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Extremal Functions and Contractive Divisors in $A^{-n}$

C. HOROWITZ - B. KORENBLUM - B. PINCHUK

1. - Introduction

For $n > 0$, $A^{-n}$ is defined as the Banach space of all analytic functions $f$ in the unit disc $U$ such that

$$\|f\|_{A^{-n}} = \sup_{z \in U} |f(z)|(1 - |z|^2)^n < \infty.$$

If $M$ is a subset of $A^{-n}$ and if $a \in U$ we consider the extremal problem

$$\sup_{f \in M, \|f\| \leq 1} |f(a)|.$$

Any function which attains the supremum will be called an extremal function for $M$ at $a$. We note that if $M$ is closed under locally uniform convergence then $M$ has extremal functions at every $a \in U$, for the functions of norm $\leq 1$ in $A^{-n}$ form a normal family. However, it should be emphasized that not every norm-closed subspace of $A^{-n}$ is closed in this sense. A simple example of such a subspace is

$$A_0^{-n} = \{f \in A^{-n}: \lim_{|z| \to 1} |f(z)|(1 - |z|^2)^n = 0\}.$$

A function $f \in M$ is called a contractive divisor for $M$ if $\|f\| = 1$ and for every $g \in M$, $g/f \in A^{-n}$ with

$$\left\|\frac{g}{f}\right\| \leq \|g\|.$$

In [2] and [3] the importance and interdependence of the above concepts were amply demonstrated in the case of the Bergman spaces which are closely related to $A^{-n}$.

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In Section 2 of the present work, we shall give a geometric characterization of extremal functions in $A^{-n}$ and some conditions relating them to contractive divisors. Section 3 is devoted to the explicit calculation of extremal functions in some simple cases. The authors wish to thank Dr. E. Beller for helpful discussions pertaining to this work.

2. - The main results

For $K \subset \mathbb{C}$ compact define the polynomial hull of $K$ by

$$\hat{K} = \{ z \in \mathbb{C} : |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all analytic polynomials } p \}.$$

The following elementary characterization of $\hat{K}$ is proved in the first chapter of [4].

**Proposition 2.1.** $\hat{K} = K$ together with all of the bounded components of $\mathbb{C} \setminus K$.

**Definition 2.2.** If $f \in A^{-n}$, if $\|f\| = 1$, and if $\varepsilon > 0$ define

$$K_\varepsilon(= K_\varepsilon(f)) = \{ z \in U : |f(z)|(1 - |z|^2)^n > 1 - \varepsilon \}$$

$$K_0(= K_0(f)) = \{ z \in U : |f(z)|(1 - |z|^2)^n = 1 \}.$$

**Definition 2.3.** Let $a \in U$. We say that $f \in A^{-n}$ has property $P_a$ if $\|f\| = 1$ and for every $\varepsilon > 0$, $a \in \hat{K}_\varepsilon$. (Roughly, this means that every $K_\varepsilon$ "surrounds" $a$.) We say that $f$ has property $P_a$ if $a \in \hat{K}_0$.

**Theorem 2.4.** Let $M$ be a subset of $A^{-n}$ invariant under multiplication by nonvanishing entire functions (e.g., a closed subspace invariant under multiplication by $z$ or the set of all functions in $A^{-n}$ vanishing precisely on a given set in $U$.) Then if $f$ is an extremal function for $M$ at some point $a \in U$, $f$ has property $P_a$.

**Proof.** Clearly, $\|f\| = 1$. Assume that $f$ does not have property $P_a$. Then for some $\varepsilon > 0$, $a \notin \hat{K}_\varepsilon$. However, $\mathbb{C} \setminus \hat{K}_\varepsilon$ is just the unbounded component of $\mathbb{C} \setminus K_\varepsilon$, which is open and connected. It follows that also $\mathbb{C} \setminus \hat{K}_\varepsilon \cup \{a\}$ is open and connected. Therefore, by Runge's theorem there exists a polynomial $p$ such that $p(a) > 0$ and $\Re p(z) < 0$ on $\hat{K}_\varepsilon$. Define

$$h(z) = \exp \left( \frac{1}{m} p(z) \right),$$

where $m$ is chosen so large that $|h(z)| \leq \frac{1}{1 - \varepsilon}$ everywhere in $U$. By
our assumption, $hf \in M$. Furthermore, on $K_\varepsilon$, $|h(z)| < 1$ so $|h(z)| |f(z)|(1 - |z|^2)^n < 1$. On $U \setminus K_\varepsilon$, $|f(z)|(1 - |z|^2)^n \leq 1 - \varepsilon$, so $|h(z)| |f(z)|(1 - |z|^2)^n \leq 1$. We conclude that $\|hf\| \leq 1$. But $|h(a)| > 1$, so $|(hf)(a)| > |f(a)|$, contradicting the extremality of $f$ at $a$.

**DEFINITION 2.5.** For $f \in A^{-n}$, $I_f$ represents the smallest $z$-invariant closed subspace containing $f$, i.e., the norm closure of all functions $pf$ such that $p$ is a polynomial. Also, define $J_f = \{g \in A^{-n}: g/f \text{ is analytic in } U\}$.

With this notation we can prove a converse to Theorem (2.4).

**THEOREM 2.6.** If $f$ has property $P_a$ for some $a \in U$ then $f$ is extremal at $a$ for the subspace $I_f$.

**PROOF.** Clearly, it suffices to show that if $p$ is a polynomial such that $\|pf\| = 1$ then $|p(a)| \leq 1$. To that end let $\varepsilon > 0$ be given. By property $P_a$ applied to $f$, $a \in \overline{K}_\varepsilon$. Thus

$$|p(a)| \leq \sup_{z \in \overline{K}_\varepsilon} |p(z)| = \sup_{z \in K_\varepsilon} |p(z)| = \sup_{z \in K_\varepsilon} \frac{|p(z)| |f(z)|(1 - |z|^2)^n}{|f(z)|(1 - |z|^2)^n} \leq \frac{1}{1 - \varepsilon}.$$  

Letting $\varepsilon \to 0$ we obtain the result. 

Next we prove strengthened versions of Theorems (2.4) and (2.6) for extremal functions with "tame" boundary behavior; i.e., for functions $f$ of norm 1 which satisfy

$$\lim_{|z| \to 1} |f(z)|(1 - |z|^2)^n = 1 - \delta, \text{ with } \delta > 0. \tag{2.7}$$

**THEOREM 2.8.** Let $M$ and $f$ be as in Theorem (2.4) with the additional assumption that $f$ satisfies (2.7). Then $f$ has property $P'_a$ (as in Definition (2.3)).

**PROOF.** By (2.7) $K_0(f)$ is a compact subset of $U$. If $f$ does not have property $P'_a$ then by the geometric characterization of $\overline{K}_0$ there must be a curve $\gamma$ connecting $a$ to the boundary of $U$ which does not intersect $K_0$. This together with (2.7) implies that for some $\varepsilon$, with $0 < \varepsilon < \delta$,

$$|f(z)|(1 - |z|^2)^n < 1 - \varepsilon \text{ on } \gamma.$$  

It follows that $\gamma$ is disjoint from $\overline{K}_\varepsilon$. But this means that $f$ does not have property $P_a$. In view of Theorem (2.4) we have arrived at a contradiction to the assumption that $f$ is extremal. 

**THEOREM 2.9.** Assume that $\|f\| = 1$ and $f$ satisfies (2.7). Then $f$ is extremal at every point of $\overline{K}_0$ both for $I_f$ and for the generally larger subspace $J_f$. 

PROOF. The condition (2.7) implies that $K_0$ is a compact subset of $U$. Now if $a \in \hat{K}_0$ and if $g \in J_f$ while $\|g\| = 1$ then

$$\left| \frac{g(a)}{f(a)} \right| \leq \max_{z \in K_0} \left| \frac{g(z)}{f(z)} \right| = \max_{z \in K_0} \left| \frac{g(z)(1 - |z|^2)^n}{f(z)(1 - |z|^2)^n} \right| \leq 1.$$ 

Therefore, $f$ is extremal at $a$. □

Our next two results concern the contractive property.

**Theorem 2.10.** Let $f$ be a contractive divisor for some set $M \subset A^n$. Then $f$ is extremal for $M$ at zero.

**Proof.** If $g \in M$ and if $\|g\| \leq 1$, then

$$\left| \frac{g(0)}{f(0)} \right| \leq \left| \frac{g}{f} \right| \leq \|g\| \leq 1,$$

so $|g(0)| \leq |f(0)|$. □

**Theorem 2.11.** Assume that $f \in A^n$ is continuous on $\overline{U}$ and is a contractive divisor for the subspace $J_f$. Then $f$ has property $P_0'$ and $|f(z)| \geq 1$ on the circle $T = \{z: |z| = 1\}$. Conversely, if $f$ is continuous on $\overline{U}$, $|f(z)| \geq 1$ on $T$, and if all of the zeros of $f$ are contained in $\hat{K}_0$, then $f$ is a contractive divisor on $J_f$.

**Proof.** If $f$ is a contractive divisor on $J_f$, Theorem (2.10) gives that $f$ is extremal for $J_f$ at zero. If, moreover, $f$ is continuous on $\overline{U}$, then it certainly satisfies (2.7). Thus we can conclude from Theorem (2.8) that $f$ has property $P_0'$. For the next assertion we argue by contradiction. If at some $z_0 \in T$, $|f(z_0)| < 1$, then there is a neighborhood $S$ of $z_0$ and a number $\epsilon > 0$ such that

$$|f(z)| \leq 1 - \epsilon \text{ in } S \cap \overline{U}.$$ 

Now for any positive integer $m$ the function

$$g(z) = \frac{1}{(1 - \overline{z}_0z)^n} \left( \frac{z + z_0}{2} \right)^m$$

has norm 1, and if we choose $m$ sufficiently large we can arrange that $|g(z)| \leq 1 - \epsilon$ outside of $S$. It follows immediately that $\|fg\| \leq 1 - \epsilon$ while $\|g\| = 1$. Since $fg \in J_f$ we have contradicted the contractive property of $f$. For the converse assertion, if $g \in J_f$ and $\|g\| = 1$ then Theorem 2.9 implies that

$$|g(z)| \leq |f(z)| \text{ for all } z \in \hat{K}_0,$$

or

$$\left| \frac{g(z)}{f(z)} \right| \leq 1 \text{ on } \hat{K}_0. \quad (2.12)$$
On the other hand, \( U \setminus \hat{K}_0 \) is an open connected set whose boundary is contained in \( T \cup K_0 \). By hypothesis \( f \) is nonvanishing in \( U \setminus \hat{K}_0 \) and \( |f(z)| \geq 1 \) on \( T \). Furthermore, on \( K_0 \)
\[ |f(z)| = \frac{1}{(1 - |z|^2)^n} \geq 1, \]
and so we conclude by the minimum principle that
\[ |f(z)| \geq 1 \text{ for all } z \in U \setminus \hat{K}_0. \]
Thus if \( g \in J_f \) and \( \|g\| = 1 \)
\[ \left| \frac{g}{f} \right| \leq |g| \text{ in } U \setminus \hat{K}_0. \]
This together with (2.12) proves that \( f \) is a contractive divisor on \( J_f \).

We conclude this section with the observation that although we have presented our results in the context of \( A^{-n} \) they apply essentially verbatim to spaces of analytic functions in \( U \) which are bounded by arbitrary weight functions.

### 3. Construction of extremal functions for finite zero sets

An important special case of the theory presented in Section 2 is obtained by choosing a finite set \( P = \{z_1 \ldots z_m\} \subset U \) and positive integers \( \{k_1 \ldots k_m\} \) and letting
\[
M = \{ f \in A^{-n} : f \text{ vanishes at each } z_i \in P \text{ with multiplicity at least } k_i \},
\]
\[
M' = \{ f \in A^{-n} : f \text{ vanishes at each } z_i \in P \text{ with multiplicity } k_i \text{ and nowhere else in } U \}.
\]

By a normal families argument, \( M \) and \( M' \) both contain extremal functions at each point of \( U \). Now if \( h \) is an extremal function for \( M' \) at some point \( a \in U \), and if \( h \) satisfies (2.7) then by Theorem (2.8) \( h \) has property \( P_{a'} \). Theorem (2.9) then implies that \( h \) is extremal at \( a \) for \( J_h \), which in this case is just \( M \). In particular, we obtain an extremal function for \( M \) which has no extraneous zeros. These considerations together with the results of Section 2 accentuate the importance of the following question which we have been unable to resolve.

**QUESTION.** With \( M \) and \( M' \) as above, what can be said about the boundary behavior of their extremal functions?

We turn to the problem of explicit construction of extremal functions for \( M \) and \( M' \) at a point \( a \in U \) in some simple cases. If we presume that these
functions satisfy (2.7) then they must have property $P_a$. It is convenient to hypothesize the slightly stronger property

\[ P_a^*: \|f\| = 1, \ f \text{ satisfies (2.7), and there exists an analytic Jordan curve } \gamma \text{ surrounding } a \text{ on which } |f(z)|(1 - |z|^2)^n \equiv 1. \]

Clearly, any function $f$ satisfying $P_a^*$ is extremal for $J_f$ at $a$. Moreover, $P_a^*$ implies that

\[ \nabla(|f(z)|(1 - |z|^2)^n) = 0 \text{ on } \gamma \]

or

\[ \frac{\partial}{\partial z} (|f(z)|^2(1 - |z|^2)^{2n}) = 0 \text{ on } \gamma. \]

which gives the differential equation

\[ (3.1) \quad \frac{f'(z)}{f(z)} = \frac{2n\bar{z}}{1 - |z|^2} \text{ on } \gamma. \]

Thus is $S(z)$ is the “Schwarz function” (see [1] and [6]) analytic near $\gamma$ and satisfying $S(z) = \bar{z}$ on $\gamma$, we have

\[ (3.2) \quad \frac{f'(z)}{f(z)} = \frac{2nS(z)}{1 - zS(z)} \text{ on } \gamma, \]

and by analytic continuation this must persist in the whole unit disc.

Conversely, if $\gamma$ is an analytic Jordan curve in $U$ which surrounds $a$ and if the Schwarz function $S$ of $\gamma$ is such that

\[ \frac{2nS(z)}{1 - zS(z)} \]

is analytic in $U$ except for simple poles at $z_i (i = 1, 2, \ldots m)$ with positive integral residues $k_i$ then we can integrate (3.2) to produce $f \in M'$ (as defined above) such that $|f(z)|^2(1 - |z|^2)^{2n}$ has zero gradient on $\gamma$. Now if this $f$ also satisfies (2.7) and if $S(z) = \bar{z}$ only on $\gamma$ (so that $|f(z)|^2(1 - |z|^2)^{2n}$ has no other critical points in $U$) we can conclude that $|f(z)|(1 - |z|^2)^n$ takes its maximum identically on $\gamma$, so that if we normalize $f$ to have norm 1 it has property $P_a^*$, and in particular $f$ is extremal for $M'$ and for $M$.

Let us apply these ideas in the case of a single zero of order $k \geq 1$ at the origin. Here

\[ M_k = \{ f \in A^{-n}: f(z)/z^k \text{ is analytic in } U \} \]

\[ M'_k = \{ f \in A^{-n}: f(z)/z^k \text{ is analytic and nonvanishing in } U \}. \]

If $f \in M'_k$, the function $zf'(z)/f(z)$ is analytic in $U$. If this $f$ satisfies (3.1) we have

\[ (3.3) \quad \frac{zf'(z)}{f(z)} = \frac{2n|z|^2}{1 - |z|^2} \text{ on } \gamma. \]
In particular, the analytic function \( zf'(z)/f(z) \) is positive on \( \gamma \) which is possible only if this function is identically constant. Since \( f \in M'_k \) we must have

\[
\frac{zf'(z)}{f(z)} = k; \quad f(z) = cz^k.
\]

By (3.3)

\[
\frac{2n|z|^2}{1 - |z|^2} = k \text{ on } \gamma,
\]

from which we deduce that \( \gamma \) must be the circle \( |z| = \sqrt{\frac{k}{2n+k}} \). Letting

\[
\alpha_{k,n} = \left( \sqrt{\frac{k}{2n+k}} \right)^k \left( 1 - \frac{k}{2n+k} \right)^n,
\]

we find that \( f(z) = \frac{z^k}{\alpha_{k,n}} \) is a function of norm 1 which attains its norm identically on the circle \( \gamma \). By the reasoning outlined at the beginning of this section we conclude that \( f \) is extremal both for \( M' \) and for \( M \) at every point inside or on \( \gamma \). Clearly, \( |f(z)| > 1 \) on the boundary so by Theorem (2.11) \( f \) is a contractive divisor in \( M \). All of this can easily be checked by a direct argument using the maximum principle.

At points outside of \( \gamma \) we have yet to determine an extremal function for \( M'_k \). Since (3.3) cannot be fulfilled on any curve surrounding such points, one might expect that for each \( a \) outside of \( \gamma \) there are extremal functions which attain the value \( \frac{1}{(1 - |a|^2)^n} \) at \( a \), as is indeed the case on \( \gamma \) itself where \( f \) attains this value identically. In fact, we can construct such functions explicitly, as follows: Consider the two parameter family of functions

\[
h(z) = h_{k,n}(z) = \frac{z^k}{(1 - z)^{n-\varepsilon}} \quad (0 < \varepsilon < n).
\]

If we substitute this \( h \) into formula (3.1) for critical points of \( |h(z)|^2(1 - |z|^2)^{2n} \) we find that critical points occur when

\[
k + \frac{(n-\varepsilon)z}{1 - z} = \frac{2n|z|^2}{1 - |z|^2}.
\]

By inspection of the function \( |h(z)|^2(1 - |z|^2)^{2n} \) it is clear that its maximum must occur at some point \( z = r > 0 \). Conversely, if we choose \( r > 0 \) such that \( \frac{2nr^2}{1 - r^2} > k \) we find that (3.4) is satisfied uniquely at \( r \) if we choose

\[
(n - \varepsilon) = \frac{2nr}{1 + r} - \frac{k(1 - r)}{r}.
\]
This choice is always possible since the right side of (3.5) increases precisely from 0 to \( n \) as \( r \) proceeds from the circle \( \gamma \) to 1. Finding \( \varepsilon \) as indicated by (3.5) and normalizing, we find that the function \( \frac{h_{k,\varepsilon}(z)}{h_{k,\varepsilon}(r)(1-r^2)^n} \) is extremal at \( r \), where it attains the value \( \frac{1}{(1-r^2)^n} \). By a rotation we can extend the above construction to arbitrary points outside of \( \gamma \). For the case \( k = 1 \) we can summarize our results in the following “Schwarz Lemma for \( A^{-n} \)”.

**Lemma 3.6.** If \( f \in A^{-n}, \|f\| = 1, \) and if \( f(0) = 0 \) then the following estimates are sharp:

\[
|f(z)| \leq \begin{cases} 
\frac{(2n+1)^{n+1/2}}{(2n)^n} |z| & ; \ |z| < \left( \frac{1}{2n+1} \right)^{1/2} \\
\frac{1}{(1-|z|^2)^n} & ; \ |z| \geq \left( \frac{1}{2n+1} \right)^{1/2}.
\end{cases}
\]

Equality is attained in the first estimate only by the functions

\[
f(z) = e^{i\theta} \frac{(2n+1)^{n+1/2}}{(2n)^n} z.
\]

Next we consider extremal functions vanishing at an arbitrary single point \( \alpha \in U \). We use the important fact that for every such \( \alpha \) the transformation

\[
(Tf)(z) = \frac{1-|\alpha|^2}{(1-\overline{\alpha} z)^2} f \left( \frac{\alpha - z}{1-\overline{\alpha}z} \right)
\]

is an isometry on \( A^{-n} \).

\((T\) was used extensively in [5].\) In particular, if \( f \) has norm 1 and vanishes at \( \alpha \), \( Tf \) has norm 1 and vanishes at zero. Thus we can generalize the Schwarz Lemma as follows:

**Lemma 3.7.** If \( f \in A^{-n}, \|f\| = 1, \) and \( f(\alpha) = 0 \) then the following estimates are sharp:

\[
|f(z)| \leq \begin{cases} 
\frac{(2n+1)^{n+1/2}}{(2n)^n} \frac{1-|\alpha|^2}{|1-\overline{\alpha} z|^2} \left| \frac{\alpha - z}{1-\overline{\alpha} z} \right| & ; \ \left| \frac{\alpha - z}{1-\overline{\alpha} z} \right| \leq \frac{1}{\sqrt{2n+1}} \\
\frac{1}{(1-|z|^2)^n} & ; \ \text{otherwise}.
\end{cases}
\]

Equality is obtained in the first estimate only by the extremal functions

\[
G_\alpha(z) = e^{i\theta} \frac{(2n+1)^{n+1/2}}{(2n)^n} \frac{(1-|\alpha|^2)^n}{(1-\overline{\alpha} z)^2n} \frac{\alpha - z}{1-\overline{\alpha} z}.
\]

It is interesting to note that the extremal function \( G_\alpha \) is not always a contractive divisor. By Theorem (2.11) it is contractive if and only if \( |G_\alpha(z)| \geq 1 \).
whenever $|z| = 1$. One easily sees that this occurs if and only if

$$\frac{1 - |\alpha|}{1 + |\alpha|} \geq \frac{2n}{(2n + 1)^{1/2n}} \quad \text{on} \quad |\alpha| \leq \frac{(2n + 1)^{1+1/2n} - 2n}{(2n + 1)^{1+1/2n} + 2n}.$$ 

We can show further that if

$$\frac{(2n + 1)^{1+1/2n} - 2n}{(2n + 1)^{1+1/2n} + 2n} < |\alpha| < \frac{1}{\sqrt{2n + 1}}$$

there are no contractive divisors for the space $M_a = \{ f \in A^{-n}: f(\alpha) = 0 \}$.

Indeed, suppose that $\alpha$ lies in the indicated region, $g \in M_a$, and $g$ is contractive. Then by (3.7) $|g(0)| \leq |G_a(0)|$, and by the contractive property

$$\left| \frac{G_a(0)}{g(0)} \right| \leq \left\| \frac{G_a}{g} \right\| \leq 1.$$

Hence $|G_a(0)| = |g(0)|$ which by Lemma 3.7 implies that $G_a(z) = g(z)$, which contradicts the assumption that $g$ is contractive.

This situation is in sharp contrast with the case of the Bergman spaces, see [2].

As a final example we consider the problem of extremal functions for the set of functions in $A^{-n}$ which have simple zeros at two symmetric points $\pm z_0$ and are nonvanishing elsewhere. By the remarks following equation (3.1) the main problem is to produce an appropriate curve $\gamma$ and an appropriate Schwarz function $S$. To that end we use some ideas from Shapiro’s notes [6].

Let $A$ and $R$ be positive numbers such that

$$R < A < \frac{R^2 - 1}{2}.$$ 

Then the function

$$z = \varphi(w) = \frac{2Aw}{w^2 + R^2}$$

maps $\overline{U}$ univalently into $U$. Specifically, the inverse is given by

$$w = \frac{A - \sqrt{A^2 - R^2z^2}}{z},$$

where we choose that branch of the square root which makes $w = 0$ correspond to $z = 0$. Now when $|w| = 1$, $z$ traces a Jordan curve $\gamma$ on which

$$\bar{z} = \frac{2A\bar{w}}{\bar{w}^2 + R^2} = \frac{2Aw}{1 + R^2w^2}.$$
so by inserting (3.10) into (3.11) we obtain a Schwarz function $S(z)$ for $\gamma$. Equation (3.2) now becomes

$$\frac{f'(z)}{f(z)} = \frac{4nAw}{1 + R^2w^2 - 2Awz} = \frac{4An}{w + R^2w - 2Az}.$$

By (3.10) $\frac{1}{w} = \frac{A + \sqrt{A^2 + R^2z^2}}{R^2z}$ so

$$\frac{f'(z)}{f(z)} = \frac{4nAR^2z}{A(1 + R)^4 + [(1 - R^4)\sqrt{A^2 - R^2z^2}] - 2AR^2z^2} = \frac{4nz}{\left(\frac{1}{R^2} + R^2\right) - 2z^2 + \left(\frac{1}{R^2} - R^2\right) \sqrt{1 - \frac{R^2}{A^2} z^2}}.$$

At this point it is convenient to define new parameters:

$$(3.12) \quad a = R^2 + \frac{1}{R^2}; \quad b = R^2 - \frac{1}{R^2}; \quad c = \frac{R^2}{A^2}.$$

It follows from (3.8) that $a$, $b$ and $c$ are positive and $c < 1$. Clearly, $a^2 - b^2 = 4$. In these parameters we have the equation

$$\frac{f'(z)}{f(z)} = \frac{4nz}{a - 2z^2 - b\sqrt{1 - c}z^2}.$$

Thus if $g$ satisfies

$$(3.13) \quad \frac{g'(z)}{g(z)} = \frac{2n}{a - 2z - b\sqrt{1 - c}z}$$

and has a single zero in $U$, we can take $f(z) = g(z^2)$ to obtain a solution of (3.2) having two symmetric zeros. Using $a^2 - b^2 = 4$ we observe that

$$\frac{2n}{a - 2z - b\sqrt{1 - c}z} + \frac{2n}{a - 2z + b\sqrt{1 - c}z} = \frac{n(a - 2z)}{z^2 - \left(a - \frac{b^2c}{4}\right)z + 1}.$$

The expression on the right has two reciprocal poles, say at $r$ with $|r| \leq 1$ and at $1/r$. However, by a calculation one deduces from (3.8) and (3.12) that $a - \frac{b^2c}{4} < -2$ which implies that $-1 < r < 0$. Since $a$, $b$, and $c$ are positive, $a - 2z + b\sqrt{1 - c}z$ cannot vanish when $z$ is negative, so we conclude that the
expression on the right side of (3.13) has exactly one simple pole in $U$, namely at $r$, and one additional pole at $1/r$. By partial fractions

$$
\begin{align*}
\frac{n(a-2z)}{z^2 - \left(a - \frac{b^2 c}{4}\right)z + 1} &= \frac{\alpha}{z - r} + \frac{\beta}{z - 1/r},
\end{align*}
$$

where

$$
\alpha = \frac{n(a-2r)}{r - 1/r},
$$

which we make equal to one by an appropriate choice of $a$. Equating coefficients of $z$ in (3.14) we then find that

$$
\alpha + \beta = -2n, \quad \text{or} \quad \beta = -2n - 1.
$$

Solving (3.13) and inserting $f$ in place of $g$ we conclude that

$$
\begin{align*}
f(z) &= e \left(\frac{z^2 - r}{z^2 - 1/r}\right) \frac{1}{(z^2 - 1/r)^{2n}} h(z^2),
\end{align*}
$$

where

$$
\begin{align*}
h(w) &= \exp \left(\int_0^w \frac{-2ndz}{1 - 2z + b\sqrt{1 - cz}}\right).
\end{align*}
$$

The integration for $h$ can be carried out explicitly to obtain a closed expression for $f$. We prefer a different approach. Namely, going back to (3.13) we note that

$$
\begin{align*}
\int \frac{2ndz}{a - 2z - b\sqrt{1 - cz}} &= (w = \sqrt{1 - cz}) \int \frac{-2nwdw}{w^2 - \frac{bc}{2}w + \left(\frac{ac}{2} - 1\right)}
\end{align*}
$$

where

$$
\begin{align*}
&= \int \frac{\alpha}{w - w_1} + \frac{\beta}{w - w_2} \, dw.
\end{align*}
$$

From (3.13) we obtain

$$
f(z) = c_1(\sqrt{1 - cz^2} - w_1)^\alpha(\sqrt{1 - cz^2} - w_2)^\beta.
$$

Comparing with (3.16) we conclude that

$$
\begin{align*}
f(z) &= c_1 \frac{\sqrt{1 - cz^2} - \sqrt{1 - cr}}{\left(\sqrt{1 - cz^2} - \sqrt{1 - c/r}\right)^{2n+1}},
\end{align*}
$$

where $c_1$ is chosen to give $\|f\| = 1$. 

It remains only to verify that the function we have constructed really is extremal. Since this function is analytic in a neighborhood of $\overline{U}$ our remarks at the beginning of this section imply that $f$ will be proved extremal if we can verify that in the above construction

$$S(z) = \bar{z} \text{ only on } \gamma.$$ 

But this can easily be checked via the parametric equation

$$\frac{2Aw}{1 + R^2w^2} = S(z) = \bar{z} = \frac{2A\overline{w}}{w^2 + R^2}$$

which one readily sees is satisfied only if $|w| = 1$; i.e., on $\gamma$, or if $w = 0$, which corresponds to $z = 0$. However, the point $z = 0$ is in general an extraneous critical point of the function $|f(z)|^2(1 - |z|)^{2n}$, introduced by the fact that this is a smooth function of $z^2$. One sees this clearly in the case where $|r| < \frac{1}{2n + 1}$, for then the Schwarz Lemma (3.7) prevents any function of $f$ of norm 1 which vanishes at the point $\sqrt{r}$ from taking the value 1 at the origin. Thus zero cannot be a maximum point of $|f(z)|^2(1 - |z|)^{2n}$ in this case, and all the more so if $f$ also vanishes at $-\sqrt{r}$. So in general the function constructed in (3.17) really attains its norm on $\gamma$, and we can conclude that it is extremal at all points inside or on $\gamma$ for the subspace of functions vanishing at the points $z = \pm \sqrt{r}$.

Finally, we compute the range of $r$ and $n$ for which our last example is applicable. Now if $n > 0$ and $r \in (-1, 0)$ are given, formula (3.15) shows that we must choose the parameter $a$ so that

$$a = 2r + \frac{1}{n} \left( r - \frac{1}{r} \right).$$

By (3.12) $b = \sqrt{4 - a^2}$, and by (3.4) $a - \frac{b^2c}{4} = r + \frac{1}{r} \Rightarrow c = \left( r + \frac{1}{r} - a \right) / 1 - a^2 / 4$.

The restrictions (3.8) together with (3.12) imply that $a > 1$, $c < 1$ and $c > \frac{4}{a - 2}$. However, the last inequality is an automatic consequence of our explicit formula for $c$, together with the fact that $-1 < r < 0$. So really the only restrictions are

$$a > 6, \quad c < 1,$$

from which one can find the exact range of applicability of the example. Qualitatively one sees that as $n \to \infty$ we can accept $r$'s only from a progressively smaller neighborhood of zero, and as $n \to 0$ the range of $r$ expands to the whole interval $(-1, 0)$. 
REFERENCES


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