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An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations

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Consider for instance the model problem

$$-\Delta_p u = F(x,u) \qquad \text{on} \quad \Omega \tag{1.1}$$

$$u(x) = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $1 < p < \infty$, $Du = (\partial_1 u, \cdots, \partial_N u)$ denotes the gradient of u, the expression $\Delta_p(u)$ means $\operatorname{div}(|Du|^{p-2}Du)$ and F is a continuous function which is nonincreasing in u and such that $F(x,0) = L^1(\Omega)$ and $F(x,c) \in L^1_{\text{loc}}(\Omega)$ if $c \neq 0$.

Many authors have considered this problem, specially in the case $p = 2$, in the form

$$-\Delta u + \beta(u) = f(x),$$

cf. e.g. [BBC], [BS], [BG1]. We are interested here in the case $1 < p < N$. The case $p > N$ offers less difficulties and for bounded Ω can be found in [LL]. Indeed, the solution u is bounded and the gradient Du belongs to $L^p(\Omega)$, so that variational methods apply. This is not the case when $p \leq N$, so that we have to use a different approach to obtain existence and uniqueness.

There are two difficulties associated with the study of equation (1.1), even in a bounded domain, which are not solved in former works. The first is to give a sense to the solutions of an equation of the form $-\Delta_p(u) = f \in L^1(\Omega)$ for p close to 1, precisely for $p \leq p_0 = 2 - (1/N)$. In fact, we cannot expect the solution to be in $W^{1,1}_{\text{loc}}(\Omega)$. This can be seen by direct inspection of the fundamental solution, i.e. the solution of (1.1) when F equals a Dirac mass, which takes the form

$$U(x) = C|x|^{-\alpha}, \qquad \alpha = \frac{N-p}{p-1}. \tag{1.3}$$

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We see that $|DU| \in L^p_{\text{loc}}(\mathbb{R}^N)$ if $p > N$ and also that $|DU| \in L^1_{\text{loc}}(\mathbb{R}^N)$ if $p > p_0$. More generally, the same conclusion holds for L^1 data, see Appendix I at the end of the paper (cf. also the remarks in [BS] or [BGV]). Therefore, we cannot take the gradient of u appearing in the p-Laplacian operator in the usual distribution sense. We solve this difficulty by introducing a new space $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ in which we can naturally give a sense to the gradient of u which in general is not locally integrable. The idea consists in considering truncatures of the solution u, $T_k(u)$, and working instead of Du with the derivatives $DT_k(u)$, which turn out to be locally integrable. Precise definitions are given in Section 2. Then the first term in equation (1.1) makes sense when $|Du|^{p-2}Du \in L^1_{\text{loc}}(\Omega)$. In order to take into account condition (1.2) we seek the solution in a proper subspace of $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$, $\mathcal{T}^{1,p}_0(\Omega)$. Of course, when $u \in W^{1,1}_{\text{loc}}(\Omega)$, and this happens for the solutions of (1.1), (1.2) when $p > 2 - (1/N)$, the new derivative concept reduces to the usual one.

A second difficulty appears with the question of uniqueness of solutions. We obtain existence and uniqueness of a special class of solutions of (1.1)-(1.2) that satisfy an extra condition that we call the *entropy condition* (formula (3.3) below). The use of such conditions is rather common in conservation laws, cf. [La], [Kr], but is novel to elliptic equations.

Let us state next our precise framework. We will pose a slightly more general equation

$$-\operatorname{div}(\mathbf{a}(x, Du)) = F(x, u) \qquad \text{in} \quad \mathcal{D}'(\Omega). \tag{1.4}$$

The following assumptions are made on Ω, $\mathbf{a}$ and F:

(H1) Ω is an open set, not necessarily bounded, in $\mathbb{R}^N$, $N \geq 2$.

(H2) The function $\mathbf{a} : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function (continuous in ξ for a.e. x and measurable in x for every ξ) and there exist $p \in (1, N)$ and $\lambda > 0$ such that

$$\langle \mathbf{a}(x, \xi), \xi \rangle \geq \lambda |\xi|^p$$

holds for every ξ and a.e. x. There is no restriction in assuming that $\lambda = 1$.

(H3) For every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$\langle \mathbf{a}(x, \xi) - \mathbf{a}(x, \eta), \xi - \eta \rangle > 0,$$

where $\langle , \rangle$ means scalar product in $\mathbb{R}^N$.

(H4) There exists $\Lambda > 0$ such that

$$|\mathbf{a}(x, \xi)| \leq \Lambda(j(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$ with $j \in L^{p'}(\Omega)$, $p' = p/(p-1)$.

(H5) F is a Carathéodory function, continuous and nonincreasing in u for fixed x, and measurable in x for fixed u. Moreover, $F(x,0) \in L^1(\Omega)$, and if

$$G_c(x) = \sup_{\{|u|\leq c\}} \{|F(x,u)|\},$$

then $G_c \in L^1_{\text{loc}}(\Omega)$ for every $c > 0$.

Let us briefly summarize the contents of the paper: after a section devoted to developing the necessary functional setting we introduce the concept of entropy solution and derive the main properties of such solutions (Section 3). In Section 4 we derive the basic a priori estimates on the measure of their level sets. We are then ready to establish uniqueness (Section 5) and existence (Section 6) of entropy solutions for the Dirichlet problem (1.4), (1.2). We gather in Section 7 some properties of the solution and their relation to the theory of accretive operators and the generation of semigroups. Extensions to more general settings will be commented upon and partially worked out in Section 8. We treat in particular the case where $F(x,u) = f(x) - \beta(u)$, with f a bounded measure and β a maximal monotone graph. We note that our paper contains new results even for linear growth, i.e. $p = 2$, for equations of the form $-\operatorname{div} \mathbf{a}(x, Du) = F(x,u)$ posed in arbitrary domains. Finally, four appendices contain technical results. The first one comments on the need of a new functional setting when $p \leq p_0$. Appendix II gives different characterizations of the basic space $\mathcal{T}_0^{1,p}(\Omega)$. Appendix III is also related to spaces of truncated functions. Finally, Appendix IV discusses the need of entropy conditions.

For reasons of concision and clarity of exposition we have chosen not to include the study of the limit case $p = N$ in the present work. The reader will easily check that most of the theory developed below still applies though it has some particular features which may deserve separate attention. In particular, the uniqueness theory is unchanged and the estimates of Section 4 are easily adapted. The supercritical case $p > N$ is easier since solutions turn out to be continuous. We give some more precise details and results in Section 8.

Let us mention some parallel developments. First, the works of P.L. Lions and F. Murat [LM] (see also [M]) on the equation $\operatorname{div}(A(x)Du + \phi(u)) + \lambda u = f$ with $f \in L^1(\Omega)$, where ϕ is locally Lipschitz-continuous with any growth at infinity; they prove existence and uniqueness of a renormalized solution, a notion introduced in [DL] in the study of the Boltzmann equations. The existence of a renormalized solution for $f \in H^{-1}(\Omega)$ was proved in [BGDM]. *Entropy* solutions and *renormalized* solutions are different approaches to the definition of a suitable generalized solution which will make the problem well-posed. Let us also mention the work of Dall'Aglio [D] who constructed solutions for equations of the form $-\Delta_p(u) + g(x,u) = f$ with $f \in L^1(\Omega)$, defined as limits of variational solutions, and proved uniqueness of the limit solution thus obtained. This notion of solution is related to the abstract development of [BC]. The works of Rakotoson [R1], [R2] and [R3] address equations of the form $-\operatorname{div} a(x,u,Du)+g(x,u) = \mu$ where μ is an L^1 function or a bounded measure on

Ω; he also introduces a space of functions similar to our $\mathcal{T}^{1,p}_{\text{loc}}(\Omega)$ (while smaller) and proves existence of generalized solutions; in [R3] he proves existence and uniqueness of renormalized solutions when $\mu \in L^1(\Omega)$. In all the aforementioned works the open set Ω is assumed bounded. Some of the difficulties below will be related to the consideration of unbounded domains. Finally, the parabolic equation $u_t = \Delta_p(u)$ has been treated among others by DiBenedetto and Herrero [DBH1,2]. For small p they also deal with truncated solutions. In concluding we would like to point out that the basic ideas of this paper, including the introduction of T-spaces to account for the unusual derivatives, and the a priori estimates of the distribution function of u and Du, were announced years ago (see [B2] and the reference [1] in [BGDM]).

2. Functional spaces

Before we discuss the concept of solution we need to go into the functional setting in some detail. First, some notation. As usual, for $1 \le p \le \infty$ $L^p(\Omega)$ and $W^{1,p}(\Omega)$ will denote the standard Lebesgue and Sobolev spaces and $W^{1,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$. $\|\cdot\|_p$ denotes the L^p-norm in Ω. We shall also use the local spaces $L^p_{\text{loc}}(\Omega)$ and $W^{1,p}_{\text{loc}}(\Omega)$. By $L_0(\Omega)$ we denote the set of measurable functions $u : \Omega \to \mathbb{R}$ such that the sets $\{|u| > \varepsilon\}$ have finite measure for every $\varepsilon > 0$. This expresses the fact that the functions go to 0 as $|x| \to \infty$ in measure. We have $L^p(\Omega) \subset L_0(\Omega)$ for every $1 \le p < \infty$. For a measurable set $A \subset \mathbb{R}^N$ we use the notation $\text{meas}(A) = |A|$ to denote its measure.

We begin by introducing the truncature operator. For a given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ k \,\text{sign}\,(s) & \text{if } |s| > k. \end{cases} \tag{2.1}$$

For a function $u = u(x)$, $x \in \Omega$, we define the truncated function $T_k u = T_k(u)$ pointwise: for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$. We now introduce the functional spaces we will need in our theory:

i) $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ is defined as the set of measurable functions $u : \Omega \mapsto \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{\text{loc}}(\Omega)$.

ii) For $p \in (1, \infty)$ we define $\mathcal{T}^{1,p}_{\text{loc}}(\Omega)$ as the subset of $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ consisting of the functions u such that $D(T_k(u)) \in L^p_{\text{loc}}(\Omega)$ for every $k > 0$. Likewise, $\mathcal{T}^{1,p}(\Omega)$ is the subset of $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ consisting of the u such that moreover $DT_k(u) \in L^p(\Omega)$ for every $k > 0$.

iii) Finally, $\mathcal{T}^{1,p}_0(\Omega)$ will be the subset of $\mathcal{T}^{1,p}(\Omega)$ consisting of the functions that can be approximated by smooth functions with compact support in Ω in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}^{1,p}_0(\Omega)$ if for every

$k > 0$ there exists a sequence $\phi_n \in C_0^\infty(\Omega)$ such that

$$D\phi_n \to DT_k(u) \qquad \text{in} \qquad L^p(\Omega),$$

$$\phi_n \to T_k(u) \qquad \text{in} \qquad L^1_{\text{loc}}(\Omega).$$

This space will play an important role in what follows. Alternative characterizations of it are given in Appendix II at the end of the paper.

Let us now devote some space to consider the properties of these spaces. To begin with, it is clear that for every $p \in [1, \infty)$ we have the inclusions $W^{1,p}_{\text{loc}}(\Omega) \subset \mathcal{T}^{1,p}_{\text{loc}}(\Omega)$ and $W^{1,p}_0(\Omega) \subset \mathcal{T}^{1,p}_0(\Omega)$ and in these cases we have

$$DT_k(u) = 1_{\{|u|<k\}} Du,$$

where 1_A denotes the characteristic function of a measurable set $A \subset \mathbb{R}^N$. It is also clear that $\mathcal{T}^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) = W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. Moreover, we can easily convince ourselves that the inclusions are strict, i.e. the new spaces are strict extensions. In fact, $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ is not even a vector space, as the following example in one space dimension shows: consider in $\Omega = (-1, 1)$ the functions $u(x) = x \sin(1/x)$ and $v(x) = x^{-2}$. Then v and $u+v$ belong to $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$, but u does not. However, it is true for instance that if $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ and $v \in W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ then $u + v \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$. Let us remind the reader that in defining $\mathcal{T}^{1,p}(\Omega)$ we did not impose the condition $T_k(u) \in L^p(\Omega)$. Of course, this condition follows immediately when Ω has finite measure (since $T_k(u)$ is bounded), but for unbounded Ω it makes a real difference.

We want to give a sense to the derivative Du of a function $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$, generalizing the usual concept of weak derivative in $W^{1,1}_{\text{loc}}(\Omega)$, cf. [GT]. The following result paves the way in this direction.

LEMMA 2.1. *For every $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ there exists a unique measurable function $v : \Omega \mapsto \mathbb{R}^N$ such that*

$$DT_k(u) = v 1_{\{|v|<k\}} \qquad \text{a.e.} \tag{2.2}$$

Furthermore, $u \in W^{1,1}_{\text{loc}}(\Omega)$ if and only if $v \in L^1_{\text{loc}}(\Omega)$, and then $v \equiv Du$ in the usual weak sense.

Here unique is understood in the almost everywhere sense. The *proof* of this result is as follows: We have seen that formula (2.2) is true for $u \in W^{1,1}_{\text{loc}}(\Omega)$ with $v = Du$. Note also that for $k, \varepsilon > 0$ we have $T_k(T_{k+\varepsilon}(u)) = T_k(u)$. Therefore, we get in $\Omega_k = \{|u| < k\}$ the a.e. equality $DT_{k+\varepsilon}(u) = DT_k(u)$. But, $\bigcup_{k>0} \Omega_k = \Omega$, hence the assertion (2.2) follows.

We are left with the proof that $u \in W^{1,1}_{\text{loc}}(\Omega)$ if $v \in L^1_{\text{loc}}(\Omega)$. Indeed, in that case $DT_k(u) \to v$ in $L^1_{\text{loc}}(\Omega)$. We still have to see that $u \in L^1_{\text{loc}}(\Omega)$. If this were not true there would exist a closed ball $B \subset \Omega$ such that

$$t_k = \|T_k(u)\|_{L^1(B)} \to \infty$$

as $k \to \infty$. Normalize $v_k = T_k(u)/t_k$. Then $v_k \to 0$ a.e., $\|v_k\|_{L^1(B)} = 1$ and $\|Dv_k\|_{L^1(B)} \to 0$. This is a contradiction to the compactness of the embedding $W^{1,1}(B) \subset L^1(B)$.

Thanks to this result *we define the derivative Du of a function* $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ *as the unique function* v *which satisfies* (2.2). This notation will be used throughout in the sequel. We recall that in general the derivative of a function $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega) \cap L^1_{\text{loc}}(\Omega)$ *need not* be a locally integrable function, and that this definition of derivative is *not* a definition in the sense of distributions.

The following straightforward result will be useful.

LEMMA 2.2. *If* $u \in \mathcal{T}^{1,p}_0(\Omega)$ *and* $1 < p < N$ *then* $DT_k(u) \in L^p(\Omega)$ *and* $T_k(u) \in L^{p^*}(\Omega)$ *for* $p^* = pN/(N-p)$. *If* Ω *is bounded then for every* $1 < p < \infty$ *we have* $u \in \mathcal{T}^{1,p}_0(\Omega)$ *if and only if* $T_k(u) \in W^{1,p}_0(\Omega)$ *for every* $k > 0$. *Finally, for bounded* Ω $u \in W^{1,p}_0(\Omega)$ *if and only if* $u \in \mathcal{T}^{1,p}_0(\Omega)$ *and* $Du \in L^p(\Omega)$.

Observe that if $1 < p < N$ then $\mathcal{T}^{1,p}_0(\Omega) \subset L_0(\Omega)$. Indeed, since $T_k(u) \in L^{p^*}(\Omega)$ for $k > 0$, $u \to 0$ in measure as $|x| \to \infty$. This will be used later on.

It is sometimes useful to replace the truncation $T_k u$ introduced above by smoother truncations: in this sense, it is worth noticing the following result.

LEMMA 2.3. *If* $u \in \mathcal{T}^{1,p}_{\text{loc}}(\Omega)$ *then* $T(u) \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ *for every Lipschitz-continuous function* $T : \mathbb{R} \mapsto \mathbb{R}$ *satisfying*

$$T'(s) = 0 \qquad \text{for } |s| \text{ large enough.} \tag{2.3}$$

Moreover, $DT(u) = P(u)Du$ *where* P *is a measurable function defined a.e. by* $P(u) = T'(u)$. *Finally, if* $u \in \mathcal{T}^{1,p}_0(\Omega)$ *and* $T(0) = 0$ *then* $T(u) \in \mathcal{T}^{1,p}_0(\Omega)$.

The proof of this lemma is straightforward since $T(u) = T(T_k(u))$ for large enough k. We must notice that the sole assumptions $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ and $T : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous (resp. Lipschitz continuous and bounded) do not in general imply that $T(u) \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ (resp. $W^{1,1}_{\text{loc}}(\mathbb{R}^N)$). See Appendix III for a counterexample.

3. Entropy solutions

DEFINITIONS. Let us consider now the concept of solution for our kind of equations in the new functional setting. Thus, given the equation

$$-\operatorname{div}(\mathbf{a}(x, Du)) = f(x), \tag{3.1}$$

under the assumptions (H1)–(H4) and with $f \in L^1(\Omega)$, by a **solution** we will understand a function $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ such that $\mathbf{a}(Du(x))$ belongs to $L^1_{\text{loc}}(\Omega)$ and

the equation is satisfied in $\mathcal{D}'(\Omega)$, i.e.

$$\int_\Omega \langle \mathbf{a}(x, Du), D\phi \rangle \, dx = \int_\Omega f\phi dx, \tag{3.2}$$

for every test function $\phi \in C_0^\infty(\Omega)$. In this paper we will deal with special solutions of the homogeneous Dirichlet problem (3.1)-(1.2). Thus, if in (3.2) we allow as test function $T_k(u - \phi)$, $k > 0$, we obtain

$$\int_{\{|u-\phi|<k\}} \langle \mathbf{a}(x, Du), Du - D\phi \rangle \, dx = \int T_k(u - \phi) f \, dx. \tag{3.3}$$

Notice that both integrals in (3.3) are well defined. The second member offers no difficulty since $f \in L^1(\mathbb{R}^N)$. As to the first member we observe that

$$\langle \mathbf{a}(x, Du), Du - D\phi \rangle \, 1_{\{|u-\phi|<k\}} \geq -|\mathbf{a}(x, Du)| \, |D\phi| \, 1_{\{|u|<K\}}, \tag{3.4}$$

where $K = k + \|\phi\|_\infty$. Since the second member in (3.4) is integrable in Ω, the integral in the first member of (3.3) is well-defined.

It must be observed at this stage that (3.3) cannot be derived in general from (3.2). We will briefly discuss this issue in Appendix IV. We will in fact see that we cannot even derive the inequalities

$$\int_{\{|u-\phi|<k\}} \langle \mathbf{a}(x, Du), Du - D\phi \rangle \, dx \leq \int T_k(u - \phi) f \, dx, \quad k > 0. \tag{3.5}$$

This family of inequalities is precisely the basis of our theory.

Indeed, we *define* an **entropy solution** of problem (3.1)-(1.2) as a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ satisfying the family of inequalities (3.5) for every $\phi \in \mathcal{D}(\Omega)$ and $k > 0$. This will be referred to as the *entropy condition.*

As above the integrals in (3.5) are well defined. On the other hand, using the fact that $Du - D\phi = 0$ a.e. on the set where $|u - \phi| = k$, it is clear that replacing the integration set $\{|u - \phi| < k\}$ in the first member of (3.3) by $\{|u - \phi| \leq k\}$ does not change the value of the integral, so the latter set can be used in (3.5) instead of $\{|u - \phi| < k\}$.

While a priori it is not clear, we will prove below that an entropy solution is always a solution of (3.1) in the standard sense defined above. This will be done in Section 4 after deriving convenient a priori estimates for the entropic solutions.

PROPERTIES. We are going to derive some properties of entropy solutions. Firstly, setting $\phi = 0$ we obtain an immediate consequence of the definition

LEMMA 3.1. *If* $u \in T_0^{1,p}(\Omega)$ *is an entropy solution of* (3.1)-(1.2) *then for every* $k > 0$

$$\frac{1}{k} \int_{\{|u|<k\}} \langle \mathbf{a}(x, Du), Du \rangle \, dx \leq \int |f| \, dx = \|f\|_1. \tag{3.6}$$

Hence, under hypothesis (H2) *we obtain the following bound in* $L^p(\Omega)$:

$$\|DT_k(u)\|_p^p \leq \frac{k}{\lambda} \|f\|_1. \tag{3.7}$$

It is technically useful to extend the entropy condition to more general truncations than T_k and more general test functions than $\phi \in C_0^\infty(\mathbb{R}^N)$. To begin with, we introduce the class $\mathcal{T}$ of functions $T \in C^2(\mathbb{R} : \mathbb{R}) \cap L^\infty(\mathbb{R} : \mathbb{R})$ satisfying:

$$T(0) = 0, \quad T' \geq 0, \quad T'(s) = 0 \quad \text{for all } s \text{ large enough,}$$

$$T(-s) = -T(s), \quad \text{and } T''(s) \leq 0 \quad \text{for} \quad s \geq 0.$$

For $T \in \mathcal{T}$ we write $k(T) = \inf\{k : T(s) = \|T\|_\infty\}$. Then we have

LEMMA 3.2. *The entropy condition* (3.5) *is equivalent to the statement that*

$$\int \langle \mathbf{a}(Du), DT(u - \phi) \rangle \, dx \leq \int_\Omega fT(u - \phi) \, dx \tag{3.8}$$

holds for every test function $\phi \in C_0^\infty(\Omega)$ *and every function* $T \in \mathcal{T}$.

PROOF. Suppose that (3.8) holds and let us prove (3.5). Take a $k > 1$. We may use an approximation of the standard cut T_k by an increasing sequence of functions $S_n \in \mathcal{T}$ chosen so that $S_n'(s) = 0$ for $|s| \geq k$, $S_n'(s) = 1$ for $|s| \leq k - (1/n)$ and $S_n' \leq 1$ everywhere. Since as $n \to \infty$ $S_n(u - \phi) \to T_k(u - \phi)$ uniformly and $S_n'(u - \phi) \to T_k'(u - \phi)$ a.e., applying (3.6) with $T = S_n$ and passing to the limit we obtain (3.5).

Conversely, if (3.5) holds consider the case where $T \in \mathcal{T}$ is just a combination of cut functions,

$$T = \sum a_j T_{k_j}, \quad a_j \geq 0.$$

In that case we apply (3.5) to the T_{k_j} and add to obtain that (3.6) holds. In the general case $T \in \mathcal{T}$ we approximate in C^1-norm by a sequence of functions of that type and pass to the limit.

Next we show that the entropy condition (3.5) holds for a much wider class of test functions. This fact will be very important below.

LEMMA 3.3. *If u is an entropy solution of* (3.1)-(1.2). *Then* (3.5) *holds for every test function* $\phi \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

PROOF. By definition there exists a sequence $\phi_n \in C_0^\infty(\Omega)$ such that $D\phi_n \to D\phi$ in $L^p(\Omega)$ and $\phi_n \to \phi$ in $L^1_{\text{loc}}(\Omega)$ and a.e. Replacing ϕ_n by $R(\phi_n)$ with $R \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $R(s) = s$ for $|s| \le \|\phi\|_\infty$ we may always assume that the ϕ_n's are uniformly bounded in Ω. We may also assume that there exists a function $w \in L^p(\Omega)$ such that $|D\phi_n| \le w$ a.e. We have

$$T_k(u - \phi_n) \to T_k(u - \phi) \quad \text{a.e}$$

and $|DT_k(u - \phi_n)| \le |DT_K(u)| + w$, with $K = k + \sup \|\phi_n\|_\infty$. It is not difficult to see that

$$DT_k(u - \phi_n) \to DT_k(u - \phi) \quad \text{weakly in} \quad L^p(\Omega).$$

Assuming now the definition of entropy solution we have

$$\int \langle \mathbf{a}(x, Du), DT_k(u - \phi_n) \rangle \, dx \le \int T_k(u - \phi_n) f \, dx.$$

We may pass to the limit in both sides; the right-hand side is clear since $f \in L^1(\Omega)$. As for the left-hand side, observe that the integrand equals $\langle \mathbf{a}(x, DT_K(u)), DT_k(u - \phi_n) \rangle$ and $\mathbf{a}(x, DT_K(u)) \in L^{p'}(\Omega)$.

Observe that for given a and $k > 0$ the function $T_{k,a}(s) = T_a(s - T_k(s))$ takes the values

$$T_{k,a}(s) = \begin{cases} s - k \operatorname{sign}(s) & \text{for} \quad k \le |s| < s + a, \\ a & \text{for} \quad |s| \ge k + a, \\ 0 & \text{for} \quad |s| \le k. \end{cases}$$

Now, if $v \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ the expression $T_{k,a}(u - v)$ can be written in the form $T_a(u - w)$ with $w = v + T_k(u - v) \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Applying Lemma 3.3 we get:

COROLLARY 3.4. *If u is an entropy solution of* (3.1)-(1.2) *then*

$$\int_{\{k<|u|<k+a\}} \langle \mathbf{a}(x, Du), Du \rangle \, dx \le \int f T_{k,a}(u) \, dx \le a \int_{\{|u|>k\}} |f| \, dx, \tag{3.9}$$

so that under hypothesis (H2)

$$\frac{1}{a} \int_{\{k<|u|<k+a\}} |Du|^p dx \le \frac{1}{\lambda} \int_{\{|u|\ge k\}} f \, dx. \tag{3.10}$$

This L^p-estimate for Du will play a fundamental role in the sequel.

4. A priori estimates

As another preliminary to the existence and uniqueness theory we derive estimates for a function u that satisfies the inequalities of previous section and for its gradient $|Du|$. The estimates consist of controlling the measure of the level sets, i.e. we work in Marcinkiewicz spaces. We recall, cf. [BBC], that for $0 < q < \infty$ the Marcinkiewicz space $M^q(\Omega)$ can be defined as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that the corresponding distribution functions

$$\phi_f(k) = \text{meas}\,\{x \in \Omega : |f(x)| > k\} \quad k > 0, \tag{4.1}$$

satisfy an estimate of the form

$$\phi_f(k) \le C\,k^{-q}, \qquad C < \infty. \tag{4.2}$$

It is immediate that $L^q(\Omega) \subset M^q(\Omega) \subset L_0(\Omega)$ and that for bounded Ω we have $M^q(\Omega) \subset M^{\hat{q}}(\Omega)$ if $q \ge \hat{q}$. We begin with the estimate for u.

LEMMA 4.1. *Let $1 < p < N$, let Ω be as above and let $u \in T_0^{1,p}(\Omega)$ be such that*

$$\frac{1}{k} \int\limits_{\{|u|<k\}} |Du|^p\,dx \le M \tag{4.3}$$

for every $k > 0$. Then $u \in M^{p_1}(\Omega)$ with $p_1 = \dfrac{N(p-1)}{N-p}$. More precisely, there exists $C = C(N,p) > 0$ such that

$$\text{meas}\,\{|u| > k\} \le C\,M^{\frac{N}{N-p}}\,k^{-p_1}. \tag{4.4}$$

PROOF. For $k > 0$ one has by Sobolev's embedding

$$\|T_k(u)\|_{p^*} \le c(N,p)\|DT_k(u)\|_p \le c(N,p)(Mk)^{1/p}.$$

For $0 < \varepsilon \le k$ we have $\{|u| \ge \varepsilon\} = \{|T_k(u) \ge \varepsilon|\}$. Hence

$$\text{meas}\,\{|u| > \varepsilon\} \le \left(\frac{\|T_k(u)\|_{p^*}}{\varepsilon}\right)^{p^*} \le c_1(N,p)M^{N/(N-p)}k^{N/(N-p)}\varepsilon^{-pN/(N-p)}.$$

Setting $\varepsilon = k$ we obtain (4.4).

REMARK. Such estimates are not new for solutions of elliptic equations. They have been proved by Talenti [Ta] for quasilinear equations using rearrangement theory. However, this elementary proof is new.

We now proceed with the derivative estimates.

LEMMA 4.2. *Let* $1 < p < N$ *and assume that* $u \in T_0^{1,p}(\Omega)$ *satisfies* (4.3) *for every* k. *Then for every* $h > 0$

$$\text{meas}\{|Du| > h\} \leq C(N,p)\, M^{\frac{N}{N-1}}\, h^{-p_2}, \qquad p_2 = \frac{N(p-1)}{N-1}. \tag{4.5}$$

PROOF. For $k, \lambda > 0$ set

$$\Phi(k,\lambda) = \text{meas}\{|Du|^p > \lambda, |u| > k\}.$$

From Lemma 4.1 we have

$$\Phi(k,0) \leq C(N,p) M^{N/(N-p)} k^{-p_1}. \tag{4.6}$$

Using the fact that the function $\lambda \mapsto \Phi(k,\lambda)$ is nonincreasing we get for $k, \lambda > 0$

$$\Phi(0,\lambda) \leq \frac{1}{\lambda}\int_0^\lambda \Phi(0,s)ds \leq \Phi(k,0) + \int_0^\lambda (\Phi(0,s) - \Phi(k,s))ds. \tag{4.7}$$

Now, observe that since

$$\Phi(0,s) - \Phi(k,s) = \text{meas}\{|u| < k, |Du|^p > s\},$$

we have thanks to (4.3)

$$\int_0^\infty (\Phi(0,s) - \Phi(k,s))ds = \int_{\{|u|<k\}} |Du|^p dx \leq Mk. \tag{4.8}$$

Going back to (4.7) and using (4.6) and (4.8) we arrive at

$$\Phi(0,\lambda) \leq \frac{Mk}{\lambda} + C(N,p) M^{N/(N-p)} k^{-p_1}. \tag{4.9}$$

Minimization of (4.9) in k and setting $\lambda = h^p$ give (4.5).

As a corollary we have:

COROLLARY 4.3. *Under assumptions* (H1)–(H4), *if* u *is an entropy solution of* (3.1)-(1.2) *then* $\mathbf{a}(x,Du) \in L^1(\Omega) + L^\infty(\Omega)$ *and* u *is a solution of* (3.1), *i.e.* (3.2) *holds for every* $\phi \in C_0^\infty(\Omega)$.

PROOF. Using Corollary 3.4 and Lemma 4.2 we obtain (4.5). Using (H4) and $p < N$ it follows that

$$\text{meas}\{|\mathbf{a}(x,Du)| > h\} \leq Ch^{-N/(N-1)}$$

for some $C > 0$ depending on N, p, λ, Λ and $\|f\|_1$. Therefore, $\mathbf{a}(x, Du) \in L^1(\Omega) \cap L^\infty(\Omega)$.

Let now $\phi \in C_0^\infty(\Omega)$. Applying Lemma 3.3 with test function $T_h(u) - \phi$ instead of ϕ we get

$$\int_{\{|u-T_h u+\phi|<k\}} \langle \mathbf{a}(Du), 1_{\{|u|>h\}} Du + D\phi \rangle \, dx \leq \int T_k(u - T_h(u) + \phi) f \, dx,$$

and then

$$\int_{\{|u-T_h u+\phi|<k\}} \langle \mathbf{a}(Du), D\phi \rangle \, dx \leq \int T_k(u - T_h(u) + \phi) f \, dx.$$

Choosing $k > \|\phi\|_\infty$ at the limit $h \to \infty$ we have

$$\int \langle \mathbf{a}(Du), D\phi \rangle \, dx \leq \int \phi f \, dx.$$

Replacing ϕ by $-\phi$ we get the converse inequality. Hence, equality holds.

In this way we have shown that an entropy solution is indeed a solution in the standard distribution sense. This result would follow in any case from the existence and uniqueness of sections 6 and 5. Indeed, we will prove that entropy solutions are unique and then we will construct a standard solution of the problem that is also an entropy solution.

5. Uniqueness

We settle here the question of uniqueness of entropy solutions in the spirit of Section 3.

DEFINITION OF SOLUTION. By a solution of (1.4)-(1.2) we understand a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ such that $F(x, u(x)) \in L^1(\Omega)$ and which is a solution of equation (3.1) with second member $f(x) = F(x, u(x))$. The definition of entropy solution is similar to (3.5).

Our main result is:

THEOREM 5.1. *Let u_1 and u_2 be two functions in $\mathcal{T}_0^{1,p}(\Omega)$ which are entropy solutions of the equation*

$$-\operatorname{div}(\mathbf{a}(x, Du)) = F(x, u)$$

under assumptions (H1)–(H5). *Then* $u_1 = u_2$.

PROOF. (i) Let $f_i(x) = F(x, u_i(x))$, $i = 1, 2$. We are assuming that $f_i \in L^1(\Omega)$. We will write $\mathbf{a}(Du)$ instead of $\mathbf{a}(x, Du)$ for convenience. We write the entropy inequality corresponding to solution u_1 with test function $T_h u_2$ and u_2 with test function $T_h u_1$ (use Lemma 3.3). Adding up both results we get

$$\begin{aligned} &\int_{\{|u_1 - T_h u_2| < k\}} \langle \mathbf{a}(Du_1), Du_1 - DT_h u_2 \rangle \, dx \\ &+ \int_{\{|u_2 - T_h u_1| < k\}} \langle \mathbf{a}(Du_2), Du_2 - DT_h u_1 \rangle \, dx \\ &\leq \int_\Omega f_1 T_k(u_1 - T_h(u_2)) \, dx + \int_\Omega f_2 T_k(u_2 - T_h(u_1)) \, dx. \end{aligned} \tag{5.1}$$

(ii) The conclusion $u_1 = u_2$ will be reached after passing to the limit $h \to \infty$ in this formula and disregarding some positive but uninteresting terms. We proceed by splitting the integrals above into the contributions corresponding to different integration sets. Thus, if we put

$$A_0 = \{x \in \Omega : |u_1 - u_2| < k, \ |u_1| < h, \ |u_2| < h\}$$

when restricted to A_0 the first member of (5.1) gives the following main contribution that we will keep:

$$I_0 = \int_{A_0} \langle \mathbf{a}(Du_1) - \mathbf{a}(Du_2), Du_1 - Du_2 \rangle \, dx.$$

The remaining first member integral is estimated as follows. Take the first term. On the set

$$A_1 = \{x \in \Omega : |u_1 - T_h u_2| < k, |u_2| \geq h\}$$

we have

$$\int_{A_1} \langle \mathbf{a}(Du_1), Du_1 - DT_h u_2 \rangle \, dx = \int_{A_1} \langle \mathbf{a}(Du_1), Du_1 \rangle \, dx \geq 0,$$

while on the remaining set

$$A_2 = \{x \in \Omega : |u_1 - T_h u_2| < k, |u_2| < h, |u_1| \geq h\}$$

we get

$$\int_{A_2} \langle \mathbf{a}(Du_1), Du_1 - DT_h u_2 \rangle \, dx = \int_{A_2} \langle \mathbf{a}(Du_1), Du_1 - Du_2 \rangle \, dx$$

$$\geq - \int_{A_2} \langle \mathbf{a}(Du_1), Du_2 \rangle \, dx.$$

In the same way we estimate the second integral in the sets A_1', where $|u_1| \geq h$, and A_2', where $|u_1| < h$ and $|u_2| \geq h$. All these sets and integrals depend of course on k and h. Summing up we estimate the first member of (5.1) in the form $I \geq I_0 - I_3$, where

$$I_3 = \int_{A_2} \langle \mathbf{a}(Du_1), Du_2 \rangle \, dx + \int_{A_2'} \langle \mathbf{a}(Du_2), Du_1 \rangle \, dx.$$

Now, I_3 goes to 0 as $h \to \infty$. Indeed, the first term can be estimated by

$$\|\mathbf{a}(Du_1)\|_{L^{p'}(\{h \leq |u_1| \leq h+k\})} \|Du_2\|_{L^p(\{h-k \leq |u_2| \leq h\})}$$

$$\leq \Lambda \left(\|j\|_{L^{p'}(\{|u_1| \geq h\})} + \|Du_1\|^{p-1}_{L^p(\{h \leq |u_1| \leq h+k\})} \right) \|Du_2\|_{L^p(\{h-k \leq |u_2| \leq h\})},$$

and this converges to 0 as $h \to \infty$ for every $k > 0$ thanks to Corollary 3.4 and Lemma 4.1. Likewise the second term.

(iii) The second member of (5.1) can be worked out by the same method. The integral on $B_0 = \{x \in \Omega : |u_1| < h, \ |u_2| < h\}$ gives

$$J_0 = \int_{B_0} (F(x, u_1) - F(x, u_2)) \, T_k(u_1 - u_2) \, dx \leq 0,$$

while on the set $B_1 = \{x \in \Omega : |u_1| \geq h\}$ the integral, J_1, is estimated by

$$|J_1| \leq k \int_{B_1} (|f_1| + |f_2|) \, dx.$$

Likewise on $B_2 = \{x \in \Omega : |u_2| \geq h\}$ we have

$$|J_2| \leq k \int_{B_2} (|f_1| + |f_2|) \, dx.$$

Now, the measure of both sets, $B_1(h, k)$ and $B_2(h, k)$, goes to zero as $h \to \infty$ for fixed $k > 0$. Hence $J_1 + J_2 \to 0$.

(iv) Combining the above estimates we get from (5.1)

$$\int_{A_0(h,k)} \langle \mathbf{a}(Du_1) - \mathbf{a}(Du_2), Du_1 - Du_2 \rangle \, dx \leq \varepsilon(h),$$

where $\varepsilon(h) \to 0$ as $h \to \infty$, k fixed. Since $A_0(h,k)$ converges to $\{x \in \Omega : |u_1 - u_2| < k\}$ we conclude that

$$\int_{\{|u_1-u_2|<k\}} \langle \mathbf{a}(Du_1) - \mathbf{a}(Du_2), Du_1 - Du_2 \rangle \, dx \leq 0.$$

Since this is true for all $k > 0$ we conclude by (H3) that $Du_1 = Du_2$ a.e. Taking into account that u_1 and $u_2 \in \mathcal{T}_0^{1,p}(\Omega) \cap L_0(\Omega)$ (use Corollary 3.4 and Lemma 4.1) we conclude that $u_1 = u_2$ a.e.

6. Existence

THEOREM 6.1. *Under assumptions* $1 < p < N$ *and* (H1)–(H5) *there exists a unique entropy solution of equation* (1.4) *in* $\mathcal{T}_0^{1,p}(\Omega)$. *Moreover,*

$$u \in M^{p_1}(\Omega), \quad |Du| \in M^{p_2}(\Omega) \tag{6.1}$$

where $p_1 = \dfrac{N(p-1)}{N-p}$ *and* $p_2 = \dfrac{N(p-1)}{N-1}$. *In case* $p > 2 - (1/N)$ *the solution belongs to* $W^{1,q}_{\text{loc}}(\Omega)$ *for every* $q < p_2$, *and if* Ω *is bounded to* $W_0^{1,q}(\Omega)$.

PROOF. *Step* 1. Let us write the second member in the form

$$F(x,u) = F(x,0) - \beta(x,u). \tag{6.2}$$

Then $f(x) = F(x,0) \in L^1(\mathbb{R}^N)$ and β is monotone nondecreasing in u with $\beta(x,0) = 0$, so that

$$\beta(x,u)u \geq 0.$$

We recall that β is continuous in u for a.e. $x \in \Omega$ and measurable in x for every $u \in \mathbb{R}$. Following the classical procedure, our first step consists in approximating the second member f with a sequence of smooth functions $f_n \in C_0^\infty(\Omega)$, $f_n \to f$ in $L^1(\Omega)$. It will be also useful to ask that

$$\|f_n\|_1 \leq \|f\|_1 \tag{6.3}$$

for every $n \geq 1$. We also approximate the monotone function β by bounded functions β_n, nondecreasing in u. For instance, we take

$$\beta_n(x,s) = \max\{-n, \min\{n, \beta(x,s)\}\}.$$

In this way $|\beta_n(x,s)| \le |\beta(x,s)|$ for every $s \in \mathbb{R}$ and $x \in \Omega$. Finally, we take

$$\gamma_n(s) = \beta_n(x,s) + \frac{1}{n}|s|^{p-2}s. \tag{6.4}$$

Then it is well-known, see [LL], [Li], and [Bw] for unbounded domains, that there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$-\operatorname{div}(\mathbf{a}(Du_n)) + \gamma_n(x,u_n) = f_n \tag{6.5}$$

holds in the sense of distributions in Ω. We also point out that $u_n \in L^1(\Omega) \cap L^\infty(\Omega)$.

Multiplying (6.5) by convenient test functions and integrating one gets the following uniform estimates

$$\frac{1}{a}\int_{\{k<|u_n|<k+a\}} |Du_n|^p\,dx \le \int_{\{|u_n|>k\}} |f_n|\,dx \le \|f_n\|_1 = C_1. \tag{6.6}$$

$$\int_{\{|u_n|>k\}} |\gamma_n(u_n)|\,dx \le \int_{\{|u_n|>k\}} |f_n|\,dx \le \|f_n\|_1 \le C_1. \tag{6.7}$$

$$\int_{\{|u_n|<k\}} |Du_n|^p\,dx \le \int_{\{|u_n|<k\}} \langle \mathbf{a}(Du_n, Du_n)\rangle\,dx \le kC_1. \tag{6.8}$$

We recall that, for the sake of simplicity, we are fixing the ellipticity constant $\lambda = 1$.

Step 2. *Convergence.* Using (6.8) we see that $\{D(T_k u_n)\}$ is bounded in $L^p_{\text{loc}}(\Omega)$ for every $k > 0$. With (6.6) and Lemma 4.1, we also have that meas $\{|u_n| > k\}$ is bounded uniformly in n for every $k > 0$. Let us prove that $u_n \to u$ *locally in measure*; to begin with, we observe that for $t, \varepsilon > 0$ we have

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_k) - T_k(u_m)| > t\}$$

so that

$$\operatorname{meas}\{|u_n - u_m| > t\} \le \operatorname{meas}\{|u_n| > k\}$$
$$+ \operatorname{meas}\{|u_m| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > t\}.$$

Choosing k large enough the first two terms in the second member are less that ε. Since $\{DT_k u_n\}_n$ is bounded in $L^p(\Omega)$ for all $k > 0$ and $T_k u_n \in W_0^{1,p}(\Omega)$ we can assume that $\{T_k u_n\}$ is a Cauchy sequence in $L^q(\Omega \cap B_R)$ for any $q < p_* = pN/(N-p)$ and any $R > 0$ and

$$T_k u_n \to T_k u \qquad \text{in} \quad L^q_{\text{loc}}(\Omega) \text{ and a.e.}$$

Then

$$\operatorname{meas}(\{|T_k u_n - T_k u_m| > t\} \cap B_R) \leq t^{-q} \int_{\Omega \cap B_R} |T_k u_n - T_k u_m|^q \, dx \leq \varepsilon$$

for all $n, m \geq n_0(k, t, R)$. This proves that $\{u_n\}$ is a Cauchy sequence in measure in B_R, hence that $u_n \to u$ locally in measure.

We now prove that Du_n converges to some function v locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). To prove this we show that $\{Du_n\}$ is a Cauchy sequence in measure in any ball B_R. Let again t and $\varepsilon > 0$. Then

$$\begin{aligned} &\{|Du_n - Du_m| > t\} \cap B_R \\ &\subset \{|Du_n| > A\} \cup \{|Du_m| > A\} \cup (\{|u_n - u_m| > k\} \cap B_R) \\ &\quad \cup \{|u_n - u_m| \leq k, |Du_n| \leq A, |Du_m| \leq A, |Du_n - Du_m| > t\}. \end{aligned} \tag{6.9}$$

We first choose A large enough in order to have

$$\operatorname{meas}\{|Du_n| > A\} \leq \varepsilon \qquad \text{for all} \qquad n \in \mathbb{N}$$

(this is possible by Lemma 4.2). If $\mathbf{a}$ is a continuous function independent of x we argue as follows: then by (H3) there exists $\mu > 0$ such that $|\xi| < A$, $|\eta| < A$ and $|\xi - \eta| > t$ together imply

$$\langle \mathbf{a}(\xi) - \mathbf{a}(\eta), \xi - \eta \rangle \geq \mu.$$

This is a consequence of the continuity and strict monotonicity of $\mathbf{a}$. Then, if we set

$$h_n = f_n - \gamma_n(u_n),$$

we have (note that $\mathbf{a}(Du_n)$ and $\mathbf{a}(Du_m)$ belong to $L^{p'}(\Omega)$)

$$\begin{aligned} &\int_{\{|u_n - u_m| \leq k\}} \langle \mathbf{a}(Du_n) - \mathbf{a}(Du_m), Du_n - Du_m \rangle \, dx \\ &= \int_{\Omega} (h_n - h_m) T_k(u_n - u_m) \, dx \leq 4C_1 k. \end{aligned}$$

Then

$$
\begin{aligned}
&\operatorname{meas}\{|u_n - u_m| \le k, |Du_n| \le A, |Du_m| \le A, |Du_n - Du_m| > t\} \\
&\le \operatorname{meas}\{|u_n - u_m| \le k, (\mathbf{a}(Du_n) - \mathbf{a}(Du_m)) \cdot (Du_n - Du_m) \ge \mu\} \\
&\le \frac{1}{\mu} \int_{\{|u_n - u_m| \le k\}} \langle \mathbf{a}(Du_n) - \mathbf{a}(Du_m), Du_n - Du_m \rangle \, dx \qquad (6.10) \\
&\le \frac{1}{\mu} 4kC_1 \le \varepsilon,
\end{aligned}
$$

if k is small enough, $k \le \mu\varepsilon/(4C_1)$. Under the general assumption (H3) the argument is technically more delicate; it can be seen in [BG2] or [BeW].

Since A and k have been already chosen, if n_0 large enough we have for $n, m \ge n_0$ the estimate $\operatorname{meas}(\{|u_n - u_m| > k\} \cap B_R) \le \varepsilon$, and then $\operatorname{meas}(\{|Du_n - Du_m|\} \cap B_R) \le 4\varepsilon$. This proves that $\{Du_n\}_n$ converges locally in measure to some function v, hence also a.e. (up to extraction of a subsequence, if necessary). Finally, since $\{DT_k u_n\}_n$ is bounded in $L^p(\Omega)$ (for any $k > 0$), it converges weakly to $D(T_k u)$ in $L^1_{\text{loc}}(\Omega)$. Then, we have $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ and $Du = v$ a.e.

Summing up, we have established the following facts:

$u_n \in W^{1,p}_0(\Omega)$,	$u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$
$u_n \to u$	a.e. and in locally measure
$Du_n \to Du$	a.e and locally in measure
$\{D(T_k u_n)\}_n$	is bounded in $L^p(\Omega)$ (for fixed k).

We also have $D(T_k(u)) \in L^p(\Omega)$ and moreover $u \in \mathcal{T}^{1,p}_0(\Omega)$. Indeed, we can construct $\phi_n \in C^\infty_0(\Omega)$ such that $\|D\phi_n - D(T_k u_n)\|_{L^p} \le 1/n$ and $\|\phi_n - T_k u_n\|_{L^{p_*}} \le 1/n$. We then have $D\phi_n \to D(T_k u)$ weakly in $L^p(\Omega)$ and $\phi_n \to T_k u$ strongly in $L^q_{\text{loc}}(\Omega)$ for $q < p_*$. From ϕ_n we can construct ψ_n (convex combinations of the ϕ_n's, using Mazur's lemma) so as to have strong convergence of derivatives. We conclude that $u \in \mathcal{T}^{1,p}_0(\Omega)$. See also Appendix II.

Furthermore, using the convergence of u_n to u and Du_n to Du, we can prove for u the inequalities stated in Lemmas 4.1 and 4.2.

Step 3. In order to complete the proof of the existence of a solution we still have to show

i) that $\mathbf{a}(Du) \in L^1_{\text{loc}}(\Omega)$,

ii) that $\beta(x, u(x)) \in L^1(\Omega)$, and finally that

iii) $-\operatorname{div}(\mathbf{a}(Du)) + \beta(x, u) = f$ in $\mathcal{D}'(\Omega)$,

and also that the entropy inequality holds. We first remark that the sequence

$\{\mathbf{a}(Du_n)\}_n$ is bounded in $L^q_{\text{loc}}(\Omega)$ for all $q \in (1, N/(N-1))$. Indeed,

$$|\mathbf{a}(x, Du_n)| \leq \Lambda(j(x) + |Du_n|^{p-1})$$

with $j \in L^{p'}(\Omega) \subset L^q_{\text{loc}}(\Omega)$, and, according to Lemma 4.2, $|Du_n|^{p-1}$ is bounded in $M^{N/(N-1)}(\Omega) \subset L^q_{\text{loc}}(\Omega)$ (recall that this simply means that $\text{meas}\{|\mathbf{a}(x, Du_n)| > \lambda\} \leq C\lambda^{-\frac{N}{N-1}}$). On the other hand, according to Nemitskii's theorem [K] the convergence of Du_n to Du in measure implies that $\mathbf{a}(x, Du_n)$ converges in measure to $\mathbf{a}(x, Du)$. It follows that $\mathbf{a}(x, Du) \in M^{N/(N-1)}(\Omega) \subset L^q_{\text{loc}}(\Omega)$ for all $q \in (1, N/(N-1))$. We now use a convergence result whose easy proof is left to the reader.

LEMMA 6.1. *Let v_n be a sequence of measurable functions on a measurable space Ω with finite measure. Assume that the sequence converges in measure to a function v and is uniformly bounded in $L^p(\Omega)$ for some $p > 1$. Then $v_n \to v$ strongly in $L^1(\Omega)$.*

Applying this result to $\mathbf{a}(x, Du_n)$ we conclude that $\mathbf{a}(x, Du) \in L^1_{\text{loc}}(\Omega)$ and

$$\mathbf{a}(x, Du_n) \to \mathbf{a}(x, Du) \qquad \text{strongly in} \quad L^1_{\text{loc}}(\Omega).$$

Therefore,

$$\text{div } \mathbf{a}(x, Du_n) \to \text{div } \mathbf{a}(x, Du) \qquad \text{in} \quad \mathcal{D}'(\Omega).$$

We also have

$$f_n \to f \qquad \text{in} \quad L^1(\Omega).$$

Let $\tilde{\gamma}_n(x) = \gamma_n(x, u_n)$. The only remaining difficulty consists in proving that (i) $\tilde{\gamma}_n \to w$ in $\mathcal{D}'(\Omega)$, (ii) $w(x) = \beta(x, u(x))$ a.e., and (iii) $\beta(x, u(x)) \in L^1(\Omega)$.

We can easily establish local equi-integrability for the sequence $\{\tilde{\gamma}_n\}$. Indeed, the first part of formula (6.7) gives

$$\int_{\{|u_n|>k\}} |\beta_n(x, u_n)|\, dx + \frac{1}{n} \int_{\{|u_n|>k\}} |u_n|^{p-1}\, dx \leq \varepsilon \tag{6.11}$$

for k large enough, uniformly with respect to n. Together with assumption (H5) this implies that the sequence $\tilde{\gamma}_n$ is uniformly equi-integrable. Hence, after passing to a subsequence we can assume that

$$\tilde{\gamma}_n \to w \qquad \text{weakly in} \quad L^1_{\text{loc}}(\Omega).$$

Note moreover that $(1/n)|u_n|^{p-2}u_n \to 0$ in $L^1_{\text{loc}}(\Omega))$ since it converges a.e. and we have a uniform estimate for the u_n in a Marcinkiewicz space of higher exponent.

The other facts are easier. That $w(x) = \beta(x, u(x))$ follows from the continuity of β. Finally, by (6.7) $\{\tilde{\gamma}_n\}_n$ is uniformly bounded in $L^1(\Omega)$, hence we get $\beta(x, u(x)) \in L^1(\Omega)$.

Step 4. To complete the proof it remains to show that u is an entropy solution. In order to prove inequality (3.6) we take a function $T \in \mathcal{F}$ bounded above by k, $k > 0$, and such that $T'(s) = 0$ for $|s| \geq k$; we also choose a smooth function $\phi \in C_0^\infty(\Omega)$ and apply the test function $T(u_n - \phi)$ to equation (6.4) to get

$$\int_\Omega \langle \mathbf{a}(Du_n), D(T(u_n - \phi))\rangle \, dx = \int_\Omega (f_n - \gamma_n(x, u_n)) \, T(u_n - \phi) \, dx. \tag{6.12}$$

We can write the first member of (6.12) as

$$\int \langle \mathbf{a}(Du_n), Du_n\rangle \, T'(u_n - \phi) \, dx - \int \langle \mathbf{a}(Du_n), D\phi\rangle \, T'(u_n - \phi) \, dx. \tag{6.13}$$

Since $u_n \to u$ and $Du_n \to Du$ a.e., we have by Fatou's Lemma

$$\int \langle \mathbf{a}(Du), Du\rangle \, T'(u - \phi) \, dx \leq \liminf_{n\to\infty} \int \langle \mathbf{a}(Du_n), Du_n\rangle \, T'(u_n - \phi) \, dx.$$

The second term of (6.13) is estimated as follows. We know that

$$\langle \mathbf{a}(Du_n), D\phi\rangle \, T'(u_n - \phi) \to \langle \mathbf{a}(Du), D\phi\rangle \, T'(u - \phi)$$

a.e as $n \to \infty$. We also know that $\mathbf{a}(Du_n)$ converges strongly in L^1_{loc}, hence we may assume that it is dominated in $L^1_{\text{loc}}(\Omega)$. Then

$$\int \langle \mathbf{a}(Du_n), D\phi\rangle \, T'(u_n - \phi) \, dx \to \int \langle \mathbf{a}(Du), D\phi\rangle \, T'(u - \phi) \, dx.$$

The second member of (6.12) can be likewise split into two terms. The first, $\int_\Omega \gamma_n(x, u_n) T(u_n - \phi) \, dx$ is estimated as follows: let us consider an increasing sequence $\{K_m\}_m$ of compact subsets of Ω such that $\cup_m K_m = \Omega$. Of course, for m large, say $m \geq m_0$, the support of ϕ will be contained in K_m. Then, using the monotonicity of γ_n we have

$$\int_{K_m} \gamma_n(x, u_n) T(u_n - \phi) \, dx \leq \int_\Omega \gamma_n(x, u_n) \, T(u_n - \phi) \, dx. \tag{6.14}$$

On the other hand, we can write

$$\int_{K_m} \{w \, T(u - \phi) - \gamma_n(x, u_n) \, T(u_n - \phi)\} \, dx = I_1 + I_2,$$

where

$$I_1 = \int_{K_m} (w - \gamma_n(x, u_n)) \, T(u - \phi) \, dx$$

tends to 0 since $\gamma_n(x, u_n) \to w$ weakly in $L^1_{\text{loc}}(\Omega)$, and

$$I_2 = \int_{K_m} \gamma_n(x, u_n)\{T(u - \phi) - T(u_n - \phi)\}\, dx$$

can be split into $I_2' + I_2''$, where I_2' is the integral on the set where $|u_n| \geq L$, and I_2'' is the integral on $|u_n| < L$. On the first set we conclude that I_2' is small (uniformly in n) if L is large by (6.11), while for given L we can make I_2'' small by letting $n \to \infty$ and using the uniform bound $|F(x, u(x))| \leq G_L(x) \in L^1(K_m)$ given by (H5). Therefore,

$$\lim_{n\to\infty} \int_{K_m} \gamma_n(x, u_n)\, T(u_n - \phi)\, dx = \int_{K_m} w\, T(u - \phi)\, dx. \tag{6.15}$$

Combining (6.14) and (6.15) we get

$$\int_{K_m} w\, T(u - \phi)\, dx \leq \liminf_{n\to\infty} \int_{\Omega} \gamma_n(x, u_n)\, T(u_n - \phi)\, dx.$$

Since the second member is independent of m, passing to the limit $m \to \infty$ we get the same inequality with K_m replaced by Ω. Finally, passing to the limit in the last term of (6.12) is immediate and we have

$$\int_{\Omega} f_n\ T(u_n - \phi)\, dx \to \int_{\Omega} f\ T(u - \phi)\, dx.$$

Using these estimates we obtain (3.6) in the limit when $n \to \infty$. ∎

IMPORTANT REMARK. Actually, it is possible to prove that equality holds in (3.3) or (3.6) by proving that for all $k > 0$

$$D(T_k(u_n)) \to D(T_k(u)) \qquad \text{in} \qquad L^p(\Omega).$$

7. Properties of the solution. Semigroup generation

We gather in this section a number of properties of the solution of problem (1.4)-(1.2) we have constructed which can be of use in the applications. We write the equation in the form

$$-\operatorname{div} \mathbf{a}(x, Du) + \beta(x, u) = f, \tag{7.1}$$

where, as in Section 6, $f = F(x, 0)$ and $\beta(x, u) = F(x, 0) - F(x, u)$. Given $f \in L^1(\Omega)$ let $u = u_f$ be the entropy solution of (7.1)-(1.2) and let $w_f(x) = \beta(x, u_f(x))$. Then we have

THEOREM 7.1. *Under the assumptions* (H1)–(H5) *if* $f, \hat{f} \in L^1(\Omega)$ *and* $(u, w) = (u_f, w_f)$, $(\hat{u}, \hat{f}) = (u_{\hat{f}}, w_{\hat{f}})$ *then:*

(i) w *and* $\hat{w} \in L^1(\Omega)$ *and*

$$\int_\Omega [w - \hat{w}]_+ dx \leq \int_\Omega [f - \hat{f}]_+ dx. \tag{7.2}$$

It follows that the map $f \mapsto w_f$ *is an order-preserving contraction in* $L^1(\Omega)$. *The map* $f \mapsto u_f$ *from* $L^1(\Omega)$ *to* $M^{p_1}(\Omega)$ *is also order-preserving.*

(ii) *Assume that* $\beta(x, r)$ *depends only on* r *and let* $j : \mathbb{R} \to \mathbb{R}_+$ *be a convex function with* $j(0) = 0$. *Then*

$$\int_\Omega j(w) dx \leq \int_\Omega j(f) dx. \tag{7.3}$$

In particular, the map $f \mapsto w_f$ *is bounded from* $L^p(\Omega) \cap L^1(\Omega)$ *into itself for every* $1 \leq p \leq \infty$.

The above results have an interpretation in terms of accretive operators. Indeed, given the spatial domain Ω and the functions $\mathbf{a}$ and β as above, we *define* the (possibly multivalued) operator $\mathcal{A}$ in $L^1(\Omega)$ by the rule: "$f \in \mathcal{A}(w)$ if and only if w, $f \in L^1(\Omega)$ and there exists $u \in T_0^{1,p}(\Omega)$ such that

$$w(x) = \beta(x, u(x)) \tag{7.4}$$

and u is the entropy solution of

$$-\operatorname{div}(\mathbf{a}(x, Du)) + w = f \tag{7.5}$$

with zero boundary data". Then we have

THEOREM 7.2. *The operator* $\mathcal{A}$ *is m-accretive in* $L^1(\Omega)$.

According to Crandall and Liggett's Semigroup Generation Theorem (see [C]) such an operator generates a semigroup of (order-preserving) contractions S_t in $L^1(\Omega)$ which solves in a generalized sense, usually called *the mild sense*, the evolution problem

$$\begin{aligned} &w_t = -\operatorname{div} \mathbf{a}(x, Du) && \text{in} \quad \Omega \times (0, \infty) \\ &u = 0 && \text{on} \quad \partial\Omega \times (0, \infty) \\ &w(x, 0) = w_0 && \text{for} \quad x \in \Omega. \end{aligned} \tag{7.6}$$

with $w(\cdot, t) = S_t u_0(\cdot)$.

It is however interesting to note that in order to generate a semigroup we can restrict the operator to act on functions such that u is bounded, thus

avoiding the problems of integrability of Du, the major source of concern in the foregoing theory. Therefore, we consider $\mathcal{A}_0$ defined as follows: "$f \in \mathcal{A}_0(w)$ if and only if w, $f \in L^1(\Omega)$ and there exists $u \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that (7.4) holds and

$$-\operatorname{div}(\mathbf{a}(x, Du)) = f \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{7.5'}$$

(no entropy condition needed). Observe moreover that for Ω bounded $\mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega) = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Clearly, $\mathcal{A}_0$ is a restriction of $\mathcal{A}$. By classical monotone arguments one shows that $\mathcal{A}_0$ is accretive in $L^1(\Omega)$, as well as the range condition

$$\operatorname{Range}(I + \lambda \mathcal{A}_0) \supseteq L^1(\Omega) \cap L^\infty(\Omega) \tag{7.7}$$

for every $\lambda > 0$. According to the semigroup theory ([B1], [BCP]) this operator generates a semigroup of contractions S_t in $L^1(\Omega)$ on $\overline{D(\mathcal{A}_0)} = \{w \in L^1(\Omega) : w(x) \in \overline{R(\beta(x, \cdot))}\}$. This solves (7.6) in the mild sense.

In this respect Theorem 7.2 amounts to say that $\mathcal{A}$ *is the closure of* $\mathcal{A}_0$ *in* $L^\infty(\Omega)$. This fact completely characterizes the functional setting in the stationary problems.

We will skip further discussion of the evolution aspects since the extensive theory of mild solutions falls out of the scope of this work. Let us only say that for particular choices of $\mathbf{a}$ and β one proves that the mild solution is in fact a continuous weak solution, in the lines of standard PDE theory. Most often found in the literature are cases when $\mathbf{a}$ and β are power-like, i.e.

$$\mathbf{a}(x, Du) = |Du|^p Du, \quad \beta(x, s) = |s|^{r-1} s. \tag{7.8}$$

Then $\mathcal{A}$ is a realization of the sometimes called *doubly-nonlinear Laplacian* and we solve the evolution problem

$$u_t = \Delta_p(|u|^{m-1}u), \quad u(0) = u_0 \in L^1(\Omega), \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{7.9}$$

with $m = 1/r$. Especially well-known cases are, apart from (i) the classical heat equation ($r = 1$, $p = 2$), (ii) the case $r = 1$, $p \neq 2$, which gives the p-Laplacian equation, and (iii) the case $p = 2$, $r \neq 1$, which gives the so-called *porous medium equation*.

8. Extensions

8.1. Maximal monotone graphs. There are a number of interesting generalizations that can be considered in the above existence and uniqueness results. One of the most common variations of equation (1.4) found in the

literature concerns the possibility of including functions $F(x,u)$ which are monotone but discontinuous in u. To simplify matters, we will consider functions F of the uncoupled form

$$F(x,u) = f(x) - \beta(u), \tag{8.1}$$

where, according to (H5) we assume that $f \in L^1(\Omega)$. We also assume that:

(H6) β is maximal monotone graph in $\mathbb{R}^2$ with $0 \in \beta(0)$.

Therefore, we allow the term $\beta(u)$ to be multivalued, not necessarily defined in the whole of $\mathbb{R}$. The reader interested in the properties of maximal monotone graphs can consult the monograph [Br]. This leads to the differential inclusion

$$-\operatorname{div}\ \mathbf{a}(x,Du) + \beta(u) \ni f. \tag{8.2}$$

But for the complications of taking care of the multiplicity of β, and replacing equations by inclusions, nothing essential changes in the proofs of the uniqueness result (Theorem 5.1) and the existence result (Theorem 6.1), if we assume the form (8.2) with (H1)–(H4) and the extra hypothesis (H6). We leave the details to the interested reader. Notice in particular that a complete specification of the solution involves a pair (u,w) where w is an integrable function such that $w(x) \in \beta(u(x))$ for a.e. $x \in \Omega$ and u is a solution of

$$-\operatorname{div}\ \mathbf{a}(x,Du) = f - w \tag{8.3}$$

in the sense of Section 3. Both u and w are unique.

Actually, using the tools of [BC] we can deduce directly the results for $F(x,u) = f(x) - \beta(u)$ from the results for $F(x,u) = f(x)$. See also [BeW] for the case $F(x,u) = f(x) - \beta(x,u)$ when β is maximal monotone in u with $0 \in \beta(x,0)$.

8.2. Existence for measures. Another interesting extension direction concerns the possibility of replacing the integrable functions of the second member by bounded measures. We consider again an equation of uncoupled form, but this time we avoid the complications of dealing with graphs and take the equation

$$-\operatorname{div}\ \mathbf{a}(x,Du) + \beta(u) = f. \tag{8.4}$$

THEOREM 8.1. *Let* $1 < p < N$, *let the assumptions* (H1)–(H4) *hold and let* $f \in M_b(\Omega)$, *the space of bounded measures in* Ω. *Assume that* β *be a continuous and nondecreasing real function with* $\beta(0) = 0$ *and assume moreover that* $\operatorname{Domain}(\beta) = \mathbb{R}$ *and*

$$\beta(\pm|x|^{-\frac{N-p}{p-1}}) \in L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{8.5}$$

Then there exists a function $u \in T_0^{1,p}(\Omega)$ *such that* $w = \beta(u) \in L^1(\Omega)$ *and* u *is a solution of* $-\operatorname{div}(\mathbf{a}(Du)) = f - w$ *in the sense of distributions in* Ω. *Moreover,* $u \in M^{p_1}(\Omega)$ *and* $|Du| \in M^{p_2}(\Omega)$.

PROOF. The existence proof of Section 6 can be easily adapted to this case. The proof begins by approximating the second member $f \in M_b(\Omega)$ with a sequence of smooth functions $f_n \in C_0^\infty(\Omega)$, $f_n \to f$ in the weak* topology of $M_b(\Omega)$. Then it stays litterally the same from the beginnig to Step 3, where a different proof has to be given for the equi-integrability of the sequence $\{\gamma_n(u_n)\}$. In doing this we need to assume the restrictions on β stated above. It is clear that the term $(1/n)|u_n|^{p-2}u_n$ still converges to 0 in $L^1_{\text{loc}}(\Omega)$ since $u_n \to u$ a.e. and $\{|u_n|^{p-1}\}$ is bounded in $M^{N/(N-p)}$. The proof of local equi-integrability proceeds as follows: we use the following facts:

(i) $|\beta_n| \le |\beta|$;

(ii)

$$\int_{\{|u_n|>k\}} |\beta(u_n)|\,dx = \int_{\{|u_n|>k\}} \left(\int_0^\infty 1_{[0,\beta(u_n(x))]}(t)\,dt \right) dx$$

$$= \int_0^\infty \left(\int_{\{|u_n|>k\}} 1_{[0,\beta(u_n(x))]}(t)\,dx \right) dt$$

$$= \int_0^\infty \operatorname{meas}(\{x : |\beta(u_n(x))| \ge t, |u_n| > k\})\,dt$$

$$= \int_0^\infty (\operatorname{meas}\{x : u_n(x) \ge \sup(k, \beta^{-1}(t))\}$$

$$+ \operatorname{meas}\{x : u_n(x) \le \inf(-k, \beta^{-1}(-t))\})\,dt.$$

(Use the classical definition of β^{-1} as a multivalued function).

(iii) By Lemma 4.1 we have

$$\operatorname{meas}\{u_n \ge \beta^{-1}(t)\} \le c\,(\beta^{-1}(t))^{-\frac{N}{N-1}(p-1)}.$$

(iv) By hypothesis (H6)

$$\int^\infty (\beta^{-1}(t))^{-\frac{N}{N-p}(p-1)}\,dt < \infty.$$

Indeed, if $v(x) = |x|^{-\frac{N-p}{p-1}}$ one has

$$\infty > \int_{\{v \geq t_0\}} \beta(v(x))\,dx = \int_0^{\infty} \text{meas}\,\{x : v(x) \geq \sup(t_0, \beta^{-1}(t))\}\,dt,$$

$$= \int_{\beta(t_0)}^{\infty} \text{meas}\,\{x : |x| \leq (\beta^{-1}(t))^{-\frac{p-1}{N-p}}\}\,dt,$$

hence

$$\int_{\beta(t_0)}^{\infty} \beta^{-1}(t)^{-\frac{N}{N-p}(p-1)}\,dt < \infty.$$

We also have a similar result for meas $\{u_n \leq \beta^{-1}(-t)\}$. From all this we deduce that

$$\int_{\{|u_n|>k\}} \beta_n(u_n)\,dx \to 0$$

as $k \to \infty$ uniformly respect to n. Finally, noting that $\beta_n(\pm k)$ is bounded we then deduce the local equi-integrability of $\beta_n(u_n)$. We then have (up to extraction of a subsequence)

$$\beta_n(u_n) \to w \qquad \text{weakly in} \quad L^1_{\text{loc}}(\Omega).$$

As in the first case we then prove that $w \in \beta(u)$ a.e. and that $w \in L^1(\Omega)$. We have thus completed the proof of Theorem 8.1.

REMARK. In this case we are not able to establish a property like (3.3) or (3.6). Consequently we cannot prove uniqueness. Notice that the expressions (3.3) and (3.6) make no sense when f is just a measure, not an integrable function.

8.3. For bounded domains or, more generally, when meas $(\Omega) < \infty$ the case $p \geq N$ does not offer much difficulty.

Actually, in the case $p > N$, under assumptions (H2)–(H4) there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of (1.4) (which is continuous, indeed $u \in C_0(\Omega)$). This case can be proved by classical monotone arguments.

When $p = N$ one can prove with the same arguments developed above that there exists a unique $u \in \mathcal{T}_0^{1,N}(\Omega)$ satisfying the entropy conditions.

The case meas $(\Omega) = \infty$ is a bit trickier: to be convinced one can look at the problem

$$-\Delta u + \beta(u) = f \qquad \text{in} \quad \mathbb{R}^N$$

for $N = 1$ and $N = 2$ as studied in [BBC]. We will refrain here from entering into more details.

8.4. Another extension direction consists in dealing with more general operators A. A simple example is provided by operators of the form $A(u) = -\operatorname{div} \mathbf{a}(x, u, Du)$.

Appendix I

The introduction of a special functional setting for our problem if $p \leq 2 - (1/N)$ is motivated by the following result

PROPOSITION. *Let Ω be an open set in $\mathbb{R}^N$ and let $p \leq 2 - (1/N)$. Then there exists a function $f \in L^1(\Omega)$ such that the problem*

$$u \in W^{1,1}_{\text{loc}}(\Omega), \qquad u - \Delta_p(u) = f \quad \text{in} \quad \mathcal{D}'(\Omega),$$

has no solution.

PROOF. If a solution u exists and since $p < 2$ we have $\Delta_p(u) \in W^{-1,\frac{1}{p-1}}(\Omega)$. If this happens then for every $f \in L^1(\Omega)$ we have

$$L^1(\Omega) \subset W_0^{1,1}(\Omega) + W^{-1,\frac{1}{p-1}}(\Omega).$$

By the Closed Graph Theorem and Duality this implies that

$$W^{-1,\infty}(\Omega) \cap W_0^{1,\frac{1}{2-p}}(\Omega) \subset L^\infty(\Omega),$$

which can only hold if $1/(2-p) > N$, i.e. $p > 2 - (1/N)$.

Appendix II

We give here useful characterizations of the spaces $\mathcal{T}_0^{1,p}(\Omega)$ which played such a role in the preceding theory.

PROPOSITION. *Let $1 < p < \infty$ and let Ω be an open subset in $\mathbb{R}^N$. The following statements are equivalent for a measurable function $u : \Omega \to \mathbb{R}$:*

(i) $u \in \mathcal{T}_0^{1,p}(\Omega)$ *according to the definition of Section* 2.

(ii) $u \in \mathcal{T}^{1,p}(\Omega)$ *and there exists a sequence* $\zeta_n \in C_0^\infty(\Omega)$ *such that for any* $k > 0$

(a) $\zeta_n \to u$ *a.e. in* Ω,

(b) $D(T_k(\zeta_n)) \to D(T_k(u))$ *in* $L^p(\Omega)$.

(iii) *For any* $k > 0$ *there exists a sequence* $\phi_n \in W_0^{1,p}(\Omega)$ *such that*

(a) $\phi_n \to T_k(u)$ *a.e. in* Ω,

(b) *the sequence* $\{D\phi_n\}$ *is bounded in* $(L^p(\Omega))^N$.

(iv) $u \in \mathcal{T}^{1,p}(\Omega)$ *and for every* $k > 0$ *and every smooth cutoff function* $\zeta \in C_0^\infty(\mathbb{R}^N)$

(a) $\zeta T_k(u) \in W_0^{1,p}(\Omega)$,

(b) *if* $p < N$ *we also need the condition* $T_k(u) \in L_0(\Omega)$.

We recall that the space $L_0(\Omega)$ is defined in Section 2.

PROOF. It is immediate that (i) $\Rightarrow$ (iii). (ii) $\Rightarrow$ (i) is also clear taking $\phi_n = T_k(\zeta_n)$. To show that (iii) $\Rightarrow$ (iv) we need to prove that whenever $v \in L^\infty(\Omega)$ is an a.e. limit of a sequence ϕ_n in $W_0^{1,p}(\Omega)$ with gradient $D\phi_n$ bounded in $L^p(\Omega)^N$, then

$$(\alpha) \qquad Dv \in L^p(\Omega)^N,$$

$$(\beta) \qquad \zeta v \in W_0^{1,p}(\Omega) \quad \text{for any} \quad \zeta \in C_0^\infty(\Omega),$$

$$(\gamma) \qquad v \in L_0(\Omega) \quad \text{when} \quad 1 < p < N.$$

Notice first that we can assume the sequence ϕ_n to be bounded in $L^\infty(\Omega)$ by substituting $T_c(\phi_n)$ for ϕ_n, $c = \|v\|_\infty$. Then we have $\phi_n \to v$ in $L^1_{\text{loc}}(\Omega)$, and thus $D\phi_n \to Dv$ as distributions. Since $D\phi_n$ is bounded in $L^p(\Omega)^N$ then (α) holds. Moreover, for $\zeta \in C_0^\infty(\Omega)$ we have $\zeta\phi_n \to \zeta v$ in $L^p(\Omega)$. Since $\zeta\phi_n$ is bounded in $W_0^{1,p}(\Omega)$ (β) also holds. Finally, if $p < N$ the sequence ϕ_n is bounded in $L^{p^*}(\Omega)$, $p^* = Np/(N-p)$ by the Sobolev embedding. Therefore, $v \in L^{p^*} \subset L_0(\Omega)$, and (γ) holds.

Let us now prove that (iv) $\Rightarrow$ (ii). Assuming that (iv) holds we take k, ε and $R > 0$. We **claim** that there exists $\zeta \in C_0^\infty(\Omega)$ such that (with $\Omega_R = \Omega \cap \{|x| < R\}$)

$$(*) \qquad \|D(\zeta - T_k(u))\|_{L^p(\Omega)} + \|\zeta - T_k(u)\|_{L^p(\Omega_R)} \le \varepsilon.$$

This will prove (ii). Indeed, let $\zeta_{n,m} \in C_0^\infty(\Omega)$ be the function corresponding to the choice $k = n$, $\varepsilon = 1/m$, $R = m$ in the above estimate. Then, for fixed n,

$$D\zeta_{n,m} \to DT_n(u) \quad \text{in} \quad L^p(\Omega), \text{ and}$$
$$\|\zeta_{n,m} - T_n(u)\|_{L^p(\Omega_R)} \to 0$$

for any $R > 0$, as $m \to \infty$. It follows that for any $0 < k \le n$

$$DT_k(\zeta_{n.m}) \to DT_k(u) \quad \text{in} \quad L^p(\Omega), \quad \text{and}$$
$$\|T_k(\zeta_{n,m}) - T_k(u))\|_{L^p(\Omega_R)} \to 0$$

for any $R > 0$ as $m \to \infty$. Thus, for any n there exists $\zeta_n = \zeta_{n,m(n)} \in C_0^\infty(\Omega)$ such that

$$\|D(T_k(\zeta_n) - T_k(u))\|_{L^p(\Omega)} + \|T_k(\zeta_n) - T_k(u)\|_{L^p(\Omega_R)} \le \frac{1}{n}$$

for any $k = 1, 2, \ldots$ After extracting a suitable subsequence (ii) holds.

To prove the claim let $0 < \delta < k$ and set $v = T_k(u) - T_\delta(u)$. Consider also $\rho \in C_0^\infty(\Omega)$, $0 \le \rho \le 1$, $\rho = 1$ for $|x| \le 1$ and $\rho = 0$ for $|x| \ge 2$ and set $\zeta_m(x) = \rho(x/m)$. We have for $m > R$

$$\|\zeta_m v - T_k(u)\|_{L^p(\Omega_R)} = \|T_\delta(u)\|_{L^p(\Omega_R)} \le \delta\,|\{|x| < R\}|^{\frac{1}{p}}.$$

On the other hand,

$$\|D(\zeta_m v - T_k(u))\|_p \le I_1 + I_2 + I_3,$$

with

$$I_1 = \|DT_k(u)\|_{L^p(\Omega \cap \{|x| > m\}}$$

$$I_2 = \|DT_\delta(u)\|_p$$

$$I_3 = \frac{k}{m}\,\|D\rho\|_\infty |B|^{\frac{1}{p}},$$

where $B = \{|u| > \delta\} \cap \{m < |x| < 2m\}$. Since $DT_k(u) \in L^p(\Omega)^N$, I_1 goes to 0 as $m \to \infty$, and $I_2 \to 0$ as $\delta \to 0$. If $p < N$ we have $|B| \le |\{|T_k(u)| > \delta\}| \le \infty$ and for every $\delta > 0$, $I_3 \to 0$ as $m \to \infty$. We also have $|B| \le |\{1 < |x| < 2\}| \cdot m^N$. When $p > N$, $I_3 \to 0$ as $m \to \infty$ uniformly in $\delta > 0$, and when $p = N$, I_3 is bounded uniformly in m, δ.

From this analysis it follows that for $p \neq N$ we can choose δ and $m > 0$ in such a way that

$$\|D(\zeta_m v - T_k(u))\|_p + \|\zeta_m v - T_k(u)\|_{L^p(\Omega_R)} < \varepsilon.$$

Since $\zeta_m v \in W_0^{1,p}(\Omega)$, the claim $(*)$ holds. To show that it also holds for $p = N$ we take $\delta = 0$ and observe that

$$D(\zeta_m T_k(u)) \to DT_k(u) \quad \text{weakly in} \quad L^p(\Omega)^N$$

as $m \to \infty$. Then we may find w in the convex hull of $\{\zeta_m T_k(u); m > R\}$ in $W_0^{1,p}(\Omega)$ such that

$$\|D(w - T_k(u))\|_p < \varepsilon.$$

Since $w = T_k(u)$ in $\{|x| < R\}$, $(*)$ still holds.

Appendix III

We give an example of a function $u \in T^{1,1}_{\text{loc}}(-1,1)$ and a Lipschitz-continuous and bounded $T : \mathbb{R} \to \mathbb{R}$ such that $T(u) \notin W^{1,1}_{\text{loc}}(-1,1)$. More precisely, we show that condition (2.3) is optimal.

PROPOSITION. *Let $T : \mathbb{R} \to \mathbb{R}$ be a monotone Lipschitz-continuous and*

bounded function that does not satisfy (2.3). *Then there exists* $u \in \mathcal{T}_0^{1,1}(-1,1)$ *such that* $T(u) \notin W^{1,1}_{\mathrm{loc}}(-1,1)$.

PROOF. After replacing $T(r)$ by $-T(-r)$ if necessary we may assume that there exists a real sequence $u_0 = 0 < u_1 < \cdots < u_n < \cdots$ such that $u_n \to \infty$ and $T(u_{n+1}) > T(u_n)$ for any n. Let k_n be a sequence of integers such that the sum $\sum k_n(T(u_{n+1}) - T(u_n))$ diverges, and set $a_n = 2^{-n}$. Finally, define

$$u(x) = \sum_{n=0}^{\infty} 1_{(a_{n+1},a_n]}(|x|)(u_n + (u_{n+1} - u_n)\,\rho((2k_n+1)(2 - 2^{n+1}|x|))),$$

where

$$\rho(t) = \int_0^{\infty} \sum_{i=0}^{\infty} (-1)^i 1_{(i,i+1)}(s)ds.$$

We have $u \in W^{1,\infty}_{\mathrm{loc}}([-1,1] \setminus \{0\})$, $u(\pm 1) = 0$ and $u_n \le u(x) \le u_{n+1}$ for $a_n \le |x| \le a_{n+1}$, and in fact u goes from the value u_n at $x = a_n$ to the value u_{n+1} at $x = a_{n+1}$ by going up and down in this range $2k_n + 1$ times. It is clear that $u \in \mathcal{T}_0^{1,p}(-1,1)$ for any $p \in [1,\infty)$ since the truncation eliminates all but a finite number of terms in the sum. On the other hand, for $T(u)$ all terms count. We have

$$\frac{d}{dx}\,T(u)(x) = T'(u(x))u'(x) = (-1)^{i+1}T'(u(x))|u'(x)|.$$

for $i < (2k_n+1)(2 - 2^{n+1}|x|) < i+1$, $n = 0,\dots,2k_n$. It follows that for every $x_1 \in (0,1)$

$$\int_{-x_1}^{x_1} |(Tu)'(x)|dx \ge 2 \sum_{n=n_1}^{\infty} \sum_{i=0}^{2k_n} (-1)^i \,[T(u)]_{u_n+(u_{n+1}-u_n)(1-(-1)^i)/2}^{u_n+(u_{n+1}-u_n)(1+(-1)^i)/2}$$

$$\ge 4 \sum_{n=n_1}^{\infty} k_n(T(u_{n+1}) - T(u_n)) = \infty.$$

Appendix IV

We discuss here the question of uniqueness of solutions of (1.4), (1.2), which motivated our introduction of the concept of entropy solution, in the light of an example given by J. Serrin [S] of a solution of the linear equation

$$A(u) \equiv \sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0,$$

with

(E) $$a_{ij}(x) = \delta_{ij} + (a-1)\frac{x_i x_j}{r^2}, \quad r = |x|,$$

which is uniformly elliptic for $a > 0$. J.S. considers a solution of the form

$$u(x) = x_1 r^{-\alpha}, \quad \text{where} \quad \alpha = \frac{N}{2} + \sqrt{\left(\frac{N}{2} - 1\right)^2 + \frac{N-1}{a}}.$$

Assuming that $a > 1$ one has $N - 1 < \alpha < N$. It is clear that $u \in C^\infty(\mathbb{R}^N - \{0\}) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ for $p < N/\alpha$, and $u \notin W^{1,N/\alpha}_{\text{loc}}(\mathbb{R}^N)$. The author also verifies that u is a weak solution of (E). By an easy computation one can also see that $u \in T^{1,p}_{\text{loc}}(\mathbb{R}^N)$ for $p < p_\alpha = 1 + (N-1)/\alpha$ and $u \notin T^{1,p_\alpha}_{\text{loc}}(\mathbb{R}^N)$.

Let now Ω be the unit ball in $\mathbb{R}^N$ and let

$$f(x) = (a-1)(N-1)\frac{x_1}{r^2} = A(x_1).$$

We have $f(x) \in L^q(\Omega)$ for $q < N$. The function

$$v(x) = x_1 - u(x)$$

is a weak solution in $C^\infty(\overline{\Omega} - \{0\}) \cap W^{1,1}(\Omega)$ of the problem

$$\text{(P)} \qquad \begin{cases} A(v) = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

But $v \notin T^{1,2}_0(\Omega)$ and then v is not the entropy solution of (P).

This example shows that we cannot derive in general (3.5) from (3.2), even for bounded and smooth Ω, at least if we only assume $u \in T^{1,1}_0(\Omega)$ with $\mathbf{a}(x, Du) \in L^1(\Omega)$. However, the question of deriving (3.5) from (3.2) is still open for a solution of (3.1) in $T^{1,p}_0(\Omega)$. According to the above existence and uniqueness theory, this question is equivalent to the problem of uniqueness of a solution of (3.1) in the class $T^{1,p}_0(\Omega)$.

It is worth noting in the example that $v(x)$ is a weak solution of the equation $-A(u) = f$ in the domain $\Omega_1 = \{x : r < 1, x_1 > 0\}$. Though $v \in C^\infty(\Omega_1) \cap W^{1,1}_{\text{loc}}(\Omega_1)$, it is not an entropy solution of problem (P) in Ω_1.

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