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On Liouville Type Theorems for Second Order Elliptic Differential Equations

LAVI KARP (*)

0. - Introduction

The classical Liouville's theorem asserts that if u is a harmonic function in $\mathbb{R}^n$, and $u(x) = O(|x|^{m+\varepsilon})$ $(0 \leq \varepsilon < 1)$ as $|x| \to \infty$, then u is a harmonic polynomial of degree $\leq m$.

In this paper we prove the analogous result for the following class of second-order uniformly elliptic operators:

(i) $\mathcal{P} = \Delta + \mathcal{Q}$, where $\mathcal{Q}$ is an operator with variables coefficients which have a certain decay to zero at $|x| = \infty$,

and

(ii) $\mathcal{P}$ is subcritical in $\mathbb{R}^n$.

If the operator $\mathcal{P}$ admits a positive Green's function (fundamental solution) $G_{\mathcal{P}}(x,y)$ in $\mathbb{R}^n$, then $\mathcal{P}$ is subcritical in $\mathbb{R}^n$. In order to clarify the concept of subcritical operators, the equivalent conditions proven by Agmon [2] are given below. Let $W_0^{1,2}(\Omega)$ be the Hilbert space defined by the completion of $C_0^1(\Omega)$ in the norm of the Sobolev space $W^{1,2}(\Omega)$. The Dirichlet realization of $\mathcal{P}$ in $L^2(\Omega)$ is denoted by $\mathcal{P}_\Omega^\sim$ and is defined as follows:

$$\mathcal{P}_\Omega^\sim u = \mathcal{P}u \quad \text{for } u \in \{v \in W_0^{1,2}(\Omega) : \mathcal{P}v \in L^2(\Omega)\}.$$

Let $\Gamma(\mathcal{P},\Omega) = \inf\{\Re z : z \in \text{spectrum of } -\mathcal{P}_\Omega^\sim\}$ and $\Gamma(\mathcal{P}) = \inf\{\Gamma(\mathcal{P},\Omega) : \bar{\Omega}$ is compact in $\mathbb{R}^n\}$. In [2] Agmon proved that the following properties are equivalent (see also [17]):

(a) $\mathcal{P}$ is subcritical in $\mathbb{R}^n$.

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(*) This research was carried out during a period in which the author was a research fellow in the Department of Mathematics, Technion – Israel Institute of Technology.

(b) $\Gamma(\mathcal{P}) \geq 0$ and 0 is not an eigenvalue of $\mathcal{P}$.

(c) There exists $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u \geq 0$, $\mathcal{P}u \leq 0$ and $\mathcal{P}u \not\equiv 0$ in $\mathbb{R}^n$.

Condition (c) yields a simple way by which one can check whether an operator is subcritical. For example, if the zero-order term of $\mathcal{P}$ is non-positive and not identical to zero, then any positive constants is a subsolution. Thus $\mathcal{P}$ is subcritical by the above equivalences. We refer to [17] and [19] for further details and references concerning subcritical operators.

A Liouville type theorem for a different class of second-order elliptic operators was proved by Avellaneda and Lin [3]. They proved it for operators in the divergence form and with periodic coefficients. Moser and Struwe [15] gave an alternative proof of that theorem. Both methods use homogenization theory.

Our approach is based upon an investigation of the action of $\mathcal{P}$ on function spaces with weighted norms. For an integer m, $0 \leq \varepsilon < 1$ and $0 < \sigma < 1$, $C^{2,\sigma}_{m+\varepsilon}$ is the space of twice differentiable functions for which the second derivatives are σ-Hölder continuous. In addition, if $u \in C^{2,\sigma}_{m+\varepsilon}$, then $u(x) = O(|x|^{m+\varepsilon})$, $Du(x) = O(|x|^{m+\varepsilon-1})$ and $D^2u(x) = O(|x|^{m+\varepsilon-2})$, as $|x| \to \infty$. $C^{\sigma}_{m+\varepsilon-2}$ denotes the space of σ-Hölder continuous functions such that $u(x) = O(|x|^{m+\varepsilon-2})$.

In this paper we will prove that

$$\mathcal{P} : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2} \tag{0.1}$$

is a surjective Fredholm operator for $0 < \varepsilon < 1$. Moreover, $\mathcal{P}$ has the same Fredholm's index as the Laplace operator. We will show also that the Fredholm's index of the Laplace operator is equal to

$$d_m := \frac{(n+m-2)!}{m!(n-1)!}(2m+n-1), \tag{0.2}$$

the number of linearly independent harmonic polynomials of degrre $\leq m$ (see e.g. [23]). Begehr and Hile [5] as well as Friedman [7] showed that $\mathcal{P}$ is surjective for $m \leq -1$ and $0 < \varepsilon < 1$.

We will discuss the case $\varepsilon = 0$ separately, since in that case the operator $\mathcal{P}$ in (0.1) is not Fredholm. Nevertheless, and in spite that difficulty, we will prove that

$$\dim \ker(\mathcal{P}, C^{2,\sigma}_m) = \dim \ker(\Delta, C^{2,\sigma}_m)$$

provided that the perturbation $\mathcal{Q}$ decays faster at infinity than it does in the case $0 < \varepsilon < 1$.

We will also consider weighted Sobolev spaces. These spaces are defined as the completion of $C^{\infty}_0(\mathbb{R}^n)$ with respect to the norm

$$\|\varphi\|_{\delta,k,p} := \sum_{|\alpha| \leq k} \|(1+|x|)^{\delta+|\alpha|}\partial^{\alpha}\varphi\|_p,$$

where $\|\cdot\|$ denotes the standard L^p-norm, $1 < p < \infty$ and $\delta \in \mathbb{R}$. We denote them by $W_\delta^{k,p}$, if $k = 0$ they become the usual weighted L^p-spaces which are denoted by L_δ^p.

The operator

$$\mathcal{P} : W_\delta^{k,p} \to L_{\delta+2}^p \tag{0.3}$$

was studied by several authors, including Lockhart [9], McOwen [12], Lockhart and McOwen [10] and Murata [16]. Their main result is that if the perturbation $\mathcal{Q} = \sum_{|\alpha|\le 2} q_\alpha(x)\partial^\alpha$ satisfies the condition

$$\lim_{|x|\to\infty} \sup (1+|x|)^{|\alpha|-2}|q_\alpha(x)| = 0,$$

then $\mathcal{P} = \Delta + \mathcal{Q}$ in (0.3) is Fredholm with the same Fredholm's index as the Laplace operator. That result is true for $-\delta - n/p$ different from a positive integer if $\delta \le -n/p$, and for $\delta + 2 - n(1-1/p)$ different from a positive integer if $\delta > -n/p$.

Since in this paper we assume that $\mathcal{P}$ is subcritical, $\mathcal{P}$ has a Green's function $G_{\mathcal{P}}(x,y)$. In [20] Pinchover proved that if the coefficients of $\mathcal{Q}$ tend to zero in a certain swiftness as $|x| \to \infty$, then

$$G_{\mathcal{P}}(x,y) \le C|x-y|^{2-n}, \quad n \ge 3.$$

We use that estimate to improve the above mentioned result as follows: $\mathcal{P}$ in (0.3) is injective if $\delta > -n/p$. Hence, if in addition $\delta + 2 - n(1-1/p)$ is not a positive integer, then

$$\dim \ker(\mathcal{P}^*, L_{-2-\delta}^{p'}) = \dim \ker(\Delta, L_{-2-\delta}^{p'}),$$

where $\mathcal{P}^*$ is the formal adjoint of $\mathcal{P}$ and $1/p + 1/p' = 1$.

We prove those results in $\mathbb{R}^n$, $n \ge 3$. Yet we believe they remain true for $n = 2$. However, we refrain to treat the case $n = 2$ in the present paper.

The outline of this paper is as follows: In Section 1 we define norms and the spaces $C_\delta^{k,\sigma}$, and state the main results within them. Section 2 deals with weighted Schauder estimates. In Section 3 we study the behavior of the Laplace operator. Sections 4 and 5 deal with the null spaces of $\mathcal{P}$ and the range of $\mathcal{P}$, respectively. In Section 6 we prove the theorems which were stated in Section 1. We will treat weighted Sobolev spaces in Section 7. Finally, in Section 8 we will indicate the failure of Liouville type theorems for fourth-order elliptic operators.

List of notations

$\langle x\rangle = 1 + |x|$, where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$.
$B_r(x_0) = \{x : |x - x_0| < r\}$, $B_r = B_r(0)$.
$\Omega_r = \mathbb{R}^n \backslash \overline{B_r}$.
Δ = Laplace operator, Δ^2 = BiLaplace operator.
Du = gradient of u.
The spaces $C_\delta^{k,\sigma}$ and the norms $\|\cdot\|_{\delta,k,\sigma;\Omega}$, $\|\cdot\|_{\delta,k,\sigma}$ are defined in Section 1.
The spaces $W_{k,\delta}^p$ and the norm $\|\cdot\|_{\delta,k,p}$ are defined in Section 7.
$\ker(\mathcal{A}, \mathcal{X})$ = null space of operator $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$.
$\dim \ker(\mathcal{A}, \mathcal{X})$ = dimension of $\ker(\mathcal{A}, \mathcal{X})$.
d_m = number of linearly independent harmonic polynomials of degree $\leq m$, see (0.2).
Large generic constants will be denoted by C (even in the same sequence of inequalities).

1. - Definitions of the norms and statement of the main results

Let Ω be a domain in $\mathbb{R}^n$, $\delta \in \mathbb{R}$ and let $u \in C^k(\Omega)$. We define

$$\|u\|_{\delta,k;\Omega} = \sum_{|\alpha|\leq k} \sup_{\Omega} \frac{|\partial^\alpha u(x)|}{\langle x\rangle^{\delta-|\alpha|}}. \tag{1.1}$$

For $0 < \sigma < 1$, set

$$H_\sigma(x,u) = \sup_{\left\{y:|y-x|<\frac{1}{4}\langle x\rangle\right\}} \frac{|u(x) - u(y)|}{|x-y|^\sigma}, \tag{1.2}$$

and define

$$\|u\|_{\delta,k,\sigma;\Omega} = \|u\|_{\delta,k;\Omega} + \sum_{|\alpha|=k} \sup_{\Omega} \frac{H_\sigma(x, \partial^\alpha u)}{\langle x\rangle^{\delta-k-\sigma}}. \tag{1.3}$$

It is easy to see that for $\delta = \mu + \nu$,

$$\|uv\|_{\delta,k,\sigma;\Omega} \leq \|u\|_{\mu,k,\sigma;\Omega}\|v\|_{\nu,k,\sigma;\Omega}. \tag{1.4}$$

If $\Omega = \mathbb{R}^n$, we write $\|u\|_{\delta,k;\Omega} := \|u\|_{\delta,k}$ and $\|u\|_{\delta,k,\sigma;\Omega} := \|u\|_{\delta,k,\sigma}$. Let C_δ^k and $C_\delta^{k,\sigma}$ be the Banach spaces of all functions u such that $\|u\|_{\delta,k}$ and $\|u\|_{\delta,k,\sigma}$ are finite, respectively. Whenever $k = 0$ we will not denote it.

Now we can state the Liouville's theorem as follows:

THEOREM (Liouville). *Let m be a non-negative integer and $0 \le \varepsilon < 1$. Then the null space of the operator*

$$\Delta : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2}$$

consists of all harmonic polynomials of degree $\le m$. Therefore, $\dim \ker(\Delta, C^{2,\sigma}_{m+\varepsilon}) = d_m$ *(d_m is given in (0.2)), and if $u \in \ker(\Delta, C^{2,\sigma}_{m+\varepsilon})$, then $u \in C^{2,\sigma}_m$.*

Let

$$\mathcal{P} = \Delta + \mathcal{Q} = \Delta + \sum_{|\alpha|\le 2} q_\alpha(x)\partial^\alpha$$

and consider the following assumptions on $\mathcal{P}$:

$$\sum_{i,j}^{n} (q_{i,j}(x) + \delta_{i,j})\xi_i\xi_j \ge \lambda|\xi|^2, \quad x,\ \xi \in \mathbb{R}^n, \tag{1.5}$$

where λ is a positive constant and $\delta_{i,j}$ are the Kronecker's symbols.

$$\|q_\alpha\|_{|\alpha|-2,\sigma} \le \Lambda, \quad |\alpha| \le 2, \tag{1.6}$$

here Λ is a positive constant.

$$\limsup_{r\to\infty} \|q_\alpha\|_{|\alpha|2,\sigma;\Omega_r} = 0, \quad \Omega_r := \mathbb{R}^n \backslash \overline{B_r(0)}, \quad |\alpha| \le 2. \tag{1.7}$$

$$\mathcal{P} \text{ is subcritical in } \mathbb{R}^n. \tag{1.8}$$

$$\begin{cases} \text{There are constants } K_1 \text{ and } K_2 \text{ such that} \\ |q_\alpha(x)| \le K_1\langle x\rangle^{|\alpha|-2}h(|x|) \text{ for all } |x| \ge 1 \text{ and for all } |\alpha| \le 2, \\ \text{where } h(t) \text{ is a non-increasing function satisfying } \displaystyle\int_1^\infty t^{-1}h(t)dt \le K_2. \end{cases} \tag{1.9}$$

REMARK. $q_0(x) \le 0$ is a sufficient condition for a uniformly elliptic operator $\mathcal{P}$ to be subcritical in $\mathbb{R}^n$ under the decay assumption (1.9). Indeed, $q_0(x) \le 0$ implies that any $u \equiv \text{const} > 0$ is a subsolution, further, Lemma 5.3 in [20] yields that constants are not ground states provided that $\mathcal{P}$ satisfies (1.9). Therefore $\mathcal{P}$ is subcritical.

Our main results are:

THEOREM 1.1. *Let m be a non-negative integer and $0 < \varepsilon < 1$. Suppose $\mathcal{P}$ satisfies* (1.5)-(1.9), *then $\mathcal{P}$ is a surjective Fredholm operator. Moreover,*

$$\dim \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon}) = \dim \ker(\Delta, C^{2,\sigma}_{m+\varepsilon}) = d_m.$$

THEOREM 1.2. *Let m be a non-negative integer and assume $\mathcal{P}$ satisfies* (1.5)-(1.8) *and* (1.9) *is replaced by*

$$\|q_\alpha\|_{|\alpha|-2-\gamma,\sigma} < \infty \quad \text{for some positive } \gamma, \quad |\alpha| \le 2. \tag{1.10}$$

Then

$$\dim \ker(\mathcal{P}, C^{2,\sigma}_m) = d_m.$$

COROLLARY. *Under the hypotheses of Theorem* 1.2*: if* $u \in \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$, *then* $u \in C^{2,\sigma}_m$.

We conclude this section with a proposition which will be needed in the next sections.

PROPOSITION 1.3. *Let δ be a real number. If $Du \in C_{\delta-1}$, then*

$$\sup_{\mathbb{R}^n} \frac{H_\sigma(x,u)}{\langle x\rangle^{\delta-\sigma}} \le C\|Du\|_{\delta-1},$$

where $C = C(\delta,\sigma)$.

PROOF.

$$\frac{|u(x)-u(y)|}{|x-y|^\sigma} \le |Du(z)|\,|x-y|^\sigma,$$

where z lies on the segment $[x,y]$. For z in the ball $\left\{y : |x-y| < \frac{1}{4}\langle x\rangle\right\}$ we have $\frac{3}{4}\langle x\rangle \le \langle z\rangle \le \frac{5}{4}\langle x\rangle$. Hence,

$$\begin{aligned} \frac{H_\sigma(x,u)}{\langle x\rangle^{\delta-\sigma}} &\le \left(\frac{1}{4}\right)^{1-\sigma} \frac{1}{\langle x\rangle^{\delta-1}}\,|Du(z)| \le C(\delta,\sigma)\,\frac{|Du(z)|}{\langle z\rangle^{\delta-1}} \\ &\le C(\delta,\sigma)\|Du\|_{\delta-1}, \end{aligned} \tag{1.11}$$

with the constant $C(\delta,\sigma) = \left(\frac{1}{4}\right)^{1-\sigma} \max\left\{\left(\frac{5}{4}\right)^{\delta-1}, \left(\frac{3}{4}\right)^{\delta-1}\right\}$. □

REMARK. It follows from inequalities (1.11) that condition (1.7) is satisfied if

$$\limsup_{|x|\to\infty} \langle x\rangle^{2-|\alpha|}|q_\alpha(x)| = \limsup_{|x|\to\infty} \langle x\rangle^{3-|\alpha|}|Dq_\alpha(x)| = 0.$$

2. - Weighted Schauder estimates

The Schauder estimates play an essential role in the regularity theory of elliptic operators. In addition to the regularity, the weighted Schauder estimates

control the growth of the derivatives near infinity. Meyers proved the weighted Schauder estimates for functions which are defined in a neighborhood of infinity [13]. Only slight modifications of Meyers' methods are required in order to prove Theorem 2.1 below. A different proof is given in [5].

THEOREM 2.1 (Meyers, Begehr and Hile). *Let δ be a real number. If the operator $\mathcal{P}$ satisfies conditions* (1.5) *and* (1.6), *then*

$$\|u\|_{\delta,2,\sigma} \leq C\{\|u\|_\delta + \|\mathcal{P}u\|_{\delta-2,\sigma}\}, \tag{2.1}$$

where $C = C(\lambda, \Lambda, n, \delta, \sigma)$.

3. - The Laplace operator

In this section we study the behavior of the Laplace operator in the spaces $C_\delta^{2,\sigma}$. The results of this section are based on the following lemma.

LEMMA 3.1. *Let m be a non-negative integer and $0 < \varepsilon < 1$.*

(i) *If $f \in C_{m+\varepsilon-2}$, then there exists $u \in C_{m+\varepsilon}$ such that*

$$\Delta u = f \quad \text{and} \quad \|u\|_{m+\varepsilon} \leq C\|f\|_{m+\varepsilon-2},$$

where $C = C(m, n, \varepsilon)$.

(ii) *If $f \in C_{m-2}$, then there exists u such that*

$$\Delta u = f \quad \text{and} \quad \sup_{\mathbb{R}^n} \frac{|u(x)|}{\langle x\rangle^m \log(2+|x|)} \leq C\|f\|_{m-2},$$

where $C = C(m, n)$. Moreover, the log-factor in that estimate is indispensable.

(iii) *If $f \in C_{m-2}$ and*

$$|f(x)| \leq \|f\|_{m-2} K_1 \langle x\rangle^{m-2} h(|x|),$$

where $h(t)$ is a non-increasing function satisfying $\int_1^\infty t^{-1}h(t)dt \leq K_2$. Then there exists $u \in C_m$ such that

$$\Delta u = f \quad \text{and} \quad \|u\|_m \leq C\|f\|_{m-2},$$

where $C = C(m, n, K_1, K_2)$.

As a consequence of Lemma 3.1 (i), of the weighted Schauder estimates (2.1) and of the classical Liouville's theorem we obtain:

THEOREM 3.2. *Let m be a non-negative integer and $0 < \varepsilon < 1$. Then the*

Laplace operator

$$\Delta : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2} \tag{3.1}$$

is a surjective Fredholm operator. Its kernel consists of all harmonic polynomials of degree $\leq m$, and hence

$$\dim \ker(\Delta, C^{2,\sigma}_{m+\varepsilon}) = d_m.$$

Let

$$\Pi_m : C^{2,\sigma}_{m+\varepsilon} \to \ker(\Delta, C^{2,\sigma}_{m+\varepsilon}) \tag{3.2}$$

be a projection.

COROLLARY. *Let m be a non-negative integer and $0 < \varepsilon < 1$. Then there exists a positive constant C such that*

$$\|u\|_{m+\varepsilon,2,\sigma} \leq C\|\Delta u\|_{m+\varepsilon-2,\sigma} + \|\Pi_m u\|_{m+\varepsilon,2,\sigma} \quad \text{for all } u \in C^{2,\sigma}_{m+\varepsilon}. \tag{3.3}$$

The main device in the proof of Lemma 3.1 is a modification of the fundamental solution of the Laplace operator by subtracting finitely many terms of its Taylor expansion. That method was used in [8] as well as in many other situations.

Let $\Gamma(x) = c_n|x|^{2-n}$, where $c_n = ((2-n)\text{area}(\mathbb{S}^{n-1}))^{-1}$ and $n \geq 3$. Set

$$F_m(x,y) := \sum_{|\alpha|\leq m} \frac{(-x)^\alpha}{\alpha!}(\partial^\alpha\Gamma)(y), \tag{3.4}$$

and

$$\Gamma_m(x,y) := \Gamma(y-x) - F_m(x,y). \tag{3.5}$$

F_m is the mth-order Taylor expansion of the function $x \mapsto \Gamma(y-x)$.

PROPOSITION 3.3. *For $|y| \geq 2|x|$,*

$$|\Gamma_m(x,y)| \leq C|x|^{m+1}|y|^{2-n-(m+1)}, \tag{3.6}$$

where $C = C(m,n)$.

PROOF. Since $\Gamma_m(x,y)$ is the remainder term of the Taylor expansion (3.4),

$$\Gamma_m(x,y) = (-1)^{m+1} \sum_{|\alpha|=m+1} \frac{m+1}{\alpha!} x^\alpha \int_0^1 (1-t)^m (\partial^\alpha\Gamma)(y-tx)dt. \tag{3.7}$$

For $|y| \geq 2|x|$ and for $0 \leq t \leq 1$, $\frac{1}{2} \leq \left|\frac{y}{|y|} - \frac{tx}{|y|}\right| \leq \frac{3}{2}$ and therefore

$$\left|(\partial^\alpha \Gamma)\left(\frac{y}{|y|} - \frac{tx}{|y|}\right)\right| \leq C_\alpha.$$

Using the fact that $\partial^\alpha \Gamma$ is homogeneous of degree $2 - n - |\alpha|$ and inserting the above inequality in (3.7), we obtain (3.6). □

For any $f \in C_{m+\varepsilon-2}$, $(0 \leq \varepsilon < 1)$ with $\operatorname{supp}(f) \subset \Omega_1 := \{x : |x| > 1\}$ we put

$$K_m f(x) := \int \Gamma_m(x, y) f(y) dy.$$

(3.6) implies that $K_m f$ is well-defined. Furthermore, it is easy to show that $\Delta K_m f = f$ in the sence of distributions (and hence, $K_m f$ is a classical solution if f is σ-Hölder continuous). We turn now to estimate the growth of $K_m f$.

PROPOSITION 3.4. *Let m be a non-negative integer and $0 < \varepsilon < 1$. Assume $f \in C_{m+\varepsilon-2}$, $\operatorname{supp}(f) \subset \Omega_1$ and $\|f\|_{m+\varepsilon-2} \leq 1$. Then for $|x| \geq 1$*

$$|K_m f(x)| \leq C|x|^{m+\varepsilon},$$

where $C = C(m, n, \varepsilon)$.

PROOF. Since $\operatorname{supp}(f) \subset \Omega_1$ and $\|f\|_{m+\varepsilon-2} \leq 1$, $|f(y)| \leq C(m+\varepsilon-2)|y|^{m+\varepsilon-2}$ for any $0 \leq \varepsilon < 1$, where $C(m+\varepsilon-2) = \max\{2^{m+\varepsilon-2}, 1\}$. So

$$\begin{aligned}
|K_m f(x)| \leq & \int_{\{y:|y|\leq 2|x|\}} \Gamma(x-y)|f(y)|dy + \int_{\{y:1\leq|y|\leq 2|x|\}} |F_m(x,y)f(y)|dy \\
& + \int_{\{y:|y|\geq 2|x|\}} |\Gamma_m(x,y)f(y)|dy \\
\leq & \, C(m+\varepsilon-2) \int_{\{y:|y|\leq 2|x|\}} \Gamma(x-y)|y|^{m+\varepsilon-2}dy \\
& + C(m+\varepsilon-2) \int_{\{y:1\leq|y|\leq 2|x|\}} |F_m(x,y)|\,|y|^{m+\varepsilon-2}dy \\
& + \int_{\{y:|y|\geq 2|x|\}} |\Gamma_m(x,y)|\,|f(y)|dy \\
:= & \, C(m+\varepsilon-2)\{I + II\} + III.
\end{aligned}$$

Each of the above terms is now analyzed separately.

Estimate of I: For $m+\varepsilon-2>0$,

$$I \le \int_{\{y:|y|\le 2|x|\}} \Gamma(x-y)|y|^{m+\varepsilon-2}dy \le C|x|^{m+\varepsilon-2} \int_{\{y:|y-x|\le 3|x|\}} \Gamma(x-y)dy$$
$$\le C|x|^{m+\varepsilon}.$$

For $m+\varepsilon-2<0$,

$$I \le \left(\int_{\{y:|y|\le 2|x|,2|y|\ge|x|\}} + \int_{\{y:|y|\le 2|x|,2|y|\le|x|\}} \right) \Gamma(x-y)|y|^{m+\varepsilon-2}dy$$
$$\le C\left(\int_{\{y:|y|\le 2|x|\}} |x-y|^{m+\varepsilon-n}dy + \int_{\{y:|y|\le 2|x|\}} |y|^{m+\varepsilon-n}dy \right)$$
$$\le C|x|^{m+\varepsilon}.$$

Estimate of II: Since $\partial^\alpha\Gamma$ is homogeneous of degree $2-n-|\alpha|$, we have from (3.4)

$$|F_m(x,y)| \le C\sum_{i=1}^{m} |x|^i|y|^{2-n-i}.$$

Hence,

$$II \le C\sum_{i=1}^{m} |x|^i \int_{\{y:1\le|y|\le 2|x|\}} |y|^{m+\varepsilon-n-i}dy \le C\sum_{i=1}^{m} \frac{|x|^{m+\varepsilon}}{m+\varepsilon-i} \le C|x|^{m+\varepsilon}.$$

Estimate of III: By Proposition 3.3,

$$III \le \int_{\{y:|y|\ge 2|x|\}} |\Gamma_m(x,y)|\,|f(y)|dy$$
$$\le C(m+\varepsilon-2) \int_{\{y:|y|\ge 2|x|\}} |\Gamma_m(x,y)|\,|y|^{m+\varepsilon-2}dy$$
$$\le C(m+\varepsilon-2)C|x|^{m+1} \int_{\{y:|y|\ge 2|x|\}} |y|^{-n-1+\varepsilon}dy$$
$$= C(m+\varepsilon-2)C\,\frac{2^{\varepsilon-1}}{1-\varepsilon}\,|x|^{m+\varepsilon}.$$

We complete the proof by combining the estimates for I, II and for III. □

PROPOSITION 3.5. *Let m be a non-negative integer. Assume $f \in C_{m-2}$, $\operatorname{supp}(f) \subset \Omega_1$ and $\|f\|_{m-2} \leq 1$. Then for $|x| \geq 1$*

$$|K_m f(x)| \leq C|x|^m \log(2+|x|),$$

where $C = C(m,n)$.

PROOF. The procedure of this proof is the same as the one of Proposition 3.4. The log-factor arises in the estimation of II. Other estimates remain unchanged.

$$\begin{aligned} II &\leq \int_{\{y:1\leq|y|\leq 2|x|\}} |F_m(x,y)|\,|y|^{m-2}dy \leq C\sum_{i=1}^{m}|x|^i \int_{\{y:1\leq|y|\leq 2|x|\}} |y|^{m-n-i}dy \\ &\leq C\sum_{i=1}^{m-1}\frac{|x|^m}{m-i} + C|x|^m\int_1^{2|x|} t^{-1}dt \\ &\leq C|x|^m \log(2+|x|). \end{aligned}$$

□

PROPOSITION 3.6. *Let m be a non-negative integer. Assume $f \in C_{m-2}$, $\operatorname{supp}(f) \subset \Omega_1$, $\|f\|_{m-2} \leq 1$ and $|f(x)| \leq K_1\langle x\rangle^{m-2}h(|x|)$, where $h(t)$ is the given function in Lemma* 3.1 (iii). *Then for $|x| \geq 1$*

$$|K_{m-1}f(x)| \leq C|x|^m,$$

where $C = C(m,n,K_1,K_2)$.

PROOF. As we did in the preceding proof, the outline of this proof follows the one of Proposition 3.4. This time it differs only in the estimation of III. We have by Proposition 3.3,

$$\begin{aligned} III &\leq \int_{\{y:|y|\geq 2|x|\}} |\Gamma_{m-1}(x,y)|\,|f(y)|dy \leq CK_1|x|^m \int_{\{y:|y|\geq 2|x|\}} |y|^{-n}h(|y|)dy \\ &\leq CK_1|x|^m\int_{2|x|}^{\infty} t^{-1}h(t)dt \leq CK_1K_2|x|^m. \end{aligned}$$

□

PROOF OF LEMMA 3.1. Let $\vartheta \in C_0^\infty(\mathbb{R}^n)$, $0 \le \vartheta(x) \le 1$, $\vartheta(x) = 0$ for $|x| \le 1$ and $\vartheta(x) = 1$ for $|x| \ge 2$. Set

$$u(x) = \int \Gamma(x-y)((1-\vartheta)f)(y)dy + K_m(\vartheta f)(x). \tag{3.8}$$

(In case (iii) we put $K_{m-1}(\vartheta f)$ in (3.8)). Then $\Delta u = (1-\vartheta)f + \vartheta f = f$. The first term of the right hand side of (3.8) is bounded, while the second one has the required growth by Propositions 3.4. 3.5 and 3.6. It remains to show that in case (ii) the log-factor is indispensable. Let $p_m(x)$ be a homogeneous harmonic polynomial of degree m and put

$$u(x) = p_m(x)\log(2+|x|). \tag{3.9}$$

Then $\Delta u \in C_{m-2}$. Now, suppose there exixts a solution of the equation $\Delta u_1 = \Delta u$ which has a lower growth then u in (3.9). As a result $u_1 - u$ is equal to a harmonic polynomial of degree $\le m$. Hence u_1 has the same growth as u, but that contradicts our assumption on u_1. The proof of Lemma 3.1 is complete.□

REMARK. Let m be a non-negative integer and consider the operator

$$\Delta : C_m^{2,\sigma} \to C_{m-2}^{\sigma}. \tag{3.10}$$

We showed that if u is the function given in (3.9), then Δu does not belong to the range of operator (3.10). Thus, unlike operator (3.1) (for $0 < \varepsilon < 1$), (3.10) is not surjective. Furthermore, we can deduce from the above that the range of (3.10) has an infinite co-dimension. I do not know whether operator (3.10) has a closed range.

The lemma below will be needed in Section 5. Its proof is similar to the proof of Lemma 3.1 (iii) and it also relies on the weighted Schauder estimates (2.1). We leave the proof for the reader.

LEMMA 3.7. *Let m be a non-negative integer. Assume $f \in C_{m-2}^{\sigma}$,* $\operatorname{supp}(f) \subset \Omega_r = \{x : |x| > r\}$ *and* $|f(x)| \le M\langle x\rangle^{m-2}h(|x|)$, *where $h(t)$ is the function given in Lemma* 3.1 (iii). *Then there are positive constants K_3 and K_4, and $u \in C_m^{2,\sigma}$ such that*

$$\Delta u = f \quad \text{and} \quad \|u\|_{m,2,\sigma} \le K_3\|f\|_{m-2,\sigma} + K_4 M\rho, \tag{3.11}$$

where $\rho = \int_r^\infty t^{-1}h(t)dt.$

4. - The null spaces of elliptic operators in $\mathbb{R}^n$

Nirenberg and Walker studied the null spaces of elliptic operators in weighted Sobolev spaces [18]. Their results inspired us to show

THEOREM 4.1. *Let m be a non-negative integer and $0 \leq \varepsilon < 1$. If $\mathcal{P}$ satisfies conditions* (1.5)-(1.7), *then operator*

$$\mathcal{P} : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2}. \tag{4.1}$$

has a finite dimensional null space.

PROOF. We first prove the theorem for $0 < \varepsilon < 1$. We do it by showing that any bounded sequence $\{u_k\}_{k=1}^{\infty} \subset \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$ has a Cauchy subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ in $C^{2,\sigma}_{m+\varepsilon}$. For the rest of this section we will not differ between sequences and subsequences, and all of them will be denoted by $\{u_k\}_{k=1}^{\infty}$.

It follows from (1.7) that for any positive ρ there exists r such that

$$\|q_\alpha\|_{|\alpha|-2,\sigma;\Omega_r} \leq \rho, \quad |\alpha| \leq 2. \tag{4.2}$$

r will be determined later on.

We denote by $C^{k,\sigma}(B_r)$ the space of kth differentiable functions in the ball B_r equiped with the standard norm

$$\|u\|^*_{k,\sigma;B_r} := \sum_{|\alpha|\leq k} \sup_{B_r} |\partial^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{B_r} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\sigma}.$$

Clearly there are two constants $c(r,\delta)$ and $C(r,\delta)$ such that

$$c(r,\delta)\|u\|_{\delta,k,\sigma;B_r} \leq \|u\|^*_{k,\sigma;B_r} \leq C(r,\delta)\|u\|_{\delta,k,\sigma;B_r}. \tag{4.3}$$

By that equivalence and by the weighted Schauder estimate (2.1),

$$\begin{aligned} \|u\|^*_{2,\sigma;B_{r+1}} &\leq C(r+1, m+\varepsilon)\|u\|_{m+\varepsilon,2,\sigma} \\ &\leq C(r+1, m+\varepsilon)C\{\|\mathcal{P}u\|_{m+\varepsilon-2,\sigma} + \|u\|_{m+\varepsilon}\}. \end{aligned} \tag{4.4}$$

Let ϑ be infinitely differeniable function such that $0 \leq \vartheta(x) \leq 1$, $\vartheta(x) = 1$ for $|x| \leq r$ and $\mathrm{supp}(\vartheta) \subset B_{r+1}$. Notice that for any $u \in \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$ we have

by (2.1) and (4.3),

$$\begin{aligned}
\|\vartheta u\|^*_{2,\sigma;B_{r+1}} &\le C(r+1,m+\varepsilon)\|\vartheta u\|_{m+\varepsilon,2,\sigma} \\
&\le C(r+1,m+\varepsilon)C\{\|\mathcal{P}(\vartheta u)\|_{m+\varepsilon-2,\sigma} + \|\vartheta u\|_{m+\varepsilon}\} \\
&\le C(r+1,m+\varepsilon)C\left\|\sum_{i,j=1}^{n}(q_{i,j}+\delta_{i,j})\partial_i\vartheta\partial_j u\right\|_{m+\varepsilon-2,\sigma} \\
&\quad + C(r+1,m+\varepsilon)C\left\{\left\|\sum_{i=1}^{n} q_i(\partial_i\vartheta)u\right\|_{m+\varepsilon-2,\sigma} + \|\vartheta u\|_{m+\varepsilon}\right\} \\
&\le C_1\|u\|^*_{1,\sigma;B_r}.
\end{aligned}$$

Let now $\{u_k\}_{k=1}^{\infty}$ be a sequence in $\ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$ with $\|u_k\|_{m+\varepsilon,2,\sigma} \le 1$. Then (4.4) implies that $\{u_k\}_{k=1}^{\infty}$ is bounded in $C^{2,\sigma}(B_{r+1})$ and by the compactness of the embedding $C^{2,\sigma}(B_{r+1}) \hookrightarrow C^{1,\sigma}(B_{r+1})$ (cf. [1; Theorem 1.31]), $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $C^{1,\sigma}(B_{r+1})$. By the later inequality, $\{\vartheta u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $C^{2,\sigma}_{m+\varepsilon}$ and it is bounded in $C^{2,\sigma}(B_{r+1})$.

We turn now to show that $\{(1-\vartheta)u_k\}_{k=1}^{\infty}$ is a Cauchy in $C^{2,\sigma}_{m+\varepsilon}$. By inequality (3.3), we get that for any $u \in \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$,

$$\begin{aligned}
\|(1-\vartheta)u\|_{m+\varepsilon,2,\sigma} &\le C\|\Delta((1-\vartheta)u)\|_{m+\varepsilon-2,\sigma} + \|\Pi_m((1-\vartheta)u)\|_{m+\varepsilon,2,\sigma} \\
&\le C\ \|\Delta((1-\vartheta)u) - (1-\vartheta)\Delta u\|_{m+\varepsilon-2,\sigma} \\
&\quad + C\|(1-\vartheta)\Delta u\|_{m+\varepsilon-2,\sigma} + \|\Pi_m((1-\vartheta)u)\|_{m+\varepsilon,2,\sigma} \\
&\le C\|\Delta((1-\vartheta)u) - (1-\vartheta)\Delta u\|_{m+\varepsilon-2,\sigma} \\
&\quad + C\|(1-\vartheta)\mathcal{Q}u\|_{m+\varepsilon-2,\sigma} + \|\Pi_m((1-\vartheta)u)\|_{m+\varepsilon,2,\sigma}.
\end{aligned} \tag{4.5}$$

By Proposition 1.3, (1.4) and (4.2),

$$\begin{aligned}
\|(1-\vartheta)\mathcal{Q}u\|_{m+\varepsilon-2,\sigma} &\le \sum_{|\alpha|\le 2}\|(1-\vartheta)q_\alpha\partial^\alpha u\|_{m+\varepsilon-2,\sigma} \\
&\le \sum_{|\alpha|\le 2}\|q_\alpha\|_{|\alpha|-2,\sigma;\Omega_r}\|\partial^\alpha u\|_{m+\varepsilon-|\alpha|,\sigma} \\
&\le \rho\sum_{|\alpha|\le 2}\|\partial^\alpha u\|_{m+\varepsilon-|\alpha|,\sigma} \\
&\le \rho C_2\|u\|_{m+\varepsilon,2,\sigma} \\
&\le \rho C_2\{\|(1-\vartheta)u\|_{m+\varepsilon,2,\sigma} + \|\vartheta u\|_{m+\varepsilon,2,\sigma}\}.
\end{aligned}$$

Thus, choosing r sufficiently large so that $\rho C C_2 < 1/2$, we obtain by (4.5) that

for any $u \in \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon})$,

$$\begin{aligned}\|(1-\vartheta)u\|_{m+\varepsilon,2,\sigma} &\le 2C\|\Delta((1-\vartheta)u) - (1-\vartheta)\Delta u\|_{m+\varepsilon-2,\sigma} \\ &\quad + \|\vartheta u\|_{m+\varepsilon,2,\sigma} + 2C\|\Pi_m(1-\vartheta)u\|_{m+\varepsilon,2,\sigma}.\end{aligned} \tag{4.6}$$

The sequence $\{\Delta((1-\vartheta)u_k) - (1-\vartheta)\Delta u_k\}_{k=1}^{\infty}$ is supported in B_{r+1} and it is bounded in $C^{1,\sigma}(B_{r+1})$. Hence it is a Cauchy sequence in $C^{\sigma}(B_{r+1})$. Using the equivalence (4.3), the fact that $\{\vartheta u_k\}_{k=1}^{\infty}$ is Cauchy in $C^{2,\sigma}_{m+\varepsilon}$, and the compactness of Π_m, we obtain by (4.6) that $\{(1-\vartheta)u_k\}_{k=1}^{\infty}$ is Cauchy in $C^{2,\sigma}_{m+\varepsilon}$. Therefore, $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in that space and this completes the proof in the case $0 < \varepsilon < 1$. For $\varepsilon = 0$ we have by the inclusion $C^{2,\sigma}_m \subset C^{2,\sigma}_{m+\varepsilon}$,

$$\dim \ker(\mathcal{P}, C^{2,\sigma}_m) \le \dim \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon}) < \infty. \qquad \square$$

5. - The range of elliptic operators

In order to show that operator $\mathcal{P} = \Delta + \mathcal{Q}$ is surjective, all assumptions (1.5)-(1.9) on $\mathcal{P}$ have to be fulfilled. Condition (1.8) implies that $\mathcal{P}$ has a Green's function $G_{\mathcal{P}}(x,y)$. In [20] Pinchover proved that if in addition $\mathcal{P}$ satisfies the decay assumption (1.9), then $G_{\mathcal{P}}(x,y)$ is equivalent to the Green's function of the Laplacian, which implies the existence of a positive constant C_1 such that

$$G_{\mathcal{P}}(x,y) \le C_1|x-y|^{2-n}, \quad x,\ y \in \mathbb{R}^n, \quad n \ge 3. \tag{5.1}$$

Inequality (5.1) is the reason for considering $\mathbb{R}^n$ with $n \ge 3$ in the present paper.

THEOREM 5.1. *Let m be a non-negative integer and $0 < \varepsilon < 1$. Assume $\mathcal{P}$ satisfies* (1.5)-(1.9), *then*

$$\mathcal{P} : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2}$$

is a surjective operator.

PROOF. We write $\mathcal{P} = \Delta + \mathcal{Q}^1 + \mathcal{Q}^2$, where $\mathcal{Q}^i = \sum_{|\alpha|\le 2} q^i_\alpha(x)\partial^\alpha$ $(i = 1,2)$ $\operatorname{supp}(\mathcal{Q}^1) \subset B_{r+1}$ and $\operatorname{supp}(\mathcal{Q}^2) \subset \Omega_r$. As we did in the previous section, we have by Proposition 1.3 and (1.4) that

$$\|\mathcal{Q}^2 u\|_{m+\varepsilon-2,\sigma} \le \delta C_2 \|u\|_{m+\varepsilon,2,\sigma}, \quad 0 \le \varepsilon < 1, \tag{5.2}$$

if

$$\|q_\alpha\|_{|\alpha|-2,\sigma;\Omega_r} \le \delta, \quad |\alpha| \le 2.$$

Therefore, in view of (1.7) we may choose r so that δC_2 is sufficiently small for $\Delta + \mathcal{Q}^2$ being Fredholm with the same Fredholm's index as the Laplace

operator. Since the Laplace operator is surjective (Theorem 3.2), and since $\dim\ker(\Delta+\mathcal{Q}^2, C^{2,\sigma}_{m+\varepsilon}) \le \dim\ker(\Delta, C^{2,\sigma}_{m+\varepsilon})$ (cf. [22; page 115]), it follows that $\Delta+\mathcal{Q}^2$ is surjective. Hence, for any $f \in C^{\sigma}_{m+\varepsilon-2}$ there is $v \in C^{2,\sigma}_{m+\varepsilon}$ which is a solution of the equation $(\Delta+\mathcal{Q}^2)v = f$. Let $G_{\mathcal{P}}(x,y)$ be the Green's function of $\mathcal{P}$ and set

$$w(x) = \int G_{\mathcal{P}}(x,y)(\mathcal{Q}^1 v)(y)dy,$$

then $\mathcal{P}w = \mathcal{Q}^1 v$. Putting $u = v - w$, yields

$$\mathcal{P}u = (\Delta+\mathcal{Q}^2)v + \mathcal{Q}^1 v - \mathcal{P}w = f + \mathcal{Q}^1 v - \mathcal{Q}^1 v = f.$$

Thus, in order to complete the proof it remains to show that $w \in C^{2,\sigma}_{m+\varepsilon}$. By Proposition 1.3, (1.4) and (1.6),

$$\begin{aligned}\|\mathcal{Q}^1 v\|_{m+\varepsilon-2,\sigma} &\le \sum_{|\alpha|\le 2} \|q_\alpha \partial^\alpha v\|_{m+\varepsilon-2,\sigma} \le \sum_{|\alpha|\le 2} \|q_\alpha\|_{|\alpha|-2,\sigma} \|\partial^\alpha v\|_{m+\varepsilon-|\alpha|,\sigma} \\ &\le \Lambda C \|v\|_{m+\varepsilon,2,\sigma},\end{aligned}$$

where Λ is the constant in (1.6). Hence,

$$|(\mathcal{Q}^1 v)(y)| \le \Lambda C \|v\|_{m+\varepsilon,2,\sigma} \langle y\rangle^{m+\varepsilon-2}.$$

Using inequality (5.1) and performing similar calculations as we did in the proof of Proposition 3.4, we obtain

$$|w(x)| \le \Lambda C C_1 \|v\|_{m+\varepsilon,2,\sigma} \int_{\{y:|y|\le r+1\}} |x-y|^{2-n} \langle y\rangle^{m+\varepsilon-2} dy \le C_3 \langle x\rangle^{m+\varepsilon}.$$

The Schauder estimate (2.1) implies that $w \in C^{2,\sigma}_{m+\varepsilon}$. □

Our next aim is to prove an analogous result to Lemma 3.1 (iii) for operator $\mathcal{P}$, that is, equation $\mathcal{P}u = f \in C^{\sigma}_{m-2}$ has a solution $u \in C^{2,\sigma}_{m}$ if f satisfies

$$|f(x)| \le K_1 \langle x\rangle^{m-2} h(|x|), \quad |x| \ge 1, \tag{5.3}$$

where $h(t)$ is a non-increasing function satifying

$$\int_1^\infty t^{-1} h(t) dt \le K_2. \tag{5.4}$$

According to Lemma 3.7, there are two constants K_3 and K_4 so that if (5.3) is fulfilled, and $\operatorname{supp}(f) \subset \Omega_r$, then there exists $u \in C^{2,\sigma}_{m-2}$ such that

$$\Delta u = f \quad \text{and} \quad \|u\|_{m,2,\sigma} \le \|f\|_{m-2,\sigma} K_3 + K_1 K_4 \rho,$$

where

$$\rho = \int_r^\infty t^{-1} h(t) dt.$$

As we did in the previous proof, we write $\mathcal{P} = \Delta + \mathcal{Q}^1 + \mathcal{Q}^2$, where $\operatorname{supp}(\mathcal{Q}^1) \subset B_{r+1}$ and $\operatorname{supp}(\mathcal{Q}^2) \subset \Omega_r$. In view of (1.7), (5.2) and (5.4) we have that

$$C_2 K_3 \delta + K_1 K_4 \rho < 1, \tag{5.5}$$

if r is sufficiently large.

LEMMA 5.2. *Let m be a non-negative integer and assume $\mathcal{P}$ satisfies condition* (1.5)-(1.9). *Suppose $f \in C^\sigma_{m-2}$, f satisfies (5.3), and* $\operatorname{supp}(f) \subset \Omega_r$ *with r sufficiently large so that (5.5) holds, then equation*

$$(\Delta + \mathcal{Q}^2) v = f \tag{5.6}$$

has a solution $v \in C^{2,\sigma}_m$.

PROOF. We may assume without loss of generality that $\|f\|_{m-2,\sigma} \le 1$. We solve equation (5.6) by iteration. Define the sequence $\{u_j\}_{j=0}^\infty$ as follows:

$$\Delta u_0 = f,$$
$$\Delta u_{j+1} = \mathcal{Q}^2 u_j.$$

We claim that

$$\|u_j\|_{m,2,\sigma} \le (K_3 + K_1 K_4)(C_2 K_3 \delta + K_1 K_4 \rho)^j. \tag{5.7}$$

We show (5.7) by induction. By Lemma 3.7,

$$\|u_0\|_{m,2,\sigma} \le K_3 + K_1 K_4 \rho \le K_3 + K_1 K_4.$$

By (1.9),

$$\begin{aligned} |\mathcal{Q}^2(x) u_j(x)| &\le \sum_{|\alpha| \le 2} |q^2_\alpha(x) \partial^\alpha u_j(x)| \le \sum_{|\alpha| \le 2} \|\partial^\alpha u_j\|_{m-|\alpha|} |q_\alpha(x)| \langle x \rangle^{m-|\alpha|} \\ &\le \|u_j\|_{m,2,\sigma} K_1 \langle x \rangle^{m-2} h(|x|) \end{aligned}$$

Combining the later inequality with (5.2) and with the fact that $\operatorname{supp}(\mathcal{Q}^2) \subset \Omega_r$, we obtain by Lemma 3.7,

$$\begin{aligned} \|u_{j+1}\|_{m,2,\sigma} &\le \|u_j\|_{m,2,\sigma} (C_2 K_3 \delta + K_1 K_4 \rho) \\ &\le (K_3 + K_1 K_4)(C_2 K_3 \delta + K_1 K_4 \rho)^{j+1}. \end{aligned}$$

Thus (5.7) is proved.

Setting

$$v(x) = \sum_{j=0}^{\infty} (-1)^j u_j(x),$$

gives that v satisfies equation (5.6) and from (5.7) it follows that

$$\|v\|_{m,2,\sigma} \leq (K_3 + K_1 K_4)(1 - (C_2 K_3 \delta + K_1 K_4 \rho))^{-1}. \qquad \square$$

LEMMA 5.3. *Retain all the hypotheses of Lemma* 5.3, *then equation*

$$\mathcal{P} u = f$$

has a solution $u \in C_m^{2,\sigma}$.

PROOF. Let v be a solution of equation (5.6),

$$w(x) = \int G_{\mathcal{P}}(x,y)(\mathcal{Q}^1 v)(y)dy,$$

and set $u = v - w$. Then $\mathcal{P} u = f$. The rest of the proof is conducted precisely as it was done in the proof of Theorem 5.1. $\square$

THEOREM 5.4. *Let* m *be a non-negative integer. Suppose* $\mathcal{P}$ *satisfies* (1.5)-(1.9), $f \in C_{m-2}^{\sigma}$, *and* f *satisfies* (5.3). *Then there exists* $u \in C_m^{2,\sigma}$ *such that*

$$\mathcal{P} u = f. \tag{5.8}$$

PROOF. We write $\mathcal{P} = \Delta + \mathcal{Q}^1 + \mathcal{Q}^2$ and we choose r so that (5.5) is fulfilled. Let $\vartheta \in C_0^{\infty}(\mathbb{R}^n)$, $0 \leq \vartheta(x) \leq 1$, $\vartheta(x) = 1$ for $|x| \leq r$ and $\vartheta(x) = 0$ for $|x| \geq r+1$.

Let u_1 be the solution of equation $\mathcal{P} u_1 = (1 - \vartheta) f$, which exists according to Lemma 5.3. Let

$$u_2(x) = \int G_{\mathcal{P}}(x,y)(\vartheta f)(y)dy,$$

where $G_{\mathcal{P}}(x,y)$ is the Green's function of $\mathcal{P}$. Then $\mathcal{P} u_2 = \vartheta f$ and by inequalities (2.1) and (5.1), $\|u_2\|_{m,2,\sigma} \leq C\|f\|_{m-2,\sigma}$. Clearly, $u_1 + u_2$ is a solution of equation (5.8), and the proof is complete. $\square$

6. - Proofs of the main results

PROOF OF THEOREM 1.1. Let $q_0^-(x) = \min\{q_0(x), 0\}$, $q_0^+(x) = \max\{q_0(x), 0\}$ and

$$\widehat{\mathcal{Q}}(x) = q_0^-(x) + \sum_{1 \leq |\alpha| \leq 2} q_\alpha(x) \partial^\alpha$$

Consider a family of operators,

$$\widehat{\mathcal{P}}_t := \Delta + t\widehat{\mathcal{Q}} : C^{2,\sigma}_{m+\varepsilon} \to C^{\sigma}_{m+\varepsilon-2}, \quad 0 \le t \le 1, \quad 0 < \varepsilon < 1.$$

Then $\widehat{\mathcal{P}}_t$ is uniformly continuous in t, $\widehat{\mathcal{P}}_t = \Delta$ and $\widehat{\mathcal{P}}_t$ is subcritical in $\mathbb{R}^n$ by the first Remark in Section 1. Furthermore, conditions (1.5)-(1.9) are fulfilled for $\widehat{\mathcal{P}}_t$, where λ is replaced by $\lambda_t := \min\{1, \lambda\}$ and Λ is replaced by $\Lambda_t := \max\{1, \Lambda\}$, and all other parameters remains unchanged. Therefore, in view of Theorems 4.1 and 5.1, $\widehat{\mathcal{P}}_t$ is a surjective Fredholm operator for each $0 \le t \le 1$. Since the Fredholm's index is a continuous function of t,

$$\dim \ker(\widehat{\mathcal{P}}_1, C^{2,\sigma}_{m+\varepsilon}) = \dim \ker(\widehat{\mathcal{P}}_t, C^{2,\sigma}_{m+\varepsilon}) = \dim \ker(\Delta, C^{2,\sigma}_{m+\varepsilon}) = d_m.$$

Let now $\mathcal{P}_t = \widehat{\mathcal{P}}_1 + t\, q_0^+(\cdot)$. Then $\mathcal{P}_0 = \widehat{\mathcal{P}}_1$, $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}_t$ is subcritical for $0 \le t \le 1$ since $\mathcal{P}$ is subcritical. Using the same arguments as above we obtain

$$\dim \ker(\mathcal{P}, C^{2,\sigma}_{m+\varepsilon}) = \dim \ker(\mathcal{P}_t, C^{2,\sigma}_{m+\varepsilon}) = \dim \ker(\widehat{\mathcal{P}}_1, C^{2,\sigma}_{m+\varepsilon}) = d_m. \qquad \square$$

PROOF OF THEOREM 1.2. Set $k_m = d_m - d_{m-1}$ for $m \ge 1$ and $k_0 = d_0$. We conduct the proof by showing that for any $m \ge 0$ there are $u_1, \dots, u_{k_m}$ linearly independent functions which belong to $\ker(\mathcal{P}, C^{2,\sigma}_m) \setminus \ker(\mathcal{P}, C^{2,\sigma}_{m-1})$.

Consider the operator

$$\mathcal{Q} : \mathcal{H}_m \to C^{\sigma}_{m-2},$$

where $\mathcal{H}_m$ is the space of homogeneous harmonic polynomials of degree m. Set $l := \dim \ker(\mathcal{Q}, \mathcal{H}_m)$ and let $p_1, \dots, p_l$ be a basis for $\ker(\mathcal{Q}, \mathcal{H}_m)$. Then p_j belongs to $\ker(\mathcal{P}, C^{2,\sigma}_m)$ for $j = 1, \dots, l$. Let $p_{l+1}, \dots, p_{k_m}$ be a basis for $\mathcal{H}_m \setminus \ker(\mathcal{Q}, \mathcal{H}_m)$, then using assumption (1.10) we have, $\|\mathcal{Q}p_j\|_{m-2-\gamma,\sigma}$ is finite for $j = l+1, \dots, k_m$. Therefore, by Theorem 5.1, equation

$$\mathcal{P}v_j = \mathcal{Q}p_j$$

has a solution $v_j \in C^{2,\sigma}_{m-\gamma}$.

Setting $u_j = v_j - p_j$ for $j = l+1, \dots, k_m$, we have that u_j belongs to $\ker(\mathcal{P}, C^{2,\sigma}_m)$. Moreover, $u_j \notin \ker(\mathcal{P}, C^{2,\sigma}_{m-1})$ since $v_j \in C^{2,\sigma}_{m-\gamma}$ and hence v_j is not a polynomial of degree m. Clearly, $u_{l+1}, \dots, u_{k_m}$ are linearly independent. Thus,

$$u_j = \begin{cases} p_j, & j = 1, \dots, l \\ v_j - p_j, & j = l+1, \dots, k_m \end{cases}$$

form a basis for $\ker(\mathcal{P}, C^{2,\sigma}_m)$. $\square$

7. - Weighted Sobolev spaces

Our goal in this section is to show an analogous result to Theorem 1.1 in weighted Sobolev spaces. These spaces are defined by the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|\varphi\|_{\delta,k,p} := \sum_{|\alpha|\le k} \|(1+|x|)^{\delta+|\alpha|}\partial^\alpha\varphi\|_p, \quad -\infty < \delta < \infty, \quad 1 < p < \infty,$$

where $\|\cdot\|_p$ is the standard L^p-norm. We denote them by $W^p_{k,\delta}$. When $k = 0$ they are the weighted L^p-spaces which are denoted by L^p_δ.

The Laplacian is considered as the operator

$$\Delta : W^p_{2,\delta} \to L^p_{\delta+2}. \tag{7.1}$$

A complete description of operator (7.1) was given by McOwen [11] and Lockhart [9]. They proved that operator (7.1) is Fredholm if and only if

$$\begin{cases} -\delta - \dfrac{n}{p} \notin \mathbb{N}, & \text{if } \delta \le -\dfrac{n}{p} \\ \delta + 2 - \dfrac{n}{p'} \notin \mathbb{N}, & \text{if } \delta > -\dfrac{n}{p} \end{cases}, \tag{7.2}$$

where $1/p + 1/p' = 1$ and $1 < p < \infty$. Furthermore, if $\delta > -n/p$, then (7.1) is injective and its Fredholm's index is equal to $-d_{[\delta+2-n/p']}$ ($[\gamma]$ is the largest integer $\le \gamma$).

Let

$$\mathcal{P} = \Delta + \mathcal{Q} = \Delta + \sum_{|\alpha|\le 2} q_\alpha(x)\partial^\alpha$$

be a perturbation of the Laplacian.

THEOREM (Lockhart [9] and McOwen [12]). *Suppose $\mathcal{P}$ is uniformly elliptic operator in $\mathbb{R}^n$ and δ satisfies* (7.2). *If*

$$\lim_{|x|\to\infty} \sup \langle x\rangle^{2-|\alpha|}|q_\alpha(x)| = 0,$$

then

$$\mathcal{P} : W^p_{2,\delta} \to L^p_{\delta+2}$$

is Fredholm with the same index as the Laplace operator (7.1).

Lockhart and McOwen assumed $q_\alpha \in C_0^\infty(\mathbb{R}^n)$ for $|\alpha| = 2$, however, that condition may be removed by means of a perturbation. Imposing additional restrictions on $\mathcal{P}$, we obtain a stronger result for $\delta > -n/p$.

THEOREM 7.1. *Let $\mathcal{P}$ be uniformly elliptic and subcritical operator in $\mathbb{R}^n$. Assume q_α are σ-Hölder continuous and*

$$\langle x\rangle^{2-|\alpha|}|q_\alpha(x)| \le K_1 h(|x|), \quad |x| \ge 1,$$

where $h(t)$ is a non-increasing function satisfying $\int_1^\infty t^{-1}h(t)dt \le K_2$.

If $\delta > -n/p$, then

$$\mathcal{P} : W^p_{2,\delta} \to L^p_{\delta+2}$$

is an injective operator.

COROLLARY 1. *Retain the hypotheses of Theorem 7.1 and assume δ satisfies (7.2). Then there is a positive constant C such that*

$$\|\varphi\|_{\delta,2,p} \le C\|\mathcal{P}\varphi\|_{\delta+2,p} \quad \textit{for all } \varphi \in W^p_{2,\delta}.$$

REMARK. In [4] Avellaneda and Lin showed that if $\mathcal{A}\phi \in L^p(\mathbb{R}^n)$, then

$$\|\phi\|_{W^p_2(\mathbb{R}^n)} \le C\|\mathcal{A}\phi\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

for $\mathcal{A}$ being a second-order elliptic operator in the divergence form with periodics coefficients.

COROLLARY 2. *Let $\mathcal{P}^*$ be the formal adjoint operator of $\mathcal{P}$. Assume $\mathcal{P}$ satisfies the hypotheses of Theorem 7.1, and $\delta > -n/p$ satisfying (7.2). Then*

$$\mathcal{P}^* : L^{p'}_{-\delta-2} \to (W^p_{2,\delta})^* = W^{p'}_{-2,-\delta}$$

is a surjective Fredholm operator and its index is equal to

$$\dim\ker(\mathcal{P}^*, L^{p'}_{-\delta-2}) = \dim\ker(\Delta, L^{p'}_{-\delta-2}) = d_{[\delta+2-n/p']}.$$

As a consequence of Corollary 2, we have that if $\langle x\rangle^{-\delta} f \in L^{p'}(\mathbb{R}^n)$, then equation $\mathcal{P}^*u = f$ has a solution u such that $\langle x\rangle^{-\delta-2}u \in L^{p'}(\mathbb{R}^n)$. Moreover, any two solutions u_1, $u_2 \in L^{p'}_{-\delta-2}$, differ by a linear combination of finitely many functions from $\ker(\mathcal{P}^*, L^{p'}_{-\delta-2})$.

PROOF OF THEOREM 7.1. It suffices to show that $C_0^\infty(\mathbb{R}^n)$ annihilates $\ker(\mathcal{P}, W^p_{2,\delta})$. So fix $\varphi \in C_0^\infty(\mathbb{R}^n)$ and let $G_{\mathcal{P}^*}(x,y)$ be the Green's function of $\mathcal{P}^*$. Define

$$\mathcal{K}\varphi(x) = \int G_{\mathcal{P}^*}(x,y)\varphi(y)dy,$$

then

$$\mathcal{P}^*(\mathcal{K}\varphi) = \varphi. \tag{7.3}$$

Since the Green's function of $\mathcal{P}^*$ is given by $G_{\mathcal{P}^*}(x,y) = G_{\mathcal{P}}(y,x)$ (see e.g. [14; Theorem 20.6]), we have by Pinchover's estimate (5.1),

$$|\mathcal{K}\varphi(x)| \leq C_1 \int |x-y|^{2-n}|\varphi(y)|dy \leq C_1 C_\varphi \langle x\rangle^{2-n},$$

where the constant C_φ depents on φ. Therefore, it follows from the later inequality that if $\delta > -n/p$, then

$$\mathcal{K}\varphi \in L^{p'}_{-\delta-2}. \tag{7.4}$$

Suppose now $u \in \ker(\mathcal{P}, W^p_{2,\delta})$, then there is a sequence $\{u_k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ so that $u_k \to u$ in $W^p_{2,\delta}$ and $\mathcal{P}u_k \to 0$ in $L^p_{\delta+2}$, as $k \to \infty$. Therefore, by (7.3) and (7.4),

$$\int u\varphi dx = \lim_{k\to\infty} \int u_k\varphi dx = \lim_{k\to\infty} \int u_k \mathcal{P}^*(\mathcal{K}\varphi)dx = \lim_{k\to\infty} \int \mathcal{P}u_k(\mathcal{K}\varphi)dx = 0.$$

The proof is completed. □

8. - Failure of the Liouville type theorems for fourth-order elliptic operators

It is known that the classical Liouville's theorem holds true for the BiLaplace operator Δ^2 (cf. [6; Theorem 2.28]). Nevertheless, it is possible to construct a perturbation of BiLaplace

$$\mathcal{P} := \Delta^2 + \mathcal{Q} = \Delta^2 + \sum_{|\alpha|\leq 4} q_\alpha(x)\partial^\alpha,$$

such that all q_α are compactly supported (and hence conditions (1.7) and (1.9) are fulfilled), while $\dim\ker(\mathcal{P})$ is large as we wish in any of the spaces $C^{2,\sigma}_{m+\varepsilon}$ and $W^p_{2,\delta}$.

Our construction is similar to the one given in [18] of an elliptic operator which has an infinite dimension null space, and is based upon the following example due to Pliś. In [21] he showed the existence of an operator

$$\mathcal{P}_0 = \Delta^2 + \sum_{|\alpha|\leq 4} a_\alpha(x)\partial^\alpha$$

such that:

(i) $a_\alpha \in C_0^\infty(B_1(0))$

(ii) there is $u_0 \in C_0^\infty(B_1(0))$ satisfying $\mathcal{P}_0 u_0 = 0$.

For any integer N let $\{B_j\}_{j=1}^N$ be a disjoint sequence of balls, $B_j = \{x : |x - x_j| < r_j\}$. Set

$$q_\alpha(x) = \sum_{j=1}^{N} a_\alpha\left(x_j + \frac{x}{r_j}\right), \quad \mathcal{P} = \Delta^2 + \sum_{|\alpha|\le 4} q_\alpha(x)\partial^\alpha,$$

and

$$u_j(x) = u_0\left(x_j + \frac{x}{r_j}\right).$$

$u_1, \dots, u_N$ are linearly independent since $\operatorname{supp}(u_j) \subset B_j$ and B_j are disjoint. In addition, $\mathcal{P}u_j = 0$. Thus $\dim \ker(\mathcal{P}) \ge N$.

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