^2 \backslash L_C$ is not C-hyperbolic.*

PROOF. Since a generic smooth curve is a Plücker curve, we know that all its inflection tangents are distinct. Consider the quasiprojective variety $\mathcal{L}$ of all the configurations $l = (l_1, l_2, l_3, P_1, P_2, P_3)$, where l_1, l_2, l_3 are three distinct lines in $\mathbb{P}^2$ all passing through a common point, and $P_i \in l_i$, $i = 1, 2, 3$, are pairwise distinct points. Let $\mathcal{C}(d)$ be the quasiprojective variety of the smooth plane curves of degree d. For a given $l = (l_1, l_2, l_3, P_1, P_2, P_3) \in \mathcal{L}$ denote by $\mathcal{C}(d)_l$ the subvariety of curves in $\mathcal{C}(d)$ which have flexes at P_i with inflectional tangents l_i, $i = 1, 2, 3$.

CLAIM. *For any $d \geq 4$ and for all $l \in \mathcal{L}$ we have* $\operatorname{codim}_{\mathcal{C}(d)}\mathcal{C}(d)_l = 9$.

PROOF. It can be easily shown that there are exactly three $\mathrm{Aut}(\mathbb{P}^2)$-orbits in $\mathcal{L}$, say $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ where $\mathcal{L}_a$ is the only open orbit which consists of the configurations l such that P_1, P_2, P_3 are not at the same line and none of them coincides with the intersection point $Q \in l_1 \cap l_2 \cap l_3$; $l \in \mathcal{L}_b$ if and only if P_1, P_2, P_3 are at the same line, and $l \in \mathcal{L}_c$ if and only if $P_i = Q$ for some i. To prove the claim we may assume that $l \in \mathcal{L}_i$, $i = a, b, c$, is one of the standard configurations l_a, l_b, l_c described below. For all of them $l_1 = \{x = 0\}$, $l_2 = \{y = 0\}$, $l_3 = \{x = y\}$ in the homogeneous coordinates $(x : y : z)$ in $\mathbb{P}^2$, and, respectively,

l_a) $P_1 = (0:1:0)$, $P_2 = (1:0:0)$, $P_3 = (1:1:1)$,
l_b) $P_1 = (0:1:0)$, $P_2 = (1:0:0)$, $P_3 = (1:1:0)$,
l_c) $P_1 = (0:0:1)$, $P_2 = (1:0:0)$, $P_3 = (1:1:0)$.

Let $C \in \mathcal{C}(d)$ be given by the equation

$$\sum_{0 \le i+j \le d} a_{ij}\, x^i\, y^j\, z^{d-i-j} = 0\,.$$

Then $C \in \mathcal{C}(d)_l$ if and only if, respectively,

a) $a_{0,d} = a_{0,d-1} = a_{0,d-2} = a_{d,0} = a_{d-1,0} = a_{d-2,0} = 0$,
$\sum_{i+j \le d} a_{i,j} = 0$, $\sum_{i+j\le d} (i+j) a_{i,j} = 0$, $\sum_{i+j \le d} (i+j)(i+j-1) a_{i,j} = 0$,

b) $a_{0,d} = a_{0,d-1} = a_{0,d-2} = a_{d,0} = a_{d-1,0} = a_{d-2,0} = 0$,
$\sum_{i+j=d} a_{i,j} = 0$, $\sum_{i+j=d-1} a_{i,j} = 0$, $\sum_{i+j=d-2} a_{i,j} = 0$,

c) $a_{0,0} = a_{0,1} = a_{0,2} = a_{d,0} = a_{d-1,0} = a_{d-2,0} = 0$,
$\sum_{i+j=d} a_{i,j} = 0$, $\sum_{i+j=d-1} a_{i,j} = 0$, $\sum_{i+j=d-2} a_{i,j} = 0$.

Representing these equations on the Newton diagram, it is an easy exercise to check that, if $d \ge 4$, they impose 9 independent conditions on the coefficients a_{ij} of C in all the cases (a), (b) and (c), and the claim follows. □

To prove the proposition in the case when $d \ge 4$, note that the subvariety $\mathcal{S}(d) := \bigcup_{l\in\mathcal{L}} \mathcal{C}(d)_l \subset \mathcal{C}(d)$ consists of the orbits of the induced $\mathrm{PGL}(3;\mathbb{C})$-action on $\mathcal{C}(d)$. Moreover, it consists of the orbits of the subsets $\mathcal{C}(d)_{l_a}$, $\mathcal{C}(d)_{l_b}$, $\mathcal{C}(d)_{l_c}$. Since $\dim \mathrm{PGL}(3;\mathbb{C}) = 8$, due to the above claim, all of these three orbits have codimension at least one. Hence, the complement $\mathcal{C}(d)\backslash\mathcal{S}(d)$ contains a Zariski open subset. It remains to notice that the latter complement coincides with the set of smooth curves of degree d whose inflection tangent lines are in general position.

Consider further the remaining case $d = 3$. It is easily seen that a plane cubic which satisfies one of the conditions (a) or (c) is reducible. The only cubics which satisfy (b) are those from the linear pencil

$$\mathcal{A} = \left\{ C_{(\alpha:\beta)} = \{\alpha x y (x - y) - \beta z^3 = 0\}, \quad (\alpha : \beta) \in \mathbb{P}^1 \right\}.$$

Therefore, $\mathcal{C}(3)_{l_b} \subset \mathcal{A}$. The linear pencil $\mathcal{A}$ is invariant under the action of the one parameter group of automorphisms $(x : y : z) \mapsto (x : y : cz)$, $c \in \mathbb{C}^*$. Hence, the $\mathrm{PGL}(3;\mathbb{C})$-orbit $S(3)$ of $\mathcal{C}(3)_{l_b}$ is of dimension at most 8. Once again, the complement $\mathcal{C}(3)\backslash S(3)$ is Zariski open and it consists of the smooth cubics whose inflectional tangent lines are in general position. This completes the proof. □

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