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E. BOMBIERI

J. B. FRIEDLANDER

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# Dirichlet Polynomial Approximations to Zeta Functions

E. BOMBIERI - J.B. FRIEDLANDER\*

## Abstract

In this paper we consider  $L$ -functions satisfying certain standard conditions, their approximation by Dirichlet polynomials and, especially, lower bounds for the lengths of the polynomials that provide good approximations.

## 1. - Introduction

For  $\zeta(s)$  the Riemann zeta-function one has the Dirichlet series representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{valid for } \sigma > 1,$$

where  $s = \sigma + it$ . By the absolute convergence of this series one sees that, even for  $x$  not very large, the Dirichlet polynomial  $\sum_{n \leq x} n^{-s}$  gives a rather good approximation to  $\zeta(s)$ , with a remainder which is  $o(1)$  as  $x \rightarrow \infty$ .

This is a nice property, since one would expect the finite sum to be easier to work with for purposes of estimation. However, one is of course more interested in estimating  $\zeta(s)$  in the critical strip  $0 < \sigma < 1$ . Here the above polynomial still provides [T, §4.11] (at least away from the pole) a useful approximation to  $\zeta(s)$ , moreover the smoothed polynomials

$$\sum_{n \leq x} \frac{1}{n^s} \left(1 - \frac{n}{x}\right)^k$$

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do an even better job, but all of these only for  $x > (1+o(1)) \frac{|t|}{2\pi}$ , and this limits their usefulness for application.

Thus the question arises whether shorter approximations of the same quality to  $\zeta(s)$ , or for that matter to  $L$ -series and general zeta functions, exist.

In this paper we investigate approximations by Dirichlet polynomials to  $L$ -functions of a fairly general type, and show in many cases that it is not possible to achieve a very good level of approximation by means of polynomials essentially shorter than the known approximations. Thus we may view such a result as a first step toward understanding the analytic complexity of a zeta function.

We shall consider  $L$ -functions  $L(s)$  having the following properties (compare, for example, [S]):

(H1)  $L(s)$  is given by an absolutely convergent Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

in the half-plane  $\sigma > 1$ , with coefficients  $a_n$  satisfying  $a_1 = 1$  and  $a_n \ll n^{o(1)}$ .

(H2)  $L(s)$  is meromorphic of finite order in the whole complex plane, has only finitely many poles and satisfies a functional equation<sup>1</sup>

$$\Phi(s)L(s) = \overline{\Phi}(1-s)\overline{L}(1-s)$$

where  $\Phi(s) = wQ^s \prod_{j=1}^J \Gamma(\lambda_j s + \mu_j)$  with constants satisfying

$$|w| = 1, \quad Q > 0, \quad \lambda_j > 0, \quad \operatorname{Re} \mu_j \geq 0.$$

From the fact that  $L(s)$  is of finite order with finitely many poles and satisfies a functional equation of the above type and from the Phragmén-Lindelöf principle, it follows that  $L(s)$  has, away from the poles, polynomial growth in any fixed vertical strip. Moreover  $L(s)$ , for  $\sigma < \frac{1}{2}$ , has order not less than  $|t|^{2\Lambda(\frac{1}{2}-\sigma)}$  and for  $\sigma < 0$  has order precisely  $|t|^{2\Lambda(\frac{1}{2}-\sigma)}$ , where  $\Lambda = \sum_{j=1}^J \lambda_j$ .

It now follows by a well-known argument (cf. [T, § 9.4]) that the number  $N(T; L)$  of non-trivial zeros (that is, those not located at the poles of the  $\Gamma$  factors) of  $L(s)$  satisfying  $0 < t \leq T$ , is given asymptotically by

$$(1.1) \quad N(T; L) = \frac{\Lambda}{\pi} T \log T + c_L T + O(\log T),$$

<sup>1</sup> For a function  $f(s)$  we define  $\overline{f}(s) = \overline{f(\overline{s})}$ .

where  $c_L$  is a constant depending on  $L$ . Since we assume  $a_1 = 1$  we may compute the constants explicitly and write this in the form

$$(1.1') \quad N(T; L) = \frac{T}{2\pi} \log(C_L T^{2\Lambda}) + O(\log T),$$

where  $C_L = Q^2 \prod_j \lambda_j^{2\lambda_j}$ .

One should remark that the choice of the parameter  $Q$  and the Gamma factors in the above decomposition of  $\Phi(s)$  are not uniquely determined due to the multiplication formula for the Gamma function. However the key quantities  $\Lambda$  and  $C_L$  used in this paper are uniquely determined by  $L(s)$ .

The next assumption that we make about our  $L$ -function is that it satisfy a weak zero-density estimate. Let  $N(\sigma, T; L)$  denote the number of non-trivial zeros  $\rho = \beta + i\gamma$  of  $L$  with  $0 < \gamma \leq T$  and  $\beta \geq \sigma$ . Then we assume:

(H3) *For any fixed  $\delta > 0$ , we have*

$$N\left(\frac{1}{2} + \delta, T; L\right) = o(T \log T).$$

Our two main results place a limitation on the length of the Dirichlet polynomial

$$(1.2) \quad D_x(s) = \sum_{n \leq x} a_n(x) n^{-s}, \quad |a_1(x)| > \frac{1}{2}, \quad a_n(x) \ll n^{o(1)}$$

(actually,  $\frac{1}{2}$  may be replaced by any fixed positive constant) if it is to be a useful approximation to  $L(s)$ . Specifically, we prove

**THEOREM 1.** *Let  $L(s)$  satisfy assumptions (H1)–(H3), and let  $\varepsilon, \varepsilon' > 0$ . Suppose that we have*

$$(1.3) \quad L(s) = D_x(s) + O(T^{-\varepsilon})$$

*on the segment  $\left\{\sigma = \frac{1}{2} - \varepsilon', T \leq t \leq 2T\right\}$ . Then  $x > T^{2\Lambda - o(1)}$ .*

Our basic strategy is to use a well known lemma of Littlewood to compare, in a suitable rectangle, the number of zeros of the function  $L(s)$  with that of the approximating polynomial  $D_x(s)$ . These should be nearly equal if the approximation is sufficiently good. On the other hand we shall be able to estimate the former using (1.1). This will give a contradiction provided that we can give a smaller upper bound to the number of zeros of  $D_x(s)$  in case  $x$  is not too large. Such a result is provided by the following:

**PROPOSITION 1.** *Let  $D_x(s)$  given by (1.2) satisfy  $|a_1(x)| > \frac{1}{2}$  and  $a_n(x) \ll n$ . Then, uniformly for  $\alpha < \sigma < \infty$  we have*

$$\arg D_x(s) \ll (|\alpha| + 1) \log x.$$

*Let also  $N(\alpha, T, T+H; D_x)$  denote the number of zeros of  $D_x(s)$  satisfying  $\sigma \geq \alpha$ ,  $T \leq t \leq T+H$ , where  $H \leq T$ . Then, uniformly for  $-H < \alpha < -1$ , we have*

$$(1.4) \quad N(\alpha, T, T+H; D_x) \leq \frac{H}{2\pi} \log x + O\left(|\alpha|^{\frac{1}{2}} H^{\frac{1}{2}} \log x\right),$$

*where the implied constant is absolute.*

The exponent  $2\Lambda$  given in Theorem 1 is sharp, as will be seen in the next section. Nevertheless, a slightly different argument using Rouché's theorem shows that the bound can be made still more precise if one is willing to strengthen the assumptions to some extent. Specifically, we have:

**THEOREM 2.** *Let  $L(s)$  satisfy assumptions (H1), (H2), and also, for every  $\delta > 0$ , the strengthened zero density bound*

$$N\left(\frac{1}{2} + \delta, T; L\right) = o(T(\log T)^{-1}).$$

*Suppose that we have*

$$(1.3) \quad L(s) = D_x(s) + O(T^{-\varepsilon})$$

*on the segment  $\{\sigma = -\varepsilon', T \leq t \leq (1 + \varepsilon_0)T\}$ . Then  $x > (1 + o(1))C_LT^{2\Lambda}$ , with  $C_L$  as in (1.1').*

In Theorem 2, not only the exponent, but even the constant  $C_L$  is the best possible. We remark that the assumption of (1.3) on the segment with  $\sigma = -\varepsilon'$  is stronger than the assumption on a corresponding segment with  $\sigma = \frac{1}{2} - \varepsilon'$ ; see Proposition 3. In the event that one assumes a stronger version as in Theorem 2, but is willing to settle for the weaker conclusion of Theorem 1, then it is possible to give a somewhat simpler proof which combines the principle of the argument with the result of Proposition 1.

Throughout the paper, implied constants may depend on  $L(s)$  which is considered to be fixed. It would be of interest to have analogous results that are uniform in the parameter  $Q$ .

The paper is organized as follows. In Section 2 we give a number of examples illustrating the sharpness of our results. In Section 3 we give an alternative argument that is considerably shorter than the proof of the main

theorems, but which gives only weaker bounds except in the case  $\Lambda \leq \frac{1}{2}$ . The remaining sections are devoted to the proof of the Theorems. In Section 4, we give the proof of Proposition 1, bounding the number of zeros of Dirichlet polynomials. In Section 5 we prove Proposition 3 which shows that, given an approximation of the type hypothesized by the theorems, that approximation continues to hold for all larger values of  $\sigma$ . In Section 6 we prove a number of consequences of our hypothesis of a zero-density bound. We find, with good localization, thin horizontal strips on which there holds a Lindelöf strength bound for  $L$ , and within each of these, a horizontal line on which holds a similar bound for  $L^{-1}$ . These bounds, which are needed for our application of the Littlewood lemma and the Rouché theorem, improve earlier results which would not have sufficed. Finally, in Section 7, we combine the above preparations to complete the proofs of our results.

## 2. - Some Examples and Remarks

### EXAMPLE 1. *Zeros of Dirichlet polynomials*

The finite Euler product  $f(s) = \prod_{p \leq \log x} (1 - p^{-s})$  has length

$$\exp \left( \sum_{p \leq \log x} \log p \right) = x^{1+o(1)}$$

by the prime number theorem and has, for  $T < t \leq 2T$ , zeros on the imaginary axis at  $t = 2n\pi/\log p$ . These number

$$\sum_{p \leq \log x} \left( \left[ \frac{2T \log p}{2\pi} \right] - \left[ \frac{T \log p}{2\pi} \right] \right) \sim \frac{T}{2\pi} \log x,$$

again by the prime number theorem. Thus the bound given in Proposition 1 is asymptotically sharp.

### EXAMPLE 2. *Approximate functional equation*

As is well known, it is possible to approximate the Riemann zeta function, using two Dirichlet polynomials rather than one, in a way which allows shorter polynomials, namely:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1})$$

for  $xy = \frac{|t|}{2\pi}$ ,  $x, y \geq \frac{1}{2}$ , and  $s$  in any fixed vertical strip away from the pole at  $s = 1$ . Here  $\chi(s) = \pi^{\frac{1}{2}-s}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)^{-1}$  appears in the functional equation as  $\zeta(s) = \chi(s)\zeta(1-s)$ .

It is likely that there should be an analogue to our Theorem for approximate functional equations of this type stating that such an approximate functional equation can only hold for  $L(s)$  in the range  $T \leq t \leq 2T$  provided that  $xy > T^{2\Lambda-o(1)}$ .

At first we hoped that our method, based on counting zeros, would lead to this result, but were stopped by the following example which shows that the analogue for Proposition 1 (at least in its obvious form) does not hold.

Take  $L(s)$  to be  $\zeta(s)$ ; take  $x = y = 1$ . Then the “approximation” is  $1 + \chi(s)$  and, despite the fact that  $x$  and  $y$  are bounded, this has asymptotically (in fact, with an error term only  $O(\log T)$ ) as many zeros as  $\zeta(s)$  itself inside the rectangle  $0 \leq \sigma \leq 1$ ,  $T \leq t \leq 2T$ .

### EXAMPLE 3. *Existence of approximations*

It is well-known that smoothed truncations of a Dirichlet series can provide very good approximations. Let  $u(x)$  be a  $C^\infty$  function with compact support in  $(0, 1]$ , such that

$$\int_0^\infty u(t) dt = 1$$

and let

$$v(x) = \int_x^\infty u(t) dt.$$

Thus  $v(0) = 1$  and  $v$  has compact support in  $[0, 1]$ . The Mellin transform  $\tilde{u}(s)$  of  $u(x)$  is entire of exponential type and rapidly decreasing at  $\infty$  (i.e. faster than any negative power of  $s$ ) in any fixed vertical strip, and  $\tilde{u}(1) = 1$ . The Mellin transform of  $v(x)$  is  $s^{-1}\tilde{u}(s+1)$ .

We have the integral formula for inverse Mellin transforms

$$v(x) = \frac{1}{2\pi i} \int_{(c)} \tilde{u}(w+1) x^{-w} \frac{dw}{w}$$

valid for any  $c > 0$  and  $x > 0$ ; here  $(c)$  stands for the vertical line  $\operatorname{Re} w = c$ .

**PROPOSITION 2.** *If  $x > T^{2\Lambda+\epsilon}$  then for every fixed  $N$  and any fixed strip  $A \leq \sigma \leq B$  we have*

$$L(s) = \sum_{n \leq x} \frac{a_n}{n^s} v\left(\frac{n}{x}\right) + O(T^{-N})$$

as  $T$  tends to  $\infty$ .

PROOF. For  $c > 1$ , we may integrate term by term because of total convergence and get, noting that the support of  $v$  is in the interval  $[0, 1]$ :

$$\frac{1}{2\pi i} \int_{(c)} L(s+w) \tilde{u}(w+1) x^w \frac{dw}{w} = \sum_{n \leq x} \frac{a_n}{n^s} v\left(\frac{n}{x}\right).$$

As usual, we shift to the vertical line  $(-c)$  where  $c > 0$ , so that the poles of  $L(w)$  are all in the half-plane  $\operatorname{Re} w > -c$ . In doing so we pick up the residues of the integrand at the poles of  $L(w)$  and at  $w = 0$ . Thus (assuming for simplicity in writing that the poles of  $L(s)$  are all simple) we obtain

$$(2.1) \quad \sum_{n \leq x} \frac{a_n}{n^s} v\left(\frac{n}{x}\right) = L(s) + \sum_{\eta} \operatorname{res}_{w=\eta} L(w) \tilde{u}(\eta - s + 1) \frac{x^{\eta-s}}{\eta - s} + O(I),$$

where  $\operatorname{res}_{w=\eta}$  denotes the residue at the pole  $\eta$  and, because of the fact that  $\tilde{u}(w+1)$  is rapidly decreasing at  $\infty$ ,

$$I = \int_{(-c)} \ll x^{-c} \int_{(-c)} |L(s+w)| \frac{|dw|}{(1+|w|)^K}$$

for any fixed positive constant  $K$ . Under the assumption  $\sigma c < 0$  we have

$$L(s+w) \ll (T+|w|)^{2\Lambda(\frac{1}{2}-(\sigma-c))}.$$

If  $K$  is sufficiently large, as we may suppose, the above integral is absolutely convergent and is

$$\ll T^{2\Lambda(\frac{1}{2}-\sigma)} \left( \frac{T^{2\Lambda}}{x} \right)^c.$$

If  $x > T^{2\Lambda+\epsilon}$  we can make this  $O(T^{-N})$  for any fixed  $N$ , in any fixed strip  $A \leq \sigma \leq B$ , if we choose  $c$  sufficiently large as a function of  $\epsilon, A, B, N$ . Because of the rapid decrease of  $\tilde{u}$ , once  $x$  is specified not to exceed any particular power of  $T$ , the sum over  $\eta$  is also small for large  $T$ .  $\square$

The function  $u(x) = k(1-x)^{k-1}$  for  $0 \leq x \leq 1$ ,  $u(x) = 0$  for  $x > 1$  leads to the approximating polynomial mentioned in Section 1, since then  $v(x) = (1-x)^k$ . However  $u(x)$  does not have compact support in  $(0, 1]$  and its Mellin transform is no longer an entire function of  $s$  (it has poles at  $s = 0, -1, \dots, -k+1$ ) and decays at infinity only like  $|s|^{-k-1}$ .

This example shows that the Theorem is sharp for any function  $L(s)$  to which it applies. A non-trivial example of such is the Dedekind zeta function  $\zeta_K(s)$  of an abelian field  $K$ . In fact in this case (H1) and (H2) are well-known and (H3) is also clear because  $\zeta_K(s)$  splits into the product of the Riemann zeta function and suitable Dirichlet  $L$ -series and Hecke  $L$ -series with Grössencharacters, and for these (H3) is known (in a much stronger form).

**REMARK.** *On the Lindelöf hypothesis for  $L(s)$ , in any fixed strip  $\frac{1}{2} + \delta \leq \sigma \leq B$  we have*

$$L(s) = \sum_{n \leq x} \frac{a_n}{n^s} v\left(\frac{n}{x}\right) + O(x^{-\delta+o(1)})$$

as  $T$  tends to  $\infty$ .

The proof is the same as in Proposition 2, except that this time we shift the integral only to the line  $\frac{1}{2} - \sigma$  rather than  $-c$  with  $c$  large. See also [T, Theorem 13.3] for a result of similar nature.

This result shows that, on the Lindelöf hypothesis, it is possible to approximate  $L(s)$  just to the right of the critical line, to a degree of approximation  $o(1)$  using very short Dirichlet polynomials, whereas our theorems show a very different behaviour just to the left.

### 3. - A Weak Lower Bound

In this section we give the (much easier) proof of the following result.

**WEAK VERSION.** *Under the hypotheses (H1), (H2) and the approximation (1.3) for  $\left\{ \sigma = \frac{1}{2} - \delta, T \leq t \leq 2T \right\}$ , where  $\delta > 0$  is fixed, we have  $x > T^{\theta-o(1)}$ , with  $\theta = \min\left(2\Lambda, \frac{1+4\Lambda\delta}{1+2\delta}\right)$ .*

**PROOF.** By a well-known mean-value estimate (e.g. take  $Q = 1$  in [Bo, Théorème 10]) we have, for fixed  $\delta > 0$ ,

$$(3.1) \quad \begin{aligned} \int_T^{2T} \left| \sum_{n \leq x} \frac{a_n(x)}{n^{\frac{1}{2}-\delta+it}} \right|^2 dt &\ll T \sum_{n \leq x} \frac{|a_n(x)|^2}{n^{1-2\delta}} + \sum_{n \leq x} n^{2\delta} |a_n(x)|^2 \\ &\ll Tx^{2\delta+o(1)} + x^{1+2\delta+o(1)}. \end{aligned}$$

On the other hand, applying first our assumed approximation, and then the functional equation, we get, for each  $t$ ,  $T \leq t \leq 2T$ ,

$$(3.2) \quad \left| \sum_{n \leq x} \frac{a_n(x)}{n^{\frac{1}{2}-\delta+it}} \right| + O(T^{-\epsilon}) \gg \left| L\left(\frac{1}{2} - \delta + it\right) \right| \gg |t|^{2\Lambda\delta} \left| \overline{L}\left(\frac{1}{2} + \delta - it\right) \right|.$$

Squaring (3.2) and integrating over  $t$ , we note that the left hand side of (3.1) is

$$\gg T^{4\Lambda\delta} \int_T^{2T} \left| L\left(\frac{1}{2} + \delta + it\right) \right|^2 dt \gg T^{1+4\Lambda\delta},$$

the last inequality follows easily by the convexity theorem [T, § 7.8] applied to the lines  $\sigma = \frac{1}{2} + \delta$ ,  $\sigma = c$ ,  $\sigma = 2c$ , where  $c$  is so large that  $L(c + it) \gg 1$  uniformly in  $t$ . This gives the result.  $\square$

For  $\Lambda \leq \frac{1}{2}$  this matches the conclusion of Theorem 1. In the more general case in which  $\Lambda > \frac{1}{2}$ , one can approach but not achieve this optimal exponent by assuming the approximation to hold much further to the left. The difficulty in improving this simple-minded argument is that the error term in the square mean-value estimate (3.1) is too large if  $x > T^{1+a}$  with  $a > 0$ . This suggests replacing the square mean-value by mean-values of small fractional order. In the limit, with order tending to 0, this suggests comparing the integral of  $\log|L(s)|$  and  $\log|D_x(s)|$ . This is tantamount to counting zeros, and motivates the approach we will follow next.

#### 4. - Zeros of Dirichlet Polynomials

In this section we prove Proposition 1, that is we give an upper bound for the number of zeros of the Dirichlet polynomial

$$D_x(s) = \sum_{n \leq x} \frac{a_n(x)}{n^s}.$$

We begin by recalling Littlewood's lemma [T, § 9.9]:

**LEMMA 1.** *Let  $f(s)$  be meromorphic in a closed rectangle  $R$  with sides parallel to the coordinate axes. Let  $\nu(\sigma)$ ,  $\alpha \leq \sigma \leq \alpha'$ , denote the number of zeros less the number of poles of  $f$  in this region, having real part  $\geq \sigma$ . These are counted with multiplicity (and given weight  $\frac{1}{2}$  if they occur on the boundary). Then*

$$(4.1) \quad \int_{\alpha}^{\alpha'} \nu(\sigma) d\sigma = -\frac{1}{2\pi i} \int_{\partial R} \log f(s) ds$$

where the integral over the boundary  $\partial R$  is taken in the positive direction.

We apply the lemma to our Dirichlet polynomial  $f(s) = D_x(s)$ , taking  $H \leq T$  and

$$R = \{\alpha \leq \operatorname{Re} s \leq \alpha', \quad T \leq \operatorname{Im} s \leq T + H\}.$$

We have by partial summation,

$$\int_{\alpha}^{\alpha'} \nu(\sigma) d\sigma = \sum_{T \leq \gamma \leq T+H} (\beta - \alpha)^+$$

where  $u^+ = \max(u, 0)$ , where  $\rho = \beta + i\gamma$  runs through the zeros and, since the left hand side of (4.1) is real:

$$\begin{aligned}
 \sum_{T \leq \gamma \leq T+H} (\beta - \alpha)^+ &= \frac{1}{2\pi} \int_T^{T+H} \log |f(\alpha + it)| dt \\
 &\quad - \frac{1}{2\pi} \int_T^{T+H} \log |f(\alpha' + it)| dt \\
 (4.2) \quad &+ \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \arg f(\sigma + i(T+H)) d\sigma \\
 &\quad - \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \arg f(\sigma + iT) d\sigma \\
 &= I_1 + I_2 + I_3 + I_4, \quad \text{say.}
 \end{aligned}$$

We are going to let  $\alpha \rightarrow -\infty$ ,  $\alpha' \rightarrow \infty$ . We have  $f(\infty) = a_1(x)$  and we can assume this is a positive real, since multiplying  $f$  by a unimodular constant will not affect the number of its zeros. For  $\alpha'$  large,  $\log |f(\alpha' + it)|$  is bounded and

$$(4.3) \quad I_2 = O(H).$$

We have  $|a_n(x)| \ll n$  and so  $|f(s)| \ll x^2 \max(1, x^{-\sigma})$ . Thus

$$\log |f(\alpha + it)| \leq O(1) + \max(2, 2 - \alpha) \log x$$

so that

$$(4.4) \quad I_1 \leq O(H) + \max(2, 2 - \alpha) \frac{H}{2\pi} \log x.$$

We next estimate the horizontal integrals using a familiar argument [T, § 9.4], which we repeat here for completeness. First note that, for  $\sigma_0$  large, and  $\sigma \geq \sigma_0$  we have

$$f(\sigma + it) = a_1(x) + O\left(\sum_2^{\infty} \frac{1}{n^{\sigma-1}}\right) = a_1(x) + O(2^{-\sigma})$$

so that  $\arg f = O(2^{-\sigma})$ . This shows that for both horizontal integrals we may write  $\int_{\alpha}^{\alpha'} = \int_{\alpha}^{\sigma_0} + O(1)$ , where  $\sigma_0 \ll 1$ .

Now  $\arg f(s) = \arctan \frac{\operatorname{Im} f}{\operatorname{Re} f}$ . We estimate this on the segment  $\sigma + iT$ ,  $\alpha \leq \sigma \leq \sigma_0$  by bounding the number  $q$  of zeros of  $\operatorname{Re} f$ . Since the change in  $\arg f$  between consecutive zeros is bounded by  $\pi$  (it is either  $\pi$ , 0, or  $-\pi$ ), we have on the entire segment

$$(4.5) \quad |\arg f(\sigma + iT)| \leq \pi(q+2) + O(1).$$

Now  $q$  is the number of real zeros of  $F(z) = \frac{1}{2} \{f(z+iT) + \bar{f}(z-iT)\}$  and we bound this by the total number of zeros in the disk with centre  $z_0 = \sigma_0 + iT$  and radius  $r = 2(\sigma_0 - \alpha)$ .

By Jensen's theorem we have

$$(4.6) \quad \log \left( \prod_{\rho} \frac{r}{|\rho - z_0|} \right) + \log |F(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(z_0 + re^{i\theta})| d\theta,$$

where  $\rho$  runs over zeros in the disk. Since  $\sigma_0$  is large,  $|F(z_0)|$  is bounded away from zero so that  $\log |F(z_0)|$  is bounded. The integral on the right hand side is bounded by

$$(4.7) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |F(z_0 + re^{i\theta})| d\theta \leq O((|\alpha| + \sigma_0) \log x) \leq O((|\alpha| + 1) \log x).$$

We also have  $\prod_{\rho} (r|\rho - z_0|^{-1}) \geq 2^q$ , since we get a contribution of at least 1 for each zero in the disk and, of these, the  $q$  zeros being counted each give a contribution at least 2. Combining the last three estimates with (4.6), (4.7) we deduce that  $q \ll (|\alpha| + 1) \log x$ ; therefore from this and (4.5) we find

$$\arg f(\sigma + iT) \ll (|\alpha| + 1) \log x$$

uniformly for  $\alpha < \sigma < \infty$ , which is the first statement of Proposition 1.

It also follows that the horizontal integrals satisfy

$$(4.8) \quad I_3, I_4 \ll (|\alpha| + 1)^2 \log x.$$

Then, by combining (4.3), (4.4), and (4.8) in (4.2), we get for  $\alpha < 0$ :

$$\sum_{T \leq \gamma \leq T+H} (\beta - \alpha)^+ \leq (2 - \alpha) \frac{H}{2\pi} \log x + O(H) + O((|\alpha| + 1)^2 \log x).$$

Let  $-1 > \alpha_0 > \frac{1}{2} \alpha$ . Then *a fortiori*

$$(\alpha_0 - \alpha) N(\alpha_0, T, T + H; D_x) \leq (2 - \alpha) \frac{H}{2\pi} \log x + O(H) + O(|\alpha|^2 \log x).$$

Proposition 1 follows on choosing  $\alpha = -2\sqrt{H|\alpha_0|}$ , and then writing  $\alpha$  in place of  $\alpha_0$ .  $\square$

## 5. - Extending the Region

In this section we prove the following result.

**PROPOSITION 3.** *Assume that for  $\sigma = \sigma_1$ ,  $T \leq t \leq (1 + \alpha)T$  we have*

$$(1.3) \quad L(s) = D_x(s) + O(T^{-\varepsilon}).$$

*Then (1.3) also holds for  $\sigma \geq \sigma_1$ ,  $\left(1 + \frac{1}{3}\alpha\right)T \leq t \leq \left(1 + \frac{2}{3}\alpha\right)T$ .*

**PROOF.** Consider, for  $\sigma > \sigma_1$ , the integral

$$(5.1) \quad I = \frac{1}{2\pi i} \int_{(c)} (L(s+w) - D_x(s+w)) G(w) \frac{dw}{w}$$

where  $c = \sigma_1 - \sigma < 0$ , where  $D_x(s) = \sum_{n=1}^x a_n(x)n^{-s}$  and where  $G(w)$  is an entire function of  $w$ , with  $G(0) = 1$ , bounded by  $O(\exp(\delta \operatorname{Re} w)(1 + |w|)^{-K})$  in the half-plane  $\operatorname{Re} w > 0$ . Here  $0 < \delta < \log 2$  is a small positive constant and  $K$  is any fixed large constant. Such functions  $G(w)$  can be constructed easily taking the Fourier transform of a smooth function with compact support.

It is clear from  $c + \sigma = \sigma_1$  that  $I \ll T^{-\varepsilon}$  for  $\left(1 + \frac{1}{3}\alpha\right)T \leq \operatorname{Im} s \leq \left(1 + \frac{2}{3}\alpha\right)T$ . The approximation suffices to deal with the integrand in the range  $|\operatorname{Im} w| \leq \frac{1}{3}\alpha T$ , and beyond this range  $G(w)$  is so small as to offset the polynomial growth of  $L(s+w)$  and  $D_x(s+w)$ .

Now move the line of integration to the right, which we may because the integral is absolutely convergent. We encounter the residue  $L(s) - D_x(s)$  at the simple pole at  $w = 0$ , and possibly the residues at the finitely many poles of  $L(s+w)$ , say at points  $w = \eta - s$ . Since  $G(\eta - s)$  decreases with  $t = \operatorname{Im} s$  faster than any negative power of  $t$ , the residues from the poles contribute very little.

Hence

$$L(s) = D_x(s) - \frac{1}{2\pi i} \int_{(c')} (L(s+w) - D_x(s+w)) G(w) \frac{dw}{w} + O(T^{-\varepsilon})$$

provided  $c'$  is sufficiently large. This estimate is uniform in  $c'$ . The integral now is estimated trivially by

$$\frac{1}{2\pi i} \int_{(c')} (a_1 - a_1(x)) G(w) \frac{dw}{w} + O \left( \exp(\delta c') \sum_2^\infty |a_n - a_n(x)| n^{-(\sigma+c')} \right).$$

The sum is  $\ll 2^{-c'}$ , therefore since  $\delta < \log 2$  letting  $c' \rightarrow \infty$  we see that the integral tends to  $(a_1 - a_1(x)) \cdot \text{const.}$  as  $c'$  tends to  $\infty$ . If we next multiply (1.3) by  $G \left( w - i \left( 1 + \frac{1}{2}\alpha \right) T \right)$  and integrate over the segment  $s = \sigma_1 + it$ ,  $T < t < (1 + \alpha)T$ , we find in a similar manner that  $a_1 - a_1(x) \ll T^{-\varepsilon}$ . This completes the proof.  $\square$

## 6. - Local Estimates

In this section we prove several estimates for  $L(s)$  and  $D_x(s)$  valid in thin horizontal strips. On the assumption of a zero density hypothesis, we find many such strips where there is a local bound of the same strength as the Lindelöf Hypothesis, and this leads to a number of consequences.

There are several techniques available for dealing with thin horizontal strips, see [T, §§ 9.11–9.13], and we follow here Littlewood's and Hoheisel's method based on conformal mapping. Another elegant way of obtaining such bounds is due to Siegel [Si].

**PROPOSITION 4.** *Let  $R_2$  be a bounded, simply-connected domain with non-empty sub-domains  $R_0 \subset R_1$ ,  $\overline{R}_1 \subset R_2$ . Then, there exist constants  $c, c', C$  with  $0 < c < 1$  depending only on the regions  $R_j$ , with the following properties:*

*Suppose  $f$  is regular analytic in  $R_2$ , that  $|f| \leq M$  there, and  $M^\varepsilon \geq |f| \geq M^{-\varepsilon}$  in  $R_0$ . Suppose also  $f$  has  $\leq \varepsilon \log M$  zeros in  $R_2$  where  $(\log M)^{-1} < \varepsilon < 1$ . Then, there is a subset  $S$  of the zeros of  $f$  containing all zeros in  $R_1$ , such that the distance  $d(S, \partial R_2)$  exceeds  $c'$  and the bounds*

$$M^{-\eta} \leq \left| \frac{f(z)}{\prod_{\rho \in S} (z - \rho)} \right| \leq M^\eta$$

and

$$|f(z)| \leq M^{2\eta}$$

hold for all  $z$  in  $R_1$ , with  $\eta = C\varepsilon^c$ .

PROOF. Note, by hypothesis,  $M \geq 1$ .

By mapping  $R_2$  conformally onto a disk, and then enlarging  $R_1$  and shrinking  $R_0$  if necessary, we may assume  $R_j = \{|z| < r_j\}$  with  $0 < r_0 < r_1 < r_2$ .

We choose three further radii  $r_3, r_4, r_5$  with  $r_1 < r_3 < r_4 < r_2$ ,  $0 < r_5 < r_0$  and define  $f_1(z) = f(z) \prod_{\rho \in R_4} (z - \rho)^{-1}$  where  $R_4 = \{|z| < r_4\}$ . Clearly  $f_1 \neq 0$  in  $R_4$ . Thus  $g(z) = \log f_1(z)$ , with the principal branch for  $\log f_1(0)$ , is regular analytic in  $R_4$ .

Let  $N$  be number of zeros of  $f$  in  $R_4$ . Then, on  $\partial R_2$  we have

$$c_1^{-N} \leq \left| \prod_{\rho \in R_4} (z - \rho) \right| \leq c_1^N$$

with  $c_1 = \max((r_2 - r_4)^{-1}, r_2 + r_4)$ . By the maximum principle,  $|f_1| \leq M c_1^N$  in  $R_2$ . Also, the assumption that  $|f(z)| \geq M^{-\varepsilon}$  in  $R_0$  shows that  $f(z) \neq 0$  in  $R_0$  and therefore

$$c_2^N \geq \left| \prod_{\rho \in R_4} (z - \rho) \right| \geq c_2^{-N}, \quad \text{for } |z| \leq r_5,$$

with  $c_2 = \max((r_0 - r_5)^{-1}, r_4 + r_5)$ . Now,  $M^{-\varepsilon} c_2^{-N} \leq |f_1(z)| \leq M^\varepsilon c_2^N$  in  $R_5$ . Therefore  $\operatorname{Re} g(z) \leq c_3 \log M$  in  $R_4$  and  $|g(0)| \leq c_4 \varepsilon \log M + \pi \leq c_4 \log M + \pi$  with  $c_3 = 1 + \log c_1$  and  $c_4 = 1 + \log c_2$ . It follows from the Borel-Carathéodory theorem that

$$(6.1) \quad |g(z)| \leq c_5 (\log M + \pi)$$

in the disk  $R_3 \subset R_4$ , specifically with  $c_5 = \frac{r_4 + r_3}{r_4 - r_3} (c_3 + c_4)$ . Now, we have

$$\operatorname{Re} g(z) \leq c_4 \varepsilon \log M$$

for  $z$  in  $R_5$  so, if  $0 < r_6 < r_5$ , another application of the Borel-Carathéodory theorem shows that, in  $R_6 = \{|z| < r_6\}$ ,

$$(6.2) \quad |g(z)| \leq c_6 (\varepsilon \log M + \pi)$$

with  $c_6 = \frac{r_5 + r_6}{r_5 - r_6} (2c_4 + 1)$ . Next apply Hadamard's three circle theorem to  $g(z)$  and  $R_6 \subset R_1 \subset R_3$ . We deduce from (6.1), (6.2) that, in  $R_1$

$$|g(z)| \leq (c_6 (\varepsilon \log M + \pi))^a (c_5 (\log M + \pi))^{1-a}$$

with  $a = \log \left( \frac{r_3}{r_1} \right) \log \left( \frac{r_3}{r_6} \right)^{-1}$ .

Since, by hypothesis,  $\varepsilon \log M \geq 1$  we get

$$|g(z)| \leq c_7 \varepsilon^a \log M$$

in  $R_1$  with e.g.  $c_7 = c_6^a c_5^{1-a} (1 + \pi)$ . This gives Proposition 4, taking for  $S$  the pre-image of the zeros in  $R_4$  under the conformal map.  $\square$

**PROPOSITION 5.** *Let  $R_0, R_1, R_2$  be as in Proposition 4. Let  $f$  be regular analytic in  $R_2$  and suppose that  $|f| \leq M$  on  $R_2$  and  $|f| \geq M^{-1}$  in  $R_0$ . Then, the number of zeros of  $f$  in  $R_1$  does not exceed  $C' \log M$  with  $C'$  determined by the  $R_j$  and independent of  $f$ .*

**PROOF.** As in the previous proof, we may suppose the  $R_j$  are concentric disks. The result is then a straightforward application of Jensen's theorem as in the proof of Proposition 1.  $\square$

**LEMMA 2.** *Let  $f = f(z)$  be a regular analytic function of  $z = x + iy$  in the rectangle  $R = \{-1 \leq x \leq 1, -\lambda \leq y \leq \lambda\}$ , where  $\lambda > 0$ . Suppose further that  $|f(z)| \leq M_1$  on the horizontal sides  $y = \pm\lambda$  of  $R$ , and  $|f(z)| \leq M_2$  on the vertical sides  $x = \pm 1$  of  $R$ , and that  $M_1 \geq M_2$ .*

Let  $\alpha = \left( \cosh \left( \frac{1}{3} \pi \lambda \right) - 1 \right)^{-1}$ . Then for  $-1 \leq x \leq 1$  we have the bound:

$$|f(x)| \leq M_1^\alpha M_2^{1-\alpha}.$$

**PROOF.** The most elegant argument uses harmonic measure to majorize  $f(z)$  inside  $R$  (cf. [H, § 18.3, Th. 18.3.2]). Let  $\omega(z)$  be the function harmonic in  $R$  with boundary values 1 on the horizontal sides and 0 on the vertical sides of  $R$ . Then we have

$$|f(z)| \leq M_1^{\omega(z)} M_2^{1-\omega(z)} \quad \text{for } z \in R.$$

In practice, one can use the maximum principle as follows. Let  $B > 1$  and consider the function

$$(6.3) \quad f_0(z) = B^{-\cos(\frac{1}{3}\pi z)} f(z).$$

We have  $\operatorname{Re} \cos(x + iy) = \cos(x) \cosh(y)$ , hence

$$\frac{1}{2} \cosh \left( \frac{1}{3} \pi y \right) \leq \operatorname{Re} \cos \left( \frac{1}{3} \pi z \right) \leq \cosh \left( \frac{1}{3} \pi y \right) \quad \text{for } -1 \leq x \leq 1.$$

This gives

$$\left| B^{-\cos(\frac{1}{3}\pi z)} \right| \leq B^{-\frac{1}{2} \cosh(\frac{1}{3}\pi y)}$$

on the horizontal sides of  $R$  and

$$|B^{-\cos(\frac{1}{3}\pi z)}| \leq B^{-\frac{1}{2}}$$

in  $R$ . It follows that

$$(6.4) \quad |f_0(z)| \leq \max(B^{-\frac{1}{2}\cosh(\frac{1}{3}\pi\lambda)}M_1, B^{-\frac{1}{2}}M_2)$$

on the boundary of  $R$ , and hence inside  $R$  by the maximum principle. Lemma 2 follows from (6.3) and (6.4), choosing  $B = M_1^{2\alpha}M_2^{-2\alpha}$ .  $\square$

We are now ready to specialize the above general results to the functions  $L$  and  $D$ . We fix  $H \geq 1$ ,  $\frac{1}{100} > \delta > 0$ , and subdivide the interval  $T \leq t \leq 2T$  into  $\left[\frac{T}{H}\right]$  subintervals  $I_\nu = \left\{ |tt_\nu| \leq \frac{1}{2}H \right\}$ , plus possibly a last interval of length  $< H$ . Let  $\varepsilon = \varepsilon(T)$  be a function of  $T \rightarrow 0$  as  $T \rightarrow \infty$ . Call  $I_\nu$  ‘good’ if  $L(s)$  has at most  $\varepsilon(T) \log T$  zeros in the half strip  $\sigma \geq \frac{1}{2} + \delta$ ,  $t \in I_{\nu-1} \cup I_\nu \cup I_{\nu+1}$ . We denote by  $A \geq 1$  a constant such that  $|L(s)| \leq T^A$  ( $= M$ , say) for all sufficiently large  $T$ , say  $T \geq T_0$ . We fix  $\sigma_0$  sufficiently large that both  $L$  and  $D$  are uniformly bounded (away from zero and infinity) for  $\sigma > \sigma_0$ . For each good  $I_\nu$ , we apply Proposition 4, choosing  $f(s) = L(s)$  and

$$\begin{aligned} R_2 &= \left\{ s : \frac{1}{2} + \delta < \sigma < \sigma_0 + 2, |t - t_\nu| < \frac{3}{2}H \right\} \\ R_1 &= \left\{ s : \frac{1}{2} + 2\delta < \sigma < \sigma_0 + \frac{3}{2}, |t - t_\nu| < \frac{5}{4}H \right\} \\ R_0 &= \left\{ s : \sigma_0 < \sigma < \sigma_0 + 1, |t - t_\nu| < \frac{1}{2}H \right\}. \end{aligned}$$

It follows that for  $T \geq T_1$  where  $T_1$  depends on  $H, \delta, \sigma_0, A$ , we have for  $s \in R_1$ ,  $|L(s)| \leq T^{2CA\varepsilon^c}$ . Here  $C$  and  $c$  are given by Proposition 4 as functions of  $H, \delta$ , and  $\sigma_0$ .

We abbreviate  $\eta = CA\varepsilon^c$ , keeping in mind that  $\eta = \eta(T) \rightarrow 0$  as  $T \rightarrow \infty$ . By the functional equation and conjugation we deduce

$$(6.5) \quad |L(s)| \ll T^{2\eta + 2\Lambda \max(\frac{1}{2} - \sigma, 0)}$$

for  $\sigma \leq \frac{1}{2} - 2\delta$ ,  $|t - t_\nu| \leq H$ . For each  $\tau$  with  $|\tau - t_\nu| \leq H$ , we apply Lemma 2, after re-scaling, to the function  $L(s)$  in the rectangle  $\frac{1}{2} - 2\delta \leq \sigma \leq \frac{1}{2} + 2\delta$ ,  $|t - \tau| \leq \frac{1}{4}H$ . We obtain, after a simple majorization,

$$L(s) \ll T^{2\eta + c_1 \delta},$$

with some constant  $c_1 = c_1(\Lambda, A)$  for

$$\frac{1}{2} - 2\delta \leq \sigma \leq \frac{1}{2} + 2\delta, \quad |t - t_\nu| \leq H.$$

We conclude that locally we have the Lindelöf hypothesis bound, namely:

$$(6.6) \quad L(s) \ll T^{2\Lambda \max(\frac{1}{2} - \sigma, 0) + c_1 \delta + 2\eta}$$

in the whole strip  $|t - t_\nu| \leq H$ , where  $\eta \rightarrow 0$  as  $T \rightarrow \infty$ .

**REMARK.** Actually the lemma allows us to replace  $c_1 \delta$  by  $c_2 \exp\left(-\frac{1}{3\delta}\right)$ . This may be useful in other situations.

We now apply Proposition 4, again to  $L(s)$ , but (since we shall soon be needing (6.6)) we now use the regions

$$\begin{aligned} R_2 &= \left\{ s : \frac{1}{2} + \delta < \sigma < \sigma_0 + 2, \quad |t - t_\nu| < H \right\} \\ R_1 &= \left\{ s : \frac{1}{2} + 2\delta < \sigma < \sigma_0 + \frac{3}{2}, \quad |t - t_\nu| < \frac{3}{4}H \right\} \\ R_0 &= \left\{ s : \sigma_0 < \sigma < \sigma_0 + 1, \quad |t - t_\nu| < \frac{1}{2}H \right\}. \end{aligned}$$

With  $L_1(s) = L(s) \prod_{\rho \in S} (s - \rho)^{-1}$  we find that, for any  $s^* = \frac{1}{2} + 3\delta + it^*$  with  $|t^* - t_\nu| \leq \frac{1}{2}H$ ,

$$(6.7) \quad T^{-\eta'} \leq |L_1(s^*)| \leq T^{\eta'},$$

with some  $\eta' = \eta'(T) \rightarrow 0$  as  $T \rightarrow \infty$ . Let  $R'_2$  and  $R'_1$  be the expanded rectangles

$$\begin{aligned} R'_2 &= \left\{ s : \frac{1}{2} - 21\delta < \sigma < \sigma_0 + 2, \quad |t - t_\nu| < H \right\} \\ R'_1 &= \left\{ s : \frac{1}{2} - 21\delta < \sigma < \sigma_0 + \frac{3}{2}, \quad |t - t_\nu| < \frac{3}{4}H \right\}. \end{aligned}$$

Then  $d(S, \partial R'_2) \geq d(S, \partial R_2) \geq c' > 0$ . Therefore,  $\max |L_1(s)| \leq \max |L(s)| c'^{-N}$  on  $\partial R'_2$  where  $N \leq \varepsilon \log M = \varepsilon A \log T$  is the number of zeros in  $S$ . By the local Lindelöf estimate (6.6),  $\max |L(s)| \ll T^{c_8 \delta}$  on  $\partial R'_2$ , and so, by the maximum principle,

$$(6.8) \quad |L_1(s)| \ll T^{2c_8 \delta}$$

in all of  $R'_2$ , as soon as  $T$  is sufficiently large.

We apply Jensen's theorem to the circles centred at  $s^*$  with radii  $24\delta$  and  $12\delta$ , obtaining, by (6.7), (6.8) that the number  $N'$  of zeros of  $L_1(s)$ , hence also the number  $N + N'$  of zeros of  $L(s)$ , in the disk  $|s - s^*| \leq 12\delta$  satisfy

$$(6.9) \quad N', N + N' \ll \delta \log T.$$

Note that  $|\rho - s^*| \geq \delta$  by our conditions on  $s^*$ . Therefore

$$\left| L_1(s^*) \prod_{|\rho - s^*| \leq 12\delta} (s^* - \rho)^{-1} \right| \geq |L_1(s^*)| \delta^{-N'}.$$

Thus, with  $L_2(s) = L_1(s) \prod (s - \rho)^{-1}$ , we have, by (6.9),

$$(6.10) \quad T^{c_9 \delta \log \frac{1}{\delta}} \gg |L_2(s^*)| \gg T^{-c_9 \delta \log \frac{1}{\delta}}.$$

Apply the Landau lemma [T, § 3.9 Lemma  $\alpha$ ] to  $L_1$  getting

$$\frac{L'_1(s)}{L_1(s)} - \sum'_{|\rho - s^*| \leq 12\delta} \frac{1}{s - \rho} \ll \log T$$

for  $|s - s^*| \leq 6\delta$ . Here  $\sum'$  restricts the summation to zeros of  $L_1$ . Integrating and taking real parts we get

$$\log |L_1(s)| - \sum'_{|\rho - s^*| \leq 12\delta} \log |s - \rho| \log |L_2(s^*)| \ll \delta \log T$$

and therefore, (6.10) gives, for  $|s - s^*| \leq 6\delta$ ,

$$\log |L_1(s)| - \sum'_{|\rho - s^*| \leq 12\delta} \log |s - \rho| \ll \delta \log \frac{1}{\delta} \log T$$

The zeros in  $S$  but not in  $|s - s^*| \leq 12\delta$  have distance  $\geq 6\delta$  from the disk  $|s - s^*| \leq 6\delta$  and their number is  $\leq \varepsilon A \log T$  with  $\varepsilon \rightarrow 0$  as  $T \rightarrow \infty$ . We conclude that

$$(6.11) \quad \log |L(s)| - \sum_{|\rho - s^*| \leq 12\delta} \log |s - \rho| \ll \delta \log \frac{1}{\delta} \log T$$

holds in the disk  $|s - s^*| \leq 6\delta$ , for any  $s^* = \frac{1}{2} + 3\delta + i\tau$  with  $|\tau - t_\nu| \leq \frac{1}{2} H$ .

**REMARK.** *The above may be interpreted as a local version of [T, Theorem 14.15] which gives a result that is both global and sharper, but only under the assumption of the Riemann Hypothesis.*

Now we need to do the same with  $D_x$  in place of  $L$ . We begin by observing that, provided  $\delta \leq \frac{1}{21} \varepsilon'$ , the approximation yields, for  $\sigma \geq \frac{1}{2} - 21\delta$ , the same bounds (6.5) and (6.6) for  $D_x$  as for  $L$ .

Apply Proposition 5 with

$$\begin{aligned} R_2 &= \left\{ s : \frac{1}{2} + 2\delta < \sigma < \sigma_0 + 2, |t - t_\nu| < H \right\} \\ R_1 &= \left\{ s : \frac{1}{2} + \frac{5}{2}\delta < \sigma < \sigma_0 + \frac{3}{2}, |t - t_\nu| < \frac{3}{4}H \right\} \\ R_0 &= \left\{ s : \sigma_0 < \sigma < \sigma_0 + 1, |t - t_\nu| < \frac{1}{2}H \right\}. \end{aligned}$$

By the bound (6.5) we see that the number of zeros of  $D_x$  in  $R_1$  does not exceed  $2C'\eta \log T$  and  $\eta \rightarrow 0$  as  $T \rightarrow \infty$ .

With very minor changes the argument now proceeds as before giving the bound (6.9) and also the formula (6.11) as stated, but with  $L$  replaced by  $D_x$ . We have proven:

**LEMMA 3.** *In a good interval  $I_\nu$  we have, with  $f(s) = L(s)$  or  $= D_x(s)$  (provided, in the latter case, we have (1.3)), and for any  $s^* = \frac{1}{2} + 3\delta + i\tau$  having  $|\tau - t_\nu| \leq \frac{1}{2}H$ :*

$$(6.11) \quad \log |f(s)| = \sum_{|\rho - s^*| \leq 12\delta} \log |s - \rho| + O\left(\delta \log \frac{1}{\delta} \log T\right),$$

for  $|s - s^*| \leq 6\delta$  where  $\rho$  runs over zeros of  $f$ ; these number  $\ll \delta \log T$ .

**COROLLARY 1.** *Let  $\theta > 0$  and let  $E$  be the set of those  $t \in [T, 2T]$  for which  $\left|L\left(\frac{1}{2} - \delta + it\right)\right| \leq T^{-\theta}$ . Then, for every good interval  $I_\nu$  and  $\tau \in I_\nu$ , we have*

$$(6.12) \quad \text{meas}(E \cap [\tau - \delta, \tau + \delta]) \ll \theta^{-1} \delta^2 \log \frac{1}{\delta}.$$

**PROOF.** For any set  $E$  of reals having measure  $\mu$ ,

$$(6.13) \quad \int_E \log |t| dt \geq \int_{-\mu/2}^{\mu/2} \log |y| dy = -\mu \left(1 + \log \frac{2}{\mu}\right).$$

Denote by  $E_\delta = E_\delta(\tau)$  the set in question. By Lemma 3,

$$\begin{aligned} \text{meas}(E_\delta) \theta \log T &\geq \int_{E_\delta} \log \left| L\left(\frac{1}{2} - \delta + it\right) \right| dt \\ &= \sum_{|\rho - s^*| \leq 12\delta} \int_{E_\delta} \log \left| \frac{1}{2} - \delta + it - \rho \right| dt \\ &\quad - O\left(\delta^2 \log \frac{1}{\delta} \log T\right) \\ &\geq -O(\delta^2 \log \frac{1}{\delta} \log T), \end{aligned}$$

using

$$(6.14) \quad |s - \rho| \geq |t - \gamma|,$$

then (6.13), then (6.9). The conclusion follows.  $\square$

**COROLLARY 2.** (Safe tracks through the strip). *Let  $N$  denote the number of zeros of  $L(s)$  with  $\sigma \geq \frac{1}{2} + \delta$ , and  $T < t < 2T$ . Assume  $N \leq \frac{1}{2} T$ . Then any subinterval of  $[T, 2T]$  of length  $3(N+1)$  contains a point  $t^*$  such that, for all  $\sigma$ ,*

$$(6.15) \quad L(\sigma + it^*) \gg T^{2\Lambda \max(\frac{1}{2}\sigma, 0) - C\delta \log \frac{1}{\delta}}.$$

**PROOF.** We subdivide the subinterval into  $3(N+1)$  further subintervals of length one and choose three consecutive ones for which  $L$  has none of the  $N$  zeros. Let  $J$  denote the middle subinterval of the three. Apply Proposition 4 with  $\varepsilon = \frac{1}{A \log T}$  to deduce that, for  $\sigma \geq \frac{1}{2} + 2\delta$ ,  $t \in J$  we have

$$T^{-o(1)} \ll L(s) \ll T^{o(1)}.$$

By the functional equation and conjugation, this proves the desired bound for all  $s$  with  $t \in J$  except for the part of the strip where  $\frac{1}{2} - 2\delta \leq \sigma \leq \frac{1}{2} + 2\delta$ .

We choose  $\tau$  as the midpoint of  $J$ ,  $s^* = \frac{1}{2} + 3\delta + i\tau$  and note that

$$\left\{ \frac{1}{2} - 2\delta \leq \sigma \leq \frac{1}{2} + 2\delta, |t - \tau| \leq \delta \right\} \subset \{|s - s^*| \leq 6\delta\}.$$

By Lemma 3 and (6.14),

$$\min_{|\sigma - \frac{1}{2}| \leq 2\delta} |\log L(\sigma + it)| \geq \sum_{|\rho - s^*| \leq 12\delta} \log |t - \gamma| - O\left(\delta \log \frac{1}{\delta} \log T\right).$$

Integrating this, then using (6.13), (6.9)

$$\int_{\tau-\delta}^{\tau+\delta} \min_{|\sigma - \frac{1}{2}| \leq 2\delta} |\log L(\sigma + it)| dt \geq -O\left(\delta^2 \log \frac{1}{\delta} \log T\right),$$

guaranteeing the existence of the required  $t^*$ .  $\square$

## 7. - Proofs of the Theorems

PROOF OF THEOREM 1. Let  $T_1 = \frac{4}{3}T$ ,  $T_2 = \frac{5}{3}T$ . We subdivide  $[T_1, T_2]$  into  $\left[\frac{1}{3}T\right]$  intervals  $I_\nu = \left\{t : |t - t_\nu| \leq \frac{1}{2}\right\}$  plus a last interval of length at most 1. Let  $0 < \delta < \frac{1}{100}\varepsilon'$ . By hypothesis (H3) we have,

$$(7.1) \quad N\left(\frac{1}{2} + \delta^2, T_1, T_2; L\right) \leq \eta(T)T \log T$$

for a certain  $\eta(T) \rightarrow 0$  as  $T \rightarrow \infty$ . We say that  $I_\nu$  is ‘good’ if the number of zeros of  $L(s)$  in  $\left\{\sigma \leq \frac{1}{2} + \delta^2, t \in I_{\nu-1} \cup I_\nu \cup I_{\nu+1}\right\}$  does not exceed  $\sqrt{\eta(T)} \log T$  (and that it is “bad” otherwise). This clearly implies that  $I_\nu$  is also good in the sense of § 6 with  $H = 1$ ,  $\varepsilon(T) = \sqrt{\eta(T)}$ , and  $\delta$  replaced by  $\delta^2$ .

We apply Littlewood’s lemma to  $D_x$  in each good interval, with  $\alpha = \frac{1}{2} - \delta$ ,  $\alpha' \rightarrow \infty$ , obtaining

$$(7.2) \quad \begin{aligned} \sum_{\gamma \in I_\nu}^{(D)} (\beta - \alpha)^+ &= \frac{1}{2\pi} \int_{I_\nu} \log |D_x(\alpha + it)| dt \\ &\quad - \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \arg D_x \left( \sigma + i \left( t_\nu - \frac{1}{2} \right) \right) d\sigma \\ &\quad + \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \arg D_x \left( \sigma + i \left( t_\nu + \frac{1}{2} \right) \right) d\sigma + O(1). \end{aligned}$$

Note that the sum is over zeros of  $D_x$  although the goodness of the interval is relative to zeros of  $L$ . By hypothesis (1.3) holds on the segment  $\sigma = \frac{1}{2} - \varepsilon'$ ,

$T \leq t \leq 2T$ , and hence, by Proposition 3, also holds in the region  $\sigma \geq \frac{1}{2} - 21\delta$ ,  $T_1 < t < T_2$ . Let

$$E = \left\{ t : |L(\alpha + it)| \leq T^{-\frac{1}{2}\varepsilon} \right\}.$$

We split the integral  $\int_{I_\nu}$  as follows:

$$\int_{I_\nu} \log |D_x(\alpha + it)| dt = \int_{I'_\nu} + \int_{I''_\nu}$$

where  $I''_\nu = I_\nu \cap E$  and  $I'_\nu$  is the complement. In  $I'_\nu$  we have

$$\log |D_x| = \log |L| + \log |1 + L^{-1}(D_x - L)|, \quad L^{-1}|D_x - L| \ll T^{-\frac{1}{2}\varepsilon},$$

and therefore

$$(7.3) \quad \int_{I'_\nu} \log |D_x(\alpha + it)| dt = \int_{I'_\nu} \log |L(\alpha + it)| dt + O\left(T^{-\frac{1}{2}\varepsilon}\right).$$

To treat  $I''_\nu$  we further subdivide the interval  $I_\nu$  into  $\ll \delta^{-1}$  subintervals  $I_{\nu,m}$  of length  $\delta$  (plus possibly one that is shorter) and write  $I''_{\nu,m} = I_{\nu,m} \cap E$ . We have

$$\int_{I''_\nu} \log |D_x(\alpha + it)| dt = \sum_m \int_{I''_{\nu,m}} \log |D_x(\alpha + it)| dt.$$

Let  $s_m^* = \frac{1}{2} + 3\delta + i\tau_m$  where  $\tau_m$  is the midpoint of  $I_{\nu,m}$ . By Lemma 3 and (6.14),

$$\log |D_x(\alpha + it)| \geq \sum_{|\rho - s_m^*| \leq 12\delta} \log |t - \gamma| - O\left(\delta \log \frac{1}{\delta} \log T\right)$$

for  $t \in I_{\nu,m}$  where  $\rho$  runs through zeros of  $D_x(s)$ . Denoting  $\mu = \text{meas}(I''_{\nu,m})$ ,

$$\int_{I''_{\nu,m}} \log |D_x(\alpha + it)| dt \geq \sum_{|\rho - s_m^*| \leq 12\delta} \int_{I''_{\nu,m}} \log |t - \gamma| dt - O\left(\mu \delta \log \frac{1}{\delta} \log T\right).$$

Using (6.13) and the bound  $\ll \delta \log T$  of Lemma 3 for the number of zeros,

$$\int_{I''_{\nu,m}} \log |D_x(\alpha + it)| dt \geq O\left(\mu \delta \log \frac{1}{\mu} \log T\right) - O\left(\mu \delta \log \frac{1}{\delta} \log T\right).$$

By (6.12) we have  $\mu \ll \varepsilon^{-1} \delta^2 \log \frac{1}{\delta}$  whence

$$\int_{I''_{\nu,m}} \log |D_x(\alpha + it)| dt \geq -O\left(\delta^3 \left(\log \frac{1}{\delta}\right)^2 \log T\right),$$

the implied constant depending on  $\varepsilon$ . Summing over  $m$ ,

$$(7.4) \quad \int_{I''_\nu} \log |D_x(\alpha + it)| dt \geq -O\left(\left(\delta \log \frac{1}{\delta}\right)^2 \log T\right).$$

Now we find an upper bound for the analogous integral for  $L$ . We note that, since the local Lindelöf estimate holds for  $L$  in  $I_\nu$ ,

$$\int_{I''_{\nu,m}} \log |L(\alpha + it)| dt \leq O(\mu \delta \log T) = O\left(\delta^3 \log \frac{1}{\delta} \log T\right)$$

and hence

$$(7.5) \quad \int_{I''_\nu} \log |L(\alpha + it)| dt \leq O\left(\delta^2 \log \frac{1}{\delta} \log T\right).$$

From the decomposition

$$\begin{aligned} \int_{I_\nu} \log |D_x| dt &= \int_{I_\nu} \log |L| dt + \int_{I'_\nu} (\log |D_x| - \log |L|) dt \\ &\quad + \int_{I''_\nu} \log |D_x| dt - \int_{I''_\nu} \log |L| dt, \end{aligned}$$

it follows that

$$\int_{I_\nu} \log |D_x| dt \geq \int_{I_\nu} \log |L| dt - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 \log T\right),$$

on estimating the last three integrals by (7.3), (7.4), (7.5). We insert this in

(7.2) and sum over all good intervals  $I_\nu$ , obtaining

$$\begin{aligned} \sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta\alpha)^+ &\geq \sum_{\text{good}} \frac{1}{2\pi} \int_{I_\nu} \log |L(\alpha + it)| dt \\ &\quad - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 T \log T\right) - O(T) \\ &\quad + \sum_{\text{good}} \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \left( \arg D_x \left( \sigma + i \left( t_\nu + \frac{1}{2} \right) \right) \right. \\ &\quad \left. - \arg D_x \left( \sigma + i \left( t_\nu - \frac{1}{2} \right) \right) \right) d\sigma. \end{aligned}$$

The last sum telescopes into

$$\begin{aligned} \frac{1}{2\pi} \int_{\alpha}^{\alpha'} & \left( \arg D_x(\sigma + iT_2) - \arg D_x(\sigma + iT_1) \right) d\sigma \\ & - \sum_{\text{bad}} \frac{1}{2\pi} \int_{\alpha}^{\alpha'} \left( \arg D_x \left( \sigma + i \left( t_\nu + \frac{1}{2} \right) \right) \right. \\ & \quad \left. - \arg D_x \left( \sigma + i \left( t_\nu - \frac{1}{2} \right) \right) \right) d\sigma \end{aligned}$$

with an obvious notational adjustment required in the sum for the term corresponding to the last subinterval in case it is shorter than one (it is bad by definition). The total contribution from these horizontal integrals is, due to (4.8), bounded by

$$O(\log x (\#(\text{bad } I_\nu) + 1)).$$

The number of bad  $I_\nu$  is estimated as follows:

$$\eta(T)T \log T \geq N\left(\frac{1}{2} + \delta^2, T_1, T_2; L\right) \geq \frac{1}{3} \sum_{\text{bad}} \sqrt{\eta(T)} \log T.$$

Hence the number of bad  $I_\nu$  is  $\leq 3\sqrt{\eta(T)}T = o(T)$ .

We have shown

$$\begin{aligned} \sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta\alpha)^+ &\geq \sum_{\text{good}} \frac{1}{2\pi} \int_{I_\nu} \log |L(\alpha + it)| dt \\ &\quad - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 T \log T\right). \end{aligned}$$

By the upper bound  $L \ll T^A$  and the bound for the number of bad  $I_\nu$ ,

$$\sum_{\text{bad}} \frac{1}{2\pi} \int_{I_\nu} \log |L(\alpha + it)| dt \leq o(T \log T)$$

so that

$$(7.6) \quad \sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta - \alpha)^+ \geq \frac{1}{2\pi} \int_{T_1}^{T_2} \log |L(\alpha + it)| dt - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 T \log T\right).$$

Now, by the Littlewood lemma, again with  $\alpha = \frac{1}{2} - \delta$ , but this time applied to  $L$ ,

$$(7.7) \quad \frac{1}{2\pi} \int_{T_1}^{T_2} \log |L(\alpha + it)| dt = \sum_{T_1 \leq \gamma \leq T_2}^{(L)} (\beta - \alpha)^+ + O(\log T),$$

where, since  $L \ll T^A$ , the error term follows from the same reasoning that yielded (4.8). We obtain a lower bound for the left hand side of (7.7) if, in the sum on the right, we keep the contribution from only those zeros with  $\beta > \frac{1}{2} - \delta^2$ . By the functional equation, this lower bound is

$$\geq (\delta - \delta^2) \left( N(-\infty, T_1, T_2; L) - N\left(\frac{1}{2} + \delta^2, T_1, T_2; L\right) \right) + O(\log T),$$

and, by (1.1),

$$\frac{1}{2\pi} \int_{T_1}^{T_2} \log |L(\alpha + it)| dt \geq (\delta - \delta^2) \frac{\Lambda}{3\pi} T \log T - o(T \log T).$$

This, together with (7.6), gives the lower bound

$$(7.8) \quad \sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta - \alpha)^+ \geq (\delta - \delta^2) \frac{\Lambda}{3\pi} T \log T - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 T \log T\right).$$

We obtain an upper bound as follows:

$$\begin{aligned} \sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta\alpha)^+ &= \sum_{\text{good}} \left( \sum_{\substack{\gamma \in I_\nu \\ \beta \leq \frac{1}{2} + 3\delta^2}} + \sum_{\substack{\gamma \in I_\nu \\ \beta > \frac{1}{2} + 3\delta^2}} \right) (\beta - \alpha)^+ \\ &\leq (\delta + 3\delta^2) N(\alpha, T_1, T_2; D_x) + \sum_{\text{good}} \sum_{\substack{\gamma \in I_\nu \\ \beta > \frac{1}{2} + 3\delta^2}} O(1). \end{aligned}$$

Since  $I_\nu$  is good, we have the local Lindelöf bound  $L(s) \ll T^{o(1)}$  in the half strip  $\sigma \geq \frac{1}{2} + 2\delta^2$ ,  $|t - t_\nu| \leq 1$ ; hence  $D_x(s) \ll T^{o(1)}$  by the approximation hypothesis.

Now Proposition 5 shows that  $D_x(s)$  has  $o(\log T)$  zeros for  $\sigma \geq \frac{1}{2} + 3\delta^2$ ,  $t \in I_\nu$ . This gives

$$\sum_{\text{good}} \sum_{\gamma \in I_\nu}^{(D)} (\beta - \alpha)^+ \leq (\delta + 3\delta^2) N(\alpha, T_1, T_2; D_x) + o(T \log T).$$

Comparison with the lower bound (7.8) and Proposition 1, finally gives,

$$\begin{aligned} &(\delta + 3\delta^2 + o(1)) \frac{1}{6\pi} T \log x + o(T \log T) \\ &\geq (\delta - \delta^2) \frac{\Lambda}{3\pi} T \log T - O\left(\left(\delta \log \frac{1}{\delta}\right)^2 T \log T\right) \end{aligned}$$

whence

$$\log x \geq \left( \frac{1 - \delta}{1 + 3\delta} 2\Lambda - O\left(\delta \left(\log \frac{1}{\delta}\right)^2\right) \right) \log T,$$

giving Theorem 1, on letting  $\delta \rightarrow 0$ . □

PROOF OF THEOREM 2. We give the proof for the Riemann zeta function and just mention the changes for the general case. For  $\sigma = -\varepsilon'$ ,  $T \leq t \leq (1+\varepsilon_0)T$ , we have  $\zeta = D_x + O(T^{-\varepsilon})$  by hypothesis. Since there are no zeros of  $\zeta$  nearby, we have  $|\zeta| > |\zeta - D_x|$  on this line segment, provided that  $T > T_0(\varepsilon')$ .

Let  $T_1 = \left(1 + \frac{1}{3}\varepsilon_0\right)T$ ,  $T_2 = \left(1 + \frac{2}{3}\varepsilon_0\right)T$ . Near the horizontal sides  $t = T_1, T_2$  we get “safe tracks” by Corollary 2, here inputting the stronger zero density bound  $N\left(\frac{1}{2} + \delta, T; L\right) \leq \eta T(\log T)^{-1}$  for a certain  $\eta = \eta(T) \rightarrow 0$  as  $T \rightarrow \infty$ . This allows us to obtain  $T'_1, T'_2$  satisfying

$$T_1 < T'_1 < T_1 + 4\eta T(\log T)^{-1}$$

$$T_2 - 4\eta T(\log T)^{-1} < T'_2 < T_2,$$

and such that, on the half-lines,  $t = T'_1, T'_2, \sigma \geq -\varepsilon'$ , we have  $|\zeta| > |\zeta - D_x|$ . The same inequality holds also on  $\sigma = 2$  and so, by Rouché's theorem,  $\zeta$  and  $D_x$  have the same number of zeros within the rectangle  $\{-\varepsilon' \leq \sigma \leq 2, T'_1 \leq t \leq T'_2\}$ . But, the number of zeros of  $\zeta$  in this region is, by (1.1), just

$$\begin{aligned} \int_{T'_1/2\pi}^{T'_2/2\pi} \log y dy + O(\log T) &= \int_{T_1/2\pi}^{T_2/2\pi} \log y dy + O(\eta T) \\ &\geq \frac{\varepsilon_0 T}{6\pi} \log \frac{T}{2\pi} + O(\eta T). \end{aligned}$$

On the other hand, an upper bound for the number of zeros of  $D_x$  is provided by Proposition 1, giving

$$\begin{aligned} &\leq \frac{T'_2 - T'_1}{2\pi} \log x + O\left(T^{\frac{1}{2}} \log x\right) \\ &\leq \frac{\varepsilon_0 T}{6\pi} \log x + O(\eta T), \end{aligned}$$

if we take (as we may),  $x \leq T^{3\Lambda}$ ,  $\eta \geq T^{-\frac{1}{4}}$ , say. Combining these, we have

$$\log \frac{T}{2\pi} - O(\varepsilon_0^{-1} \eta) \leq \log x,$$

and so

$$x > (1 - o(1)) \frac{T}{2\pi},$$

completing the proof.  $\square$

**REMARKS.** As already noted, approximations exist [T, § 4.18] with  $x$  as short as  $(1 + o(1)) \frac{T}{2\pi}$ , so that the above bound is sharp.

In the general case the proof is virtually the same, apart from the fact that one requires a little work to show at the outset that there are no zeros of  $L$  to the right of  $\sigma = 1$ . This follows from the stronger zero density hypothesis, since, by almost periodicity (cf. [B]), the existence of one zero to the right of  $\sigma = 1$  implies the existence of  $\gg T$  of them with  $T < t < (1 + \varepsilon_0)T$ , for all sufficiently large  $T$ . Then, following the above argument, one gets the bound  $x \gtrsim Q^2 \prod_j (\lambda_j T)^{2\lambda_j}$

and, as for  $\zeta$ , this is also sharp. The existence of approximations of this quality follows by a refinement of Proposition 2. The latter is obtained by transforming the integral  $I$  in (2.1), using the functional equation for  $L(s)$ , expanding  $L(s)$  and integrating termwise, then showing that the corresponding integrals are rapidly decreasing as soon as  $x \geq (1 + \varepsilon)Q^2 \prod_j (\lambda_j T)^{2\lambda_j}$ . Evaluating these integrals

for smaller  $x$  by stationary phase, one is led to the standard proof of the approximate functional equation.

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School of Mathematics  
Institute for Advanced Study  
Princeton, N.J. 08540

Scarborough College  
University of Toronto  
Scarborough, Ont. M1C 1A4