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1. – Introduction

Let D be a bounded domain of $\mathbb{C}^n$ and let ω be a positive measurable function on D. We denote by

$$L_l^p(D,\omega), \quad 1 \leq p < \infty, \quad l = 0, 1, \ldots$$

the weighted Sobolev space with norm

$$\|f\|_{p,l,\omega} = \sum_{|\alpha|\leq l} \int_D |D^\alpha f|^p \omega \, dz\,, \tag{1.1}$$

where dz denotes the volume element on D. We will denote by $A_l^p(D,\omega)$ the space of holomorphic functions on D which belong to $L_l^p(D,\omega)$.

This paper deals with the $\bar\partial$–equation and some problems of division and extension. To precise the object of our work, we begin stating some known results in this direction.

The first problem is to obtain sharp Sobolev estimates of the solution of $\bar\partial g = f$. For a strictly pseudoconvex domain D with smooth boundary, N. Kerzman [KE] and N. Ovrelid [OV] obtained (L^p, L^p) estimates, S.G. Krantz [KR] (L^p, L^q) estimates and D.C. Greiner and E.M. Stein [GR-ST] Sobolev estimates $(L_l^p(D), L_{l+1/2}^p(D))$. If ρ is a defining function of D such that $\partial\rho(z) \neq 0$ for z in the boundary of D, then we can also obtain $(L_l^p(D,(-\rho)^{\delta+p/2}), L_l^p(D,(-\rho)^\delta))$ estimates (see [OR-FA 1]).

Of course the above results can be considered in other domains. J. Bruna and J. del Castillo [BR-CA], following the method of the integral operators started by M. Range [RA], obtained (L^p, L^p) estimates in some pseudoconvex domains with real analytic boundary and D.C. Chang, A. Nagel and E.M. Stein [CH-NA-ST] obtained Sobolev estimates for the canonical solution in smooth domains of finite type in $\mathbb{C}^2$. Estimates of type (L^p, L^q) have been obtained

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by P. Bonneau and K. Diederich [B-DI], A. Bonami and Ph. Charpentier [B-CHA] and Z. Chen, S.G. Krantz and D. Ma [CHE-KR-MA] in some particular domains.

The second object is a problem of division. To be precise, let $Y = \{z; u = 0\}$ be a complex submanifold in a neighbourhood of $\overline{D}$ and assume that $\partial u(z) \neq 0$, $z \in Y$. Let f be a holomorphic function on D vanishing on $Y \cap D$. Then we can consider the holomorphic function $\frac{f}{u}$ and it is natural to study the regularity of this function in terms of the regularity of f. The C^∞ case has been treated by E. Amar [AM] and M. Hickel [HI]. Assuming that $D = \{z; p < 0\}$ is a bounded strictly pseudoconvex domain with smooth boundary and that the function u satisfies the condition of transversality

$$(\partial u \wedge \partial \rho)(z) \neq 0\,, \quad z \in \partial D \cap Y\,,$$

P. Bonneau, A. Cumenge and A. Zériahi [BO-CU-ZE] showed that if f is in the Lipschitz class Λ_s then $\frac{f}{u}$ is in the class $\Lambda_{s-1/2}$. Under the same conditions it was proved that if f is in the class $A_l^p(D, (-\rho)^\delta)$, then the function $\frac{f}{u}$ is the class $A_l^p(D, (-\rho)^{\delta+p/2})$ [OR-FA1].

Let us consider the third problem. Let Y be a complex submanifold in a neighbourhood of $\overline{D}$ and transversal to the boundary of D. Then we can consider an extension problem from the submanifold $M = Y \cap D$ to D. If $D = \{z; \rho < 0\}$ is a strictly pseudo-convex domain G.M. Henkin [HE] proved restriction and extension theorems for bounded functions and continuous functions on $\overline{D}$ and A. Cumenge [CU] for Hardy spaces and $A^p(D, (-\rho)^\delta)$ spaces. F. Beatrous [BEA] obtained that if Y is a submanifold of codimension d then

$$A_l^p(D, (-\rho)^\delta)|_M = A_l^p(M, (-\rho)^{\delta+d})\,.$$

We point out that in this case it is possible to obtain a result of extension of jets (*i.e.* to find a extension of a function and derivatives, see [OR-FA 1]. For pseudoconvex domains E. Amar [AM] studied the C^∞ case and K. Adachi [AD-1, 2, 3] studied the problem for Hardy spaces in a generalized type of real ellipsoids.

When $D = \{z; \rho(z) < 0\}$ is a domain of finite type m, all the known estimates on the above problems just depend on the type. This have a close relation with the fact that the multiradious $\nu(z)$ of the bigger polidisc centered at z and contained in the intersection of D and the tangent complex space at the point z satisfies $c_0(-\rho)^{1/2} \leq |\nu(z)| \leq c_1(-\rho)^{1/m}$. Observe that this estimate if $m = 2$ gives $|\nu(z)| \approx (-\rho(z))^{1/2}$. Then, if we want to give precise results on the previous problems it seems convenient to introduce Sobolev spaces with weights ω depending on ρ and ν.

The aim of this paper is to consider the above problems in the context of these spaces. We will treat that for the most simple model of domain of finite type *i.e.* the complex ellipsoid of $\mathbb{C}^2$, where we can give sharp estimates.

Let D be the complex ellipsoid of $\mathbb{C}^2$

$$D = \{z; \rho(z) = |z_1|^{2k} + |z_2|^{2m} - 1 < 0\}$$

and ∂D its boundary.

We consider the spaces $L_l^p(D, (-\rho)^\delta \nu^r)$ for $1 \leq p < \infty$, $l = 0, 1, \dots$ and δ, r such that the function $(-\rho)^\delta \nu^r$ is integrable on D. The subspace of holomorphic functions will be denoted by $A_l^p(D, (-\rho)^\delta \nu^r)$.

We state two of the main results which will be obtained in this paper.

THEOREM A. *If f is a $(0, 1)$-form on D, $\bar\partial$ closed and with coefficients in $L_l^p(D, (-\rho)^\delta \nu^{r+p})$, then there exists a function g in $L_l^p(D, (-\rho)^\delta \nu^r)$ such that $\bar\partial g = f$.*

THEOREM B. *Let $Y = \{z \in V; u(z) = 0\}$ be a holomorphic submanifold defined in a neighbourhood of $\overline{D}$ and let $M = Y \cap D$. Moreover, assume that the holomorphic function u satisfies*

$$(\partial u)(z) \neq 0, \; z \in Y \quad \text{and} \quad (\partial u \wedge \partial \rho)(z) \neq 0, \; z \in \partial D \cap Y .$$

Then, if f is a function of class $A_l^p(D, (-\rho)^\delta \nu^r)$, vanishing on M the function f/u is of class $A_l^p(D, (-\rho)^\delta \nu^{r+p})$.

Before stating the first result about extension, we need to introduce some notations and results.

Let $Y = \{z; u(z) = 0\}$ be an analytic set, defined in a neighbourhood of $\overline{D}$ and we assume that $|(\partial u \wedge \partial \rho)(z)| \geq c > 0$, for z in a neighbourhood W of $Y \cap \partial D$. Let $M = D \cap Y$. Then we denote by

$$A_l^p(M, (-\rho)^\delta \nu^r)$$

the space of holomorphic functions f on M such that the restriction of f on $M \cap W$ belongs to $L_l^p(M \cap W, (-\rho)^\delta \nu^r)$. Note that this space does not depend of the neighbourhood W.

Then we obtain:

THEOREM C. *Let $Y = \{z; u(z) = 0\}$ be an analytic set, defined in a neighbourhood of $\overline{D}$ and we assume that $(\partial u \wedge \partial \rho)(z) \neq 0$, $z \in Y \cap \partial D$. Let $M = D \cap Y$. Then*

$$A_l^p(D, (-\rho)^\delta \nu^r)|_M = A_l^p(M, (-\rho)^\delta \nu^{r+2}) .$$

If Y is a complex submanifold we can obtain this result as corollary of a more general result of extension of jets. To state this result we will start given conditions of regularity of the restriction on M of the function and its derivatives. Finally, we prove that the above conditions are sufficient to obtain the extension results.

Let $d^j f$ be the j-th covariant differential of f and let $X = (X_1, \dots, X_j)$ be a j-tuple of smooth vector fields. Assume that X has n_2 tangent complex vector fields. We define the function $\theta(X) = (-\rho)^{n_1} \nu^{n_2}$, where $n_1 = j - n_2$. Then we will obtain the following theorem:

THEOREM D. *Let $Y = \{z; u(z) = 0\}$ be a complex submanifold defined in a neighbourhood of $\overline{D}$ and transversal to he boundary of D. Let $M = Y \cap D$.*

Then for every function f in $A_l^p(D; (-\rho)^\delta \nu^r)$ and every j-tuple of smooth vector fields $X = (X_1, \dots, X_j)$ we have

$$d^j f(X_1, \dots, X_j)|_M \in L_l^p(M, (-\rho)^\delta \nu^{r+2} \theta(X)^p).$$

For every holomorphic function on D, we denote by

$$J_n f = (d^0 f, \dots, d^n f).$$

Let $F = (F^0, \dots, F^n)$ be, where F_z^j, $z \in M$ and $j = 0, \dots, n$, is a j-covariant symmetric tensor. The next problem is to obtain necessary and sufficient conditions on F such that

$$F = J_n f|_M \quad \text{for some} \quad f \quad \text{of class} \quad A_l^p(D, (-\rho)^\delta \nu^r). \tag{1.2}$$

It is clear that condition (1.2) gives some necessary conditions of compatibility on F and that Theorem D gives necessary conditions of regularity of F^j. Then defining an $A_l^p(D, (-\rho)^\delta \nu^r)$-jet of order n on M as a jet which satisfies these conditions (to precise, see Definition 6.5), we obtain:

THEOREM E. *Let $Y = \{z; u(z) = 0\}$ be a complex manifold, defined in a neighbourhood of $\overline{D}$ and transversal to the boundary of D, and let $M = D \cap Y$.*

Then, if F is a $A_l^p(D, (-\rho)^\delta \nu^r)$-jet of order n on M, there exists a function f in $A_l^p(D, (-\rho)^\delta \nu^r)$ such that $J_n f|_M = F$.

This paper is organized as follows. In Section 2 we obtain a precise estimate of the function ν. In Section 3 we construct some representation formulas and we give estimates of the corresponding kernels. In Section 4 we study the properties of the integral operators which appears in Section 3 and as application we prove Theorem A. In Section 5 we prove the division result of Theorem B. In Section 6 we prove the extension results of Theorems D and E and in Section 7 we obtain the result of Theorem C. Finally, in Section 8 we give some results for complex ellipsoids in $\mathbb{C}^n$.

As usual, all the constants which appear in the inequalities will be denoted by c.

2. – Notations and geometric results on D

The aim of this section is to obtain a precise estimate of the radious $\nu(z)$ of the bigger disc of center z and included in the intersection of D and the tangent complex space at the point z. To do so we introduce some notations and definitions that we will use in this paper.

DEFINITION 2.1. *We denote by* $\{D_1, D_2, D_3\}$ *the following partition of* D.

$$D_1 = \{z \in D;\ |z_1|^{2k} \le -\rho(z)\},$$
$$D_2 = \{z \in D;\ |z_2|^{2m} \le -\rho(z) < |z_1|^{2k}\},$$
$$D_3 = \{z \in D;\ -\rho(z) < |z_1|^{2k}, |z_2|^{2m}\}.$$

DEFINITION 2.2. *For* z *in* D *and* $\eta \ge 0$, *we define the functions*

$$\Lambda(z,\eta) = |z_1|^{2k-2}|z_2|^{2m-2}\eta^2 + |z_2|^{2k(2m-1)}\eta^{2k} + |z_1|^{2m(2k-1)}\eta^{2m}$$

$$\tau(z,\eta) = \begin{cases} \eta^{1/2k} & \text{if } z \in D_1 \\ \eta^{1/2m} & \text{if } z \in D_2 \\ \dfrac{\eta^{1/2}}{|z_1|^{k-1}|z_2|^{m-1}} & \text{if } z \in D_3 . \end{cases}$$

If $\eta = -\rho(z)$, *we write* $\tau(z)$ *instead of* $\tau(z, -\rho(z))$.

The next lemmas will be used to prove that $\nu(z) \approx \tau(z)$.

DEFINITION 2.3. *For* $\zeta, z \in \mathbb{C}^2$ *we define*

$$\Phi(\zeta, z) = k\bar{\zeta}_1^k\zeta_1^{k-1}(\zeta_1 - z_1) + m\bar{\zeta}_2^m\zeta_2^{m-1}(\zeta_2 - z_2).$$

The following result has been obtained by M. Range in [RA].

LEMMA 2.4. *If* ζ, z *are in a neighbourhood of the boundary of* D *then*

$$\begin{aligned}\rho(z) - \rho(\zeta) &+ 2\Re e\,\Phi(\zeta, z) \\ &\approx |z_1|^{2k-2}|\zeta_1 - z_1|^2 + |\zeta_1 - z_1|^{2k} + |z_2|^{2m-2}|\zeta_2 - z_2|^2 + |\zeta_2 - z_2|^{2m} \\ &\approx |\zeta_1|^{2k-2}|\zeta_1 - z_1|^2 + |\zeta_1 - z_1|^{2k} + |\zeta_2|^{2m-2}|\zeta_2 - z_2|^2 + |\zeta_2 - z_2|^{2m}.\end{aligned}$$

PROPOSITION 2.5. *For* z *in* D *and near of the boundary, we have*

$$\tau(z) \approx \nu(z) = \min\{|\zeta - z|;\ \zeta \in \partial D, \zeta \in T_z\}$$

where $T_z = \left\{\zeta \in \mathbb{C}^n : \Sigma(\zeta_i - z_i)\frac{\partial\rho}{\partial z_i} = 0\right\}$ *denotes the complex tangent plane at the point* z.

PROOF. Using Lemma 2.4, we find that for $\zeta \in \partial D \cap T_z$,

$$-\rho(z) \approx |z_1|^{2k-2}|\zeta_1 - z_1|^2 + |\zeta_1 - z_1|^{2k} + |z_2|^{2m-2}|\zeta_2 - z_2|^2 + |\zeta_2 - z_2|^{2m}.$$

Since ζ is in T_z, then we have $\zeta - z = \lambda(m\bar{z}_2^m z_2^{m-1}, -k\bar{z}_1^k z_1^{k-1})$ and therefore

$$\begin{aligned}-\rho(z) &\approx |z_1|^{2k-2}|z_2|^{4m-2}|\lambda|^2 + |z_2|^{2k(2m-1)}|\lambda|^{2k} \\ &\quad + |z_2|^{2m-2}|z_1|^{4k-2}|\lambda|^2 + |z_1|^{2m(2k-1)}|\lambda|^{2m} \\ &\approx |z_1|^{2k-2}|z_2|^{2m-2}|\lambda|^2 + |z_2|^{2k(2m-1)}|\lambda|^{2k} + |z_1|^{2m(2k-1)}|\lambda|^{2m} = \Lambda(z, |\lambda|).\end{aligned}$$

Observe that

$$\inf\{|\lambda|;\,(z_1,z_2)+\lambda(m\bar{z}_2^m z_2^{m-1},-k\bar{z}_1^k z_1^{k-1})\in\partial D\}\approx\inf\{|\zeta-z|;\,\zeta\in\partial D\cap T_z\}$$

and that the polynomial

$$p(x)=|z_1|^{2m(2k-1)}x^{2m}+|z_2|^{2k(2m-1)}x^{2k}+|z_1|^{2k-2}|z_2|^{2m-2}x^2+\rho(z)$$

has a unique positive zero x_0 for every $z\in D$, $z\neq 0$.

Thus, to prove the result is sufficient to show that this zero satisfies

$$x_0\approx\begin{cases}(-\rho(z))^{1/2k} & \text{if } |z_1|^{2k}\le-\rho(z)\\ (-\rho(z))^{1/2m} & \text{if } |z_2|^{2m}\le-\rho(z)<|z_1|^{2k}\\ \dfrac{(-\rho(z))^{1/2}}{|z_1|^{k-1}|z_2|^{m-1}} & \text{if } |z_1|^{2k},\,|z_2|^{2m}>-\rho(z)\,.\end{cases}$$

First, we assume that $|z_1|^{2k}\le-\rho(z)$ and we take $x_t=t(-\rho(z))^{1/2k}$. Replacing x by x_t in the polynomial we obtain

$$\begin{aligned}p(x_t)=&|z_1|^{2m(2k-1)}(-\rho(z))^{m/k}t^{2m}+|z_2|^{2k(2m-1)}(-\rho(z))t^{2k}\\&+|z_1|^{2k-2}|z_2|^{2m-2}(-\rho(z))^{1/k}t^2+\rho(z)\,.\end{aligned}$$

Since z is near the boundary of D and $|z_1|^{2k}<-\rho(z)$, then $|z_2|$ is near to 1 and thus there exists t_0 such that $p(x_t)>0$ for every $t\ge t_0$.

On the other hand, we have

$$p(x_t)\le(-\rho(z))^{2m}t^{2m}+(-\rho(z))t^{2k}+(-\rho(z))t^2+\rho(z)$$

and therefore there exists a t_1 such that $p(x_t)<0$ if $0<t\le t_1$.

Using the two above results, we find that

$$t_1(-\rho(z))^{1/2k}<x_0<t_0(-\rho(z))^{1/2k}$$

if $|z_1|^{2k}\le-\rho(z)$.

The same method can be used to show the other cases and hence the lemma is proved. □

Remark. The functions $\Lambda(z,\eta)$ and $\tau(z,\eta)$ of Definition 2.2 are a precise estimate in our case of the functions $\Lambda(z,\delta)$ and $\tau(z,\delta)$ introduced by A. Nagel, E.M. Stein and S. Wainger in [NA-ST-WA].

Also note that for $0<\eta<\eta_0$ and z in a neighbourhood of the boundary of D we have

$$\tau(z,\Lambda(z,\eta))\approx\Lambda(z,\tau(z,\eta))\approx\eta\,. \tag{2.1}$$

3. – Some integral representation formulas and estimates

In this section, using a result of B. Berndtsson and M. Andersson [BE-AN], we obtain some representation formulas and some estimates which we will be used in the next sections to find solutions of the mentioned problems.

To do so we introduce the functions

$$\begin{aligned} s(\zeta, z) &= \bar{\zeta} - \bar{z}\,, \\ a(\zeta, z) &= -\rho(\zeta) + \Phi(\zeta, z) \\ &= 1 + (k-1)|\zeta_1|^{2k} + (m-1)|\zeta_2|^{2m} - k\bar{\zeta}_1^k \zeta_1^{k-1} z_1 - m\bar{\zeta}_2^m \zeta_2^{m-1} z_2 \end{aligned} \tag{3.1}$$

and the forms

$$\begin{aligned} \tilde{s}(\zeta, z) &= (\bar{\zeta}_1 - \bar{z}_1)(d\zeta_1 - dz_1) + (\bar{\zeta}_2 - \bar{z}_2)(d\zeta_2 - dz_2) \\ \widetilde{P}(\zeta, z) &= k\bar{\zeta}_1^k \zeta_1^{k-1}(d\zeta_1 - dz_1) + m\bar{\zeta}_2^m \zeta_2^{m-1}(d\zeta_2 - dz_2) \\ \widetilde{Q}(\zeta, z) &= \frac{\widetilde{P}(\zeta, z)}{a(\zeta, z)}\,. \end{aligned}$$

For every $t \geq 0$, we consider the kernels

$$K^t(\zeta, z) = \frac{1}{(2\pi i)^2}\left(\left(\frac{-\rho(\zeta)}{a(\zeta, z)}\right)^{2+t} \frac{\tilde{s} \wedge d\tilde{s}}{|\zeta - z|^4} + (2+t)\left(\frac{-\rho(\zeta)}{a(\zeta, z)}\right)^{1+t} \frac{\tilde{s} \wedge d\widetilde{Q}}{|\zeta - z|^2}\right)$$

$$R^t(\zeta, z) = \frac{(2+t)(1+t)}{2!(2\pi i)^2} \frac{(-\rho(\zeta))^t}{a(\zeta, z)^t}(d\widetilde{Q})^2$$

These kernels satisfy $d_{\zeta,z} K^t = R^t$ outside the diagonal, and R^t is holomorphic in the variable z.

THEOREM 3.1 (Koppelman Formulas [BER-AN]). *Let $K^t_{p,q}$ be the component of K^t of bidegree (p, q) in z, $(2-p, 1-q)$ in ζ, and let $R^t_{p,q}$ be the component of R^t of bidegree (p, q) in z, and $(2-p, 2-q)$ in ζ. Then, if f is a (p, q) form with coefficients in $C^1(\overline{D})$, we have*

$$\begin{aligned} f(z) = (-1)^{p+q+1}\bigg(&\int_D \bar{\partial} f(\zeta) \wedge K^t_{p,q}(\zeta, z) \\ &- \bar{\partial}_z \int_D f(\zeta) \wedge K^t_{p,q-1}(\zeta, z)\bigg)\,, \qquad \text{if } q \geq 1\,. \\ f(z) = (-1)^{p+1} \bigg(&\int_D \bar{\partial} f(\zeta) \wedge K^t_{p,0}(\zeta, z) - \int_D f(\zeta) R^t_{p,0}(\zeta, z)\bigg)\,, \ \text{if } q = 0\,. \end{aligned}$$

Note that for every $t \geq 0$, the above formulas give explicit integral operators to solve the $\bar{\partial}$-equation. For (0, 1) forms $\bar{\partial}$-closed this operator is given by the kernel

$$K^t_{0,0}(\zeta,z)=\frac{1}{(2\pi i)^2}\left(\left(\frac{-\rho(\zeta)}{a(\zeta,z)}\right)^{2+t}\frac{\tilde{s}\wedge\bar{\partial}\tilde{s}}{|\zeta-z|^4}+(2+t)\frac{(-\rho(\zeta))^{1+t}}{a(\zeta,z)^{1+t}}\frac{\tilde{s}\wedge d\widetilde{Q}}{|\zeta-z|^2}\right). \tag{3.2}$$

Also, we obtain an integral representation formula for holomorphic functions using the kernel

$$R^t_{0,0}=\frac{(2+t)(1+t)}{2!(2\pi i)^2}\frac{(-\rho(\zeta))^t}{a(\zeta,z)^t}(d\widetilde{Q})^2. \tag{3.3}$$

The following lemmas will be used to obtain estimates of these kernels.

LEMMA 3.2. *The kernels K^t and R^t satisfy the estimates*

i) $$|K^t(\zeta,z)|\le c\frac{\big(-\rho(\zeta)\big)^{2+t}}{|a(\zeta,z)|^{2+t}|\zeta-z|^3}+\frac{\big(-\rho(\zeta)\big)^{1+t}}{|a(\zeta,z)|^{3+t}|\zeta-z|}$$

ii) $$|R^t(\zeta,z)|\le c\frac{\big(-\rho(\zeta)\big)^t|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}}{|a(\zeta,z)|^{3+t}}.$$

PROOF. It is clear that $|s(\zeta,z)|=|\zeta-z|$ and that

$$d\widetilde{Q}(\zeta,z)=\frac{d\widetilde{P}(\zeta,z)}{a(\zeta,z)}-\frac{da(\zeta,z)\wedge\widetilde{P}(\zeta,z)}{a(\zeta,z)^2}.$$

Thus i) follows from the expression (3.2) of $K^t(\zeta,z)$.

To obtain ii) observe that

$$\big(d\widetilde{Q}(\zeta,z)\big)^2=\frac{\big(d\widetilde{P}(\zeta,z)\big)^2}{a(\zeta,z)^2}-2\frac{da(\zeta,z)\wedge\widetilde{P}(\zeta,z)\wedge d\widetilde{P}(\zeta,z)}{a(\zeta,z)^3}.$$

Therefore, to prove the result of the lemma is sufficient to show that

$$\big|(d\widetilde{P})^2\big|\le c|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}$$

$$\big|da(\zeta,z)\wedge\widetilde{P}(\zeta,z)\wedge d\widetilde{P}(\zeta,z)\big|\le c|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}$$

and this result follows trivially from

$$\widetilde{P}(\zeta,z)=k\bar{\zeta}_1^k\zeta_1^{k-1}(d\zeta_1-dz_1)+m\bar{\zeta}_2^m\zeta_2^{m-1}(d\zeta_2-dz_2)$$

$$\begin{aligned}d\widetilde{P}(\zeta,z)&=k^2|\zeta_1|^{2k-2}d\bar{\zeta}_1\wedge(d\zeta_1-dz_1)+m^2|\zeta_2|^{2m-2}d\bar{\zeta}_2\wedge(d\zeta_2-dz_2)\\&\quad-k(k-1)\bar{\zeta}_1^k\zeta_1^{k-2}d\zeta_1\wedge dz_1-m(m-1)\bar{\zeta}_2^m\zeta_2^{m-2}d\zeta_2\wedge dz_2\end{aligned}$$

$$\begin{aligned}da(\zeta,z)&=k(k-1)\bar{\zeta}_1^k\zeta_1^{k-2}(\zeta_1-z_1)d\zeta_1+m(m-1)\bar{\zeta}_2^m\zeta_2^{m-2}(\zeta_2-z_2)d\zeta_2\\&\quad+\big(k(k-1)\bar{\zeta}_1^{k-1}\zeta_1^k-k^2|\zeta_1|^{2k-2}z_1\big)d\bar{\zeta}_1\\&\quad+\big(m(m-1)\bar{\zeta}_2^{m-1}\zeta_2^m-m^2|\zeta_2|^{2m-2}z_2\big)d\bar{\zeta}_2\\&\quad-k\bar{\zeta}_1^k\zeta_1^{k-1}dz_1-m\bar{\zeta}_2^m\zeta_2^{m-1}dz_2.\end{aligned}$$

□

LEMMA 3.3. *The representation kernel* $R^t_{0,0}$ *is given by*

$$R^t_{0,0}(\zeta, z) = \frac{(2+t)(1+t)}{2!(2\pi i)^2} \frac{(-\rho(\zeta))^t |\zeta_1|^{2k-2}|\zeta_2|^{2m-2}}{a(\zeta, z)^{3+t}} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 .$$

PROOF. The proof follows easily from the computations of $(d\widetilde{Q})^2$, $d\widetilde{P}$ and da given in the proof of the above lemma. □

The next proposition gives some estimates of these kernels.

PROPOSITION 3.4. *Let* s, r *be real numbers such that the function* $(-\rho)^s\tau^r$ *is a function of class* $L^1(D)$ *and let* $t > 0$ *be large enough. Then we have*

i) $$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta \le c(-\rho(z))^{s-t+1}\tau(z)^r .$$

ii) $$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^{2+t} |\zeta - z|} d\zeta \le c(-\rho(z))^{s-t}\tau(z)^{r+1} .$$

iii) $$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^{2+t}} d\zeta \le c(-\rho(z))^{s-t}\tau(z)^{r+2} .$$

iv) $$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r |\zeta_1|^{2k-2}|\zeta_2|^{2m-2}}{|a(\zeta, z)|^{3+t} |\zeta - z|} d\zeta \le c(-\rho(z))^{s-t}\tau(z)^r .$$

To prove this result we need some preliminary lemmas.

LEMMA 3.5. *If* $\zeta, z \in D$ *then*

i) $$\Re e\, a(\zeta, z) \approx -\rho(z) - \rho(\zeta) + |\zeta_1|^{2k-2}|\zeta_1 - z_1|^2 + |\zeta_1 - z_1|^{2k} + |\zeta_2|^{2m-2}|\zeta_2 - z_2|^2 + |\zeta_2 - z_2|^{2m} .$$

ii) $$|a(\zeta, z)| \approx |a(z, \zeta)| \approx -\rho(z) - \rho(\zeta) + |\Im m\, a(\zeta, z)| + |z_1|^{2k-2}|\zeta_1 - z_1|^2 + |\zeta_1 - z_1|^{2k} + |z_2|^{2m-2}|\zeta_2 - z_2|^2 + |\zeta_2 - z_2|^{2m} .$$

PROOF. This result follows trivially from the definition of $a(\zeta, z)$ and the result of Lemma 2.4. □

LEMMA 3.6. *For* $0 \le \delta, \varepsilon \le 1$, $j \ge 0$ *and* $t > \frac{j+1}{2}$, *we have*

$$I = \int_0^1 \frac{s^j}{(\varepsilon + \delta^{2k-2}s^2 + s^{2k})^t} ds \approx \begin{cases} \varepsilon^{(j+1)/2-t}\delta^{-(k-1)(j+1)} & \text{if } \varepsilon \le \delta^{2k} \\ \varepsilon^{(j+1)/(2k)-t} & \text{if } \varepsilon \ge \delta^{2k} . \end{cases}$$

PROOF. It is clear that

$$I \approx \int_0^\delta \frac{s^j}{(\varepsilon + \delta^{2k-2}s^2)^t} ds + \int_\delta^1 \frac{s^j}{(\varepsilon + s^{2k})^t} ds .$$

Finally, using the change $r = \frac{\delta^{k-1}}{\varepsilon^{1/2}} s$ in the first integral and $r = \frac{1}{\varepsilon} s^{2k}$ in the second one, we obtain the estimate. □

LEMMA 3.7. *Let* $n = 0, 1, 3$, $i, j = 0, 1, \ldots$, $\delta > -1$, $t > \max(\frac{i+2}{2k}, \frac{j+2}{2m}) + \delta$ *and*

$$I(z) = \int_D \frac{(-\rho(\zeta))^\delta |\zeta_1|^i |\zeta_2|^j}{|a(\zeta, z)|^{3-n+t} |\zeta - z|^n} d\zeta .$$

Then:

If $|z_1|^{2k} \leq -\rho(z)$, *we have*

$$I(z) \leq c(-\rho(z))^{i/2k+\delta-t+1} \qquad n = 3$$
$$I(z) \leq c(-\rho(z))^{(i+2-n)/2k+\delta-t+n-1} \qquad n = 0, 1 .$$

If $|z_2|^{2m} \leq -\rho(z)$, *we have*

$$I(z) \leq c(-\rho(z))^{j/2m+\delta-t+1} \qquad n = 3$$
$$I(z) \leq c(-\rho(z))^{(j+2-n)/2m+\delta-t+n-1} \qquad n = 0, 1 .$$

If $|z_1|^{2k}$, $|z_2|^{2m} \geq -\rho(z)$, *we have*

$$I(z) \leq c(-\rho(z))^{\delta-t+1} |z_1|^i |z_2|^j \qquad n = 3$$
$$I(z) \leq c(-\rho(z))^{\delta+n/2-t} |z_1|^{i-(2-n)(k-1)} |z_2|^{j-(2-n)(m-1)} \qquad n = 0, 1 .$$

PROOF. By Lemma 3.5 we find that $|a(\zeta, z)| \geq c > 0$ if $|\zeta - z| \geq \varepsilon > 0$ or if ζ or z are in a compact subset of D.

Thus, to prove the lemma is sufficient to obtain the above estimates for

$$\int_{D \cap \{\zeta; |\zeta - z| \leq \varepsilon\}} \frac{(-\rho(\zeta))^\delta |\zeta_1|^i |\zeta_2|^j}{|a(\zeta, z)|^{3-n+t} |\zeta - z|^n} d\zeta \tag{3.4}$$

with z in a neighbourhood of ∂D.

First we consider the case $\delta = 0$.

To compute this integral we consider three cases.

a) $|z_1|^{2k} \leq -\rho(z)$.

In this case we have $|\zeta_2| \geq c > 0$ and therefore using the usual change of coordinates

$$\eta_1 = \rho(\zeta) - \rho(z) + i \Im m\, \Phi(\zeta, z)$$
$$\eta_2 = \zeta_1 - z_1$$

and the estimates of Lemma 3.5, we obtain

$$I(z) \leq c \int_{|\eta| \leq R} \frac{|\eta_2|^i + |z_1|^i}{(-\rho(z) + |\eta_1| + |z_1|^{2k-2} |\eta_2|^2 + |\eta_2|^{2k})^{3-n+t} |\eta|^n} .$$

Now, using polar coordinates $|\eta_1| = r$, $|\eta_2| = s$ we get

$$I(z) \leq c \int_0^R \int_0^R \frac{(s^i + |z_1|^i) rs}{(-\rho(z) + r + |z_1|^{2k-2} s^2 + s^{2k})^{3-n+t} (r^2 + s^2)^{n/2}} ds\, dr .$$

To finish, we consider the cases $n = 0, 1, 3$.

$n = 3$. In this one we have

$$I(z) \le c \int_0^R \frac{r}{(-\rho(z) + r)^{t-i/2k}} \int_0^R \frac{s}{(r^2+s^2)^{3/2}} ds\, dr \le c(-\rho(z))^{i/2k-t+1} .$$

$n = 1$. Using the same method and Lemma 3.6, we obtain

$$\begin{aligned} I(z) &\le c \int_0^R \int_0^R \frac{r}{(-\rho(z) + r + |z_1|^{2k-2}s^2 + s^{2k})^{2+t-i/2k}} ds\, dr \\ &\le c \int_0^R \frac{1}{(-\rho(z) + |z_1|^{2k-2}s^2 + s^{2k})^{t-i/2k}} ds \le c(\rho(z))^{(i+1)/2k-t} . \end{aligned}$$

$n = 0$. The same argument gives

$$\begin{aligned} I(z) &\le c \int_0^R \int_0^R \frac{rs}{(-\rho(z) + r + |z_1|^{2k-2}s^2 + s^{2k})^{3+t-i/2k}} ds\, dr \\ &\le c \int_0^R \frac{s}{(-\rho(z) + |z_1|^{2k-2}s^2 + s^{2k})^{1+t-i/2k}} ds \le c(-\rho(z))^{(i+2)/2k-1-t} . \end{aligned}$$

The cases *b)* $|z_2|^{2m} \le -\rho(z) < |z_1|^{2k}$ and *c)* $|z_1|^{2k}, |z_2|^{2m} > -\rho(z)$, follow in the same way.

Now, we consider the case $\delta \ne 0$.

If $\delta > 0$, using $-\rho(\zeta) \le c|a(\zeta, z)|$ the result follows trivially from the above case.

If $-1 < \delta < 0$, using $(-\rho(\zeta) + \Phi(\zeta, z))/a(\zeta, z) = 1$, $|\partial\rho(\zeta)| \ge c > 0$ for ζ in a neighbourhood of ∂D and an integration by parts we obtain

$$I(z) \le c \int_D (-\rho(\zeta))^{\delta+1} \frac{1 + i|\zeta_1|^{i-1}|\zeta_2|^j|a| + j|\zeta_1|^i|\zeta_2|^{j-1}|a| + |\zeta_1|^i|\zeta_2|^j}{|a|^{4-n+t}|\zeta - z|^n} d\zeta .$$

Hence, the result follows from the above case. □

LEMMA 3.8. *For $\zeta, z \in D$ we have*

$$|z_1|^{k-1}|z_2|^{m-1}\tau(\zeta) \le c|z_1|^{k-1}|z_2|^{m-1}\tau(\zeta, |a(\zeta, z)|) \le c|a(\zeta, z)|^{1/2} .$$

PROOF. The first inequality follows from the definitions of τ and $-\rho(\zeta) \le c|a(\zeta, z)|$.

Now we prove the second inequality. By Lemma 3.5 we have

$$\begin{aligned} |z_1|^{k-1} &\le c\big(|\zeta_1|^{k-1} + |\zeta_1 - z_1|^{k-1}\big) \le c\big(|\zeta_1|^{k-1} + |a(\zeta, z)|^{(k-1)/2k}\big) \\ |z_2|^{m-1} &\le c\big(|\zeta_2|^{m-1} + |\zeta_2 - z_2|^{m-1}\big) \le c\big(|\zeta_2|^{m-1} + |a(\zeta, z)|^{(m-1)/2m}\big) . \end{aligned}$$

Finally, using the above estimates and the definition of $\tau(\zeta, |a(\zeta, z)|)$ we obtain the result. □

PROOF OF PROPOSITION 3.4:

Estimate i). We consider the following cases:

a) $z \in D_1$. If $r \geq 0$, using the definition of τ and the estimates of Lemma 3.7, we have

$$\int_D \frac{(-\rho)^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta \leq c + \int_{D \cap \{|\zeta_1| < 1/2\}} \frac{(-\rho)^{s+r/2k}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta$$
$$\leq c(-\rho(z))^{s-t+1+r/2k} = c(-\rho(z))^{s-t+1} \tau(z)^r .$$

If $r < 0$ we obtain

$$\int_D \frac{(-\rho)^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta \leq \int_{D_1} \frac{(-\rho)^{s+r/2k}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta$$
$$+ \int_{D_2} \frac{(-\rho)^{s+r/2m}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta$$
$$+ \int_{D_3} \frac{(-\rho)^{s+r/2} |\zeta_1|^{-r(k-1)}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta$$
$$\leq c(-\rho(z))^{s-t+1+r/2k} + c + c(-\rho(z))^{s-t+1+r/2k}$$
$$\leq c(-\rho(z))^{s-t+1} \tau(z)^r .$$

b) $z \in D_2$. This case follows in the same way.

c) $z \in D_3$. If $r \geq 0$ using Lemma 3.8 and the estimates of Lemma 3.7 we obtain

$$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta \leq \frac{1}{(|z_1|^{k-1} |z_2|^{m-1})^r} \int_D \frac{(-\rho(\zeta))^s}{|a(\zeta, z)|^{t-r/2} |\zeta - z|^3} d\zeta$$
$$\leq c \frac{(-\rho(z))^{s-t+1+r/2}}{(|z_1|^{k-1} |z_2|^{m-1})^r} = c(-\rho(z))^{s-t+1} \tau(z)^r .$$

If $r < 0$, we have

$$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta = \int_{D_1} \frac{(-\rho(\zeta))^{s+r/2k}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta$$
$$+ \int_{D_2} \frac{(-\rho(\zeta))^{s+r/2m}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta + \int_{D_3} \frac{(-\rho(\zeta))^{s+r/2} (|\zeta_1|^{k-1} |\zeta_2|^{m-1})^{-r}}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta .$$

Assume for example $|z_1| \leq \frac{1}{2}$. Then, by Lemma 3.7 we obtain

$$\int_D \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^t |\zeta - z|^3} d\zeta \leq c(-\rho(z))^{s-t+1+r/2k} + c + \frac{(-\rho(z))^{s-t+1+r/2}}{(|z_1|^{k-1} |z_2|^{m-1})^r}$$
$$\leq c(-\rho(z))^{s-t+1} \tau(z)^r .$$

Thus estimate i) is proved. Estimates ii) and iii) follow in the same way and their proofs will be omitted. Estimate iv) follows from part iii) and from the estimate

$$|\zeta_1|^{2k-2}|\zeta_2|^{2m-2} \leq c\frac{-\rho(\zeta)}{\tau(\zeta)^2}$$

obtained in Lemma 3.8. □

4. – Integral operators and resolution of $\overline{\partial}$-equation

The aim of this section is to give some properties of the integral operators K^t and R^t defined in the above section. Also we prove the following result:

THEOREM 4.1. *Let $1 \leq p < \infty$ be and let δ, r be such that $(-\rho)^\delta \tau^r$ belongs to $L^1(D)$. Then, for every $(0, 1)$ form f on D, $\bar{\partial}$ closed and with coefficients in $L_l^p(D, (-\rho)^\delta \tau^{r+p})$, there exists a function g in $L_l^p(D, (-\rho)^\delta \tau^r)$ such that $\bar{\partial} g = f$.*

To prove this theorem we will use the solution of the $\bar{\partial}$-equation given by the operator $K_{0,0}^t$ for some $t > 0$ large enough (see (3.2)), the estimates of Section 3 and the following integration by parts formulas.

The first lemma can be found in [BR-BU].

LEMMA 4.2. *Let f be a $(0, 1)$ form $\bar{\partial}$-closed with coefficients of class $C^\infty(\overline{D})$. Then*

$$\begin{aligned}\frac{\partial^j}{\partial z^\alpha}\int_D f(\zeta) \wedge K_{0,0}^t(\zeta, z) &= \int_D \frac{\partial^j}{\partial \zeta^\alpha} f(\zeta) \wedge K_{0,0}^t(\zeta, z)\\ &\quad - \sum_{i,\beta,\gamma} \frac{\partial^{|\gamma|}}{\partial z^\gamma}\int_D \frac{\partial^{|\beta|}}{\partial \zeta^\beta} f(\zeta) \wedge R_{0,1,1}^{t,i}(\zeta, z)\end{aligned}$$

where γ, β are multiindexes with $|\gamma| + |\beta| = j - 1$ and $R_{0,1,1}^{t,i}$, $i = 1, 2$ denotes the coefficient in dz_i in the component of the kernel R^t of degree $(1, 0)$ in z and $(2, 1)$ in ζ.

To find estimates of the terms which appear $R_{0,1,1}^{t,i}$ we need to introduce the following operators:

DEFINITION 4.3. *For positive integers i, j, u, v, ν, η and $s, t \geq 0$ we define the kernel*

$$R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z) = \frac{(-\rho(\zeta))^s \zeta_1^i \bar{\zeta}_1^j \zeta_2^u \bar{\zeta}_2^v z_1^\nu z_2^\eta}{a(\zeta, z)^{3+t}}$$

and the differential operator $R_\zeta = \zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2}$.

REMARK. It is clear that the coefficients of $R^t(\zeta, z)$ are linear combination of these kernels.

LEMMA 4.4. *If $s > 0$ and f is a function of class $C^1(\overline{D})$, then*

$$R_z \int_D R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z) f(\zeta) d\zeta$$
$$= \int_D R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z)\big((i - j + \nu + u - v + \eta)I + (R_\zeta - \overline{R}_\zeta)\big) f(\zeta) d\zeta$$

where I denotes the Identity operator.

PROOF. First, note that Stokes' theorem implies

$$\int_D (R_\zeta - \overline{R}_\zeta) g(\zeta) d\zeta = 0$$

for every function of class $C^1(\overline{D})$ which vanishes in the boundary of D. Also, note that from the definition of $\rho(\zeta)$, and $a(\zeta, z)$ (see (3.1)), we have

$$(R_z - \overline{R}_\zeta + R_\zeta)\rho(\zeta) = 0$$
$$(R_z - \overline{R}_\zeta + R_\zeta) a(\zeta, z) = 0$$
$$(R_z - \overline{R}_\zeta + R_\zeta)\big(\zeta_1^i \bar{\zeta}_1^j \zeta_2^u \bar{\zeta}_2^v z_1^\nu z_2^\eta\big) = (i - j + \nu + u - v + \eta)\big(\zeta_1^i \bar{\zeta}_1^j \zeta_2^u \bar{\zeta}_2^v z_1^\nu z_2^\eta\big).$$

Hence, we obtain

$$R_z \int_D R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z) f(\zeta) d\zeta$$
$$= \int_D (R_z - \overline{R}_\zeta + R_\zeta) R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z) f(\zeta) d\zeta$$
$$= \int_D R^{s,t,i,j,u,v,\nu,\eta}(\zeta, z)\big((i - j + \nu + u - v + \eta)I + (R_\zeta - \overline{R}_\zeta)\big) f(\zeta) d\zeta$$

and the lemma is proved. □

The next lemma is well-known (see for instance [OK] Theorem 4.1.2).

LEMMA 4.5. *Let (X, μ_x), (Y, μ_y) be σ-finite measure spaces and let $1 < p < \infty$, and p' its conjugate exponent. Suppose that there exist μ_x, μ_y measurable functions $\varphi_1(x)$, $\varphi_2(y)$ and $\psi(x, y)$ such that*

i) $$\int_X \varphi_1^{p'}(x) |\psi(x, y)| d\mu_x \le c \varphi_2^{p'}(y)$$

ii) $$\int_Y \varphi_2^{p}(y) |\psi(x, y)| d\mu_y \le c \varphi_1^{p}(x).$$

Then the operator

$$Tf(y) = \int_X f(x) \psi(x, y) d\mu_x$$

is continuous from $L^p(X, \mu_x)$ to $L^p(Y, \mu_y)$.

If $p = 1$ and condition ii) *is satisfied for some $\varphi = \varphi_1 = \varphi_2$, then the operator T maps $L^1(X, \mu_x)$ to $L^1(Y, \mu_y)$.*

PROPOSITION 4.6. *For δ, t, r satisfying the conditions of Theorem 4.1 and K an integral operator have:*

i) *If $|K(\zeta, z)| \le c(-\rho(\zeta))^t/(|a(\zeta, z)|^t|\zeta - z|^3)$ then the integral operator K is continuous from $L^p(D, (-\rho)^{\delta+p}\tau^r)$ to $L^p(D, (-\rho)^\delta\tau^r)$.*

ii) *If $|K(\zeta, z)| \le c(-\rho(\zeta))^t/(|a(\zeta, z)|^{2+t}|\zeta - z|)$ then the integral operator K is continuous from $L^p(D, (-\rho)^\delta\tau^{r+p})$ to $L^p(D, (-\rho)^\delta\tau^r)$.*

iii) *If $|K(\zeta, z)| \le c(-\rho(\zeta))^t|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}/(|a(\zeta, z)|^t|\zeta - z|^{3+t})$ then the integral operator K is continuous from $L^p(D, (-\rho)^\delta\tau^r)$ to $L^p(D, (-\rho)^\delta\tau^r)$.*

PROOF. Let $1 \le p < \infty$ be and let p' be its conjugate exponent.
To prove i) we define

$$\psi(\zeta, z) = \frac{(-\rho(\zeta))^{t-\delta-p}\tau(\zeta)^{-r}}{|a(\zeta, z)|^t|\zeta - z|^3}$$

$$\varphi_1(\zeta) = (-\rho(\zeta))^{-s-1/p'}, \qquad \varphi_2(z) = (-\rho(z))^{-s}$$

$$\mu_\zeta = (-\rho(\zeta))^{\delta+p}\tau(\zeta)^r d\zeta, \qquad \mu_z = (-\rho(z))^\delta\tau(z)^r dz$$

with $s > 0$ small enough such that $(-\rho)^{\delta-s}\tau^r$ belongs to $L^1(D)$.
Then we have $\varphi_1 = \varphi_2$ if $p = 1$ and

$$\left|\int_D f(\zeta)K(\zeta, z)d\zeta\right| \le c\int_D |f(\zeta)|\psi(\zeta, z)d\mu_\zeta.$$

Moreover, by proposition 3.4 we have:

$$\text{a)} \qquad \int_D \varphi_1(\zeta)^{p'}\psi(\zeta, z)d\mu_\zeta \le \varphi_2(z)^{p'}, \quad 1 < p$$

$$\text{b)} \qquad \int_D \varphi_2(z)^p\psi(\zeta, z)d\mu_z \le \varphi_1(\zeta)^p, \quad 1 \le p$$

and thus applying the result of Lemma 4.5 we obtain i). □

As final result about these operators we give the following lemma:

LEMMA 4.7. *Let $R_{t,z}$ the differential operator $R_{t,z} = I + \frac{1}{3+t}R_z$, where I is the Identity operator and R_z is the differential operator $R_z = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}$.*
Then for $t > 0$ large enough, the differential operator

$$R^l_{t,z} = R_{t+l-1,z}\cdots R_{t,z} : A^p_l(D, (-\rho)^\delta\tau^r) \longrightarrow A^p(D, (-\rho)^\delta\tau^r)$$

gives an isomorphism between these spaces.
The inverse operator is given by the integral operator

$$R^{t+l,t}(\zeta, z) = c_{t,l}\frac{(-\rho(\zeta))^{t+l}|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}\left(1 + (k-1)|\zeta_1|^{2k} + (m-1)|\zeta_2|^{2m}\right)^{-l}}{a(\zeta, z)^{3+t}}$$

with $c_{t,l} = (2 + t + l)(1 + t + l)/[2!(2\pi i)^2]$.

PROOF. A direct computation gives

$$R^l_{t,z}R^{t+l,t}(\zeta,z)=R^{t+l}_{0,0}=I \quad \text{on} \quad A^p\big(D,(-\rho(\zeta))^\delta\tau^r\big)$$

and thus the lemma is proved. □

REMARK. Using this lemma we can obtain the known result

$$\|f\|_{p,l,(-\rho)^\delta\tau^r}\approx\int_D|R^l_{t,\zeta}f(\zeta)|^p(-\rho(\zeta))^\delta\tau(\zeta)^r d\zeta\,. \tag{4.1}$$

PROOF OF THEOREM 4.1: We define the function g by

$$g(z)=\int_D f(\zeta)\wedge K^t_{0,0}(\zeta,z)$$

for some $t>0$ large enough. Then by the Koppelman formulas (Theorem 3.1) it is clear that $\bar\partial g=f$.

Now, we prove that g is of class $L^p_l(D,(-\rho)^\delta\tau^r)$. By integration by parts in the formula of Lemma 4.2, it is sufficient to show that the integral operator $K^t_{0,0}$ maps the space of $(0,1)$-forms with coefficients of class $L^p(D,(-\rho)^\delta\tau^{r+p})$ to the space $L^p(D,(-\rho)^\delta\tau^r)$ and that the operator $R^t_{0,1,1}$ maps the space of $(0,1)$-forms with coefficients of class $L^p_{l-j}(D,(-\rho)^\delta\tau^{r+p})$ to the space $A^p_{l-j-1}(D,(-\rho)^\delta\tau^r)$, $j=0,\dots,l-1$.

By Lemma 4.4 we have that the last condition is equivalent to show that the operators $R^{t,i}_{0,1,1}$ maps the space $L^p_1(D,(-\rho)^\delta\tau^{r+p})$ to the space $A^p(D,(-\rho)^\delta\tau^r)$. Moreover we have $-\rho(\zeta)\le\tau(\zeta)$ and thus $L^p_1(D,(-\rho)^\delta\tau^{r+p})\subset L^p_1(D,(-\rho)^{\delta+p}\tau^r)$. Finally using $A^p_1(D,(-\rho)^{\delta+p}\tau^r)=A^p(D,(-\rho^\delta\tau^r)$ (see [GR], Theorem C) and Lemma 4.4 we find that to prove the theorem is sufficient to show that:

The operator $K^t_{0,0}$ is continuous from $L^p(D,(-\rho)^\delta\tau^{r+p})$ to $L^p(D,(-\rho)^\delta\tau^r)$.

The operators $R^{t,i}_{0,1,1}$ are continuous from $L^p(D,(-\rho)^{\delta+p}\tau^r)$ to $L^p(D,(-\rho)^{\delta+p}\tau^r)$. Thus the theorem follows from Lemma 3.2 and Proposition 4.6. □

REMARK. This result is sharp in the sense that there exists a $(0,1)$-form $\bar\partial$-closed with coefficients in $L^p_l(D,(-\rho)^\delta\tau^r)$ such that the solutions g of the equation $\bar\partial g=f$ are not in $L^p_l(D,(-\rho)^\delta\tau^{r+p-\varepsilon})$ for all $\varepsilon>0$. As usual this form can be taken of type

$$f(z)=\bar\partial\left(\frac{\bar z_1^k}{(1-z_2)^{s+1/2}\log^2(1-z_2)}\right).$$

See [CHE-KR-MA] for more details.

Also, note that the same estimates and a more general version of Lemma 4.5 (see [OK] Th. 4.1.2) permit to obtain $\big(L^p_l(D,(-\rho)^\delta\tau^r),L^q_l(D,(-\rho)^{\delta'}\tau^{r'})\big)$-estimates for the $\bar\partial$-problem.

5. – Division in the $A^p_l(D, (-\rho)^\delta \tau^r)$ spaces

The goal of this section is to prove the following theorem:

THEOREM 5.1. *Let $Y = \{z \in V; u(z) = 0\}$ be a holomorphic submanifold defined in a neighbourhood Y of $\overline{D}$ and let $M = Y \cap D$.*
Assume that the function $u(z)$ satisfies the following conditions:

$$\text{i)} \quad \partial u(z) \neq 0, \qquad z \in Y$$
$$\text{ii)} \quad (\partial u \wedge \partial \rho)(z) \neq 0, \quad z \in Y \cap \partial D.$$

Then, if f is a function of class $A^p_l(D, (-\rho)^\delta \tau^r)$ vanishing on M, the function f/u belongs to $A^p_l(D, (-\rho)^\delta \tau^{r+p})$.

To prove this theorem, we need the following lemmas.

LEMMA 5.2. *For ζ, z in D*

$$\text{i)} \quad |\bar\zeta_1^k \zeta_1^{k-1} - \bar z_1^k z_1^{k-1}| \le c|z_1|^{k-1}\big(\Re e\, a(\zeta, z)\big)^{1/2} + c\big(\Re e\, a(\zeta, z)\big)^{1-1/2k}$$

$$\text{ii)} \quad |\bar\zeta_2^m \zeta_2^{m-1} - \bar z_2^m z_2^{m-1}| \le c|z_2|^{m-1}\big(\Re e\, a(\zeta, z)\big)^{1/2} + c\big(\Re e\, a(\zeta, z)\big)^{1-1/2m}.$$

PROOF. Using

$$|\bar\zeta_1^k \zeta_1^{k-1} - \bar z_1^k z_1^{k-1}| \le c\big(|z_1|^{2k-2}|\zeta_1 - z_1| + |\zeta_1 - z_1|^{2k-1}\big)$$

and Lemma 3.5, we obtain the result i).
The result ii) follows in the same way. □

LEMMA 5.3. *Let $Y = \{z; u(z) = 0\}$ be a holomorphic submanifold with u satisfying properties* i) *and* ii) *of Theorem 5.1. Then for every point η in the boundary of $M = Y \cap D$, there exists a neighbourhood W of ζ and a holomorphic projection of class $C^\infty(\overline{W})$,*

$$w : W \longrightarrow W \cap Y$$

such that

$$\text{i)} \quad w(z) = z + u(z)\varphi(z)(m\bar w_2^m w_2^{m-1}, -k\bar w_1^k w_1^{k-1}),\ \varphi(z) \in C^\infty(\overline{W}),\ |\varphi(z)| \approx 1;$$
$$\text{ii)} \quad -\rho(w) \approx -\rho(z) + \Lambda(z, |u(z)|),\ c > 0;$$
$$\text{iii)} \quad |a(\zeta, w)| \approx |a(\zeta, z)| + \Lambda(z, |u(z)|).$$

REMARK. Observe that condition ii) implies that if $z \in V \cap D$ then $w \in V \cap M$.

PROOF. We define $w = w(z)$ by

$$(5.1) \quad \left.\begin{aligned} \Phi(w, z) &= 0 \\ u(w) &= 0 \end{aligned}\right\} \quad i.e. \quad \left.\begin{aligned} k\bar w_1^k w_1^{k-1}(w_1 - z_1) + m\bar w_2^m w_2^{m-1}(w_2 - z_2) &= 0 \\ u(w) &= 0 \end{aligned}\right\}$$

By property ii) we have

$$|(\partial\rho \wedge \partial u)(\eta)| > 0 \tag{5.2}$$

and thus, by the implicit function theorem, we find that w is well defined in a small neighbourhood W of η.

Moreover, using 5.1, 5.2 and

$$\begin{aligned} u(w) = u(z) &+ \left(\frac{\partial u(z)}{\partial z_1} + O(|z-w|)\right)(w_1 - z_1) \\ &+ \left(\frac{\partial u(z)}{\partial z_2} + O(|z-w|)\right)(w_2 - z_2) \end{aligned}$$

we obtain

$$\begin{aligned} w_1 - z_1 &= u(z)\psi(z,w)m\bar{w}_2^m w_2^{m-1} \\ w_2 - z_2 &= -u(z)\psi(z,w)k\bar{w}_1^k w_1^{k-1} \end{aligned}$$

where $\psi(z, w)$ is a function of class $C^\infty(\overline{W} \times \overline{W})$ and $|\psi| \approx 1$ if W is small enough. Taking $\varphi(z) = \psi(z, w)$ then i) is proved.

Using (5.1), Lemma 2.4 and i), we obtain

$$\begin{aligned} &\rho(z) - \rho(w) \\ \approx &|w_1|^{2k-2}|w_1 - z_1|^2 + |w_1 - z_1|^{2k} + |w_2|^{2m-2}|w_2 - z_2|^2 + |w_2 - z_2|^{2m} \\ \approx &|w_1|^{2k-2}|w_2|^{2m-2}|u(z)|^2 + |w_2|^{2k(2m-1)}|u(z)|^{2k} + |w_1|^{2m(2k-1)}|u(z)|^{2m} \\ = &\Lambda(w, |u(z)|)\,. \end{aligned} \tag{5.3}$$

Assume that $|w_1| \leq 1/2$. Then

$$\begin{aligned} \Lambda(w, |u(z)|) &\approx |w_1|^{2k-2}|u(z)|^2 + |u(z)|^{2k} \\ &\leq |z_1|^{2k-2}|u(z)|^2 + |u(z)|^{2k} + |w_1 - z_1|^{2k-2}|u(z)|^2\,. \end{aligned}$$

Now, since $x^{2k-2}y^2 \leq \varepsilon x^{2k} + \frac{y^{2k}}{\varepsilon^{2k-2}}$ for all $\varepsilon > 0$ and $|w_1 - z_1|^{2k} \leq c|a(w,z)| = c(-\rho(w))$ we obtain

$$\begin{aligned} \Lambda(w, |u(z)|) &\leq \Lambda(z, |u(z)|) + \varepsilon|w_1 - z_1|^{2k} + \frac{1}{\varepsilon^{2k-2}}|u(z)|^{2k} \\ &\leq c\Lambda(z, |u(z)|) + \varepsilon(-\rho(w))\,. \end{aligned} \tag{5.4}$$

Thus, by (5.3) and (5.4) we obtain

$$-\rho(w) \leq c\big(-\rho(z) + \Lambda(z, |u(z)|)\big)\,.$$

The same argument gives

$$\Lambda(z, |u(z)|) \leq c\big(\Lambda(w, |u(z)|) - \rho(w)\big) \approx c\big(\rho(z) - \rho(w)\big) - \rho(w)$$

and thus

$$-\rho(z) + \Lambda(z, |u(z)|) \leq c(-\rho(w)).$$

Using the same method we can show the case $|w_2| \leq 1/2$ and hence ii) is proved.

Finally we will prove iii). Using $\Phi(w, z) = 0$ we obtain

$$\begin{aligned} &|a(\zeta, z) - a(\zeta, w)| \\ &= |k(\bar{\zeta}_1^k \zeta_1^{k-1} - \bar{w}_1^k w_1^{k-1})(w_1 - z_1) + m(\bar{\zeta}_2^m \zeta_2^{m-1} - \bar{w}_2^m w_2^{m-1})(w_2 - z_2)| \\ &\leq c\big(|w_1|^{2k-2}|\zeta_1 - w_1| + |\zeta_1 - w_1|^{2k-1}\big)|w_1 - z_1| \\ &+ c\big(|w_2|^{2m-2}|\zeta_2 - w_2| + |\zeta_2 - w_2|^{2m-1}\big)|w_2 - z_2|. \end{aligned} \tag{5.5}$$

Thus, by Lemma 3.5 and $\Phi(w, z) = 0$ we have

a) $$|w_1|^{2k-2}|z_1 - w_1|^2 + |z_1 - w_1|^{2k} + |w_2|^{2m-2}|z_2 - w_2|^2|z_2 - w_2|^{2m} \leq |a(w, z)| = -\rho(w) \leq c\Re e\, a(\zeta, w)$$

b) $$|w_1|^{2k-2}|\zeta_1 - w_1|^2 + |\zeta_1 - w_1|^{2k} + |w_2|^{2m-2}|\zeta_2 - w_2|^2 + |\zeta_2 - w_2|^{2m} \leq c\Re e\, a(\zeta, w)$$

and therefore using (5.5) we obtain

$$|a(\zeta, z) - a(\zeta, w)| \leq c\Re e\, a(\zeta, w). \tag{5.6}$$

Then from ii) and (5.6) we obtain

$$|a(\zeta, z)| + \Lambda(z, |u(z)|) \leq c\big(|a(\zeta, w)| + (-\rho(w))\big) \leq c|a(\zeta, w)|.$$

To finish we prove the converse inequality. Using (5.5) and

$$xy \leq \varepsilon x^2 + \frac{1}{\varepsilon} y^2, \quad x^{2k-1}y \leq \varepsilon x^{2k} + \frac{1}{\varepsilon^{2k-1}} y^{2k}$$

for all $\varepsilon > 0$ and $x, y \geq 0$ we obtain

$$\begin{aligned} &|a(\zeta, z) - a(\zeta, w)| \\ &\leq c\varepsilon\big(|w_1|^{2k-2}|\zeta_1 - w_1|^2 + |\zeta_1 - w_1|^{2k} + |w_2|^{2m-2}|\zeta_2 - w_2|^2 + |\zeta_2 - w_2|^{2m}\big) \\ &+ c\left(\frac{1}{\varepsilon}|w_1|^{2k-2}|z_1 - w_1|^2 + \frac{1}{\varepsilon}|w_2|^{2m-2}|z_2 - w_2|^2\right. \\ &\left. + \frac{1}{\varepsilon^{2k-1}}|w_1 - z_1|^{2k} + \frac{1}{\varepsilon^{2m-1}}|w_2 - z_2|^{2m}\right). \end{aligned}$$

Then using Lemma 3.5, $\Phi(w, z) = 0$ and ii) we obtain

$$\begin{aligned} |a(\zeta, z) - a(\zeta, w)| &\leq c\varepsilon|a(\zeta, w)| + c_\varepsilon|a(w, z)| \\ &\leq c\varepsilon|a(\zeta, w)| + c\big(-\rho(z) + \Lambda(z, |u(z)|)\big) \end{aligned}$$

where c_ε is a constant which depends of ε.

Hence, it is clear that iii) follows from the above inequalities. □

Using the estimates of the proof of Lemma 5.3 we can obtain the two following properties of the projection $w(z)$.

LEMMA 5.4. *Using the same notations as in the above lemma we have:*

$$\text{i)} \qquad |a(\zeta, z) - a(\zeta, w)| \leq c\Re e\, a(\zeta, w)$$

$$\text{ii)} \qquad |a(\zeta, z)| \approx |a(\zeta, w)|\,, \quad \zeta \in M\,.$$

PROOF. Part i) is the estimate (5.6) and part ii) follows from part iii) of Lemma 5.3 and from

$$\Lambda(z, |u(z)|) \leq \Lambda(z, |\zeta - z|) \leq |a(\zeta, z)|\,. \tag{5.7}$$

□

LEMMA 5.5. *If ζ, z are in D, $w = w(z)$ is the projection of Lemma 5.3 and X_ν is the tangent complex vector field*

$$X_\nu = m\bar{\nu}_2^m \nu_2^{m-1} \frac{\partial}{\partial \nu_1} - k\bar{\nu}_1^k \nu_1^{k-1} \frac{\partial}{\partial \nu_2}$$

then we have

$$\frac{\tau(z)\left|(X_\nu a(\zeta, \nu))\right|_{\nu=w}}{|a(\zeta, w)|} \leq c\,.$$

PROOF. First note that

$$\left|X_\nu a(\zeta, \nu)\right|_{\nu=w} = km\left|\bar{w}_2^m w_2^{m-1} \bar{\zeta}_1^k \zeta_1^{k-1} - \bar{w}_1^k w_1^{k-1} \bar{\zeta}_2^m \zeta_2^{m-1}\right|.$$

The first step is to obtain estimates of $|X_\nu a(\zeta, \nu)|_{\nu=w}$ in terms of $|z_1|$, $|z_2|$ and $|a(\zeta, w)|$.

It is clear that

$$\begin{aligned} |X_\nu a(\zeta, \nu)|_{\nu=w} \leq {} & |w_2|^{2m-1} \left|\bar{\zeta}_1^k \zeta_1^{k-1} - \bar{w}_1^k w_1^{k-1}\right| \\ & + |w_1|^{2k-1} \left|\bar{\zeta}_2^m \zeta_2^{m-1} - \bar{w}_2^m w_2^{m-1}\right|. \end{aligned}$$

Thus, using Lemma 5.2 and the estimates

$$|w_1|^{2k-1} \leq c\left(|z_1|^{2k-1} + |w_1 - z_1|^{2k-1}\right) \leq c|z_1|^{2k-1} + c(-\rho(w))^{1-1/2k}$$

$$|w_2|^{2m-1} \leq c\left(|z_2|^{2m-1} + |w_2 - z_2|^{2m-1}\right) \leq c|z_2|^{2m-1} + c(-\rho(w))^{1-1/2m}$$

we obtain

$$\begin{aligned} \left|X_\nu a(\zeta, \nu)\right|_{\nu=w} \leq c\big(& |z_2|^{2m-1} |z_1|^{k-1} |a(\zeta, w)|^{1/2} + |z_1|^{2k-1} |a(\zeta, w)|^{1-1/2m} \\ & + |z_2|^{2m-1} |a(\zeta, w)|^{1-1/2k} + |a(\zeta, w)|\big)\,. \end{aligned}$$

Next, we will prove the lemma. From the above estimate we get

$$\frac{\tau(z)\left|X_\nu a(\zeta,\nu)\right|_{\nu=w}}{|a(\zeta,w)|} \le c\frac{|z_2|^{2m-1}\tau(z)}{|a(\zeta,w)|^{1/2k}} + c\frac{|z_1|^{2k-1}\tau(z)}{|a(\zeta,w)|^{1/2m}} + c\frac{|z_1|^{k-1}|z_2|^{m-1}\tau(z)}{|a(\zeta,w)|^{1/2}} + c\tau(z).$$

Now, we consider the three usual cases.

a) $z \in D_1$. In this case we have $|z_1|^{2k} \le -\rho(z)$ and $\tau(z) = (-\rho(z))^{1/2k}$. Hence

$$I \le c|z_2|^{2m-1} + c\big(-\rho(z)\big)^{1-1/2m} + c|z_2|^{m-1} + c\big(-\rho(z)\big)^{1/2k} \le c.$$

The cases *b*) $z \in D_2$ and *c*) $z \in D_3$ follow in the same way. □

LEMMA 5.6. *Let $w = w(z)$ be a local projection of Lemma 5.3. Then, for ζ in D and $0 \le s \le 1$, we have*

$$\left|a(\zeta, sz + (1-s)w)\right| \approx s|a(\zeta,z)| + (1-s)|a(\zeta,w)|.$$

PROOF. First note that $a(\zeta, sz+(1-s)w) = sa(\zeta,z) + (1-s)a(\zeta,w)$. Therefore we have

$$\left|a(\zeta, sz + (1-s)w)\right| \le s|a(\zeta,z)| + (1-s)|a(\zeta,w)|.$$

Also by Lemma 3.5 we obtain $\Re e\, a(\zeta,z)\,, \Re e\, a(\zeta,w) \ge 0$ and

$$s\Re e\, a(\zeta,z) + (1-s)\Re e\, a(\zeta,w) = \Re e\, a\big(\zeta, sz+(1-s)w\big) \le \left|a\big(\zeta, sz+(1-s)w\big)\right|.$$

On the other hand, using part i) of Lemma 5.4 we have

$$\begin{aligned}|\Im m\, a(\zeta,z)| &\le \left|\Im m\, a\big(\zeta, sz+(1-s)w\big)\right| + (1-s)|a(\zeta,z) - a(\zeta,w)| \\ &\le \left|\Im m\, a\big(\zeta, sz+(1-s)w\big)\right| + c(1-s)\Re e\, a(\zeta,w) \\ &\le c\left|a\big(\zeta, sz+(1-s)w\big)\right|\end{aligned}$$

and

$$\begin{aligned}|\Im m\, a(\zeta,w)| &\le \left|\Im m\, a\big(\zeta, sz+(1-s)w\big)\right| + cs\Re e\, a(\zeta,w) \\ &\le c\left|a\big(\zeta, sz+(1-s)w\big)\right|.\end{aligned}$$

Hence, the lemma is proved. □

PROOF OF THEOREM 5.1: We consider a covering $\{U_i\}_{i=0}^{i_0}$ of D such that:

1) $U_0 = \{z; \rho(z) < -\varepsilon < 0\}$.

2) If $1 \le i < i_1$ then $u(z) \ne 0$ on $\overline{U}_i$.

3) If $i_1 \le i \le i_0$ then there exists a projection w_i as the one in the Lemma 5.2.

Let $\{\chi_i\}$ a partition of the unity for this covering.

We want to show that the functions $g_i(z) = \chi_i(z)\frac{f(z)}{u(z)}$ are of class $L_l^p(D, (-\rho)^\delta \tau^p)$. We consider the three following cases.

1) $i = 0$. In this case, using $U_0 \subset\subset D$, we have that the function $\frac{f(z)}{u(z)}$ is of class $C^\infty(\overline{U}_0)$ and therefore the result is true.

2) $1 \le i < i_1$. In this case is clear that the function g_i is of class $L^p(D, (-\rho)^\delta)$.

3) $i_1 \le i \le i_0$. We will write w instead w_i. Thus for $z \in U_i$

$$\begin{aligned} f(z) = f(z) - f(w) &= \int_0^1 \frac{d}{ds} f\big(w + s(z-w)\big)ds \\ &= \int_0^1 \left((z_1 - w_1)\left(\frac{\partial f}{\partial \nu_1}\right)_{\nu = w+s(z-w)} + (z_2 - w_2)\left(\frac{\partial f}{\partial \nu_2}\right)_{\nu=w+s(z-w)} \right) ds\,. \end{aligned}$$

Then, using part i) of Lemma 5.3 and the differential operator

$$X_\nu = m\bar{w}_2^m w_1^{m-2}\frac{\partial}{\partial \nu_1} - k\bar{w}_1^k w_1^{k-1}\frac{\partial}{\partial \nu_2}$$

we obtain

$$\frac{f(z)}{u(z)} = -\varphi(z)\int_0^1 (X_\nu f)_{\nu=w+s(z-w)}ds$$

and moreover

$$\begin{aligned} \frac{\partial^{|\alpha|}}{\partial z^\alpha}\left(\frac{f(z)}{u(z)}\right) &= \int_0^1 \varphi_\alpha(s,z)\left(X_\nu \frac{\partial^{|\alpha|} f}{\partial \nu^\alpha}\right)_{\nu=w+s(z-w)} ds \\ &\quad + \sum_{|\gamma|\le|\alpha|}\int_0^1 \varphi_\gamma(s,z)\left(\frac{\partial^{|\gamma|} f}{\partial \nu^\gamma}\right)_{\nu=w+s(z-w)} ds \end{aligned} \tag{5.8}$$

where all the functions $\varphi_\alpha, \varphi_\gamma$ are of class $C^\infty([0,1]\times\overline{D})$.

Next, we will prove that the function

$$g_i(z) = \chi_i(z)\int_0^1 \varphi_\alpha(s,z)\Big(X_\nu \frac{\partial^{|\alpha|} f}{\partial \nu^\alpha}\Big)_{\nu=w+s(z-w)} ds$$

is of class $L^p(D, (-\rho)^\delta \tau^p)$.

Since $\frac{\partial^{|\alpha|} f}{\partial \nu^\alpha}$ is holomorphic on D, using the representation kernel $R^t_{0,0}$, we obtain that $g_i(z)$ is given by:

$$c_t \int_0^1 \varphi_\alpha(s,z) \left(X_\nu \int_D \frac{\partial^{|\alpha|} f}{\partial \zeta^\alpha} \frac{(-\rho(\zeta))^t |\zeta_1|^{2k-2} |\zeta_2|^{2m-2}}{a(\zeta,\nu)^{3+t}} d\zeta \right)_{\nu = w+s(z-w)} ds$$

$$= c'_t \int_0^1 \varphi_\alpha(\zeta,z) \int_D \frac{\partial^{|\alpha|} f}{\partial \zeta^\alpha} \frac{(-\rho(\zeta))^t |\zeta_1|^{2k-2} |\zeta_2|^{2m-2} (X_\nu a(\zeta,\nu))_{\nu=w}}{a(\zeta, w+s(z-w))^{4+t}} d\zeta\, ds\,.$$

Therefore, by Lemma 5.6 and integrating over the variable s, we obtain

$$|g_i(z)| \leq c \int_D \left| \frac{\partial^{|\alpha|} f(\zeta)}{\partial \zeta^\alpha} \right| \frac{(-\rho(\zeta))^t |\zeta_1|^{2k-2} |\zeta_2|^{2m-2} |X_\nu a(\zeta,\nu)|_{\nu=w}}{|a(\zeta,z)|^{3+t} |a(\zeta,w)|} d\zeta\,.$$

Moreover, by Lemma 5.5 we have

$$|g_i(z)|\tau(z) \leq c \int_D \left| \frac{\partial^{|\alpha|} f(\zeta)}{\partial \zeta^\alpha} \right| \frac{|\zeta_1|^{2k-2} |\zeta_2|^{2m-2}}{|a(\zeta,z)|^{3+t}} d\zeta\,.$$

and finally, applying Proposition 4.6 iii) we obtain the result.

Using the same method we can prove that the other functions which appear in the expression (5.8) are of class $L^p(D, (-\rho)^\delta \tau^r)$ and thus the theorem is shown. □

Remark. The estimate of Lemma 5.1 is sharp in the sense that we cannot take $r' < r + p$ in the hypothesis of the theorem. The result can be proved taking $Y = \{z; z_1 = 0\}$ and functions of type

$$f(z) = \frac{z_1}{(1-z_2)^{2/p+(p+2)/2kp}} \frac{1}{\log^2(1-z_2)} \in A^p(D)\,.$$

6. – Extension of jets from holomorphic submanifolds

The aim of this section is to prove the results of Theorems D and E.

Using $A^p_l(D, (-\rho)^\delta \tau^r) = A^p_{l+l'}(D, (-\rho)^{\delta + l'p} \tau^r)$ we can assume, without lost of generality, that δ, r satisfy $\delta \geq 0$ and $2\delta > -r$. This is a technical condition which gives $(-\rho)^\delta \tau^r|_{\partial D} = 0$.

As in Section 5 we consider a submanifold $Y = \{z; u(z) = 0\}$ defined in a neighbourhood of $\overline{D}$ and transversal to the boundary of D. Then we can assume that the function $u(z)$ satisfies:

$$(\partial u)(z) \neq 0\,, \quad z \in Y \quad \text{and} \quad (\partial u \wedge \partial \rho)(z) \neq 0\,, \quad z \in \partial D \cap Y\,.$$

To prove the restriction and extension Theorems D and E we will use the extension operator given by B. Berndtsson ([BER], Th. 2.) We recall briefly this result.

We find holomorphic functions $g_1(\zeta, z), g_2(\zeta, z)$ such that

$$u(z) - u(\zeta) = g_1(\zeta, z)(z_1 - \zeta_1) + g_2(\zeta, z)(z_2 - \zeta_2) \tag{6.1}$$

and we define the $(1, 0)$-form $\tilde{g}(\zeta, z) = g_1(\zeta, z)d\zeta_1 + g_2(\zeta, z)d\zeta_2$ and the $(1, 1)$ current

$$\mu = c_p \frac{\tilde{g}(\zeta, z) \wedge \bar{\partial}\bar{u}(\zeta)}{|\partial u(\zeta)|^2} d\sigma(\zeta)$$

where $d\sigma$ denotes the surface measure on M. Observe that the coefficients of μ are measures supported on M, depending holomorphically on $z \in D$.

Then for every $t > 0$ there exists a constant $c = c(t)$ such that for every holomorphic function f on M of class $C^1(\overline{M})$, the formula

$$E^t f(z) = c \int_M f(\zeta) \frac{(-\rho(\zeta))^{t+2}}{a(\zeta, z)^{t+2}} \partial\bar{\partial} \left(\log \frac{-1}{\rho(\zeta)} \right) \wedge \mu \tag{6.2}$$

defines a holomorphic function on D which coincides with f on M.

Note that the operator E^t is given by an integral operator of type

$$E^t(\zeta, z) = \frac{(-\rho(\zeta))^t \varphi(\zeta, z)}{a(\zeta, z)^{t+2}}$$

with $\varphi(\zeta, z)$ holomorphic in z and of class $C^\infty(\overline{M} \times \overline{D})$.

The next lemmas give some properties of these kernels.

Lemma 6.1. *For $z \in D$, $t > 0$ large enough and $s \geq 0$, $2s > -r$, we have*

$$\int_M \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^{2+t}} d\sigma(\zeta)$$

$$\leq c\big(- \rho(z) + \Lambda(z, |u(z)|)\big)^{s-t} \tau\big(z, -\rho(z) + \Lambda(z, |u(z)|)\big)^r .$$

Proof. Let η be in $\partial D \cap Y$ and $\zeta \in M$, $z \in D$ satisfying $|\zeta - \eta|$, $|z - \eta| \leq \varepsilon$, with ε sufficiently small such that there exists the projection $w = w(z)$ of Lemma 5.3.

First, we will prove

$$|a(\zeta, z)| \approx |a(\zeta, w(z))| \approx -\rho(w(z)) + |\zeta - w(z)| . \tag{6.3}$$

By Lemma 5.4 we have $|a(\zeta, w)| \approx |a(\zeta, z)|$.

On the other hand by (6.1) we have

$$0 = u(w) - u(\zeta) = (w_1 - \zeta_1)g_1(\zeta, w) + (w_2 - \zeta_2)g_2(\zeta, w) .$$

Moreover, since $\partial u(z) \neq 0$ on Y, we have $|g_1| + |g_2| \geq c > 0$ for $|\zeta - \eta|$, $|\zeta - z| \leq \varepsilon$.

For example we assume $|g_2(\zeta, w)| \geq c > 0$. Then, we have

$$\begin{aligned}|\Phi(\zeta, w)| &= |k\bar{\zeta}_1^k\zeta_1^{k-1}(\zeta_1 - w_1) + m\bar{\zeta}_2^m\zeta_2^{m-1}(\zeta_2 - w_2)| \\ &= \left|\frac{k\bar{\zeta}_1^k\zeta_1^{k-1}g_2(\zeta, w) - m\bar{\zeta}_2^m\zeta_2^{m-1}g_1(\zeta, w)}{g_2(\zeta, w)}\right| |\zeta_1 - w_1|.\end{aligned}$$

Hence, using the transversality of Y and taking ε small enough, we obtain

$$|a(\zeta, w)| \approx -\rho(w) + |\Phi(\zeta, w)| \approx -\rho(w) + |\zeta_1 - w_1| \approx -\rho(w) + |\zeta - w|.$$

Next, we will obtain the estimates of the lemma. Note that from (6.3) and part ii) of Lemma 5.3 we obtain

$$\int_M \frac{1}{|a(\zeta, z)|^{2+t}} d\sigma(\zeta) \leq c\big(-\rho(z) + \Lambda(z, |u(z)|)\big)^{-t}, \quad \text{for} \quad t > 0.$$

This is the estimate of the lemma for $s = r = 0$. Now we prove the case s, $r \geq 0$. We consider the three usual cases.

a) $z \in D_1$. In this case we have

$$\begin{aligned}\int_M \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^{2+t}} d\sigma(\zeta) &\leq c + c\int_{|\zeta_1| \leq 1/2} \frac{1}{|a(\zeta, z)|^{2+t-s-r/2k}} d\sigma(\zeta) \\ &\leq c + c\big(-\rho(z) + \Lambda(z, |u(z)|)\big)^{s+r/2k-t} \\ &\leq c\big(-\rho(z) + \Lambda(z, |u(z)|)\big)^{s-t} \tau\big(z, -\rho(z) + \Lambda(z, |u(z)|)\big)^r.\end{aligned}$$

b) $z \in D_2$. The result in this case can be obtained in the same way as in the above case.

c) $z \in D_3$. Using Lemma 3.8 we have

$$\begin{aligned}\int_M \frac{(-\rho(\zeta))^s \tau(\zeta)^r}{|a(\zeta, z)|^{2+t}} d\sigma(\zeta) &\leq c\frac{1}{(|z_1|^{k-1}|z_2|^{m-1})^r} \int_M \frac{1}{|a(\zeta, z)|^{2+t-s-(r/2)}} d\sigma(\zeta) \\ &\leq c\big(-\rho(z) + \Lambda(z, |u(z)|)\big)^{s-t} \tau\big(z, -\rho(z) + \Lambda(z, |u(z)|)\big)^r.\end{aligned}$$

The case $r < 0$, $2s > -r$ follows in the same way and thus the lemma is proved. □

DEFINITION 6.2. *Let $X = (X_1, \dots, X_n)$ be a n-tuple of smooth vector fields with coefficients in a neighbourhood of $\overline{D}$. Let n_2 be the number of X_i such that are tangent complex.*

We define the function

$$\theta(X) = \theta(X_1, \dots, X_n) = (-\rho(z))^{n_1} \tau(z)^{n_2}, \quad n_1 = n - n_2 .$$

LEMMA 6.3. *Let X be as in Definition 6.2. Then*

$$\theta(X) \left| X_n \dots X_1 \frac{1}{a(\zeta, z)^t} \right| \leq c \frac{1}{|a(\zeta, z)|^t} .$$

PROOF. Assume for example that z is in a neighbourhood of ∂D and that $|z_1| \leq \frac{1}{2}$.

Let $X_1 = a_1(z)\frac{\partial}{\partial z_1} + a_2(z)\frac{\partial}{\partial z_2}$ be a differential operator with coefficients of class C^∞ in a neighbourhood of $\overline{D}$. By direct computation, we have

$$\begin{aligned} X_1 a(\zeta, z) &= a_1(z) k \bar{\zeta}_1^k \zeta_1^{k-1} + a_2(z) m \bar{\zeta}_2^m \zeta_2^{m-1} \\ &= k a_1(z)(\bar{\zeta}_1^k \zeta_1^{k-1} - \bar{z}_1^k \bar{z}_1^{k-1}) + m a_2(z)(\bar{\zeta}_2^m \zeta_2^{m-1} - z_2^m \bar{z}_2^{m-1}) \\ &\quad + a_1(z) k \bar{z}_1^k z_1^{k-1} + a_2(z) m \bar{z}_2^m z_2^{m-1} . \end{aligned} \tag{6.4}$$

If X_1 is not tangent complex, then $|X_1 a(\zeta, z)| \leq c$ and thus $\theta(X_1)|X_1 a(\zeta, z)| \leq c(-\rho(z))$.

If X_1 is tangent complex we have

$$k a_1(z) \bar{z}_1^k z_1^{k-1} + m a_2(z) \bar{z}_2^m z_2^{m-1} = 0 . \tag{6.5}$$

Thus, using (6.4), (6.5), Lemma 5.2 and the definition of τ (Definition 2.2), we have

$$\begin{aligned} \theta(X_1)|X_1 a(\zeta, z)| &\leq \tau(z)\big(|z_1|^{k-1}|a(\zeta, z)|^{1/2} + |a(\zeta, z)|^{1-1/2k} \\ &\quad + |z_1|^{2k-1}|z_2|^{m-1}|a(\zeta, z)|^{1/2} + |z_1|^{2k-1}|a(\zeta, z)|^{1-1/2m}\big) \\ &\leq c|a(\zeta, z)| . \end{aligned}$$

Therefore it is clear that

$$\theta(X_1) \left| X_1 \frac{1}{a(\zeta, z)^t} \right| \leq \frac{c}{|a(\zeta, z)|^t} .$$

To prove the lemma we proceed by induction on n.

Then, if X_1 is tangent complex, by (6.5), we have that for z in a neighbourhood of the boundary of D

$$|D^\alpha a_2(z)| \leq c|z_1|^{2k-1-|\alpha|}$$

for all differential operator D^α of order $|\alpha| \le 2k-1$. Hence, by (6.4), for $|\alpha| \le 2k-1$ we obtain

$$\tau(z)^{|\alpha|+1}|D^\alpha X_1 a(\zeta,z)| \le c\tau(z)^{|\alpha|+1}\left(|\zeta_1|^{2k-1}+|z_1|^{2k-1-|\alpha|}\right) \le c|a(\zeta,z)|\,.$$

If $|\alpha| \ge 2k$ then $\tau(z)^{2k} \le c(-\rho(z))$ and thus $\tau(z)^{2k}|D^\alpha a(\zeta,z)| \le c(-\rho(z))$.

Finally, using these results, the equality

$$X_n \dots X_1 \frac{1}{a(\zeta,z)^t} = -tX_n \dots X_2\left(\frac{X_1 a(\zeta,z)}{a(\zeta,z)^{t+1}}\right)$$

and an induction argument, we obtain the result. □

THEOREM 6.4. *Let $Y = \{z; u(z)=0\}$ be a complex submanifold in a neighbourhood of $\overline{D}$ and transversal to the boundary of D. Let f be a function of class $A^p_l(D,(-\rho)^\delta\tau^r)$.*

Then for every n-tuple of smooth vector fields $X=(X_1,\dots,X_n)$, the function $X_n\dots X_1 f|_M$ belongs to $L^p_l(M,(-\rho)^\delta\tau^{r+2}\theta(X)^p)$.

PROOF. We write Xf instead $X_l \dots X_1 f$. Then, taking $t>0$ large enough and applying Lemma 4.7, we have

$$f(z) = c_t \int_D R^l_t f(\zeta) R^{t+l,t}(\zeta,z)\,.$$

Also by Lemma 6.2 we obtain

$$\theta(X)\big|D^\alpha(Xf(z))\big| \le c\int_D |R^l_t f(\zeta)|\frac{(-\rho(\zeta))^{t+l}|\zeta_1|^{2k-2}|\zeta_2|^{2m-2}}{|a(\zeta,z)|^{3+t+|\alpha|}}d\zeta\,.$$

Hence, using $|\zeta_1|^{2k-2}|\zeta_2|^{2m-2} \le -\rho(\zeta)/\tau(\zeta)^2$ (see Lemma 3.8) we obtain for $|\alpha| \le l$

$$\big|D^\alpha(Xf(z))\big|\theta(X) \le c\int_D |R^l_t f(\zeta)|\frac{(-\rho(\zeta))^{t+l+1}\tau(\zeta)^{-2}}{|a(\zeta,z)|^{3+t+l}}d\zeta\,.$$

Finally using the estimates of Lemma 6.1 and Proposition 3.4 and applying Lemma 4.5 with

$$\psi(\zeta,z) = \frac{(-\rho(\zeta))^{t+l+1-\delta}\tau(\zeta)^{-2-r}}{|a(\zeta,z)|^{3+t+l}}\,,$$

$$\varphi_1(\zeta) = (-\rho(\zeta))^{-s}\,, \qquad \varphi_2(z) = (-\rho(z))^{-s}\,,$$

$$\mu_X = (-\rho(\zeta))^\delta\tau(\zeta)^r d\zeta\,, \qquad \mu_Y = (-\rho(z))^\delta\tau(z)^{r+2}d\sigma(z)$$

for some $0<s$ small enough, we obtain the result. □

Now, as in [OR-FA 1] we introduce the following definition of $A_l^p(D, (-\rho)^\delta\tau^r)$-jet.

DEFINITION 6.5. *$F = (F^0, \dots, F^n)$ is an $A_l^p(D, (-\rho)^\delta\tau^r)$-jet of order n on M if for all $0 \leq j \leq n$ it satisfies the following conditions:*

I-1) *At every point $z \in M$, F_z^j is a j-covariant symmetric tensor.*

I-2) $F^j\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \dots, \frac{\partial}{\partial z_n}\right)$ *are holomorphic functions on M.*

I-3)
$$F^j(X_1, \dots, X_j) = X_j F^{j-1}(X_1, \dots, X_{j-1}) - \sum_{i=1}^{j-1} F^{j-1}(X_1, \dots, \nabla_{X_j} X_i, \dots, X_{j-1})$$
for every tangent vector field X_j at M.

I-4) $F^j(X_1, \dots, X_j) \in L_l^p(D, (-\rho)^\delta\tau^{r+2}\theta(X)^p)$ *for every j-tuple of smooth vector fields $X = (X_1, \dots, X_j)$.*

The conditions I-1), I-2) and I-3) just gives a relation of coherence between the tensors F^j and the condition I-4) gives a condition of regularity.

From Theorem 6.4 the following result is clear:

THEOREM 6.6. *If f is a holomorphic function of class $A_l^p(D, (-\rho)^\delta\tau^r)$ then the restriction on M of the jet $J_n f = (d^0 f, \dots, d^n f)$ is an $A_l^p(D, (-\rho)^\delta\tau^r)$-jet of order n on M.*

The next step is to prove that every $A_l^p(D, (\rho)^\delta\tau^r)$- jet of order n on M is of type $J_n f_{|M}$ for some f of class $A_l^p(D, (-\rho)^\delta\tau^r)$. To do so we need the following lemmas.

LEMMA 6.7. *Let f be of class $C^\infty(\overline{M})$ and $t > 0$. Then, for every $n \geq 0$ there exist integral operators $E^\gamma(\zeta, z)$, whose kernels are holomorphic in z and satisfy the estimates*
$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha} E^\gamma(\zeta, z)\right| \leq c \frac{(-\rho(\zeta))^{t+n}}{|a(\zeta, z)|^{2+t+|\alpha|}},$$
such that
$$E^t f(z) = \sum_{|\gamma| \leq n} E^\gamma\left(\frac{\partial^{|\gamma|} f}{\partial \zeta^\gamma}\right).$$

PROOF. The proof of this result can be obtained using $\partial\rho(z) \neq 0$ in the boundary of D and an integration by parts. See Lemma 2.5 of [OR-FA 1] for more details. □

LEMMA 6.8. *If* $f \in L^p_l(M, (-\rho)^\delta \tau^{r+2+jp})$, $j \geq 0$ *and* t *is large enough, then the function* $u^j(E^t f)$ *is in* $A^p_l(D, (-\rho)^\delta \tau^r)$.

PROOF. Using Lemma 6.7 is sufficient to show that the integral operators defined by the kernels

$$E^t_{s_1,s_2} = \frac{|u(z)|^{j-s_1}(-\rho(\zeta))^{t+l}}{|a(\zeta, z)|^{2+t+s_2}} d\sigma(\zeta),\ s_1, s_2 \geq 0, s_1 + s_2 \leq l, s_1 \leq j$$

maps $L^p(M, (-\rho)^\delta \tau^{r+2+jp})$ to $L^p(D, -\rho + \Lambda(z, |u|))^\delta \tau(z, -\rho + \Lambda(z, |u|)^r)$.

First, note that by (2.1) we have

$$|u(z)| \approx \tau\big(z, \Lambda(z, |u(z)|)\big) \leq c\tau\big(z, -\rho(z) + \Lambda(z, |u(z)|)\big)\,.$$

Moreover, the same method used to prove part iii) of Proposition 3.5 gives:

$$\int_D \frac{\big(-\rho(z)\big)^\delta \tau\big(z, -\rho(z) + \Lambda(z, |u(z)|)\big)^v}{|a(\zeta, z)|^{2+t}} dz \leq c\big(-\rho(\zeta)\big)^{\delta-t} \tau(\zeta)^{2+v}$$

for t large enough.

Thus taking $s \geq 0$ sufficiently small,

$$\psi(\zeta, z) = \frac{\big(-\rho(\zeta)\big)^{t+l-\delta} \tau(\zeta)^{-r-2-jp} |u(z)|^{j-s_1}}{|a(\zeta, z)|^{2+t+s_2}},$$

$$\varphi_1(\zeta) = \big(-\rho(\zeta)\big)^{-s-(l-s_2)/p'} \tau(\zeta)^{s_1-j/p'},\quad \varphi_2(z) = \big(-\rho(z)\big)^{-s},$$

$$\mu_X = \big(-\rho(\zeta)\big)^\delta \tau(\zeta)^{r+2+jp} d\sigma(\zeta), \qquad \mu_Y = \big(-\rho(z)\big)^\delta \tau^r(z) dz$$

and applying Lemma 4.5, with the above estimates and those of Lemma 6.1, we obtain the result. □

LEMMA 6.9. *Let* $Y = \{z \in V; u(z) = 0\}$ *be a complex submanifold in a neighbourhood of* $\overline{D}$ *and suppose that* $\partial u(z) \neq 0$ *for every* z *of* Y. *Then there exist a vector field* G *with holomorphic coefficients on a neighbourhood of* $\overline{D}$ *such that* $Gu(z) = 1$ *for* $z \in Y$.

PROOF. Let $D_e = \{z; |z_1|^{2k} + |z_2|^{2m} + \varepsilon|z_1|^2 + \varepsilon|z_2|^2 < 1 + 2\varepsilon\}$. It is clear that for every $\varepsilon > 0 D_\varepsilon$ is a strictly pseudoconvex domain with smooth boundary and that $D \subset D_\varepsilon$. Also, note that

$$\left|\left(\frac{\partial u(z)}{\partial z_1}, \frac{\partial u(z)}{\partial z_2}, u(z)\right)\right| \geq c > 0$$

for every $z \in D_\varepsilon$ and ε small enough.

Then using the Bezout's theorem we can find holomorphic functions g_1, g_2, g_3 on D_ε such that

$$g_1 \frac{\partial u}{\partial z_1} + g_2 \frac{\partial u}{\partial z_2} + g_3 u = 1\,.$$

Taking $G = g_1 \frac{\partial}{\partial z_1} + g_2 \frac{\partial}{\partial z_2}$ we obtain the result. □

THEOREM 6.10. *If* $F = (F^0, \dots, F^n)$ *is a* $A_l^p(D, (-\rho)^\delta \tau^r)$*-jet of order* n *on* M *then there exists a function* f *of class* $A_l^p(D, (-\rho)^\delta \tau^r)$ *such that* $J_n f = F$ *on* M.

PROOF. Let G be the vector field of Lemma 6.9 and let E^t be the extension operator defined in (6.2). Then, we define the function f by induction in the following way:

$$
\begin{aligned}
f_0 &= E^t F^0 \\
f_j &= f_{j-1} + \frac{u^j}{j!} E^t\left((F^j - d^j f_{j-1})(G, \dots, G)|_M\right), \quad j = 1, \dots, n \\
f &= f_n .
\end{aligned}
$$

We want to prove that f satisfies $J_n f|_M = F$ and that f is of class $A_l^p(D, (-\rho)^\delta \tau^r)$.

To prove that $d^j f|_M = F^j$ observe that $d^j f|_M = d^j f_j|_M$ and that

$$\frac{1}{j!} d^j u^j (G, \dots, G)\big|_M = 1 .$$

Then, using property I-2 of Definition 6.5, it is clear that

$$d^j f(G, \dots, G)\big|_M = F^j(G, \dots, G) . \tag{6.6}$$

To show that

$$d^j f(X_1, \dots, X_j)\big|_M = F^j(X_1, \dots, X_j) \tag{6.7}$$

for every $X_1, \dots, X_j$ we proceed by induction on j and on the number of X_j such that are equal to G.

The case $j = 0$ is obvious because E^t is an extension operator.

Now, assume that $d^i f|_M = F^j$ for every $0 \le i \le j-1$. By (6.6) it is clear that (6.7) is true if $X_1 = \dots = X_j = G$. We assume that (6.7) is true if $X_1 = \dots = X_{i+1} = G$ and we prove (6.7) in the case $X_1 = \dots = X_i = G$.

Since $Gu(z) = 1$ on M we can find a decomposition $X_{i+1} = \alpha_1 G + \alpha_2 T$ where T is a tangent field on Y. Then by properties I-1), I-3) and the hypothesis of induction, we have

$$
\begin{aligned}
d^j f(G, \dots, G, X_{i+1}, \dots, X_j)\big|_M &= \alpha_1 d^j f(G, \dots, G, X_{i+2}, \dots, X_j)|_M \\
&+ \alpha_2 T d^{j-1} f(G, \dots, G, X_{i+2}, \dots, X_j)\big|_M \\
&- \alpha_2 \sum_{i=1}^{j-1} d^{j-1} f(G, \dots, G, \nabla_T X_{i+2}, \dots, X_j)\big|_M \\
&= F^j(G, \dots, G, X_{i+1}, \dots, X_j) .
\end{aligned}
$$

Thus the result is proved.

Next we prove that f is of class $A_l^p(D, (-\rho)^\delta \tau^r)$.

Note that by properties I-3) and $d^j f|_M = F^j$, $j = 0, \ldots, n$, we have

$$(F^j - d^j f_{j-1})(X_1, \ldots, X_{j-1}, T) = 0$$

if T is a tangent field on Y. Since Y is transversal, decomposing G as sum of a tangent complex field X and a tangent field on Y and applying property I-4) we obtain that the function

$$(F^j - d^j f_{j-1})(G, \ldots, G)\big|_M = (F^j - d^j f_{j-1})(X, \ldots, X)\big|_M$$

is of class $L_l^p(M, (-\rho)^\delta \tau^{r+2+jp})$.

Finally by Lemma 6.8 and induction on j we obtain that the functions $f_0, \ldots, f_n$ are of class $A_l^p(D, (-\rho)^\delta \tau^r)$ and hence the theorem is proved. □

7. – Extension of functions from analytic sets

In all this section we consider an analytic set $Y = \{z; u(z) = 0\}$ defined in a neighbourhood of $\overline{D}$ with $u(z)$ satisfying the condition of transversality

$$(\partial u \wedge \partial \rho)(z) \neq 0, \quad \text{for} \quad z \in \partial D \cap Y. \tag{7.1}$$

Note that Y is an analytic submanifold in a neighbourhood of the boundary of D.

As in the above chapters we denote by $M = D \cap Y$ and we assume that $\delta \geq 0$, $2\delta > -r$.

The aim of this section is to prove Theorem C.

Observe that the methods used in the above section give the restriction part of Theorem C:

$$A_l^p(D, (-\rho)^\delta \tau^r)\big|_M \subset A^p(M, (-\rho)^\delta \tau^{r+2}).$$

Thus it is sufficient to prove the following theorem:

THEOREM 7.1. *Let Y be an analytic set with u satisfying* (7.1) *and let g be of class $A_l^p(M, (-\rho)^\delta \tau^{r+2})$.*

Then, there exists a function f in $A_l^p(D, (-\rho)^\delta \tau^r)$ such that $f|_M = g$.

The first step is to construct locally extension operators. To do so, we define the convex function

$$v_\varepsilon(t) = \begin{cases} \exp\left(\dfrac{-1}{t^2 - \varepsilon^2} + \dfrac{1}{\varepsilon^2}\right) & \varepsilon < t < 1/2 \\ 0, & t \leq \varepsilon. \end{cases}$$

Then for $\eta \in Y \cap \partial D$ and for some fixed $\varepsilon > 0$ small enough we define the convex domain with smooth boundary

$$\Omega_\eta^\varepsilon = \{z; \rho_\eta(z) = |z_1|^{2k} + |z_2|^{2m} + v_\varepsilon(|z-\eta|) - 1 < 0\}.$$

Note that $B(\eta, \varepsilon) \subset \Omega_\eta^\varepsilon \subset B(\eta, 2\varepsilon)$ where $B(\eta, \varepsilon)$ denotes the euclidean ball of center η and radious ε.

Also, we define the functions

$$\Phi_\eta(\zeta, z) = \frac{\partial \rho_\eta(\zeta)}{\partial \zeta_1}(\zeta_1 - z_1) + \frac{\partial \rho_\eta(\zeta)}{\partial \zeta_2}(\zeta_2 - z_2)$$
$$a_\eta(\zeta, z) = -\rho_\eta(\zeta) + \Phi_\eta(\zeta, z).$$

Observe that for $|\zeta - \eta| \leq \varepsilon$ the functions Φ_n and a_η coincide with the functions Φ of Definition 2.3 and with the function a defined in (3.1).

Then, as in Section 6 we consider the decomposition

$$u(z) - u(\zeta) = g_1(\zeta, z)(z_1 - \zeta_1) + g_2(\zeta, z)(z_2 - \zeta_2),$$

the $(1,0)$-form $\tilde{g}(\zeta, z) = g_1(\zeta, z)d\zeta_1 + g_2(\zeta, z)d\zeta_2$ and the $(1,1)$ current

$$\mu = c_p \frac{\tilde{g}(\zeta, z) \wedge \bar{\partial}\bar{u}(\zeta)}{|\partial u(\zeta)|^2} d\sigma(\zeta).$$

For every $t > 0$ we define the extension operator

$$E_\eta^t f(z) = c_t \int_{\Omega_\eta^\varepsilon \cap Y} f(\zeta) \frac{(-\rho_\eta(\zeta))^{t+2}}{a_\eta(\zeta, z)^{t+2}} \partial\bar{\partial} \log \frac{-1}{\rho_\eta(\zeta)} \wedge \mu. \tag{7.2}$$

Then we obtain

LEMMA 7.2. *For $\varepsilon > 0$ small enough the operator E_η^t maps the space $A_l^p(\Omega_\eta^\varepsilon \cap Y, (-\rho)^\delta \tau^{r+2})$ to the space $A_l^p(\Omega_\eta^\varepsilon \cap B(\eta, \varepsilon/2), (-\rho)^\delta \tau^r)$.*

PROOF. As in Lemma 6.7, using $(\partial \rho_\eta)(z) \neq 0$ on the boundary of $\partial \Omega_\eta$ we obtain:

$$(E_\eta^t f)(z) = \sum_{|\gamma| \leq l} E_\eta^\gamma \left(\frac{\partial^{|\gamma|} f}{\partial \zeta^\gamma} \right), \quad \text{with} \quad \left| \frac{\partial^{|\gamma|}}{\partial z^\alpha} E_\eta^\gamma \right| \leq c \frac{(-\rho_\eta)^{t+l}}{|a_\eta(\zeta, z)|^{2+t+|\alpha|}}.$$

Therefore, to prove the lemma, it is sufficient to show that the integral operator

$$E_\eta^{t,l}(\zeta, z) = \frac{(-\rho_\eta(\zeta))^{t+l}}{|a_\eta(\zeta, z)|^{2+t+l}}$$

maps $L^p(\Omega_\eta^\varepsilon \cap Y, (-\rho)^\delta \tau^{r+2})$ to $L^p(\Omega_\eta^\varepsilon \cap B(\eta, \varepsilon/2), (-\rho)^\delta \tau^r)$.

But for $\zeta \in B(\eta, \varepsilon)$ we have $E_\eta^{t,l}(\zeta, z) = (-\rho)^{t+l}/|a(\zeta, z)|^{2+t+l}$ and for $\zeta \in \Omega_\eta^\varepsilon \setminus B(\eta, \varepsilon)$ and $z \in B(\eta, \varepsilon/2)$ we have $E_\eta^{t,l}(\zeta, z) \le c(-\rho_\eta(\zeta))^{t+l} \le c(-\rho(\zeta))^{t+l}$. Thus, for $z \in B(\eta, \varepsilon/\eta)$, we obtain

$$\int_{\Omega_\eta^\varepsilon \cap Y} |f(\zeta)|\,|E_\eta^{t,\gamma}(\zeta, z)| d\sigma(\zeta) \le c \int_{\Omega_\eta^\varepsilon \cap Y \cap B(\eta,\varepsilon)} |f(\zeta)| \frac{(-\rho(\zeta))^{t+l}}{|a(\zeta, z)|^{2+t+l}} d\sigma(\zeta) + \int_{\Omega_\eta^\varepsilon \cap Y \setminus B(\eta,\varepsilon)} |f(\zeta)|(-\rho(\zeta))^{t+l} d\sigma(\zeta).$$

Then taking t large enough and applying the argument used in the proof of Lemma 6.8 we obtain

$$\int_{B(\eta,\varepsilon/2)} \left(\int_{\Omega_\eta^\varepsilon \cap Y} |f(\zeta)| E_\eta^{t,\gamma}(\zeta, z) d\sigma(\zeta) \right)^p (-\rho(z))^\delta \tau(z)^r dz \le c \int_{\Omega_\eta^\varepsilon \cap Y} |f(\zeta)|^p (-\rho(\zeta))^\delta \tau(\zeta)^{r+2} d\sigma(\zeta).$$

Hence the lemma is proved. □

Proof of Theorem 7.1. We consider a finite set of points $\{\eta_i\}_{i=1}^{i_0}$ of $\partial D \cap Y$ such that $\partial D \subset \cup_i B(\eta_i, \varepsilon)$ for some ε small enough. Also we can assume that $|(\partial\rho \wedge \partial u)(z)| \ge c > 0$ for $z \in B(\eta_i, 4\varepsilon)$ and that there exists a local projection of Lemma 5.3 in a neighbourhood of $\overline{\Omega}_\eta^{4\varepsilon}$. Moreover, we consider a domain $\Omega_0 \subset\subset D$ such that $\overline{D} \subset \Omega_0 \cup_i B(\eta_i, \varepsilon)$.

By Lemma 7.2 we can find holomorphic functions f_i on $\Omega_{\eta_i}^{4\varepsilon}$ of class $A_l^p(\omega_{\eta_i}^{4\varepsilon} \cap B(\eta_i, 2\varepsilon), (-\rho)^\delta \tau^r)$ such that $f_i|_Y = g$. Also, we have a holomorphic function f_0 on D such that $f_0|_M = g$.

For $\varepsilon \le \lambda \le 4\varepsilon$, we define the covering $V_{i,\lambda}$ of D by

$$V_{0,\nu} = \Omega_0, \quad V_{i,\lambda} = \Omega_\eta^{4\varepsilon} \cap B(\eta, \lambda), \quad i = 1, \dots, i_0$$

and we consider the Cousin data

$$f_{ij} = f_i - f_j, \quad \text{on} \quad V_{i,2\varepsilon} \cap V_{j,2\varepsilon}.$$

Then, using the local projection $w_i = w_i(z)$ of Lemma 5.3 and an argument like the one of the proof of Theorem 5.1, we obtain

$$f_{ij} = u h_{ij}, \quad h_{ij} \in A_l^p\left(V_{i,\varepsilon} \cap V_{j,\varepsilon}, (-\rho)^\delta \tau^{r+p}\right).$$

It is clear that h_{ij} is a Cousin data. Using the standard proceeding to solve the first Cousin problem, we take a partition of the unity χ_i respect the covering $V_{i,\varepsilon}$ and we consider the (0, 1)-form ν on D defined by

$$\nu(z) = \sum_s \bar{\partial}\chi_s(z) h_{is}(z), \quad z \in V_{i,\varepsilon}.$$

Then it is clear that ν is a $\bar{\partial}$-closed form with coefficients in $L_l^p(D, (-\rho)^\delta \tau^{r+p})$. Thus, by Theorem 4.1 there exists a function h of class $L_l^p(D, (-\rho)^\delta \tau^r)$ such that $\bar{\partial} h = \nu$.

Finally defining

$$f(z) = f_i(z) - u(z)\Big(\sum_s \chi_s(z) h_{is}(z) - h(z)\Big), \quad z \in V_{i,\varepsilon}$$

we end the proof. □

8. – Final remarks

The same methods used in the above sections permit us to obtain some analogous results for complex ellipsoids D in $\mathbb{C}^n$, $n > 2$. However, in this case the results are less complete.

Let D be the complex ellipsoid

$$D = \Big\{ z \in \mathbb{C}^n \, ; \, \rho(z) = \sum_{i=1}^n |z_i|^{2q_i} - 1 < 0 \Big\} .$$

Let us start stating some geometric results. We will need to know the distance from z to the boundary of D in the different complex tangent directions. Let $T(z) = \sum_{j=1}^n c_j(z) \partial/\partial z_j$ be a complex tangent vector field satisfying $|T(z)| \geq c_0 > 0$. We define

$$d_T(z) = \inf \big\{ |\lambda| ; \, \rho(z + \lambda T(z)) = 0 \big\} .$$

In this case, as in Proposition 2.5, we have

$$-\rho(z) \approx \sum_{j=1}^n |z_j|^{2q_j - 2} \big|c_j(z)\big|^2 d_T(z)^2 + \sum_{j=1}^n \big|c_j(z)\big|^{2q_j} d_T(z)^{2q_j}$$

and thus we have that $d_T(z)$ is equivalent to the unique positive root of the equation

$$-\rho(z) = \sum_{j=1}^n |z_j|^{2q_j - 2} \big|c_j(z)\big|^2 x^2 + \sum_{j=1}^n \big|c_j(z)\big|^{2q_j} x^{2q_j} .$$

For $z \neq 0$ we denote by $T_{i,j}$ the following vector fields which generate the complex tangent space at the point z:

$$T_{i,j} = q_i z_i^{q_i - 1} \bar{z}_i^{q_i} \frac{\partial}{\partial z_j} - q_j z_j^{q_j - 1} \bar{z}_j^{q_j} \frac{\partial}{\partial z_i}, \quad i \leq i, \; j \leq n .$$

It is clear that if $z_i \neq 0$ then $T_{i,j}$, $1 \leq j \leq n$, $j \neq i$ is a base of this space. For $|z_i| \geq 1/2$ we can obtain easily that

$$d_{T_{i,j}}(z) \approx \begin{cases} \dfrac{(-\rho(z))^{1/2}}{|z_j|^{q_j-1}}, & \text{if } |z_j|^{2q_j} \geq -\rho(z) \\ (-\rho(z))^{1/2q_j} & \text{if } |z_j|^{2q_j} \leq -\rho(z). \end{cases}$$

Let us first consider the $\bar{\partial}$-*problem*. We will obtain some solutions with L^p-estimates in terms of a weight depending of ρ and $d_{T_{i,j}}$. Some results of this type were obtained by A. Bonami and Ph. Charpertier in [B-CHA].

The result reflects the different behavior of the solution in the different complex tangential directions and in the normal direction:

THEOREM 8.1. *Let $1 \leq p < \infty$ and let ω be a $\bar{\partial}$-form on D which satisfies*

i) $$\sum_{|\alpha|\leq k} \int_D \left| \left(\frac{\partial^{|\alpha|}}{\partial \zeta_\alpha} \omega \right)(\zeta) \right|^p \left(-\rho(\zeta)\right)^{p+\delta-1} d\zeta < \infty .$$

ii) *For every $1 \leq i, j \leq n$*

$$\sum_{|\alpha|\leq k} \int_{|\zeta_i|>1/2} \left| \left(\frac{\partial^{|\alpha|}}{\partial \zeta_\alpha} \omega \right)(\overline{T}_{i,j})(\zeta) \right|^p \left(-\rho(\zeta)\right)^{\delta-1} d_{T_{i,j}}(\zeta)^p d\zeta < \infty .$$

Then there exists u which satisfies $\bar{\partial} u = \omega$ and

$$\sum_{|\alpha|\leq k} \int_D \left| \frac{\partial^{|\alpha|} u(\zeta)}{\partial \zeta_\alpha} \right|^p \left(-\rho(\zeta)\right)^{\delta-1} d\zeta < \infty .$$

The proof follows as in Section 4. We can obtain kernels $K^t(\zeta, z)$, $R^t(\zeta, z)$ which satisfy Koppelman formulas (Theorem 3.1).

Moreover, for a $\bar{\partial}$-closed $(0,1)$-form ω we have

$$\begin{aligned} |\omega(\zeta) \wedge K^t_{0,0}(\zeta, z)| \leq & \sum_{k=0}^{n-1} \frac{(-\rho(\zeta))^t \sum_{|I|=k} \prod_{i\in I} |\zeta_i|^{2q_i-2} |\omega(\zeta)|}{|a(\zeta, z)|^{t+k} |\zeta - z|^{2n-2k-1}} \\ & + \sum_{k=1}^{n-1} \frac{(-\rho(\zeta))^t \sum_{i,j=1}^{n-1} |\omega(\overline{T}_{i,j})| \sum_{|I|=k-1, i,j\notin I} \prod_{l\in I} |\zeta_l|^{2q_l-2}}{|a(\zeta, z)|^{t+k+1} |\zeta - z|^{2n-2k-1}} \end{aligned} \tag{8.1}$$

and

$$|\omega(\zeta) \wedge R^t_{0,1,1}(\zeta, z)| \leq c \frac{(-\rho(\zeta))^t \prod_{i=1}^n |\zeta_i|^{2q_1-2} |\omega(\zeta)|}{|a(\zeta, z)|^{t+n+1}} \tag{8.2}$$

where

$$a(\zeta, z) = -\rho(\zeta) + \frac{\partial \rho(\zeta)}{\partial \zeta}(\zeta - z).$$

As in Lemma 3.5, for z in a neighbourhood of the boundary of D we have

$$|a(\zeta,z)| \approx -\rho(z) - \rho(\zeta) + |\Im m\, a(\zeta,z)| + \sum_{i=2}^{n} |z_i|^{2q_i-2}|\zeta_i - z_i|^2 + |\zeta_i - z_i|^{2q_i}\,.$$

Finally, using the integration by parts formula of Lemma 4.1, a similar version of Lemma 4.4 (see Lemma 3.5 of [OR-FA 2]) and the usual changes of coordinates we obtain the theorem.

The division and extension problems are more delicate. We will only consider some particular cases of the first problem. Let $z' = (z_1, \dots, z_1, 0, \dots, 0)$, $z'' = z - z'$, $Y = \{z \in \mathbb{C}^n; z' = 0\}$ and f is a holomorphic function on D which vanishes on $Y \cap D$. Then using the projection $w(z) = z''$ and following the same steps of Section 5, we can find a decomposition

$$f(z) = \sum_{j=1}^{t} z_j f_j(z)$$

where the functions $f_j(z)$ satisfy the estimate

$$\sum_{|\alpha|\le k} \int_D \left|\frac{\partial^{|\alpha|} f_j(z)}{\partial z_\alpha}\right|^p \big(-\rho(z)\big)^{\delta-1}\big(-\rho(z'')\big)^{p/2q_j} dz$$
$$\le c \sum_{|\alpha|\le k} \int_D \left|\frac{\partial^{|\alpha|} f(z)}{\partial z_\alpha}\right| \big(-\rho(z)\big)^{\delta-1} dz\,.$$

Note that $(-\rho(z''))^{1/2q_j} = d_{\partial/\partial z_j}(z'')$.

An analogous result can be obtained for some submanifolds of codimension one. In this case we have:

THEOREM 8.2. *Let $Y = \{z \in V; u(z) = 0\}$ a complex submanifold defined in a neighbourhood of $\overline{D}$, of codimension 1 and transversal to the boundary of D.*

Then, if f is a holomorphic function on D which vanishes on $Y \cap D$, and $L(z)$ is a complex tangent vector field which satisfies $|(Lu)(z)| \ge c_0 > 0$ for $z \in \partial D \cap Y$, we have

$$\sum_{|\alpha|\le k} \int_D \left|\frac{\partial^{|\alpha|}}{\partial z_\alpha}\left(\frac{f(z)}{u(z)}\right)\right|^p \big(-\rho(z)\big)^{\delta-1}\big(d_L(z) + |u(z)|\big)^p dz$$
$$\le c \sum_{|\alpha|\le k} \int_D \left|\frac{\partial^{|\alpha|} f(z)}{\partial z_\alpha}\right| \big(-\rho(z)\big)^{\delta-1} dz\,.$$

In this case the projection $w = w(z)$ used to prove the theorem can be obtained from

$$\left.\begin{aligned} w + \lambda L(w) - z &= 0 \\ u(w) &= 0 \end{aligned}\right\}\,.$$

This projection satisfies some analogous properties to the one defined in Lemma 5.3, replacing the function $\Lambda(z, x)$ by the function

$$\Lambda(z, T, x) = \sum_{j=1}^{n} |z_j|^{2q_j-2} |c_j(w)|^2 x^2 + \sum_{j=1}^{n} |c_j(w)|^{2q_j} x^{2q_j}$$

where $T(w) = \sum_{j=1}^{n} c_j(w) \frac{\partial}{\partial w_j}$, $x \geq 0$. Now, following like in Section 5, we can finish the proof of the theorem.

REFERENCES

[AD 1] K. Adachi, *Extending bounded holomorphic functions from certain subvarieties of a weakly pseudoconvex domain*, Pacific J. Math. **110** (1984), 9-19.

[AD 2] K. Adachi, *Continuation of bounded holomorphic functions from certain subvarieties to weakly pseudoconvex domains*, Pacific J. Math. **130** (1987), 1-8.

[AD 3] K. Adachi, *Extending H^p functions from subvarieties to real ellipsoids*, Trans. Amer. Math. Soc. **317** (1990), 351-359.

[AM] E. Amar, *Cohomologie complexe et applications*, J. London Math. Soc. **29** (1984), 127-140.

[BEA] F. Beatrous, *Estimates for Derivatives of Holomorphic Functions in Pseudoconvex Domains*, Math. Z. **191** (1986), 91-116.

[BER-AN] B. Berndtsson - M. Andersson, *Henkin-Ramirez Formulas with Weight Factors*, Ann. Inst. Fourier (Grenoble) **32** (1983), 91-110.

[BER] B. Berndtsson, *A formula for Interpolation and Division in $\mathbb{C}^n$*, Math. Ann. **263** (1983), 399-412.

[B-CHA] A. Bonomi - Ph. Crarpentier, *Solutions de l'equation $\bar{\partial}$ et zeros de la classe de Nevalinna dans certains domaines faiblement pseudo-convexes*, Ann. Inst. Fourier (Grenoble) **32** (1982), 53-89.

[BO-CU-ZE] P. Bonneau - A. Cumenge - A. Zériahi, *Division dans les espaces de Lipschitz de fonctions holomorphes*, C.R. Acad. Sci. Paris **297** (1983), 517-520.

[BO-DI] P. Bonneau - K. Diederich, *Integral solution operators for the Cauchy-Riemann equations on pseudoconvex domains*, Math. Ann. **286** (1990), 77-100.

[BR-CA] J. Bruna - J. Castillo, *Hölder and L^p-estimates for the $\bar{\partial}$-equation in some convex domains with real analytic boundary*, Mat. Ann. **269** (1984), 527-539.

[BR-OR] J. Bruna - J.M. Ortega, *Interpolation by Holomorphic Functions Smooth to the Boundary in the Unit Ball of $\mathbb{C}^n$*, Math. Ann. **274** (1986), 527-575.

[CH-NA-ST] D.C. Chang - A. Nagel - E.M. Stein, *Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$*, Acta Math. **169** (1992), 153-228.

[CHE-KR-MA] Z. Chen - S.G. Krantz - D. Ma, *Optimal L^p estimates for the $\bar{\partial}$-equation in $\mathbb{C}^2$*, Manuscripta Math. **80** (1993), 131-150.

[CU] A. Cumenge, *Extension dans les classes de Hardy de fonctions holomorphes et estimations de type "Mesures de Carleson" pour l'equations $\bar{\partial}$*, Ann. Inst. Fourier (Grenoble) **33** (1983), 59-97.

[DI-FO] K. Diederich - J.E. Fornaess - J. Wiegerinck, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids*, Manuscripta Math. **56** (1986), 399-417.

[GR-ST] P.C. Greiner - E.M. Stein, Estimates for the $\bar{\partial}$-Neumann problem, Princeton University Press, 1977.

[GRE] S. Grellier, *Behavior of holomorphic functions in complex tangential directions in a domain of finite type in $\mathbb{C}^n$*, Publ. Mat. **36** (1992), 251-292.

[HE] G.M. Henkin, *Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains*, Math. USSR-Izv. **6** (1972), 536-563.

[HI] M. Hickel, *Sur un probleme de division dans l'algebre $A^\infty(D)$ d'un ouvert faiblemment pseudoconvexe de $\mathbb{C}^2$*, Math. Z. **197** (1988), 201-218.

[KE] N. Kerzman, *Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301-379.

[KO] J.J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geometry **6** (1972), 523-542.

[KO] S.G. Krantz, *Optimal Lipschitz and L^p regularity for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains*, Math. Ann. **219** (1976), 233-260.

[NA-ST-WA] A. Nagel - E.M. Stein - S. Wainger, *Boundary behavior of functions holomorphic in domains of finite type*, Proc. Nat. Acad. Sci. U.S.A. **78** (1980), 6596-6599.

[OK] G.O. Okikiolu, Aspects of the Theory of Bounded integral operators in L^p-spaces, Academic Press. London - New York, 1971.

[OR-FA 1] J.M. Ortega - J. Fàbrega, *Division and Extension in weighted Bergman-Sobolev spaces*, Publ. Mat. **36** (1992), 837-859.

[OR-FA 2] J.M. Ortega - J. Fàbrega, *Mixed-norm spaces and interpolation*, Studia Math. **109** (1994), 233-254.

[OV] N. Ovrelid, *Integral representations formulas and L^p estimates for the $\bar{\partial}$ equation*, Math. Scand. **29** (1971), 137-160.

[RA] R.M. Range, *On Hölder estimates for $\bar{\partial}u = f$ on weakly pseudoconvex domains*, Proceedings of International Conferences. Cortona, Italy 1976-1977. Scuola Norm. Sup. Pisa (1978), 247-267.

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