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Asymptotic Morse Inequalities for Pseudoconcave Manifolds

GEORGE MARINESCU [1]

1. - Introduction

The question of asymptotically estimating the dimensions of cohomology groups with coefficients in the high tensor powers of a fixed line bundle arised in connection to the conjecture of Grauert and Riemenschneider [12] which says that a compact complex space Y of dimension n is Moishezon if and only if there exists a proper non-singular modification $\pi: X \to Y$ and a line bundle E on X such that the curvature form of E is positive definite on an open dense set. Let us denote by $K(E)$ the Kodaira dimension of E. If $K(E)$ is maximal, that is, $K(E)$ equals the dimension of X, then there are many sections in $\Gamma(X, E^k)$ and by taking quotients of elements of $\Gamma(X, E^k)$ we get a large field of meromorphic functions $\mathcal{K}(X)$ so that X is Moishezon. Thus, it is sufficient to show that $K(E) = n$. This follows from Demailly's asymptotic inequalities [8] and from the general fact that $\dim H^0(X, O(E^k)) \leq C_2 k^{K(E)}$. Indeed, for $p = 1$ the Strong Morse inequality of [8] gives:

$$\text{(1.1)} \qquad \dim H^0(X, O(E^k)) \geq \frac{k^n}{n!} \int\limits_{X(\leq 1,h)} \left(\frac{i}{2\pi} c(E) \right)^n - o(k^n), \quad k \to \infty.$$

In this statement E is supposed to carry a C^∞ hermitian metric h, $ic(E) = ic(E,h)$ being its curvature form. Also, $X(p,h)$ are the p-index sets, i.e., $X(p,h) = \{x \in X: ic(E,h)$ has p negative eigenvalues and $n-p$ positive ones$\}$ and $X(\leq p, h) = \bigcup\limits_{1 \leq j \leq p} X(j,h)$. The symbol $o(k^n)$ is the Landau symbol denoting a term of order less than that of k^n.

T. Bouche [5] extended the holomorphic Morse inequalities to some class of non-compact manifolds: q-convex manifolds and weakly 1-complete Kähler

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manifolds possesing a holomorphic line bundle which is semi-positive of type q. Our main purpose is to extend the Morse inequalities, as well as some of their consequences, to q-concave manifolds. We will prove that if E and F are holomorphic vector bundles of rank 1 and r over the n-dimensional q-concave manifold X then the dimensions of the groups $\dim H^p(X, O(E^k \otimes F))$ are at most of polynomial growth of degree n with respect to k, provided $p \leq n-q-2$ (cf. § 4, Theorem 4.2). In particular we obtain the following.

THEOREM 1.1. *Let X be an n-dimensional q-concave manifold such that $q \leq n-2$. Assume that X carries a holomorphic line bundle (E, h) which is semi-negative outside a compact set and satisfies the following condition*

$$\int_{X(\leq 1, h)} \left(\frac{i}{2\pi} c(E, h)\right)^n > 0. \tag{1.2}$$

Then

$$\dim H^0(X, O(E^k)) \approx k^n \quad as\ k \to \infty \tag{1.3}$$

(that is $\dim H^0(X, O(E^k))/k^n$ *is bounded from above and from below by positive constants).*

This enables us to prove that, like in the case of compact manifolds, there are $n = \dim X$ independent meromorphic functions on X. Indeed, (1.3) shows that the Kodaira dimension $K(E)$ of E is then maximal since we can extend the inequality $\dim H^0(X, O(E^k)) \leq C_2 k^{K(E)}$ to concave manifolds. There is a large class of concave manifolds possesing a maximal number of independent meromorphic functions. For example we can consider the complements of suitable analytic sets in compact Moishezon manifolds. We shall prove that in fact 1-concave manifolds satisfying the hypothesis of Theorem 1.1 arise like those in these examples. Section 5 is devoted to the proof of the following.

THEOREM 1.2. *Let X be a connected* 1*-concave manifold of dimension at least three carrying a line bundle which satisfies the hypothesis of Theorem* 1.1. *There exists an embedding of X as an open subset of a compact Moishezon manifold.*

Non-trivial examples which satisfy these hypotheses are as follows (cf. Proposition 4.5). Consider the regular part X^* of a compact complex space X with isolated singularities carrying a holomorphic line bundle (which extends to the singular points) satisfying (1.2). Assume moreover that X^* has finite volume with respect to a suitable complete metric on X^*, called Grauert metric (see (4.10); cf. H. Grauert [11]). Then we can modify the metric on the given line bundle so that the new metric still satisfies (1.2) and its curvature form is semi-negative outside a compact set. Of course, the general result one would expect is as follows.

CONJECTURE. *A connected* 1*-concave manifold of dimension at least three is isomorphic to an open subset of a compact Moishezon manifold if and only if it carries a torsion free quasi-positive coherent analytic sheaf.*

As in the case of the Grauert-Riemenschneider conjecture by compactification and desingularisation we can reduce this conjecture to a statement involving a quasi-positive line bundle. It is then difficult to prove in general that we can modify the hermitian metric on the line bundle such that its curvature form is semi-negative outside a compact set and still satisfies (1.2). The proof of Theorem 1.1 is a slight modification of the proof of the embedding theorem of Andreotti and Siu [3]: if a connected 1-concave manifold of dimension at least three carries a line bundle which gives local coordinates on a sub-level set then it is isomorphic to an open set of a projective manifold. In our situation the positivity assumption on the curvature implies via the Morse inequalities that the line bundle gives local coordinates on an open dense subset of X.

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2. - Preliminaries

Let X be a complex paracompact manifold of dimension n endowed with a hermitian metric ds^2 and let F be a holomorphic vector bundle on X with a hermitian metric h. For integers s, $t \geq 0$ and an open set $Y \subseteq X$ we define the following notations:

$C^{t,s}_{\text{comp}}(Y, F)$: the space of smooth, compactly supported, F-valued (t, s)-forms on Y.

$L^{t,s}(Y, F, ds^2, h)$: the Hilbert space obtained by completing $C^{t,s}_{\text{comp}}(Y, F)$ with respect to the L^2-norm $\|\cdot\|_{ds^2,h}$ with respect to ds^2 and h.

$L^{t,s}(Y, F, \text{loc})$: the space of locally square integrable E-valued (t, s)-forms.

On the space of smooth F-valued forms we have the differential operators $\overline{\partial}$ and ϑ, the formal adjoint of $\overline{\partial}$. These operators have weak maximal extensions as closed linear operators with dense domain on $L^2(X, F)$. If T is one of the preceding differential operators we denote by $\mathcal{D}^{t,s}(T)$ the domain, by $N^{t,s}(T)$ the kernel and by $R^{t,s}(T)$ the range of the weak maximal extension of T in L^2. We introduce also the Hilbert space adjoint $\overline{\partial}^*$ of the closure of $\overline{\partial}$. In general

$\overline{\partial}^*$ does not coincide with the weak maximal extension of the formal adjoint ϑ, because X may have boundary. One can bypass this difficulty using the following fundamental result due to Andreotti-Vesentini [4], [23]: if ds^2 is a complete hermitian metric then $C^{t,s}_{\text{comp}}(X,F)$ is dense for the graph norm topology in the domains $\mathcal{D}^{t,s}(\overline{\partial})$, $\mathcal{D}^{t,s}(\vartheta)$ and $\mathcal{D}^{t,s}(\overline{\partial})\cap\mathcal{D}^{t,s}(\vartheta)$ of the weak maximal extensions of $\overline{\partial}$, ϑ and $\overline{\partial}+\vartheta$ respectively. From this we easily infer that if ds^2 is a complete metric then $\overline{\partial}^*$ is the weak maximal extension of the formal adjoint ϑ of $\overline{\partial}$. We denote by $\mathcal{H}^{t,s}(X,F)=N^{t,s}(\overline{\partial})\cap N^{t,s}(\vartheta)$ the space of harmonic forms. The L^2-Dolbeault cohomology groups are defined as $H^{t,s}_D(X,F)=N^{t,s}(\overline{\partial})/R^{t,s-1}(\overline{\partial})$. There is a canonical map $\mathcal{H}^{t,s}(X,F)\to H^{t,s}_D(X,F)$, which is isometric, with dense range.

$\mathcal{H}^{t,s}(X,F)$ is isomorphic to $H^{t,s}_D(X,F)$, if and only if the range of the weak maximal extension of $\overline{\partial}+\vartheta$ is closed. This is always the case if it is finite dimensional, and in particular if from every sequence $u_k\in\mathcal{D}^{t,s}(\overline{\partial})\cap\mathcal{D}^{t,s}(\vartheta)$ with $\|u_k\|\le 1$ and $\overline{\partial}u_k\to 0$, $\overline{\partial}^*u_k\to 0$, one can select a L^2 convergent subsequence. Let us consider the antiholomorphic Laplace-Beltrami operator $\Delta''=\overline{\partial}\vartheta+\vartheta\overline{\partial}$ acting on $C^{t,s}_{\text{comp}}(X,F)$. We can extend Δ'' to a densely defined, self-adjoint operator on $L^{t,s}(X,F)$. We put

$$\mathcal{D}^{t,s}(\Delta'')=\{u\in\mathcal{D}^{t,s}(\overline{\partial})\cap\mathcal{D}^{t,s}(\vartheta)\colon\overline{\partial}u\in\mathcal{D}^{t,s}(\vartheta),u\in\mathcal{D}^{t,s}(\overline{\partial})\}$$

and $\Delta''u=\overline{\partial}\vartheta v+\vartheta\overline{\partial}u$, for $u\in\mathcal{D}^{t,s}(\Delta'')$. If Δ'' has closed range one can define the Green operator as the bounded operator $\mathcal{G}$ on $L^{t,s}(X,F)$ such that $\Delta''\mathcal{G}=\mathrm{Id}-\mathcal{H}$, $\mathcal{H}\mathcal{G}=0$, $\mathcal{H}$ being the orthogonal projection on $\mathcal{H}^{t,s}(X,F)$. A sufficient condition for Δ'' to have closed range is that the graph norm of Δ'' to be completely continuous with respect to the L^2-norm (i.e., the unit ball B of $\mathcal{D}^{t,s}(\overline{\partial})\cap\mathcal{D}^{t,s}(\vartheta)$ in the graph norm is relatively compact in $L^{t,s}(X,F)$). In this case Δ'' has discrete spectrum and the norm of $\mathcal{G}$ does not exceed λ_1^{-1}, where λ_1^{-1} is the lowest non-zero eigenvalue of Δ''. Finally, let us notice that

$$H^s(X,\Omega^t(F))=\{u\in L^{t,s}(X,F,\mathrm{loc})\colon\overline{\partial}u=0\}/$$
$$\{u\in L^{t,s}(X,F,\mathrm{loc})\colon\overline{\partial}v=u,\ \text{for some } v\in L^{t,s-1}(X,F,\mathrm{loc})\}$$

where $\Omega^t(F)$ is the sheaf of germs of holomorphic F - valued t - forms.

3. - Abstract Morse Inequalities for the L^2- cohomology of Complex Manifolds

We shall examine a general situation which permits to prove asymptotic Morse inequalities for the L^2-Dolbeault cohomology groups. Our approach is based on the seminal article of J.-P. Demailly [8] and generalizes that of T. Bouche [5] which shows that the basic estimate (3.2) holds for q-convex manifolds and for weakly 1-complete Kähler manifolds possesing a semi-positive line bundle of type q. Let us consider a complex manifold X with a complete hermitian metric ds^2 and (E,h) and F holomorphic hermitian bundles over X of rank 1 and r, respectively. Suppose we are given: (i) a compact subset M of X and (ii) a real-valued continuous function ψ on X. If X is non-compact we assume that ψ is bounded bellow on the complement of M by a positive constant and converges to $+\infty$ at infinity on X (that is, X admits an exhaustion with compact sets X_l, $l = 1, 2, \ldots$ such that $\psi \geq l$ on the complement of X_l, for $l = 1, 2, \ldots$). We consider the following estimate:

$$(3.1) \qquad \|\overline{\partial}u\|^2 + \|\vartheta u\|^2 \geq k \int \psi |u|^2 dV, \quad u \in C^{s,t}_{\text{comp}}(X - M, E^k \otimes F).$$

REMARK (A). The estimate (3.1) implies that there exists $C_0 > 0$ such that, for sufficiently large k

$$(3.2) \qquad \|u\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}u\|^2 + \|\vartheta u\|^2) + C_0 \int_K |u|^2 dV, \quad u \in C^{s,t}_{\text{comp}}(X, E^k \otimes F)$$

where K is any compact set containing M in its interior. Indeed, let ρ be a smooth function on X, $0 \leq \rho \leq 1$, which vanishes in a neighbourhood of M and equals 1 in the complement of K. Then (3.2) follows by applying (3.1) to ρu for any $u \in C^{s,t}_{\text{comp}}(X, E^k \otimes F)$ and by using the following simple estimate:

$$\|\overline{\partial}(\rho u)\|^2 + \|\vartheta(\rho u)\|^2 \leq \frac{3}{2}(\|\overline{\partial}u\|^2 + \|\vartheta u\|^2) + 6(\sup |d\rho|^2)\|u\|^2$$

Estimates of type (3.2) were introduced by Morrey and used by Kohn to solve the $\overline{\partial}$-Neumann problem. In this form they appear for the first time in Hörmander [14] and Ohsawa [17] in order to prove isomorphism and finiteness theorems. Estimate (3.2) implies that the space of harmonic forms $\mathcal{H}^{s,t}(X,E)$ is finite dimensional and is isomorphic to the L^2-Dolbeault cohomology group $H^{s,t}_D(X,E)$.

REMARK (B). If the estimate (3.1) holds then the antiholomorphic Laplace-Beltrami operator Δ'' acting on $L^{s,t}(X, E^k \otimes F)$ has discrete spectrum. In fact we have that if G is a holomorphic hermitian vector bundle on X and M, ψ satisfy (i), (ii) then it is easily seen that

$$(3.1)' \qquad \|\overline{\partial}u\|^2 + \|\vartheta u\|^2 \geq \int \psi |u|^2 dV, \quad u \in C^{s,t}_{\text{comp}}(X - M, G)$$

implies that Δ'' acting on $L^{s,t}(X, G)$ has compact resolvent.

PROPOSITION 3.1. *Let X be an n-dimensional complex manifold with a complete hermitian metric ds^2 and (E,h) and F holomorphic hermitian bundles over X of rank 1 and r, respectively. Assume that here exists an integer $m > 0$ such that the estimate* (3.1) *holds for any $u \in C^{0,p}_{\text{comp}}(X, E^k \otimes F)$ and $p \leq m$. Let*

Ω be a relatively compact open set with smooth boundary such that $M \subset\subset \Omega$. Then the following inequalities hold:

$$\dim H_D^{0,p}(X, E^k \otimes F) \leq r \frac{k^n}{n!} \int\limits_{\Omega(p,h)} (-1)^p \left(\frac{i}{2\pi} c(E,h)\right)^n + o(k^n) \tag{3.3}$$

$$\begin{aligned}&\sum_{j=0}^{p} (-1)^{p-j} \dim H_D^{0,j}(X, E^k \otimes F) \\ &\qquad \leq r \frac{k^n}{n!} \int\limits_{\Omega(\leq p,h)} (-1)^p \left(\frac{i}{2\pi} c(E,h)\right)^n + o(k^n)\end{aligned} \tag{3.4}$$

for $p \leq m$ and $k \to \infty$.

PROOF. By Remark (B), Δ'' has discrete spectrum. We denote by $\mathcal{E}^{0,t}(\mu)$ the direct sum of all eigenspaces of the laplacian Δ'' acting on $L^{0,t}(X, E^k \otimes F)$ corresponding to eigenvalues $\leq \mu$. By Remark (A) the basic estimate (3.2) holds: it implies easily that, for some $M \subset\subset K \subset\subset \Omega$:

$$\|u\|^2 \leq 2C_0 \int\limits_K |u|^2 dV, \tag{3.5}$$

for any $u \in \mathcal{E}^{0,p}(k\lambda)$, if $\lambda < 1/(2C_0)$ and $p \leq m$, since $u \in \mathcal{E}^{0,p}(k\lambda)$ implies $\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \leq k\lambda \|u\|^2$. Let $\mathcal{E}^{0,t}_{\text{comp}}(\mu)$ be the direct sum of all eigenspaces of the laplacian Δ'' acting on $L^{0,t}(\Omega, E^k \otimes F)$ with Dirichlet boundary conditions on $b\Omega$, corresponding to eigenvalues $\leq \mu$. Let P_μ be the orthogonal projection from the closure of $C^{0,t}_{\text{comp}}(\Omega, E^k \otimes F)$ in $L^{0,t}(X, E^k \otimes F)$ onto $\mathcal{E}^{0,t}_{\text{comp}}(\mu)$:

$$\mathsf{P}_\mu \colon [C^{0,t}_{\text{comp}}(\Omega, E^k \otimes F)] \to \mathcal{E}^{0,t}_{\text{comp}}(\mu)$$

Our aim is to establish a link between the spectral spaces $\mathcal{E}^{0,p}(k\lambda)$ and $\mathcal{E}^{0,p}_{\text{comp}}(k\mu)$ for $p \leq m$ for suitable μ, since Demailly's spectral theorem (see Lemma 3.3 bellow) gives the precise asymptotic behaviour of $\mathcal{E}^{0,p}_{\text{comp}}(k\mu)$ when $k \to \infty$. This is done by the following lemma of T. Bouche. Let $\beta \in C^\infty_{\text{comp}}(\Omega)$ such that $0 \leq \beta \leq 1$ and $\beta = 1$ in a neighbourhood of K. Let $C_1 = 4 \sup |d\beta|^2$.

LEMMA 3.2. (Bouche [5]). *Let $\lambda < 1/(2C_0)$. There exists a constant C_2 depending only on C_0 and C_1 such that the maps $\mathcal{E}^{0,p}(k\lambda) \to \mathcal{E}^{0,p}_{\text{comp}}(3C_0 k\lambda + C_2)$, $p \leq m$, $u \to \mathsf{P}_{3C_0 k\lambda + C_2}(\beta u)$ are injective.*

We recall now Demailly's spectral theorem. For this purpose we denote by $\lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_n(x)$ the eigenvalues of $ic(E,h)(x)$ with respect to ds^2 and by $s = s(x)$ the rank of $ic(F,h)(x)$. If J is a multiindex we put $\lambda_J = \sum_{j \in J} \lambda_j$

and we make the conventions $\lambda_+^0 = 0$ if $\lambda < 0$ and $\lambda_+^0 = 1$ if $\lambda \geq 0$. For each multiindex J we define the function on $\mathbb{R}_+ \times X$:

$$\nu_J(\lambda) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} |\lambda_1\lambda_2 \dots \lambda_s| \sum_{(p_1,\dots,p_s)\in\mathbb{N}^s} \Big\{2\lambda + \lambda_{CJ} - \lambda_J - \sum (2p_j+1)|\lambda_j|\Big\}_+^{n-s}$$

where $CJ = \{1, 2, \dots, n\} - J$.

LEMMA 3.3 (cf. [8], Théorème 3.14). *There exists a countable set $\mathcal{D} \subset \mathbb{R}_+^*$ such that for any $\lambda \in \mathbb{R}_+^* - \mathcal{D}$ and any $p = 0, 1, \dots, n$ we have that*

$$\dim \mathcal{E}^{0,p}_{\mathrm{comp}}(k\lambda) = rk^n \sum_{|J|=p} \int_\Omega \nu_J(\lambda) dV + o(k^n) \tag{3.6}$$

when $k \to \infty$.

For the sake of simplicity we shall put from now on $I(p,\lambda) = \sum_{|J|=p} \int_\Omega \nu_J(\lambda)dV$. So, by means of of Lemma 3.2 and Demailly's spectral theorem, we are able to obtain informations about the asymptotic behaviour of $\dim \mathcal{E}^{0,p}(k\lambda)$ when $k \to \infty$ and fixed λ. On the other hand by applying Witten's technique one can show that the family of Dolbeault complexes $(\mathcal{E}^{0,p}(k\lambda), \overline{\partial})$ for varying $\lambda > 0$ have the same cohomology as the usual Dolbeault complex.

LEMMA 3.4 ([8], [5]). *If the basic estimate* (3.1) *holds for any $u \in \mathcal{C}^{0,p}_{\mathrm{comp}}(X, E^k \otimes F)$ and $p \leq m$, then the complex:*

$$0 \to \mathcal{E}^{0,0}(k\lambda) \to \mathcal{E}^{0,1}(k\lambda) \to \dots \to \mathcal{E}^{0,m}(k\lambda) \to \mathcal{E}^{0,m+1}(k\lambda) \tag{3.7}$$

is a subcomplex, quasi-isomorphic to the L^2-Dolbeault complex $(\mathcal{D}^{0,p}(\overline{\partial}), \overline{\partial})_{p\leq m+1}$.

Indeed, Remark (B) shows that Δ'' has compact resolvent, so that the Green operator $\mathcal{G}$ is bounded and the operator $\vartheta\mathcal{G}$ is bounded, too. Therefore $\vartheta\mathcal{G}$ $(\mathrm{Id} - \mathsf{P}_{k\lambda})$ is a homotopy operator(*) between Id and $\mathsf{P}_{k\lambda}$ on $(\mathcal{D}^{0,p}(\overline{\partial}), \overline{\partial})_{p\leq m+1}$. We need the following simple algebraic result. Let $0 \to C^0 \to C^1 \to \dots \to C^n \to 0$ be a complex of vector spaces of dimension c^p and let $h^p = \dim H^p(C^\bullet)$. If $c^p < \infty$ for $p \leq m$, then $h^p \leq c^p$ and

$$\sum_{\nu=0}^{p} (-1)^{\nu-p} h^\nu \leq \sum_{\nu=0}^{p} (-1)^{\nu-p} c^\nu, \quad \text{for } p \leq m. \tag{3.8}$$

(*) See the separate sheets.

We are now able to end the proof of the Proposition 3.1. By Lemma 3.2 for $\lambda > 0$ the following estimates hold, provided $k >> 0$: $\dim \mathcal{E}^{0,p}_{\mathrm{comp}}(k\lambda) \leq \dim \mathcal{E}^{0,p}(k\lambda) \leq \dim \mathcal{E}^{0,p}_{\mathrm{comp}}(3C_0k\lambda + C_2) \leq \dim \mathcal{E}^{0,p}_{\mathrm{comp}}(C_3k\lambda)$ for $p \leq m$, where $C_3 = 3C_0 + C_2$ does not depend on k or λ (k must be $> \lambda^{-1}$). By (3.8) applied to the complex (3.7) and Lemmas 3.3 and 3.4 we get that:

$$\lim_{k\to\infty} k^{-n} \dim H^{0,p}_D(X, E^k \otimes F) \leq I(p, C_3\lambda)$$

$$\lim_{k\to\infty} k^{-n} \sum_{\nu=0}^{p} (-1)^{\nu-p} \dim H^{0,\nu}_D(X, E^k \otimes F)$$
$$\leq I(0, C_3\lambda) - I(1,\lambda) + \ldots + (-1)^p I(p, \lambda')$$

for $p \leq m$ and any $\lambda \in \mathbb{R}^*_+ - \mathcal{D}$; we have denoted $\lambda' = \lambda$ if p is odd and $\lambda' = C_3\lambda$ if p is even. We let now $\lambda \to 0$ for $\lambda \in \mathbb{R}^*_+ - \mathcal{D}$ and we obtain in the right-hand side $I(p,0)$ and a sum of such integrals. To conclude we use the following relation:

$$I(p,0) = \frac{1}{n!} \int_{\Omega(p,h)} (-1)^p \left(\frac{i}{2\pi} c(E,h)\right)^n \tag{3.9}$$

Indeed, $I(p,0) = \displaystyle\int_{\Omega(p,h)} (2\pi)^{-n}(-1)^p \lambda_1\lambda_2 \ldots \lambda_n dV$ which equals the desired integral. □

4. - Estimates for the Cohomology of q-concave Manifolds

In this section we will apply the preceding results to the case of q-concave manifolds.

DEFINITION (Andreotti-Grauert [2]). *A complex manifold is said to be q-concave if there exists a smooth function* $\psi: X \to \mathbb{R}$ *such that the sets* $\{\psi \geq c\}$ *are compact in X for any* $c > \inf \psi$ *and* ψ *is q-convex outside a compact set of X.*

We will use the following equivalent definition (by putting $\varphi = -\psi$):

DEFINITION 4.1. *A complex manifold X of dimension n is said to be q-concave if there exists a smooth function* $\varphi: X \to \mathbb{R}$ *such that the sub-level sets* $X_c = \varphi^{-1}(-\infty, c)$ *are relatively compact in X for any* $c < \sup\varphi$ *and the Levi form of* φ, $i\partial\bar\partial\varphi$, *has at least* $n - q + 1$ *negative eigenvalues outside a compact set of X;* φ *is called an exhaustion function.*

Examples of q-concave manifolds

EXAMPLE (1) (T. Ohsawa [16]). Let X be a compact Kähler space of pure dimension n and Y an analytic subset of pure dimension q containing the singular locus of X. Then $X - Y$ is a $(q+1)$-concave manifold. In particular, the regular locus of a projective algebraic variety with isolated singularities is 1-concave.

EXAMPLE (2) (V. Vâjâitu [22]). Let X be a compact complex manifold and Y an analytic subset of pure dimension q. Then $X-Y$ is a $(q+1)$-concave manifold.

EXAMPLE (3) (V. Vâjâitu [22]). If $\pi: X \to Y$ is a proper holomorphic map between the manifold X and the q-concave manifold Y such that the dimension of its fibers does not exceed r, then X is $(q+r)$-concave (this holds for complex spaces, too).

According to Andreotti-Grauert theory all cohomology groups $H^p(X, \mathcal{F})$ with values in a locally free sheaf $\mathcal{F}$ are finite dimensional and the natural restriction maps $H^p(X, \mathcal{F}) \to H^p(X_c, \mathcal{F})$ are bijective for $p \leq n - q - 1$. Furthermore, T. Ohsawa has shown that every cohomology class in $H^p(X_c, \mathcal{F})$, $p \leq n - q - 2$, is represented uniquely by a harmonic form with respect to suitable hermitian metrics on X_c and $\mathcal{F}$. We will prove the following theorem:

THEOREM 4.2. *If E and F are holomorphic vector bundles of rank* 1 *and r over the n-dimensional q-concave manifold X $(n \geq 3)$ then the dimensions of the groups* $\dim H^p(X, O(E^k \otimes F))$ *are at most of polynomial growth of degree n with respect to k, provided $p \leq n - q - 2$.*

PROOF. The first step is to show that the estimate (3.1) holds on $X_c \supset K$ for $p \leq n - q - 1$ and to apply Proposition 3.1 in order to obtain Morse inequalities for the L^2-cohomology groups. Let $d < c$ such that $X_d \supset K$. We may assume that ds^2 is so chosen that outside some $X_e (e < d, X_e \supset K)$ the following assumptions holds:

(*) At least $n - q + 1$ eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to ds^2 are less than $-2q - 1$ and all the others are less than 1.

We shall denote these eigenvalues by $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$.

Let $\chi_\varepsilon: (-\infty, c) \to \mathbb{R}$ be smooth functions with the following properties:

$$\chi_\varepsilon'(t)^2 \leq 4\chi(t)^3; \tag{4.1}$$

$$\int_{c-1}^{c} \chi_\varepsilon(t)^{1/2} dt = \infty; \tag{4.2}$$

$$\chi_\varepsilon(t) > 4 \text{ and } \chi_\varepsilon'(t) \geq 0 \text{ everywhere.} \tag{4.3}$$

In fact, let us consider for sufficiently small $\varepsilon > 0$ the function $f_\varepsilon\colon (-\infty, c) \to \mathbb{R}$ such that $f_\varepsilon(t) = \dfrac{1}{(d+\varepsilon-t)^2} + 4$ for $t \in (-\infty, d]$, $f_\varepsilon(t) = 2\varepsilon^{-3}(t-d) + \varepsilon^{-2} + 4$ for $t \in \left(d, \dfrac{c+d}{2}\right]$ and $f_\varepsilon(t) = \dfrac{(c-d)^3\varepsilon^{-3}}{8(c-t)^2} + \dfrac{(c-d)\varepsilon^{-3}}{2} + \varepsilon^{-2} + 4$ for $t \in \left(\dfrac{c+d}{2}, c\right)$ which is differentiable and satisfy (4.1)-(4.3). We approximate f_ε by smooth functions to obtain a smooth function χ_ε with properties (4.1)-(4.3). Moreover, $\chi_\varepsilon(t) = \dfrac{1}{(d+\varepsilon-t)^2} + 4$, for $t \in (-\infty, d]$. We set:

$$(4.4) \qquad ds_\varepsilon^2 = ds^2 + \chi_\varepsilon(\varphi)\partial\varphi \otimes \overline{\partial}\varphi, \quad h_\varepsilon = h \exp\left(-A \int_{\inf\varphi}^{\varphi} \chi_\varepsilon(t)dt\right)$$

where A is a positive constant and h is some hermitian metric along the fibres of E. We denote by $\|\cdot\|_\varepsilon$ the L^2-norm with respect to ds_ε^2 and h_ε and by dV_ε the volume element with respect to ds_ε^2 and by Λ_ε is the adjoint of the left multiplication with the fundamental $(1,1)$-form associated to ds_ε^2. By (4.2), ds_ε^2 is a complete hermitian metric on X_c. If h' is a metric along the fibres of F then:

$$ic(E^k \otimes F, h_\varepsilon^k \otimes h') = ikc(E,h) + ikA(\chi_\varepsilon(\varphi)\partial\overline{\partial}\varphi + \chi'_\varepsilon(\varphi)\partial\varphi \wedge \overline{\partial}\varphi) + ic(F, h').$$

By examining the eigenvalues with respect to ds_ε^2 of the right hand-side terms in the above formula, as well as the torsion operators of ds_ε^2 we shall be able to derive the basic estimate by applying Nakano's inequality:

$$\frac{3}{2}(\|\overline{\partial}u\|^2 + \|\vartheta u\|^2) \geq ([ic(E,h), \Lambda]u, u) - \frac{1}{2}(\|\tau u\|^2 + \|\tau^* u\|^2 + \|\overline{\tau}u\|^2 + \|\overline{\tau}^* u\|^2),$$

for any $u \in C^{s,t}_{\mathrm{comp}}(X, E^k \otimes F)$. Here $\tau = [\Lambda, e(\partial\omega)]$ is the torsion operator of some hermitian metric ds^2 and ω is the fundamental $(1,1)$-form associated to ds^2, Λ is the interior product with ω, $e(\partial\omega)$ is the left multiplication with $\partial\omega$ (see [17] and [9]).

(i) Let us examine the eigenvalues of $i\chi_\varepsilon(\varphi)\partial\overline{\partial}\varphi + i\chi'_\varepsilon(\varphi)\partial\varphi \wedge \overline{\partial}\varphi$. To begin with, let us denote by γ_j^ε, $j = 1, 2, \ldots, n$, the eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to ds_ε^2. It is easily seen that the rank of $\partial\varphi \wedge \overline{\partial}\varphi$ is less than one. By the minimum-maximum principle we get that:

$$\begin{cases} \gamma_1 \leq \gamma_1^\varepsilon \leq \gamma_2 \leq \ldots \leq \gamma_{n-q+1} \leq \gamma_{n-q+1}^\varepsilon < 0 \\ \gamma_j^\varepsilon \leq \max(\gamma_n, 0) \text{ for every } j \end{cases}$$

so by (*) we have that $\gamma_1^\varepsilon \leq \gamma_2^\varepsilon \leq \ldots \leq \gamma_{n-q}^\varepsilon < -2q - 1$. Finally, if $\Gamma_1^\varepsilon \leq \Gamma_2^\varepsilon \leq \ldots \leq \Gamma_n^\varepsilon$ are the eigenvalues of $i\chi_\varepsilon(\varphi)\partial\overline{\partial}\varphi + i\chi'_\varepsilon(\varphi)\partial\varphi \wedge \overline{\partial}\varphi$ with respect to ds_ε^2, the minimum-maximum principle gives that Γ_j^ε is equal to the minimum over all subspaces $F \subset T_xX$ of dimension j of the expression:

$$\max\left\{\frac{i[\chi_\varepsilon(\varphi)\partial\overline{\partial}\varphi + \chi_\varepsilon'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi](\nu,\nu)}{[ds^2+\chi_\varepsilon(\varphi)\partial\varphi\otimes\overline{\partial}\varphi](\nu,\nu)} : \nu\in F,\ \nu\neq 0\right\}$$

But by (4.1) and (4.3),

$$\frac{i(\chi_\varepsilon'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi)(\nu,\nu)}{(ds^2+\chi_\varepsilon(\varphi)\partial\varphi\otimes\overline{\partial}\varphi)(\nu,\nu)} \leq \chi_\varepsilon(\varphi)/\chi_\varepsilon'(\varphi) \leq (4\chi_\varepsilon(\varphi))^{1/2}$$

$$\leq \chi_\varepsilon(\varphi) \text{ for } \nu\in T_xX, \quad \nu\neq 0$$

and hence $\Gamma_j^\varepsilon \leq \chi_\varepsilon(\varphi)\gamma_j^\varepsilon + \chi_\varepsilon(\varphi)$, on $X_c - X_e$. Thus, $\Gamma_1^\varepsilon \leq \ldots \leq \Gamma_{n-q}^\varepsilon \leq -2q\chi_\varepsilon(\varphi)$, $\Gamma_{n-q+1}^\varepsilon \leq \chi_\varepsilon(\varphi)$ (since $\gamma_{n-q+1}^\varepsilon < 0$) and $\Gamma_{n-q+2}^\varepsilon \leq \ldots \leq \Gamma_n^\varepsilon \leq 2\chi_\varepsilon(\varphi)$ (since $\gamma_j^\varepsilon < \gamma_n \leq 1$). Therefore any sum of $(q+1)$ eigenvalues Γ_j^ε is less than $-\chi_\varepsilon(\varphi)$ on $X_c - X_e$.

(ii) Applying the minimum-maximum principle again one obtains that $\alpha_j^\varepsilon \leq \max(\alpha_n, 0)$ where α_j are the eigenvalues of $ic(E,h)$ with respect to ds^2 and α_j^ε are the eigenvalues of the same curvature form with respect to ds_ε^2. Let us denote by $C_1' = \sup\{\alpha_n(x) : x\in\overline{X}_c\}$ and $C_1 = \max\{G_1', 0\}$. Hence $\alpha_j^\varepsilon \leq C_1$ and on $\overline{X}_c$ for every j. Also, we infer that there exists a constant C_2 such that $|\prec [ic(F,h'),\Lambda_\varepsilon]u, u\succ_\varepsilon| \leq C_2|u|_\varepsilon^2$.

(iii) As for the torsion operators, we let ω and ω_ε be the fundamental $(1,1)$-forms of ds^2 and ds_ε^2. We have that $d\omega_\varepsilon = d\omega + \chi_\varepsilon(\varphi)\partial\overline{\partial}\varphi\wedge(\partial\varphi+\overline{\partial}\varphi)$ hence the pointwise norms of the torsion operators of ds_ε^2 with respect to ds_ε^2: τ, $\overline{\tau}$, τ^*, $\overline{\tau}^*$, are bounded by $C_3\chi_\varepsilon(\varphi)^{1/2}$, where C_3 does not depend on ε. We apply now the results of (i)-(iii) combined with the Nakano inequality. We have that

$$\prec [ic(E^k\otimes F, h_\varepsilon^k\otimes h'),\Lambda_\varepsilon]u, u\succ_x$$
$$\geq (-k(\alpha_n^\varepsilon+\ldots+\alpha_{p+1}^\varepsilon) - kA(\Gamma_n^\varepsilon+\ldots+\Gamma_{p+1}^\varepsilon) - C_2)|u|_\varepsilon^2.$$

If $p\leq n-q-1$, then $q+1\leq n-p$, so by (i), $\Gamma_n^\varepsilon+\ldots+\Gamma_{p+1}^\varepsilon < -\chi_\varepsilon(\varphi)$ on $X_c - X_e$.

By (ii), α_j^ε are bounded by a constant on X_c.

Then, using (iii) we obtain that for $u\in C^{0,p}_{\text{comp}}(X_c - \overline{X}_e, E^k\otimes F)$, $p\leq n-q-1$

$$3(\|\overline{\partial}u\|_\varepsilon^2 + \|\vartheta u\|_\varepsilon^2) \geq 2\int(-knC_1 + kA\chi_\varepsilon(\varphi) - C_2)|u|_\varepsilon^2 dV_\varepsilon - 4C_3(\chi_\varepsilon(\varphi)u,u)_\varepsilon.$$

If we put $A = nC_1 + C_2 + 1$ then, since $\chi(t) > 4$ we obtain that

$$3(\|\overline{\partial}u\|_\varepsilon^2 + \|\vartheta u\|_\varepsilon^2) \geq 2\int k\chi_\varepsilon(\varphi)|u|_\varepsilon^2 dV_\varepsilon - 4C_3(\chi_\varepsilon(\varphi)u,u)_\varepsilon$$
$$= \int(2k - 4C_3)\chi_\varepsilon(\varphi)|u|_\varepsilon^2 dV_\varepsilon.$$

Therefore, for $k \geq 4C_3$ and $u \in C^{0,p}_{\text{comp}}(X_c - \overline{X}_e, E^k \otimes F)$, $p \leq n-q-1$ we have:

$$3(\|\overline{\partial} u\|^2_\varepsilon + \|\vartheta u\|^2_\varepsilon) \geq k \int \chi_\varepsilon(\varphi)|u|^2_\varepsilon dV_\varepsilon. \tag{4.5}$$

Thus, we have obtained a basic estimate of type (3.1) with $\psi = \dfrac{1}{3}\chi_\varepsilon(\varphi)$ and $M = \overline{X}_e$. The estimate (4.5) yields also:

(4.6) $\|\overline{\partial} u\|^2_\varepsilon + \|\vartheta u\|^2_\varepsilon \geq 4k\|u\|^2_\varepsilon$ for $u \in C^{0,p}_{\text{comp}}(X_c - \overline{X}_e, E^k \otimes F)$, $p \leq n-q-1$.

As in § 3, Remark (A), one readily cheks that the basic estimate (3.2) holds on X_c. Indeed, let $\eta > 0$ satisfying $e < e + \eta < d - \eta < d$ and $\rho \in C^\infty(X_c)$ such that $0 \leq \rho \leq 1$, $\rho = 0$ on $X_{e+\eta}$, $\rho = 1$ on $X_c - X_{d-\eta}$. By applying (4.5) to ρu, $u \in C^{0,p}_{\text{comp}}(X_c, E^k \otimes F)$, $\rho \leq n-q-1$ then we get that:

$$\frac{3}{2}(\|\overline{\partial} u\|^2_\varepsilon + \|\vartheta u\|^2_\varepsilon) + 6C_4(\varepsilon)\|u\|^2_\varepsilon \geq 4k\|u\|^2_\varepsilon - 4k\int (1-\rho)^2|u|^2_\varepsilon dV_\varepsilon$$

where $C_4(\varepsilon) = 4\sup|d\rho|^2_\varepsilon$. Since $ds^2_\varepsilon \geq ds^2$ we have that $|d\rho|^2_\varepsilon \leq |d\rho|^2$ where $|d\rho|$ is the norm with respect to ds^2. Therefore, the above estimate holds with $C_4(\varepsilon)$ replaced by $C_4 = 4\sup|d\rho|^2$, which does not depend on ε. Consequently, dividing by k the last relation we obtain:

$$\frac{1}{2k}(\|\overline{\partial} u\|^2_\varepsilon + \|\vartheta u\|^2_\varepsilon) + \frac{4}{3}\int_{\overline{X}_{d-\eta}} |u|^2_\varepsilon dV_\varepsilon \geq \|u\|^2_\varepsilon \tag{4.6}$$

for $u \in C^{0,p}_{\text{comp}}(X_c, E^k \otimes F)$, $p \leq n-q-1$, $k > k_0 = \max\{4C_3, 6C_4\}$, that is, the basic estimate (3.2) holds on X_c, with subellipticity constant $C_0 = 4/3$. Moreover, the exceptional set $\overline{X}_{d-\eta}$ and the integer k_0 are independent on ε. The estimate (4.5) shows that we can apply Proposition 3.1 (with $m = n-q-1$) to get

$$\dim H^{0,p}_D(X_c, E^k \otimes F)_\varepsilon \leq r\frac{k^n}{n!}\int_{X_f(p,h_\varepsilon)} (-1)^p\left(\frac{i}{2\pi}c(E,h_\varepsilon)\right)^n + o(k^n) \tag{4.7}$$

$$\begin{aligned} &\sum_{j=0}^{p}(-1)^{p-j}\dim H^{0,j}_D(X_c, E^k \otimes F)_\varepsilon \\ &\leq r\frac{k^n}{n!}\int_{X_f(\leq p,h_\varepsilon)} (-1)^p\left(\frac{i}{2\pi}c(E,h_\varepsilon)\right)^n + o(k^n) \end{aligned} \tag{4.8}$$

for $p \leq n-q-1$ and $d - \eta < f < d$. The L^2-Dolbeault cohomology groups in the left hand-side are with respect to ds^2_ε and h_ε.

The second step consists in finding an injective map from the ordinary cohomology groups into the L^2-Dolbeault cohomology groups. This is achieved by using Ohsawa's results on pseudo-Runge pairs (cf. [17], Chapter 2).

DEFINITION. *Let Y be a complex manifold, $Y_1 \subset Y_2$ two open subsets and G a holomorphic vector bundle on Y. The pair (Y_1, Y_2) is called a pseudo-Runge pair at bidegree (s,t) with respect to G, if there exists a family of complete hermitian metrics ds_ε^2 $(\varepsilon > 0)$ on Y_2, a family of hermitian metrics h_ε along the fibers of $G|Y_2$ satisfying:*

(1) *ds_ε^2 and h_ε and their derivatives converge uniformly on every compact subset of Y_1 to a complete hermitian metric ds_0^2 on Y_1 and to a hermitian metric h_0 on $G|Y_1$.*

(2) *The basic estimate (3.2) hold with respect to ds_ε^2 and h_ε at bidegree $(s,t+1)$ with the same subellipticity constant and the same exceptional set.*

(3) *$L^{s,t}(Y_2, G, ds_\varepsilon^2, h_\varepsilon) \subset L^{s,t}(Y_2, G, ds_\tau^2, h_\tau)$, $\varepsilon > \tau$, and there exists a constant C independent of ε such that $\|u|_{Y_1}\|_0 \leq C\|u\|_\varepsilon$, $u \in C^{s,t+i}_{\rm comp}(Y_2, G)$, $i = 0, 1$ and $\varepsilon > 0$ where $\|u\|_\varepsilon$ is the norm with respect to ds_ε^2 and h_ε.*

For pseudo-Runge pairs one can prove approximation theorems wich go back to Hörmander [14]:

For any $\overline{\partial}$-closed form $u \in L^{s,t}(Y_1, G, ds_0^2, h_0)$ and any $\delta > 0$ there exists an $\varepsilon > 0$ and a $\overline{\partial}$-closed form $\nu \in L^{s,t}(Y_2, G, ds_\varepsilon^2, h_\varepsilon)$ such that $\|\nu|_{Y_1} - u\|_0 \leq \delta$.
This readily implies the following:

WEAK ISOMORPHISM THEOREM. *If (Y_1, Y_2) is a pseudo-Runge pair at bidegrees (s,t) and $(s,t+1)$ with respect to G, then there exists an ε_0 such that the natural restriction maps $H_D^{s,t+1}(Y_2, G)_\varepsilon \to H_D^{s,t+1}(Y_1, G)_0$ are bijective for $\varepsilon \leq \varepsilon_0$ (the subscript ε means that the L^2 cohomology groups is with respect to ds_ε^2 and h_ε).*

We shall apply the results on pseudo-Runge pairs for $Y_1 = X_d$, $Y_2 = X_c$ and $G = E^k \otimes F$. We consider the family of complete hermitian metrics ds_ε^2 on X_c given by (4.4) and family of hermitian metrics $h_\varepsilon^k \otimes h'$ where h_ε are defined by (4.4). The metrics ds_ε^2 and h_ε converge together with their derivatives on every compact set of X_d to the metrics

$$ds_0^2 = ds^2 + \frac{\partial\varphi \otimes \overline{\partial}\varphi}{(d-\varphi)^2} + 4\partial\varphi \otimes \overline{\partial}\varphi,$$

$$h_0 = h \exp\left(-\frac{A}{d-\varphi} + \frac{A}{d-\inf\varphi} - 4A(\varphi - \inf\varphi)\right).$$

Obviously, ds_0^2 is complete. Thus, the first condition in the definition of pseudo-Runge pairs is verified. Estimate (4.6) shows that the second is also fullfiled: the basic estimate (3.2) holds with respect to ds_ε^2 and $h_\varepsilon^k \otimes h'$ at bidegrees $(0, p+1)$ for $p \leq n-q-2$, with a common subellipticity constant and a common exceptional set. The third condition is also easily verified. Consequently, (X_d, X_c) is a pseudo-Runge pair in bidegree $(0,p)$, $p \leq n-q-2$, with respect to $E^k \otimes F$, for sufficiently large k. By the Weak Isomorphism Theorem, $H_D^{0,p}(X_c, E^k \otimes F)_\varepsilon \cong H_D^{0,p}(X_d, E^k \otimes F)_0$, $p \leq n-q-2$ for sufficiently small ε. Thus (4.7) and (4.8) are valid if we replace $H_D^{0,j}(X_c, E^k \otimes F)_\varepsilon$ by $H_D^{0,j}(X_d, E^k \otimes F)_0$. Since the metrics h_ε converge uniformly together with their derivatives to h_0 on X_f, by letting $\varepsilon \to 0$ in the above inequalities we obtain in the right-hand side curvature integrals with h_ε replaced by h_0. On the other hand, the representation theorem of Ohsawa ([17], Th. 4.6, (33)) shows that, for $p \leq n-q-2$, every cohomology class in $H^p(X_d, O(E^k \otimes F))$ is represented uniquely by a form in the harmonic space $\mathcal{H}^{0,p}(X_d, E^k \otimes F)_0$ with respect to ds_0^2 and h_0. But $\mathcal{H}^{0,p}(X_d, E^k \otimes F)_0 \cong H_D^{0,p}(X_d, E^k \otimes F)_0$ hence $\dim H^p(X_d, O(E^k \otimes F)) = \dim H_D^{0,p}(X_d, E^k \otimes F)_0$ so the preceding Morse inequalities (4.7)-(4.8) imply that:

$$\lim_{k\to\infty} k^{-n} \dim H^p(X_d, E^k \otimes F) \leq \frac{r}{n!} \int\limits_{X_f(p,h_0)} (-1)^p \left(\frac{i}{2\pi}\, x(E,h_0)\right)^n$$

$$\lim_{k\to\infty} \sum_{j=0}^{p} (-1)^{p-j} \dim H^j(X_d, E^k \otimes F) \leq r\,\frac{k^n}{n!} \int\limits_{X_f(\leq p,h_0)} (-1)^p \left(\frac{i}{2\pi}\, c(E,h_0)\right)^n \tag{4.9}$$

for $p \leq n-q-2$. Finally, we invoke the isomorphism $H^p(X, O(E^k \otimes F)) \cong H^p(X_d, O(E^k \otimes F))$ to conclude. □

In the proof of Theorem 4.2 we have obtained Morse inequalities in which the coefficient of k^n is an integral depending on the modified metric h_0. Under certain hypothesis on the curvature form $ic(E,h)$ of the initial metric h we can prove that the leading coefficient depends only on h.

COROLLARY 4.3. *If E and F are holomorphic vector bundles of rank 1 and r over the n-dimensional q-concave manifold X ($n \geq 3$) and the curvature form $ic(E,h)$ is negative semi-definite outside a compact set K then*

$$\dim H^p(X, O(E^k \otimes F)) \leq r\,\frac{k^n}{n!} \int\limits_{X(p,h)} (-1)^p \left(\frac{i}{2\pi}\, c(E)\right)^n + o(k^n)$$

$$\sum_{j=0}^{p} (-1)^{p-j} \dim H^j(X, O(E^k \otimes F)) \leq r\,\frac{k^n}{n!} \int\limits_{X(\leq p,h)} (-1)^p \left(\frac{i}{2\pi}\, c(E)\right)^n + o(k^n) \tag{4.9$_{\text{bis}}$}$$

as $k \to \infty$ and $p \leq n-q-2$.

PROOF (We use the same notations as in the preceding proof). Let us consider a compact set K such that (E, h) is negative semi-definite outside K. Let m be a fixed positive integer and X_e a sublevel set such that, with respect to some hermitian metric ds^2_m the following assumptions holds outside X_e $(e < d, X_e \supset K)$:

$$(*)_m \qquad \begin{array}{c} \text{At least } n-q+1 \text{ eigenvalues of } i\partial\overline{\partial}\varphi \text{ with respect to } ds^2_m \\ \text{(denoted by } \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n) \text{ are less than } -(m+1)q-1 \\ \text{and all the others are less than } 1. \end{array}$$

Let us consider the functions $\chi_{\varepsilon,m}(t) = \dfrac{1}{m}\chi_\varepsilon(t)$ where $\chi_\varepsilon(t)$ are defined as above. We set

$$(4.4)_m \qquad ds^2_{\varepsilon,m} = ds^2 + \chi_{\varepsilon,m}(\varphi)\partial\varphi \otimes \overline{\partial}\varphi, \;\; h_{\varepsilon,m} = h \exp\left(-2\int_{\inf\varphi}^{\varphi} \chi_{\varepsilon,m}(t)dt\right)$$

(i) If we denote by $\gamma_1^{\varepsilon,m} \leq \gamma_2^{\varepsilon,m} \leq \ldots \leq \gamma_n^{\varepsilon,m}$ the eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to $ds^2_{\varepsilon,m}$ then by $(*)_m$ we have that $\gamma_1^{\varepsilon,m} \leq \gamma_2^{\varepsilon,m} \leq \ldots \leq \gamma_{n-q}^{\varepsilon,m} < -(m+1)q-1$. If $\Gamma_1^{\varepsilon,m} \leq \Gamma_2^{\varepsilon,m} \leq \ldots \leq \Gamma_n^{\varepsilon,m}$ are the eigenvalues of $i\chi_{\varepsilon,m}(\varphi)\partial\overline{\partial}\varphi + i\chi'_{\varepsilon,m}(\varphi)\partial\varphi \wedge \overline{\partial}\varphi$ with respect to $ds^2_{\varepsilon,m}$ the minimum-maximum principle yield

$$\Gamma_j^{\varepsilon,m} \leq \chi_{\varepsilon,m}(\varphi)\gamma_j^{\varepsilon,m} + m\chi_{\varepsilon,m}(\varphi) = (\gamma_j^{\varepsilon,m} + m)\chi_{\varepsilon,m}(\varphi), \text{ on } X_c - X_e.$$

Thus, as in the preceding proof any sum of $(q+1)$ eigenvalues $\Gamma_j^{\varepsilon,m}$ is less than $-\chi_{\varepsilon,m}(\varphi)$ on $X_c - X_e$.

(ii) If α_j are the eigenvalues of $ic(E, h)$ with respect to ds^2_m and $\alpha_j^{\varepsilon,m}$ are the eigenvalues of the same curvature forms with respect to $ds^2_{\varepsilon,m}$ then $\alpha_j^{\varepsilon,m} \leq \max(\alpha_n, 0) \leq 0$ on $X_c - K$, for every j since (E, h) is semi-negative outside K. We have also that there exists a constant C_2 such that $| \prec [ic(F, h'), \Lambda_{\varepsilon,m}]u, u \succ_{\varepsilon,m} | \leq C_2|u|^2_{\varepsilon,m}$.

(iii) The pointwise norms of the torsion operators of $ds^2_{\varepsilon,m}$ with respect to $ds^2_{\varepsilon,m}$ are bounded by $C_3(m)(\chi_{\varepsilon,m}(\varphi))^{1/2}$, where $C_3(m)$ does not depend on ε.

We apply as above the results of (i)-(iii) combined with the Nakano inequality. Since E is semi-positive on $X_c - X_e$ we have that for $u \in C^{0,p}_{\text{comp}}(X_c - \overline{X}_e, E^k \otimes F)$

$$3(\|\overline{\partial}u\|^2_{\varepsilon,m} + \|\vartheta u\|^2_{\varepsilon,m}) \geq 2\int (2k\chi_{\varepsilon,m}(\varphi) - C_2)|u|^2_{\varepsilon,m}dV_{\varepsilon,m}$$
$$- 4C_3(m)(\chi_{\varepsilon,m}(\varphi)u, u)_{\varepsilon,m}.$$

for $p \le n-q-1$. Since $\chi_{\varepsilon,m}(t) > 4/m$, for $k \ge mC_2/4$ we obtain that

$$3(\|\overline{\partial}u\|^2_{\varepsilon,m} + \|\vartheta u\|^2_{\varepsilon,m}) \ge \int (2k\chi_{\varepsilon,m}(\varphi) - 4C_3(m))|u|^2_{\varepsilon,m} dV_{\varepsilon,m}.$$

Therefore, for $k \ge 4C_3(m)$ and $u \in C^{0,p}_{\text{comp}}(X_c - \overline{X}_e, E^k \otimes F)$, $p \le n-q-1$ we have:

$$(4.5)_m \qquad 3(\|\overline{\partial}u\|^2_{\varepsilon,m} + \|\vartheta u\|^2_{\varepsilon,m}) \ge k \int \chi_{\varepsilon,m}(\varphi)|u|^2_{\varepsilon,m} dV_{\varepsilon,m}$$

so have obtained a basic estimate of type (3.1) with $\psi = \frac{1}{3}\chi_{\varepsilon,m}(\varphi)$ and $M = \overline{X}_e$. For $k \ge 6C_4(m)$ where $C_4(m) = 4 \sup |d\rho|^2$ and the supremum is taken with respect to the metric ds^2_m, we get the basic estimate of type (3.2):

$$(4.6)_m \qquad \frac{1}{2k}(\|\overline{\partial}u\|^2_{\varepsilon,m} + \|\vartheta u\|^2_{\varepsilon,m}) + \frac{4}{3}\int_{\overline{X}_{d-\eta}} |u|^2_{\varepsilon,m} dV_{\varepsilon,m}) \ge \|u\|^2_{\varepsilon,m}$$

for $u \in C^{0,p}_{\text{comp}}(X_c, E^k \otimes F)$, $p \le n-q-1$, $k > k_0(m) = \max\{4C_3(m), 6C_4(m)\}$ (the exceptional set $\overline{X}_{d-\eta}$ and the integer $k_0(m)$ independent on ε). The estimate $(4.5)_m$ shows that we can apply Proposition 3.1 to get

$$(4.7)_m \qquad \begin{aligned} &\lim_{k\to\infty} k^{-n} \dim H^{0,p}_D(X_c, E^k \otimes F)_{\varepsilon,m} \\ &\le \frac{r}{n!} \int_{X_f(p,h_{\varepsilon,m})} (-1)^p \left(\frac{i}{2\pi} c(E, h_{\varepsilon,m})\right)^n \end{aligned}$$

$$(4.8)_m \qquad \begin{aligned} &\lim_{k\to\infty} k^{-n} \sum_{j=0}^{p} (-1)^{p-j} \dim H^{0,j}_D(X_c, E^k \otimes F)_{\varepsilon,m} \\ &\le \frac{r}{n!} \int_{X_f(\le p,h_{\varepsilon,m})} (-1)^p \left(\frac{i}{2\pi} c(E, h_{\varepsilon,m})\right)^n \end{aligned}$$

for $p \le n-q-1$ and $d-\eta < f < d$. The L^2-Dolbeault cohomology groups in the left hand-side are with respect to $ds^2_{\varepsilon,m}$ and $h_{\varepsilon,m}$. The estimate $(4.6)_m$ shows that, for fixed m, (X_d, X_c) is a pseudo-Runge pair in bidegree $(0,p)$, $p \le n-q-2$, with respect to $E^k \otimes F$, for sufficiently large k, when we endow E and X_c with the metrics $h_{\varepsilon,m}$ and $ds^2_{\varepsilon,m}$. The metrics $ds^2_{\varepsilon,m}$ and $h_{\varepsilon,m}$ converge together with their derivatives on every compact set of X_d to the metrics $ds^2_{0,m} = ds^2_m + \frac{1}{m}\frac{\partial\varphi \otimes \overline{\partial}\varphi}{(d-\varphi)^2} + \frac{4}{m}\partial\varphi \otimes \overline{\partial}\varphi$ and

$h_m = h \exp\left(-\frac{2}{m(d-\varphi)} + \frac{2}{m(d-\inf\varphi)} - \frac{8}{m}(\varphi - \inf\varphi)\right)$. By the same arguments as in the proof of Theorem 4.2 we infer that:

$$\lim_{k\to\infty} k^{-n} \dim H^p(X_d, E^k \otimes F) \leq \frac{r}{n!} \int\limits_{X_f(p,h_m)} (-1)^p \left(\frac{i}{2\pi} c(E, h_m)\right)^n$$

$$(4.9)_m \qquad \lim_{k\to\infty} k^{-n} \sum_{j=0}^{p} (-1)^{p-j} \dim H^j(X_d, E^k \otimes F) \leq r\frac{k^n}{n!} \int\limits_{X_f(\leq p,h_m)} (-1)^p \left(\frac{i}{2\pi} c(E, h_m)\right)^n$$

for $p \leq n-q-2$. The curvature integrals in the right-hand side are independent on the hermitian metric on the base manifold; they depend only on h_m. When $m \to \infty$ these metric converge uniformly to h on $X_f \subset\subset X_d$. Therefore, by letting $m \to \infty$ in the above inequalities, we obtain the desired conclusion, since, by hypothesis, $X_f(p, h) = X(p, h)$ for $p < n$. $\square$

Corollary 4.3 gives estimates for the Monge-Ampère operator along the lines of Siu [21].

COROLLARY 4.4. *Let X be a q-concave manifold of dimension $n \geq 3$ and let ψ a smooth real function which is plurisubharmonic outside a compact set $K \subset X$. Denote by $X(p)$ the set where the complex hessian of ψ, $i\partial\overline{\partial}\psi$, is non-degenerate and has exactly p negative eigenvalues. Then for any $p \geq q+2$ we have*

$$\int\limits_{X(p)} (i\partial\overline{\partial}\psi)^n \text{ has the same sign as } (-1)^p.$$

PROOF. We consider the line bundle $E = X \otimes \mathbb{C}$ the trivial bundle equipped with the metric $h = \exp(\psi)$. Then $ic(E, h) = -i\partial\overline{\partial}\psi$ so that E satisfies the conditions of Corollary 4.3. Since the tensor powers of E are equal to the trivial bundle we immediately obtain the desired conclusion dividing by k^n the relation (4.9) and letting $k \to \infty$. $\square$

PROOF OF THEOREM 1.1 (We use the same notations as in the preceding proof). Since we are in the hypothesis of Corollary 4.3 and $q \leq n-2$, i.e., $1 \leq n-q-1$, we can apply the strong Morse inequalities $(4.8)_m$ for $p = 1$ and we obtain

$$\dim H^0(X_c, E^k) \geq \dim H^{0,0}_D(X_c, E^k \otimes F)_{\varepsilon,m} \geq \frac{k^n}{n!} \int\limits_{X_f(\geq 1,h_{\varepsilon,m})} \left(\frac{i}{2\pi} c(E, h_{\varepsilon,m})\right)^n - o(k^n).$$

For fixed $\varepsilon > 0$, the metrics $h_{\varepsilon,m}$ converge uniformly with their derivatives on any compact subset of X_c, in particular on $\overline{X}_f$, to the metric h, as $m \to \infty$. Thus, by letting $m \to \infty$ we obtain

$$\dim H^0(X, E^k) = \dim H^0(X_c, E^k) \geq \frac{k^n}{n!} \int\limits_{X_f(\geq 1,h)} \left(\frac{i}{2\pi} c(E,h)\right)^n - o(k^n)$$

Since E is semi-negative outside the compact set $K \subset\subset X_f$, we have that $X(\leq 1,h) \subset X_f$ so that $X(\leq 1,h) = X_f(\leq 1,h)$ hence the curvature integral in the right-hand term is positive. Therefore $\dim H^0(X,E^k)/k^n$ is bounded bellow by a positive constant. On the other hand, we know that it is also bounded above by Theorem 4.2. □

Examples

EXAMPLE (a). Let X be a compact complex manifold which carries a semi-positive line bundle which is positive on an open dense set. The complement Y of a finite set F is a 1-concave manifold and we can change the metric on the line bundle in the neighbourhood of F such that the hypothesis of theorem 1.1 are fulfilled.

EXAMPLE (b). Let X be an analytic space of pure dimension n, which is compact and has only isolated singularities. We denote the regular part of X by X^*. Recall that a hermitian metric of X is by definition a smooth hermitian metric ds^2 on X^* such that for each $x \in X$ one can find a neighbourhood U of x, a holomorphic embedding $\iota: U \hookrightarrow \mathbb{C}^N$ for some N and a smooth hermitian metric $d\sigma^2$ on $\mathbb{C}^N$ so that $ds^2 = \iota^* d\sigma^2$ on $U \cap X^*$. In the sequel let ds^2 be a fixed but arbitrary hermitian metric of X. Assume that p is an isolated singular point of X. We have a holomorphic embedding of the germ $(X,p) \hookrightarrow (\mathbb{C}^N, 0)$ and we fix holomorphic coordinates $z = (z_1, z_2, \dots, z_N)$ of $\mathbb{C}^N$ and the euclidean norm $\|z\|$ of z. We denote by $B_a^*(p) = \{z \in \mathbb{C}^N : 0 < \|z\|^2 < a\} \cap X^*$. For $a < e^{-1}$ let us consider the function $F(z) = -\ln(-\ln(\|z\|^2))$ defined on B_a^*. If p_1, $p_2, \dots, p_m$ are the singular ponts of X, let F_1, $F_2, \dots, F_m$ be the corresponding functions defined in the neighbourhoods of the singular points, which we may suppose mutually disjoint. By patching the functions F_1, $F_2, \dots, F_m$ and the function $F = 1$ on X^* by a smooth partition of unity on X^* we obtain a smooth function $\psi: X^* \to (-\infty, 0]$ such that $\psi = F_j$ on $B_a^*(p_j)$ for $j = 1, 2, \dots, m$ and sufficiently small a and $\psi \geq \ln(-\ln(a))$ on $X^* - \bigcup B_a^*(p_j)$. Then, for large A,

$$ds_0^2 = i\partial\overline{\partial}(\psi) + A ds^2 \tag{4.10}$$

is a hermitian metric on X^* which is quasi-isometric to $i\partial\overline{\partial}(\psi)$ on $B_a^*(p_j)$ for each singular point p_j. Metrics like (4.10) are called *Grauert metrics* (cf. H. Grauert [11], T. Ohsawa [16]).

CLAIM. *The function $\psi: X^* \to (-\infty, 0]$ is proper and $i\partial\overline{\partial}(\psi)$ is positive definite outside a compact set of X^*. There exists a neighbourhood W of the singular points of X such that*

(i) *The eigenvalues of ds_0^2 with respect to ds^2 in W converge to $+\infty$ as one approaches the singular points.*

(ii) *The pointwise norm of $i\partial\psi\overline{\partial}\psi$ with respect to ds_0^2 is bounded on X^*.*

PROOF. We have the relation:

$$i\partial\overline{\partial}(\psi) = \frac{-i\partial\overline{\partial}\ln(\|z\|^2)}{\ln(\|z\|^2)} + \frac{i\partial\ln(\|z\|^2)\overline{\partial}\ln(\|z\|^2)}{(\ln(\|z\|^2))^2} \geq i\partial\psi\overline{\partial}\psi$$

in the neighbourhood of the singular points, which shows that (ii) is satisfied. Also

$$i\partial\overline{\partial}(\psi) = \frac{\sum\limits_{j<k}(z_j dz_k - z_k dz_j)(\overline{z}_j d\overline{z}_k - \overline{z}_k d\overline{z}_j)}{(-\ln(\|z\|^2))^2\|z\|^4} + \frac{\left(\sum\limits_j \overline{z}_j dz_j\right)\left(\sum\limits_j z_j d\overline{z}_j\right)}{(\ln(\|z\|^2))^2\|z\|^4}.$$

This shows that the eigenvalues of $i\partial\overline{\partial}(\psi)$ with respect to the metric ds^2 (which is the pull-back of the euclidian metric) go to $+\infty$ as one approaches the singular points, so that (i) is satisfied, too. □

REMARK. Since $\psi(x) \to -\infty$ as $x \to p$ where p is a singular point, the function $\phi = -\psi$ is an exhaustion function $\phi: X^* \to [0, +\infty)$ whose complex hessian is negative definite outside a compact set and which makes X^* a 1-concave manifold. In the sequel we denote $X_c^* = \{x \in X^*: \phi(x) < c\}$.

PROPOSITION 4.5. *Assume that X^* has finite volume with respect to the Grauert metric ds_0^2 and there exists a holomorphic line bundle $E \to X$ (i.e., E extends to the singular locus) which satisfies $\int_{X^*(\leq 1,h)} (ic(E,h))^n > 0$ for some smooth hermitian metric h on E restricted to X^* and such that $ic(E,h)$ is bounded with respect to some smooth hermitian metric ds^2 on X (in particular, if h extends smoothly in the neighbourhood of the singular points). Then, there exists $c > 0$ and a hermitian metric h' on $E|X_c^*$ such that $ic(E,h')$ is negative definite outside a compact set of X_c^* and*

$$\int_{X_c^*(\leq 1,h')} (ic(E,h'))^n > 0.$$

PROOF. Let us denote by $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ the eigenvalues of $ic(E,h)$ with respect to the Grauert metric ds_0^2. We denote by K a compact set such that in the complement of K the form $i\partial\overline{\partial}(\phi)$ is negative definite and by $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$ the eigenvalues of $i\partial\overline{\partial}(\phi)$ with respect to ds_0^2. The minimum-maximum principle and the fact that the eigenvalues of ds^2 with respect to $ds_0^2 = i\partial\overline{\partial}(\psi) + Ads^2$ converge to zero when $x \to \mathrm{Sing}(X)$ (by (i) of the Claim) show that $\alpha_j(x) \to 0$ when $x \to \mathrm{Sing}(X)$. Let $\varepsilon > 0$ and choose g_ε a real number such that $\alpha_j < \varepsilon$ on $X^* - X^*_{g_\varepsilon}$. Let us denote by C_1 be an upper bound for the eigenvalues of of the eigenvalues of $i\partial\phi\overline{\partial}\phi$ with respect to ds_0^2 (since $\partial\psi\overline{\partial}\psi = \partial\phi\overline{\partial}\phi$ we have $C_1 < \infty$ by (ii) of the Claim). Since ds_0^2 is quasi-isometric to $i\partial\overline{\partial}\psi$ in the neighbourhood of Sing(X) there exists a constant C_2 such that $-C_2$ is an upper bound for the eigenvalues of $i\partial\overline{\partial}\phi$ with respect to ds_0^2. Let $e_\varepsilon \in \mathbb{R}$ such that $e_\varepsilon > g_\varepsilon + 3\varepsilon$ and the smooth function $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that:

$$\chi_\varepsilon(t) = \begin{cases} 3\varepsilon^2/2C_2, & t \leq e_\varepsilon - 2\varepsilon \\ \varepsilon t/C_2, & e_\varepsilon - \varepsilon < t \leq e_\varepsilon \\ \varepsilon(t - e_\varepsilon)/C_2, & e_\varepsilon < t. \end{cases}$$

Let us remark that $|\chi_\varepsilon'| \leq \varepsilon/C_2$, $|\chi_\varepsilon''| \leq 1/C_2$ and $\chi_\varepsilon'' = 0$ for $e_\varepsilon - \varepsilon \leq t < e_\varepsilon + 1$.

We consider the metric $h_\varepsilon = h\exp(-\chi_\varepsilon(\phi))$ with the curvature form $ic(E, h_\varepsilon) = ic(E, h) + i\chi_\varepsilon'(\phi)\partial\overline{\partial}\phi + i\chi_\varepsilon''(\phi)\partial\phi\overline{\partial}\phi$. We denote by $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_n$ the eigenvalues of $ic(E, h_\varepsilon)$ with respect to ds_0^2. By the above relations we have $\delta_n(x) \leq \alpha_n(x) - C_2\chi_\varepsilon'(\phi) + C_1\chi_\varepsilon''(\phi) < \varepsilon - C_2\chi_\varepsilon'(\phi) \leq 0$ on $X^*_{c_\varepsilon} - X^*_{e_\varepsilon}$ where $c_\varepsilon = e_\varepsilon + 1/2$. Let us evaluate the integral

$$\int_{X^*_{c_\varepsilon}(\leq 1, h_\varepsilon)} (ic(E, h_\varepsilon))^n$$

$$= \int_{X^*_{e_\varepsilon - 2\varepsilon}(\leq 1, h_\varepsilon)} (ic(E, h_\varepsilon))^n + \int_{(X^*_{c_\varepsilon} - X^*_{e_\varepsilon - 2\varepsilon})(\leq 1, h_\varepsilon)} (ic(E, h_\varepsilon))^n.$$

On $X^*_{e_\varepsilon - 2\varepsilon}$ we have that $ic(E, h_\varepsilon) = ic(E, h)$ so we have to evaluate only the second integral in the right-hand side. In fact

$$\int_{(X^*_{c_\varepsilon} - X^*_{e_\varepsilon - 2\varepsilon})(\leq 1, h_\varepsilon)} (ic(E, h_\varepsilon))^n \geq \int_{(X^*_{c_\varepsilon} - X^*_{e_\varepsilon - 2\varepsilon})(1, h_\varepsilon)} (ic(E, h_\varepsilon))^n$$

the last integral being non-positive. We have $(X^*_{c_\varepsilon} - X^*_{e_\varepsilon - 2\varepsilon})(1, h_\varepsilon) \subset X^*_{e_\varepsilon} - X^*_{e_\varepsilon - 2\varepsilon}$ since $ic(E, h_\varepsilon) < 0$ on $X^*_{c_\varepsilon} - X^*_{e_\varepsilon}$. On the set of integration $\delta_1 < 0 < \delta_2 \leq \ldots \leq \delta_n$ so the minimax principle shows that $|\delta_1(x)| \leq \chi_\varepsilon'(\phi) + C_1\chi_\varepsilon''(\phi) \leq \dfrac{\varepsilon + C_1}{C_2}$. Also

$$\delta_n(x) \leq \alpha_n(x) - C_2\chi_\varepsilon'(\phi) + C_1\chi_\varepsilon''(\phi) \leq \alpha_n(x) + C_1\chi_\varepsilon''(\phi) \leq \varepsilon + \frac{C_1}{C_2}.$$

Then, if dV_0 denotes the volume form of the metric ds_0^2, we have that

$$\left|\int_{(X^*_{c_\varepsilon}-X^*_{e_\varepsilon-2\varepsilon})(1,h_\varepsilon)} (ic(E,h_\varepsilon))^n\right| = \left|\int_{(X^*_{c_\varepsilon}-X^*_{e_\varepsilon-2\varepsilon})(1,h_\varepsilon)} \delta_1\delta_2\dots\delta_n dV_0\right|$$

$$\leq \int_{(X^*_{c_\varepsilon}-X^*_{e_\varepsilon-2\varepsilon})(1,h_\varepsilon)} |\delta_1|\,|\delta_n|^{n-1}dV_0 \leq$$

$$\leq \mathrm{Vol}(X^*_{e_\varepsilon}-X^*_{e_\varepsilon-2\varepsilon})\left(\frac{\varepsilon+C_1}{C_2}\right)\left(\varepsilon+\frac{C_1}{C_2}\right)^{n-1}$$

Since X^* has finite volume with respect to dV_0 we have that the last expression goes to zero when $\varepsilon \to 0$ (and hence g_ε, e_ε, $c_\varepsilon \to +\infty$). Thus

$$\int_{X^*_{c_\varepsilon}(\leq 1,h_\varepsilon)} (ic(E,h_\varepsilon))^n \geq \int_{X^*_{e_\varepsilon-2\varepsilon}(\leq 1,h)} (ic(E,h))^n + \int_{(X^*_{c_\varepsilon}-X^*_{e_\varepsilon-2\varepsilon})(1,h_\varepsilon)} (ic(E,h_\varepsilon))^n$$

$$\to \int_{X^*(\leq 1,h)} (ic(E))^n > 0$$

for $\varepsilon \to 0$. Therefore, there exists a sufficiently small $\varepsilon > 0$, real numbers $g_\varepsilon < e_\varepsilon < c_\varepsilon$ and a hermitian metric h_ε on $E|X^*_{e_\varepsilon}$ such that (E,h_ε) is negative on $X^*_{c_\varepsilon} - X^*_{e_\varepsilon}$ and $\displaystyle\int_{X^*_{c_\varepsilon}(\leq 1,h_\varepsilon)} (ic(E,h_\varepsilon))^n > 0$. □

Let us notice that there are 1-concave Moishezon manifolds which are not projective. It suffices to consider the complement Y of a finite set F in a compact non-projective Moishezon manifold X. The resulting manifold is 1-concave, has maximal number of independent meromorphic functions and is non-projective since its minimal compactification is X. The manifold X can be chosen such that it carries a semi-positive line bundle which is positive on an open dense set so that Y has the same property. By the above example a) we know that Y and the given line bundle satisfies the hypothesis of Theorem 1.1.

A natural problem is to extend the Morse inequalities to cohomology groups associated to $\overline{\partial}_b$. This has been carried out by Getseler [10] for the operator $\overline{\partial}_b$ on compact strongly pseudoconvex integrable Caychy-Riemann manifolds. We would be interested in using the Morse inequalities for $\overline{\partial}_b$ to prove embedding theorems for Cauchy-Riemann manifolds. One should start by proving that the the high tensor powers of a "positive" line bundle E over a Cauchy-Riemann manifold have many sections. This could be done by means of the strong Morse inequalities which give a lower bound for dim $H^0_b(Y,E^k)$ as k runs to infinity (where $H^0_b(Y,E^k)$ is the kernel of $\overline{\partial}_b$ acting on smooth functions). Of course, the last cohomology groups is infinite dimensional for a strongly pseudoconvex Cauchy-Riemann manifold. On the other hand, the problem of embedding strongly pseudoconvex integrable Cauchy-Riemann manifolds is settled. A classical theorem of L. Boutet de Monvel [7] (see also H. Rossi [8]) asserts that such a manifold of dimension greater than five is the boundary of a compact analytic space Z with boundary in some numeric space which is smooth in the neighbourhood of bZ and the Cauchy-Riemann structure induced by the complex structure of Z on bZ coincides with the initial one. That is why it might be interesting to fill in the details of this plan in the case of Cauchy-Riemann manifolds for which the Levi form has some negative eigenvalues. In the sequel we wish to motivate the study of this problem. First, we prove a theorem for the "embedded case". Let X be a q-concave manifold with exhaustion function φ and exceptional set K. Consider the hypersurfaces $Y_d = \{x \in X : \varphi(x) = d\}$ for non-critical points $d \in \mathbb{R}$ and such that Y_d does not intersect K. Assume that E is a holomorphic line bundle on X satisfying the hypothesis of Theorem 1.1.

THEOREM 4.6. *We have the following estimate for the $\overline{\partial}_b$-cohomology of Y_d provided that the Levi form of φ has at least three negative eigenvalues outside K:*

$$\dim H_b^0(Y_d, F^k) \geq Ck^n \quad \textit{as } k \to \infty$$

for sufficiently high tensor powers of E.

PROOF. Denote by $X_d = \{x \in X : \varphi(x) < d\}$ and by $H^0(\overline{X}_d, E^k)$ the holomorphic sections of E^k over X_d which are smooth up to the boundary. We have the natural restriction map $H^0(\overline{X}_d, E^k) \to H_b^0(Y_d, E^k)$. This map is injective. Indeed, let $\sigma \in H^0(\overline{X}_d, E^k)$ such that $\sigma = 0$ on Y_d. If E is trivial then $\sigma = 0$ thanks to the maximum principle. In the general case we considre a smooth curve C which intersects X_d along a submanifold S. Then S is a non-compact, 1-dimensional complex manifold, thus a Stein manifold, thanks to Benkhe-Thullen's theorem. Therefore, E^k restricted to S is trivial and by the preciding case we get that σ vanishes on S. The statement results now from the fact that for any point of X_d there exists such a curve which passes through that point. Let us consider $c \in \mathbb{R}$ such that $X_d \subset X_c$.

The restriction map $H^0(X, O(E^k)) \to H^0(\overline{X}_d, E^k)$ is obviously injective. Thus we have to estimate dim $H^0(X, O(E^k))$ when $k \to \infty$. We apply, for this purpose, Theorem 1.1. This is possible since there exists at least three negative eigenvalues. We get that dim $H^0(X, O(E^k)) \approx k^n$ as $k \to \infty$ which proves the theorem. □

5. - Embedding of 1-concave manifolds

In this section we prove Theorem 1.2. Let us begin with some remarks about Moishezon manifolds. Let X a complex manifold of dimension n such that $H^0(X, O(E^k))$ is finite dimensional for any holomorphic vector bundle on X (for example X is compact or 1-concave). Let L be a holomorphic line bundle on X. Set $V_k = H^0(X, O(L^k))$, $h_k = \dim V_k$ and let $\mathcal{A}(X, L) = \bigcup_{k\geq 0} V_k$ be the graded ring associated to L.

DEFINITION. *We say that $\mathcal{A}(X, L)$ gives local coordinates at a point $x \in X$ if there exists $k > 0$ and sections $s_0, s_1, \dots, s_n \in V_k$ such that the meromorphic map $(s_1/s_0, \dots, s_n/s_0)$ gives local coordinates at x (i.e., $d(s_1/s_0)\wedge\dots\wedge d(s_n/s_0) \neq 0$ at x).*

We will also say that the sections $s_0, s_1, \dots, s_n$ give local coordinates at a point x. Remark that if X is connected then $s_0, s_1, \dots, s_n$ give local coordinates on an open dense subset. Consider the canonical holomorphic maps $\Phi_k: X - Z_k \to \mathbb{P}(V_k^*)$ where $\Phi_k(x)$ equals the hyperplane of sections of V_k which vanish at x and Z_k is the divizor of zeros of V_k. It is clear that $\mathcal{A}(X, L)$ *gives local coordinates at a point x if and only if* $\operatorname{rank}\Phi_k = \dim X$ *for some* $k > 0$.

DEFINITION. *The Kodaira dimension of L is the integer*

$$K(L) = \max\{\rho_k = \operatorname{rank}\Phi_k : k > 0, h_k \neq 0\}$$

($K(L) = -\infty$ if $h_k = 0$ for all $k > 0$).

We may reformulate the above assertion in terms of the Kodaira dimension: *$\mathcal{A}(X, L)$ gives local coordinates at a point x if and only if the Kodaira dimension equals the dimension of X.* Let $\mathcal{K}(X)$ the field of meromorphic functions on X. If X is compact or 1-concave, $\mathcal{K}(X)$ is isomorphic to a simple algebraic extension of a field of rational functions with d variables where $d \leq n = \dim X$ (Siegel [19] for the compact case and Andreotti [1] for the 1-concave one). The transcendence degree of $\mathcal{K}(X)$ over $\mathbb{C}$ is called the *algebraic dimension* of X and is denoted $a(X)$.

DEFINITION. *The compact or 1-concave manifolds for which* $a(X) = \dim X$ *are said to be Moishezon manifolds.*

By a well known theorem of Moishezon [15] the compact Moishezon manifolds are bimeromorphic to projective manifolds. Indeed, there exists a proper modification $\hat{X} \to X$ of X such that $\hat{X}$ is projective algebraic. A simple argument shows that $K(L) \leq a(X)$ for any line bundle L, thus, if $K(L) = \dim X$ then X is a Moishezon manifold. The reverse is also true. Indeed, if $a(X) = \dim X$ there exist $n = \dim X$ algebraically independent meromorphic functions; we can find a line bundle L such that these functions have the form $s_1/s_0, \dots, s_n/s_0$ where $s_0, s_1, \dots, s_n$ are sections of L. The algebraic independence implies the analytic independence [1], so that $d(s_1/s_0) \wedge \dots \wedge d(s_n/s_0) \neq 0$ on the set where the left-hand side is defined and hence $K(L) = \dim X$. We have thus proved the following.

PROPOSITION 5.1. *If X is a compact or* 1*-concave manifold the following assertions are equivalent:*

(i) *X is Moishezon.*

(ii) *There exists a line bundle L such that $K(L) = \dim X$.*

(iii) *There exists a line bundle L such that $\mathcal{A}(X, L)$ gives local coordinates at a point $x \in X$.*

The main step towards the proof of Theorem 1.2 is the subsequent theorem.

THEOREM 5.2. *Let X be a connected* 1*-concave manifold of dimension $n \geq 3$ satisfying one of the equivalent conditions of Proposition* 5.1. *Then there exists a compactification $\overline{X}$ of X which is Moishezon.*

Let us restate the definition of 1-concave spaces as folows:

DEFINITION. *A complex space Y is called* 1*-concave if there exists a smooth function ϕ from Y to $(a, +\infty)$ where $a \in \overline{\mathbb{R}}$, such that $\{\phi \geq c\}$ is compact for $c \in (a, +\infty)$ and for some $a' \in (a, +\infty)\phi$ is strongly plurisubharmonic on $\{\phi < a'\}$; ϕ is called an exhaustion function.*

We will use also the notion of $(1, 1)$-convex-concave complex spaces.

DEFINITION (a). *A complex space Y is called $(1, 1)$-convex-concave if there exists a proper smooth function ϕ from Y to (a, b), where a, $b \in \overline{\mathbb{R}}$ such that ϕ is strongly plurisubharmonic on Y. For $a \leq c < d \leq b$ put $X_c^d = \{c < \phi < d\}$ and $K_d = \{\phi \leq d\}$.*

(b) *A complex space Z is called a Stein completion of a $(1, 1)$-convex-concave space Y if Y is an open set of Z, Z is a Stein space and $K_d \cup (Z - Y)$ is compact for any $d \in (a, b)$.*

We will show how the proof of the embedding theorem of Andreotti and Siu (see the Introduction) applies to the proof of Theorem 5.2. We need a few lemmas which we shall state without proof. For all the details the reader is reffered to Andreotti-Siu's paper [3]. The first one is obtained using well known results of Andreotti-Grauert.

LEMMA 5.3. *Let Z be a Stein completion of the $(1, 1)$-convex-concave space Y and let $\mathcal{F}$ be a coherent analytic sheaf on Z with $\operatorname{prof} \mathcal{F} \geq 2$. Then the restriction map $\Gamma(Z, \mathcal{F}) \to \Gamma(Y, \mathcal{F})$ is bijective.*

We deal now with the existence of Stein completions. We are concerned with Stein completions satisfying certain normality canditions. Let Y be a complex space. We say that Y is *p-normal* at x if, given any neighbourhood U of x an analytic subset A of U of dimension $\leq p$ and a holomorphic function f on $U - A$ we can find a neighbourhood V of x and a holomorphic function g on V such that $f = g$ on $V - A$. If Y is p-normal at every point we say that Y is p-normal. If Y is an n-dimensional irreducible normal space, then Y is p-normal for any $p \leq n - 2$. In particur, a complex manifold of dimension

$n \geq 3$ is 0-normal and 1-normal. If a $(1,1)$-convex-concave space Y admits a 0-normal Stein completion, then the 0-normal Stein completion is unique up to an isomorphism which is the identity on Y. As for the existence we have the following.

LEMMA 5.4. *Let Y be a $(1,1)$-convex-concave space. Suppose that for some $a' \in (a,b)$, the set $\{\phi < a'\}$ is 1-normal. Then Y admits a 0-normal Stein completion.*

We shall use the Stein completions to construct compactifications of 1-concave manifolds. Given a complex space X, an isomorphism $i: X \to W$ onto an open set of a compact complex space $\overline{X}$ will be called a *compactification* of X.

LEMMA 5.5. *Let X be a 1-concave 0-normal complex space with exhaustion function $\phi: X \to (a, +\infty)$. Assume that for some $a' \in (a, +\infty)$ the set $\{\phi < a'\}$ is 1-normal. Then X admits a 0-normal compactification.*

PROOF. We may assume that ϕ is strongly plurisubharmonic on $Y = \{\phi < a'\}$. Then Y is a $(1,1)$-concave-concave 1-normal space. By the preceding Lemma Y admits a 0-normal Stein completion Z. By pasting together X and Z along Y we obtain a compact complex space $\overline{X}$ which is a 0-normal compactification of X with respect to the natural inclusion map. □

Let us remark that the compactification obtained in this way has the property that $\overline{X} - i(X)$ contains no compact positive-dimensional complex spaces. Such compactifications are called *minimal compactifications* for they satisfy the following condition: for any other compactification $j: X \to X'$ we can find a morphism $\alpha: X' \to \overline{X}$ such that $i = \alpha(j)$.

We need also a result of extension of coherent analytic sheaves. For this purpose, let us recall the definition of absolute gap sheaves introduced in [20]. Let Y be an unreduced complex space and let $\mathcal{F}$ be a coherent analytic sheaf on Y. For any open set $U \subset Y$ we can consider the group $\mathcal{F}^{[m]}(U) = \varinjlim_{A} \Gamma(U - A, \mathcal{F})$ where A runs over all analytic subsets of U of dimension $\leq m$. If $V \subset U$ is open we have a natural restriction map $r_U^V: \mathcal{F}^{[m]}(U) \to \mathcal{F}^{[m]}(V)$ which makes $(\mathcal{F}^{[m]}(U), r_U^V)$ a presheaf. The associated sheaf, denoted $\mathcal{F}^{[m]}$ is called the m-absolute gap sheaf of $\mathcal{F}$. If Y is a $(1,1)$-convex-concave complex space and Z is a Stein completion of Y then any coherent analytic sheaf $\mathcal{F}$ on Y which satisfies $\mathcal{F}^{[1]} = \mathcal{F}$ can be extended as a coherent analytic sheaf on Z. Using this we can immediately see that the following Lemma holds.

LEMMA 5.6. *Let X be a 1-concave complex space and $i: X \to \overline{X}$ a compactification of X such that $\overline{X} - i(X)$ contains no positive-dimensional compact subspace. Let $\mathcal{F}$ be any coherent analytic sheaf on X such that $\mathcal{F}^{[1]} = \mathcal{F}$. Then there exists a coherent analytic sheaf $\overline{\mathcal{F}}$ on $\overline{X}$ such that $i^*\overline{\mathcal{F}} = \mathcal{F}$.*

PROOF OF THEOREM 5.2. Let $\mathcal{L}$ be the sheaf associated to L. Lemma 5.5 tells us that the manifold X admits a minimal 0-normal compactification (X_1, O_1). Since the normalisation of X_1 is again a minimal 0-normal compactification of X, the space X_1 must be normal. It is easily seen that $n = \dim X \geq 3$ implies that $\mathcal{L}^{[1]} = \mathcal{L}$. By Lemma 5.6 the sheaf $\mathcal{L}$ can be extended on X_1 by a coherent analytic sheaf $\mathcal{G}$. Factoring out the torsion of $\mathcal{G}$ we may assume that $\mathcal{G}$ is torsion-free. Also we may replace $\mathcal{G}$ by $\mathcal{G}^{[1]}$ which is again coherent and consequently we may assume that $\operatorname{prof} \mathcal{G} \geq 2$ (Propositions 2.4 and 2.5 of [3]). Let $a' \in (a, +\infty)$ such that the exhaustion function $\phi: X \to (a, +\infty)$ is strongly plurisubharmonic on $\{\phi < a'\}$. Denote by $Y = \{\phi < a'\}$ and by $Y_1 = Y \cup (X_1 - X)$.

By Lemma 5.3 the restriction map $\Gamma(Y_1, \mathcal{G}) \to \Gamma(Y, \mathcal{G})$ is bijective.

Consider the sections $s_0, s_1, \dots, s_n$ of L over X, which give local coordinates on an open dense subset of X. Each s_i extends uniquely to a section $s_i' \in \Gamma(Y_1, \mathcal{G})$.

The singular set S of X_1 is contained in $X_1 - X$. By the construction of X_1, S must be a finite set $S = \{x_1, x_2, \dots, x_m\}$. We use here a result of Hironaka and Rossi (Lemma 5 and Corollary 2 to Lemma 5 of [13]). We can find an open neighbourhood U_i of x_i and a coherent ideal-sheaf J_i on such that $\{x_i\}$ is the zero set of J_i and the complex space obtained from U_i by the monoidal transformation with center at $(\{x_i\}, O_1/J_i)$ is non-singular. Let J be the ideal-sheaf on X_1 which agrees with O_1 on $X_1 - S$ and with J_i on U_i and let $\Phi: (X_2, O_2) \to (X_1, O_1)$ be the monoidal transformation with center $(S, O_1/J)$. Factoring out the torsion subsheaf of $\Phi^{-1}(\mathcal{G})$ on X_2 we obtain a the sheaf $\mathcal{G}_1$. Set $\mathcal{G}_2 = \mathcal{G}_1^{[n-2]}$. Then $\mathcal{G}_2$ is a locally free sheaf of rank 1 on X_2. Let s_i'' be the unique section of $\Gamma(X_2, \Phi^{-1}(\mathcal{G}))$ by the natural map $\Gamma(X_1, (\mathcal{G})) \to \Gamma(X_2, \Phi^{-1}(\mathcal{G}))$. We have also a sheaf homomorphism $\Phi^{-1}(\mathcal{G}) \to \mathcal{G}_2$. Let s_i''' be the image of s_i'' by this homomorphism. The sections s_0''', $s_1''', \dots, s_n'''$ are global sections of the line bundle associated to $\mathcal{G}_2$ and they give local coordinates at least at a point of $\Phi^{-1}(X)$. Thus X_2 is a compact Moishezon manifold and X is identified to the open subset $\Phi^{-1}(X)$. □

PROPOSITION 5.7. *Let X be a 1-concave manifold and let ρ_k be the maximal rank of the canonical map Φ_k, $k > 0$, associated to a holomorphic line bundle L. Then*

$$\dim H^0(X, O(L^k)) \leq C k^{\rho_k}$$

PROOF. We can choose $Y \subset X$ such that $\overline{Y} \subset\subset X$ and for any point x of $\overline{Y}$ there exists a polydisc P_x with coordinates p_i, $i = 1, 2, \dots, n$ of center x and radius r_x such that:

(i) L restricted to P_x is trivial.

(ii) The Silov boundary of P_x, $S(P_x) = \{y \in U_x: |p_i(y) - p_i(x)| = r_x\} \subset Y$ (U_x is the domain where the coordinates p_i are defined).

Indeed, we may take Y to be a sublevel set and the concavity ensures condition (ii). Let P_x' be a polydisc homothetic to P_x, with the same center and of radius $r'_x = r_x e^{-1}$. Let $a_1, a_2, \dots, a_m \in \overline{Y}$ such that $\overline{Y} \subset \bigcup_{1 \le j \le m} P'_{a_j}$ and Φ_k has rank ρ_k at each a_j. Since Φ_k is a subimmersion at a_j there exists a submanifold M_j in the neighbourhood of a_j which is transversal in a_j to the fibre $\Phi_k^{-1}(\Phi_k(a_j))$ and $\dim M_j = \rho_k$. Assume that the line bundle L is given by the trasition functions $g_{ij}: \overline{P}'_{a_i} \cap \overline{P}'_{a_j} \to \mathbb{C}^*$. Set

$$\|L\| = \sup\{|g_{ij}(x)| : x \in \overline{P}'_{a_i} \cap \overline{P}'_{a_j} \text{ for all } i, j\} = e^{\mu}.$$

Since $g_{ij} = g_{ji}^{-1}$, $\mu > 0$. Consider a section $s \in \Gamma(X, O(L^k))$ which vanishes up to order $h = k([\mu] + 1)$ at each a_j along M_j ($[\mu]$ is the integral part of μ). But s vanishes on the fibre which passes form a_j, hence s vanishes up to order h at a_j on X. Assume that s is given on P_{a_i} by $s_i: P_{a_i} \to \mathbb{C}$. Set $\|s\| = \sup\{|s_j(x)| : x \in \overline{P}'_{a_i} \text{ for all } i\}$.

There exists $q \in \{1, 2, \dots, m\}$ such that for some $w \in S(P_{a_q})$, $|s_q(w)| = \|s\|$. We can find $j \neq q$ such that $w \in P_{a_j}$. Hence $s_q(w) = g_{qj}(w)s_j(w)$ so that

$$\|s\| = |s_q(w)| = |g_{qj}(w)s_j(w)| \le \|L^k\| \, |s_j(w)|$$

By applying the Schwarz inequality to s_j in P_{a_j} we get $|s_j(w)| \le \|s\| \, |w|^h r_j^{-h}$ where $|w| = \sup |p_j(w) - p_j(a_j)| \le r_j e^{-1}$. Consequently, $\|s\| \le \|s\| \, \|L^k\| e^{-h}$. If s is not identically zero this leads to a contradiction, by our choice of h.

Consider the map $H^0(X, O(L^k)) \to \prod_{1 \le j \le m} O_{M_j, a_j} / \mathcal{M}_{M_j, a_j}$ where $\mathcal{M}_{M_j, a_j}$ is the maximal ideal of the local ring O_{M_j, a_j}, which sends every section in his Taylor developpment of order h at a_j along M_j. By the preceding argument this map is injective. Since the dimension of the target space satisfies the desired estimate we are done. □

We can end now the proof of Theorem 1.2. Theorem 5.2 shows that it suffices to show that there exists $k > 0$ such that $\rho_k = n$. By Theorem 1.1 and Lemma 5.7, $C'k^n \le \dim H^0(X, O(L^k)) \le Ck^{\rho_k}$ for sufficiently large k, which proves our contention.

REMARK. Example b) from the previous section shows that if we have a compact analytic space X with isolated singularities of finite volume with respect to some natural Grauert metric on X, carrying a holomorphic line bundle E endowed with a hermitian metric whose curvature form is bounded with respect to some smooth hermitian metric on the base and satisfying the integral condition $\int_{X^*(\le 1)} (ic(E))^n > 0$, then the manifold X^* is isomorphic with some open set in a compact Moishezon manifold. We can ask ourselves if the result could have been obtained by the resolution of the singularities of X. Let $\hat{X} \to X$ be a desingularisation of X. By taking the pull-back of the curvature form $ic(E)$ on X we obtain a current on $\hat{X}$ which satisfies the integral condition above (the integral is taken on the set of points where the current is smooth). We can apply now the following criterion of L. Bonavero [6]:

PROPOSITION 5.8. *A sufficient and necessary condition for a manifold X to be Moishezon is the existence of a current T of bi-degree $(1,1)$ such that:*

(i) $\{T\} \in H^2(X,\mathbb{Z})$

(ii) $T = \frac{i}{\pi}\, \partial\bar{\partial}\psi + \alpha$, *where α is a smooth representant of T and ψ is locally of the form* $\log\left(\sum \lambda_j |f_j|^2\right)$, *for some smooth functions λ_j and holomorphic functions f_j.*

(iii) $\int_{\hat{X}(\leq 1)} T^n > 0$, *where the integral is taken over the regular points of T.*

We can apply this result if the Lelong numbers of the curvature form $ic(E)$ vanish at the singular points of X. We conclude that if the Lelong numbers of the curvature do not vanish at the singular points and, moreover, the hypothesis of Proposition 4.5 are verified, our method gives the embedding of X^* in a Moishezon manifold, while the resolution of singularieties does not provide this result.

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