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Stability of Semilinear Equations with Boundary and Pointwise Noise

BOHDAN MASLOWSKI *

Introduction

In the present paper we consider a stochastic semilinear equation of the form (1.1) below which describes a wide range of problems such as stochastic nonlinear boundary value parabolic problems with boundary or pointwise noise, and stochastic plate equations with structural damping (see Example 3.1 and Example 3.2, respectively). The main goal of the paper is to establish an existence and uniqueness theorem for the mild solutions of the equation (1.1) and to achieve some basic results on their asymptotic behavior, such as exponential stability in the mean, existence and uniqueness of invariant measures.

There is a rather extensive list of papers dealing with the semigroup theory for stochastic evolution equations (see, for instance, an almost complete bibliography in the recent book by G. Da Prato and J. Zabczyk [8]), however, only few of them cover also systems with boundary and pointwise noise in other than linear cases. For instance, Zabczyk [33] presented a model where the boundary values satisfy a stochastic differential equation. Ichikawa [20], [21] established an existence, uniqueness and stability theorem for a semilinear problem with finite-dimensional noise. Mao and Marcus [27] investigate the one-dimensional wave equation. Sowers [30] carries out a thorough analysis of the multi-dimensional Neumann problem. Da Prato and Zabczyk [9] study the equation of reaction-diffusion type where polynomial nonlinearities are allowed, with white-noise boundary conditions. Bilinear stochastic systems have been treated, for instance, by Flandoli [13], [14], [15].

The approach adopted in the present paper is similar to the one in [20] or [9] and is based on a semigroup model described in [2], [23] for the deterministic case. The conditions on noise terms are fairly general and can cover both "genuine" Wiener processes with values in the basic state spaces

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and the "cylindrical" ones. The method of proof of existence and uniqueness of the invariant measure (Theorem 2.1 and Proposition 2.8) can be probably of interest even for homogeneous boundary conditions. Also, the type of the Itô lemma proved in Proposition 1.5 can be of some independent interest.

The paper is divided into three sections. Section 1 contains the formulations of definitions and assumptions and some basic properties of solutions to the semilinear equations of the form

$$\text{(1.1)} \qquad \mathrm{d}X_t = \big(AX_t + f(X_t) + Bh(X_t)\big)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t + Bk(X_t)\,\mathrm{d}V_t, \quad t \geq 0,$$

in a Hilbert space H, where W_t and V_t are cylindrical Wiener processes on Hilbert spaces H and U, respectively, $A: H \to H$ and $B: U \to H$ are linear unbounded operators and $f: H \to H$, $h: H \to U$, $g: H \to \mathcal{L}(H)$ and $k: H \to \mathcal{L}(U)$ are Lipschitz continuous, where $\mathcal{L}(Y)$ stands for the space of linear bounded operators on Y. Theorem 1.1 is an existence and uniqueness statement for solutions to (1.1). It is proved by a standard fixed point argument along the lines of some existing results (for instance, [17], [29]). In Proposition 1.2 a continuous dependence of the solutions on initial conditions is established, by which it is already standard to show the Markov and Feller properties. In the rest of the section a type of the Itô formula is dealt with. Since (1.1) is far from being a classical stochastic differential in the basic Hilbert space H it is clear that the Itô formula can be valid only for a rather special class of functions. Such a class, which is also useful in obtaining subsequent asymptotic results, is specified in Proposition 1.5. A similar result has been obtained in the linear case by Duncan, Maslowski and Pasik–Duncan [11].

Section 2 contains results about invariant measures for (1.1) and some asymptotic results. Theorem 2.1 states that if there exists a solution to (1.1) which is bounded in probability in time average then there exists an invariant measure. Note that the statement of Theorem 2.1 which is well known for finite-dimensional stochastic differential equations need not hold for general infinite-dimensional systems as shown by Vrkoč [32]. A result similar to Theorem 2.1 has been proved recently by Da Prato, Gątarek and Zabczyk [10] for systems with compact semigroup and homogeneous boundary conditions by factorization method. Also, Theorem 2.1 generalizes the statements on the existence of invariant measure contained in Ichikawa [18] and Manthey and Maslowski [26]. A sufficient condition for the existence of an invariant measure in terms of coefficients of the equation (1.1) is proved in Proposition 2.4 by means of verification of the "average boundedness" condition from Theorem 2.1 by the Lyapunov method. For this purpose Proposition 1.5 (the Itô lemma) is used. Exponential stability in the mean in an appropriate norm is established in Theorem 2.8. As the main tools of the proof a modification of Datko's result on equivalence of exponential and L_p-stabilites (cf. Datko [7], Ichikawa [19]) and a version of the above proved Itô formula for differences (Corollary 1.6) are used. As a consequence uniqueness of the invariant measure and its stability in a space of probability distributions with appropriately defined topology are obtained (Corollary 2.10).

Section 3 contains two examples. Example 3.1 is a model for stochastic partial differential equation of parabolic type of the order $2m$ with distributed as well as the boundary noise terms. In Example 3.2 a plate equation with structural damping and pointwise random loading is studied.

For Banach spaces Z and Y we denote by $\mathcal{L}(Z,Y)$, $\mathcal{L}_1(Z,Y)$ and $\mathcal{L}_2(Z,Y)$ the space of linear bounded, nuclear and Hilbert–Schmidt, respectively, operators $Z \to Y$ and we set $\mathcal{L}(Z) = \mathcal{L}(Z,Z)$, $\mathcal{L}_1(Z) = \mathcal{L}_1(Z,Z)$, $\mathcal{L}_2(Z) = \mathcal{L}_2(Z,Z)$. By $\mathcal{D}(K)$ and $\operatorname{Im} K$ we denote the domain and the image of a (possibly nonlinear) operator K. The symbol $|\cdot|_Z$ stands for the norm of a space Z, Cl_Z means the closure in the topology of Z. $\mathcal{B}(Z)$ and $\mathcal{P}(Z)$ denote the σ-algebra of Borel sets of Z and the space of probability measures on $\mathcal{B}(Z)$, respectively. By $\mathbb{C}([0,T],Z)$ and $\mathcal{B}([0,T],Z)$ the space of all continuous and bounded, respectively, functions $[0,T] \to Z$ is denoted. The symbol I stands for the identity operator and by X^y we denote the solution of the equation (1.1) satisfying the initial condition $X(0) = y$, where y is an element in H.

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1. - Basic properties of solutions

Let $H = (H, |\cdot|, \langle\cdot,\cdot\rangle)$ and $U = (U, |\cdot|_U, \langle\cdot,\cdot\rangle_U)$ be real, separable Hilbert spaces and $A\colon \mathcal{D}(A) \to H$, $\mathcal{D}(A) \subset H$, a densely defined, linear operator generating an analytic semigroup $S(\cdot)$ on H. It is well known that there exists a $\beta > 0$ such that the operator $\beta I - A$ is uniformly positive on H, that is,

$$\langle(\beta I - A)y, y\rangle \geq \nu|x|^2, \quad x \in \mathcal{D}(A),$$

holds for some $\nu > 0$. Denote by D_A^λ, $\lambda \in \mathbb{R}$, the domains of fractional powers $(\beta I - A)^\lambda$ equipped with the graph norms $|x|_\lambda = |(\beta I - A)^\lambda x|$, $x \in D_A^\lambda$. Let $0 < \varepsilon < 1$ be fixed and $B \in \mathcal{L}(U, D_A^{\varepsilon-1})$. Consider an equation of the form

$$\text{(1.1)} \qquad \mathrm{d}X_t = \big(AX_t + f(X_t) + Bh(X_t)\big)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t + Bk(X_t)\,\mathrm{d}V_t, \quad t \geq 0,$$

with the initial condition $X_0 = x \in H$, where $f\colon H \to H$, $h\colon H \to U$, $g\colon H \to \mathcal{L}(H)$ and $k\colon H \to \mathcal{L}(U)$ are Lipschitz continuous in the respective norms, W_t and V_t are stochastically independent standard cylindrical Wiener processes on H and U, respectively, defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$.

In the sequel we can assume with no loss of generality that A is uniformly negative on $\mathcal{D}(A)$ (otherwise we can take $A - \beta I$ and $f + \beta I$ instead of A and f, respectively). Moreover, throughout the paper we assume:

(A1) A^{-1} is a compact operator on H.

(A2) There exists a $\Delta \in (0, 1/2)$ such that the function $\tilde{g} := (-A)^{\Delta - 1/2} g$ maps H into $\mathcal{L}_2(H)$ and is Lipschitz continuous in the respective norms.

(A3) There exists a $\gamma \in (0, 1/2)$ such that $\tilde{k} := (-A)^{\gamma-1/2}Bk$ maps H into $\mathcal{L}_2(U, H)$ and is Lipschitz continuous in the respective norms.

Note that if $\operatorname{Im} g \subset \mathcal{L}_2(H)$, $\operatorname{Im} k \subset \mathcal{L}_2(U)$, and $g: H \to \mathcal{L}_2(H)$, $k : H \to \mathcal{L}_2(U)$ are Lipschitz continuous then (A2) is always satisfied while (A3) is satisfied provided $\varepsilon > 1/2$ (we can take $0 < \gamma < \varepsilon - 1/2$ and $0 < \Delta < 1/2$ arbitrary). This covers the case when "genuine" H-valued and K-valued, respectively, covariance type Wiener processes can be considered in (1.1) instead of W_t and V_t.

On the other hand, (A2) and (A3) are clearly satisfied with any g and k if $(-A)^{\Delta-1/2} \in \mathcal{L}_2(H)$ and $(-A)^{\gamma-\varepsilon-1/2} \in \mathcal{L}_2(H)$ for some $0 < \Delta < 1/2$, $0 < \gamma < \varepsilon - 1/2$, respectively. This covers to the case when "space-time" white noises are allowed to be considered in the equation (1.1) (see Example 3.1 for further specification).

By the mild solution of (1.1) we understand an H-valued, $\{\mathcal{F}_t\}$-adapted process satisfying the integral equation

$$
\begin{aligned}
X(t) = S(t)x &+ \int_0^t S(t-r) f\big(X(r)\big)\,\mathrm{d}r \\
&+ \int_0^t S(t-r) Bh\big(X(r)\big)\,\mathrm{d}r \\
&+ \int_0^t S(t-r) g\big(X(r)\big)\,\mathrm{d}W_r \\
&+ \int_0^t S(t-r) Bk\big(X(r)\big)\,\mathrm{d}V_r, \quad t \ge 0.
\end{aligned} \tag{1.2}
$$

Note that (1.1) does not make sense as an "H-valued stochastic differential" since the range of B exceeds the space H and W_r, V_r are not "genuine" H- and U-valued Wiener processes, respectively. However, the semigroup S is extendable to the space $D_A^{\varepsilon-1}$ and

$$
|S(t)|_{\mathcal{L}(D_A^{\varepsilon-1}, H)} \le M e^{-\omega t} t^{\varepsilon-1}, \quad t \ge 0, \tag{1.3}
$$

holds for some $M > 0$, $\omega > 0$, (cf. [1], Theorems 2.1 and 2.4), while the assumption (A2), (A3) guarantee the existence of H-valued versions of the stochastic integrals in (1.2). This makes the above introduced mild solution a reasonable object as established in the subsequent statement.

THEOREM 1.1. *Assume* (A1) *and* (A2) *and let x be a $\mathcal{F}_0$-adapted random variable satisfying* $\mathrm{E}|x|^p < \infty$ *for a fixed* $p > \max(1/\Delta, 1/\gamma, 1/\varepsilon)$. *Then for* *any* $T > 0$ *there exists a unique mild solution of the equation* (1.1) *in* $\mathbb{C}\big([0,T];L_p(\Omega,H)\big)$ *satisfying the initial condition* $X(0)=x$.

PROOF. The proof is based on the contraction principle in the space $\mathcal{C} := \big\{Y \in \mathbb{C}\big([0,T],L_p(\Omega,H)\big);\ Y \text{ is } \{\mathcal{F}_t\}\text{-adapted}\big\}$. Let $F\colon \mathcal{C} \to \mathcal{C}$ be the mapping defined as

$$
\begin{aligned}
F(Y)(t) &:= S(t)x + \int_0^t S(t-r)\big(f(Y(r)) + Bh(Y(r))\big)\,\mathrm{d}r \\
&\quad + \int_0^t S(t-r)g\big(Y(r)\big)\,\mathrm{d}W_r + \int_0^t S(t-r)Bk\big(Y(r)\big)\,\mathrm{d}V_r, \\
&\quad t \in [0,t],\ \ Y \in \mathcal{C}.
\end{aligned}
\tag{1.4}
$$

At first we show that F is well defined as a mapping $\mathcal{C} \to \mathcal{B}\big([0,T],L_p(\Omega,H)\big)$. By Lipschitzianity of f we have that

$$
\begin{aligned}
&\mathrm{E}|S(t)x|^p + \mathrm{E}\left|\int_0^t S(t-r)f\big(Y(r)\big)\,\mathrm{d}r\right|^p \\
&\le M_1\left(\mathrm{E}|x|^p + \int_0^T \mathrm{E}\big(1+|Y(r)|^p\big)\,\mathrm{d}r\right) < \infty
\end{aligned}
\tag{1.5}
$$

for a constant $M_1 > 0$ independent of $t \in [0,T]$. Similarly, by (1.3) we get

$$
\begin{aligned}
&\mathrm{E}\left|\int_0^t S(t-r)Bh\big(Y(r)\big)\,\mathrm{d}r\right|^p \\
&\le \mathrm{E}\left[\int_0^t \frac{M_2}{(t-r)^{1-\varepsilon}}\big(1+|Y(r)|\big)\,\mathrm{d}r\right]^p \\
&\le M_2^p\left(\int_0^T r^{(\varepsilon-1)q}\mathrm{d}r\right)^{\frac{p}{q}} \int_0^T \mathrm{E}\big(1+|Y(r)|\big)^p\mathrm{d}r < \infty
\end{aligned}
\tag{1.6}
$$

for some $M_2 > 0$, where $q = p(p-1)^{-1}$. Furthermore, from (A2) and (A3) it follows that for every $0 \le r < t \le T$ we have $S(t-r)g\big(Y(r)\big) \in \mathcal{L}_2(H)$ and

$S(t-r)Bk\big(Y(r)\big) \in \mathcal{L}_2(U,H)$, respectively, and

$$\begin{aligned} \big|S(t-r)g\big(Y(r)\big)\big|^2_{\mathcal{L}_2(H)} &= \big|S(t-r)(-A)^{1/2-\Delta}\tilde{g}\big(Y(r)\big)\big|^2_{\mathcal{L}_2(H)} \\ &\le \frac{c}{(t-r)^{1-2\Delta}}\big(1+|Y(r)|\big)^2, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \big|S(t-r)Bk\big(Y(r)\big)\big|^2_{\mathcal{L}_2(U,H)} &= \big|S(t-r)(-A)^{1/2-\gamma}\tilde{k}\big(Y(r)\big)\big|^2_{\mathcal{L}_2(U,H)} \\ &\le \frac{c}{(t-r)^{1-2\gamma}}\big(1+|Y(r)|\big)^2, \end{aligned} \tag{1.8}$$

where $c > 0$ does not depend on r, t. From (1.7), (1.8) it follows that the stochastic integrals in (1.2) are well defined H-valued processes and as a particular case we obtain from a version of Burkholder–Davis–Gundy inequality ([29], Lemma 2.2) that

$$\begin{aligned} &\mathrm{E}\left|\int_0^t S(t-r)g\big(Y(r)\big)\,\mathrm{d}W_r\right|^p + \mathrm{E}\left|\int_0^t S(t-r)Bk\big(Y(r)\big)\,\mathrm{d}V_r\right|^p \\ &\le \mathrm{E}\left(\int_0^t \big|S(t-r)g\big(Y(r)\big)\big|^2_{\mathcal{L}_2(H)}\mathrm{d}r\right)^{p/2} \\ &\quad + \mathrm{E}\left(\int_0^t \big|S(t-r)Bk\big(Y(r)\big)\big|^2_{\mathcal{L}_2(U,H)}\mathrm{d}r\right)^{p/2} \\ &\le 2c_1\left(\int_0^T r^{(2\xi-1)\eta}\mathrm{d}r\right)^{\frac{p}{2\eta}} \int_0^T \mathrm{E}\big(1+|Y(r)|\big)^p\mathrm{d}r < \infty, \end{aligned} \tag{1.9}$$

where c_1 is a constant and $\xi = \min(\Delta,\gamma)$ and $\eta = p(p-2)^{-1}$. From (1.5), (1.6) and (1.9) we conclude the F is well defined on $\mathcal{C}$ and maps $\mathcal{C}$ into $\mathcal{B}\big([0,T], L_p(\Omega,H)\big)$. Now we show that $F(\mathcal{C}) \subset \mathcal{C}$. For $h \in [0,T]$, $t \in [0,T-h]$

we have

$$
\begin{aligned}
&\mathrm{E}|F(Y)(t+h) - F(Y)(t)|^p \\
&\le M_3 \Bigg[\mathrm{E}|S(t+h)x - S(t)x|^p \\
&\quad + \mathrm{E}\left|\int_t^{t+h} S(t+h-r)\big(f(Y(r)) + Bh(Y(r))\big)\,\mathrm{d}r\right|^p \\
&\quad + \mathrm{E}\left|\int_t^{t+h} S(t+h-r)g(Y(r))\,\mathrm{d}W_r\right|^p \\
&\quad + \mathrm{E}\left|\int_t^{t+h} S(t+h-r)Bk(Y(r))\,\mathrm{d}V_r\right|^p \\
&\quad + \mathrm{E}\Bigg|(S(h)-I)\Bigg(\int_0^t S(t-r)\Big(f(Y(r)) + Bh(Y(r))\Big)\,\mathrm{d}r \\
&\quad + \int_0^t S(t-r)g(Y(r))\,\mathrm{d}W_r + \int_0^t S(t-r)Bk(Y(r))\,\mathrm{d}V_r\Bigg)\Bigg|^p\Bigg].
\end{aligned}
\tag{1.10}
$$

It is obvious that

$$
\mathrm{E}|S(t+h)x - S(t)x|^p \to 0 \qquad \text{as } h \to 0+ \tag{1.11}
$$

and

$$
\begin{aligned}
\mathrm{E}\left|\int_t^{t+h} S(t+h-r)f(Y(r))\,\mathrm{d}r\right|^p &\le M_1 \int_t^{t+h} \mathrm{E}\big(1 + |Y(r)|^p\big)\,\mathrm{d}r \\
&\le M_1\big(1 + \sup_{t\in[0,T]} \mathrm{E}|Y(r)|^p\big)h,
\end{aligned}
\tag{1.12}
$$

$$
\begin{aligned}
&\mathrm{E}\left|\int_t^{t+h} S(t+h-r)Bk(Y(r))\,\mathrm{d}r\right|^p \\
&\le M_2^p \left(\int_0^T r^{(\varepsilon-1)q}\right)^{\frac{p}{q}} \big(1 + \sup_{t\in[0,T]} \mathrm{E}|Y(r)|^p\big)h
\end{aligned}
\tag{1.13}
$$

by (1.5) and (1.6), respectively. Furthermore, (1.7) and (1.8) yield

$$
\begin{aligned}
&\mathrm{E}\left|\int_t^{t+h} S(t+h-r)g\big(Y(r)\big)\,\mathrm{d}W_r\right|^p \\
&+\mathrm{E}\left|\int_t^{t+h} S(t+h-r)Bk\big(Y(r)\big)\,\mathrm{d}V_r\right|^p \qquad (1.14)\\
&\leq 2c^{p/2}\left(\int_0^T r^{(2\xi-1)\eta}\right)^{\frac{p}{2\eta}} \sup_{t\in[0,T]} \mathrm{E}\big(1+|Y(r)|\big)^p h,
\end{aligned}
$$

where q, η and ξ have the same meaning as above. Furthermore, the last term under expectation on the right-hand side of (1.10) converges to zero as $h \to 0+$ by the dominated convergence theorem because $\big(S(h)-I\big)y \to 0$ for every $y \in H$ and the terms under expectation are majorized by

$$
\begin{aligned}
&\big(1+\sup_{t\in[0,T]} |S(t)|_{\mathcal{L}(H)}\big)\left(\left|\int_0^t S(t-r)\big(f\big(Y(r)\big)+Bk\big(Y(r)\big)\big)\,\mathrm{d}r\right|^p\right. \\
&\left.+\left|\int_0^t S(t-r)g\big(Y(r)\big)\,\mathrm{d}W_r\right|^p+\left|\int_0^t S(t-r)Bk\big(Y(r)\big)\,\mathrm{d}V_r\right|^p\right)
\end{aligned}
$$

which has finite expectation by the estimates (1.5)–(1.8). This together with (1.11)–(1.14) implies

$$
\lim_{h\to 0+} \mathrm{E}|F(Y)(t+h)-F(Y)(t)|^p = 0, \quad t\in[0,T). \tag{1.15}
$$

In order to prove the left continuity of $F(Y)$ note that for every $0 < t \leq T$, $0 \leq h < t$, we have that

$$
\mathrm{E}|S(t-h)x - S(t)x|^p \to 0, \quad h \to 0+, \tag{1.16}
$$

and by the same estimates as in (1.12)–(1.14) we obtain

$$
\begin{aligned}
&\mathrm{E}\left|\int_{t-h}^{t} S(t-r)\big(f\big(Y(r)\big)+Bh\big(Y(r)\big)\big)\,\mathrm{d}r\right|^{p} \\
&+\mathrm{E}\left|\int_{t-h}^{t} S(t-r)g\big(Y(r)\big)\,\mathrm{d}W_r\right|^{p} \\
&+\mathrm{E}\left|\int_{t-h}^{t} S(t-r)Bk\big(Y(r)\big)\,\mathrm{d}V_r\right|^{p} \to 0 \quad \text{as } h\to 0+.
\end{aligned} \tag{1.17}
$$

Furthermore, by (A2), (A3), Hölder and Burkholder–Davis–Gundy inequalities we obtain

$$
\begin{aligned}
&\mathrm{E}\left|\int_{0}^{t-h} \big(S(t-r)-S(t-r-h)\big)\big(f\big(Y(r)\big)+Bh\big(Y(r)\big)\big)\,\mathrm{d}r\right|^{p} \\
&+\mathrm{E}\left|\int_{0}^{t-h} \big(S(t-r)-S(t-r-h)\big)g\big(Y(r)\big)\,\mathrm{d}W_r\right|^{p} \\
&+\mathrm{E}\left|\int_{0}^{t-h} \big(S(t-r)-S(t-r-h)\big)Bk\big(Y(r)\big)\,\mathrm{d}V_r\right|^{p} \\
&\le M_4\left\{\left(\int_0^t |S(t-r)-S(t-r-h)|^{q}_{\mathcal{L}(D_A^{\varepsilon-1},H)}\mathrm{d}r\right)^{p/q}\right. \\
&+\left(\int_0^t |S(t-r)-S(t-r-h)|^{2\eta}_{\mathcal{L}(D_A^{\Delta-1/2},H)}\mathrm{d}r\right)^{\frac{p}{2\eta}} \\
&\left.+\left(\int_0^t |S(t-r)-S(t-r-h)|^{2\eta}_{\mathcal{L}(D_A^{\gamma-1/2},H)}\mathrm{d}r\right)^{\frac{p}{2\eta}}\right\}
\end{aligned} \tag{1.18}
$$

for a constant $M_4>0$ independent of t, h, where we set $S(t-r)-S(t-r-h)=0$ for $r>t-h$. By the analyticity of A we have that $S(\cdot)\in\mathbb{C}\big((0,T),\mathcal{L}(D_A^{\delta},H)\big)$ for any $\delta\in\mathbb{R}$. This together with (1.3) implies the convergence to zero of the right-hand side of (1.18) as $h\to 0+$. This together with (1.16) and (1.17) yields

$$
\lim_{h\to 0+}\mathrm{E}|F(Y)(t)-F(Y)(t-h)|^p=0,\quad 0<t\le T. \tag{1.19}
$$

It remains to verify that $F\colon \mathcal{C} \to \mathcal{C}$ is a contraction. Denote by $\|.\|$ the norm of $\mathcal{C}$. Using the Lipschitz continuity of f, h, $\tilde{g}$ and $\tilde{k}$ and the Hölder and Burkholder–Davis–Gundy inequalities in a similar way as above we obtain

$$
\begin{aligned}
&\|F(Y) - F(Z)\|^p \\
&\le \sup_{t\in[0,T]} 4^{p-1} \Bigg(\mathrm{E}\left| \int_0^t S(t-r)\big(f(Y(r)) - f(Z(r))\big)\,\mathrm{d}r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)B\big(h(Y(r)) - h(Z(r))\big)\,\mathrm{d}r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)\big(g(Y(r)) - g(Z(r))\big)\,\mathrm{d}W_r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)B\big(k(Y(r)) - k(Z(r))\big)\,\mathrm{d}V_r \right|^p \Bigg) \qquad (1.20)\\
&\le k_1 \Bigg(T^{\frac{p}{q}} + \left(\int_0^T r^{(\varepsilon-1)q}\mathrm{d}r \right)^{\frac{p}{q}} + \left(\int_0^T r^{(2\Delta-1)\eta}\mathrm{d}r \right)^{\frac{p}{2\eta}} \\
&\quad + \left(\int_0^T r^{(2\gamma-1)\eta}\mathrm{d}r \right)^{\frac{p}{2q}} \Bigg) \mathrm{E}\int_0^T |Y(r) - Z(r)|^p \mathrm{d}r \\
&\le \alpha(T)\|Y - Z\|^p,
\end{aligned}
$$

for Y, $Z \in \mathcal{C}$, where $k_1 > 0$ is a constant independent of T, Y and Z, and $\alpha(T) \to 0$ as $T \to 0+$. Therefore F is contractive for enough small $T > 0$. For large T we can proceed in a usual way, introducing on $\mathcal{C}$ an equivalent norm $\|Z\|_b := \sup_{t\in[0,T]} e^{-bt}(\mathrm{E}|Z(t)|^p)^{1/p}$, $Z \in \mathcal{C}$, where $b > 0$ is sufficiently large (see e.g. [29]). □

PROPOSITION 1.2. *For any $T > 0$, $p > \max(1/\gamma, 1/\Delta, 1/\varepsilon)$ there exists a constant $C = C(T) < \infty$ such that*

$$
\mathrm{E}|X^x(t) - X^y(t)|^p \le C(T)\mathrm{E}|x - y|^p \qquad (1.21)
$$

holds for all $t \in [0,T]$, H-valued and $\mathcal{F}_0$-measurable random variables x, y, such that $\mathrm{E}|x|^p + \mathrm{E}|y|^p < \infty$.

PROOF. By the Lipschitz continuity of f, $\tilde{g}$, h and $\tilde{k}$ we obtain

$$
\begin{aligned}
&\mathrm{E}|X^x(t) - X^y(t)|^p \\
&\leq k_1 \left(\mathrm{E}|x-y|^p + \mathrm{E}\left| \int_0^t S(t-r)\big(f(X^x(r)) - f(X^y(r))\big)\,\mathrm{d}r \right|^p \right. \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)B\big(h(X^x(r)) - h(X^y(r))\big)\,\mathrm{d}r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)\big(g(X^x(r)) - g(X^y(r))\big)\,\mathrm{d}W_r \right|^p \\
&\quad \left. + \mathrm{E}\left| \int_0^t S(t-r)B\big(k(X^x(r)) - k(X^y(r))\big)\,\mathrm{d}V_r \right|^p \right) \qquad (1.22)\\
&\leq k_2 \left[\mathrm{E}|x-y|^p + \left(1 + \left(\int_0^T r^{(\varepsilon-1)q}\mathrm{d}r \right)^{\frac{p}{q}} \right. \right. \\
&\quad \left. + \left(\int_0^T r^{(2\Delta-1)\eta}\mathrm{d}r \right)^{\frac{p}{2\eta}} + \left(\int_0^T r^{(2\gamma-1)\eta}\mathrm{d}r \right)^{\frac{p}{2\eta}} \right) \\
&\quad \left. \times \int_0^T \mathrm{E}|X^x(r) - X^y(r)|^p \mathrm{d}r \right] \\
&\leq k_3 \left(\mathrm{E}|x-y|^p + \int_0^t \mathrm{E}|X^x(r) - X^y(r)|^p \mathrm{d}r \right), \qquad t \in [0,T],
\end{aligned}
$$

for some constants k_1, k_2, k_3 independent of $t \in [0,T]$ and the particular choice of x, y, where $q = p(p-1)^{-1}$, $\eta = p(p-2)^{-1}$. Thus (1.22) yields (1.21) by the Gronwall lemma. □

As a consequence of Theorem 1.1 and Proposition 1.2 we obtain

PROPOSITION 1.3. *The equation* (1.1) *defines an H-valued homogenous Feller Markov process with the transition probability function*

$$P(t,y,A) = \mathrm{P}[X^y(t) \in A], \quad y \in H, \quad t \geq 0, \quad A \in \mathcal{B}(H).$$

The proof of the Markov property is a standard corollary of Proposition 1.2 and follows the lines of analogous proofs in [18] (Propositions 3.2 and 3.3), [3], [8]. Fellerness is an immediate consequence of Proposition 1.2 above.

In the rest of the present section we aim at proving an appropriate version of the Itô formula. To this end we approximate the solutions of (1.1) by strong solutions of a suitably defined sequence of equations. For $\lambda > 0$ set $R(\lambda) = \lambda(\lambda I - A)^{-1}$. It is known that $R^m(\lambda)y \to y$ in H for every $y \in H$, $m \in \mathbb{N}$, as $\lambda \to +\infty$, and $|R^m(\lambda)|_{\mathcal{L}(H)} \leq c^m$ for a constant $c < \infty$. Consider the equation

$$\begin{aligned} & \mathrm{d}X_\lambda(t) = \big[AX_\lambda(t) + R(\lambda)f\big(X_\lambda(t)\big) + R^2(\lambda)Bh\big(X_\lambda(t)\big)\big]\,\mathrm{d}t \\ & \qquad\qquad + R^2(\lambda)g\big(X_\lambda(t)\big)\,\mathrm{d}W_t + R^2(\lambda)Bk\big(X_\lambda(t)\big)\,\mathrm{d}V_t, \quad t \geq 0, \\ & X_\lambda(0) = x_\lambda = R(\lambda)x. \end{aligned} \tag{1.23}$$

Note that for any $\lambda > 0$ the mappings $R(\lambda)f : H \to D_A^1$ and $R^2(\lambda)Bh : H \to D_A^1$, $R^2(\lambda)g : H \to \mathcal{L}_2(H, D_A^1)$ and $R^2(\lambda)Bk : H \to \mathcal{L}_2(U, D_A^1)$, respectively, are Lipschitz continuous, which is sufficient for existence of the strong solution to (1.23) ([17], Proposition 2.3, [18], Proposition 3.5).

LEMMA 1.4. *For any $T > 0$, $p > \max(1/\Delta, 1/\gamma, 1/\varepsilon)$, and $\mathcal{F}_0$-measurable x, $\mathrm{E}|x|^p < \infty$, we have that*

$$\mathrm{E}|X(t) - X_\lambda(t)|^p \to 0, \quad \lambda \to +\infty, \quad t \in [0,T], \tag{1.24}$$

$$\sup_{t\in[0,T],\lambda\in\mathbb{R}_+} \mathrm{E}|X_\lambda(t)|^p < \infty, \tag{1.25}$$

where X and X_λ are solutions of (1.1) *and* (1.23), *respectively.*

PROOF. In the subsequent proof k_i, $i = 1,2,\ldots,7$, represent suitable nonrandom constants which can be chosen independently of $t \in [0,T]$ and

$\lambda \in \mathbb{R}_+$. We have

$$
\begin{aligned}
&\mathrm{E}|X(t) - X_\lambda(t)|^p \\
&\le k_1 \Bigg(\mathrm{E}\left| \int_0^t S(t-r)R(\lambda)\big(f(X(r)) - f(X_\lambda(r))\big)\,\mathrm{d}r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)R^2(\lambda)B\big(h(X(r)) - h(X_\lambda(r))\big)\,\mathrm{d}r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)R^2(\lambda)\big(g(X(r)) - g(X_\lambda(r))\big)\,\mathrm{d}W_r \right|^p \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r)R^2(\lambda)B\big(k(X(r)) - k(X_\lambda(r))\big)\,\mathrm{d}V_r \right|^p + \kappa_\lambda(t) \Bigg), \\
&t \in [0,T],
\end{aligned}
\tag{1.26}
$$

where

$$
\begin{aligned}
\kappa_\lambda(t) &= \mathrm{E}|x - x_\lambda|^p + \mathrm{E}\left| \int_0^t S(t-r)\big(R(\lambda) - I\big)f\big(X(r)\big)\,\mathrm{d}r \right|^p \\
&+ \mathrm{E}\left| \int_0^t S(t-r)\big(R^2(\lambda) - I\big)Bh\big(X(r)\big)\,\mathrm{d}r \right|^p \\
&+ \mathrm{E}\left| \int_0^t S(t-r)\big(R^2(\lambda) - I\big)g\big(X(r)\big)\,\mathrm{d}W_r \right|^p \\
&+ \mathrm{E}\left| \int_0^t S(t-r)\big(R^2(\lambda) - I\big)Bk\big(X(r)\big)\,\mathrm{d}V_r \right|^p .
\end{aligned}
$$

Since $|R^m(\lambda)|_{\mathcal{L}(D_A^\alpha)} \le c^m$ for $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, and f, h, $\tilde{g}$, and $\tilde{k}$ are Lipschitz continuous, we get

$$
\begin{aligned}
&\mathrm{E}\left| \int_0^t S(t-r)R(\lambda)\big(f(X(r)) - f(X_\lambda(r))\big)\,\mathrm{d}r \right|^p \\
&\le k_2 \int_0^t \mathrm{E}|X(r) - X_\lambda(r)|^p \mathrm{d}r,
\end{aligned}
\tag{1.27}
$$

$$
\begin{aligned}
&\mathrm{E}\left|\int_0^t S(t-r)R^2(\lambda)B\big(h(X(r))-h(X_\lambda(r))\big)\,\mathrm{d}r\right|^p \\
&\leq k_3\left(\int_0^T r^{(\varepsilon-1)q}\mathrm{d}r\right)^{p/q}\int_0^t \mathrm{E}|X(r)-X_\lambda(r)|^p\mathrm{d}r,
\end{aligned}
\tag{1.28}
$$

$$
\begin{aligned}
&\mathrm{E}\left|\int_0^t S(t-r)R^2(\lambda)\big(g(X(r))-g(X_\lambda(r))\big)\,\mathrm{d}W_r\right|^p \\
&\leq k_4\left(\int_0^T r^{(2\Delta-1)\eta}\mathrm{d}r\right)^{p/2\eta}\int_0^t \mathrm{E}|X(r)-X_\lambda(r)|^p\mathrm{d}r,
\end{aligned}
\tag{1.29}
$$

$$
\begin{aligned}
&\mathrm{E}\left|\int_0^t S(t-r)R^2(\lambda)B\big(k(X(r))-k(X_\lambda(r))\big)\,\mathrm{d}V_r\right|^p \\
&\leq k_5\left(\int_0^T r^{(2\gamma-1)\eta}\mathrm{d}r\right)^{p/2\eta}\int_0^t \mathrm{E}|X(r)-X_\lambda(r)|^p\mathrm{d}r,
\end{aligned}
\tag{1.30}
$$

where $q=p(p-1)^{-1}$, $\eta=p(p-2)^{-1}$. Hence

$$
\mathrm{E}|X(t)-X_\lambda(t)|^p \leq k_6\left(\kappa_\lambda(t)+\int_0^t \mathrm{E}|X(r)-X_\lambda(r)|^p\mathrm{d}r\right), \quad t\in[0,T]. \tag{1.31}
$$

For every $0\leq r<t\leq T$ we have by (A2), (A3) that

$$
\begin{aligned}
&\big|S(t-r)\big(R^2(\lambda)-I\big)g\big(X(r)\big)\big|^2_{\mathcal{L}_2(H)} \\
&=\big|\big(R^2(\lambda)-I\big)S(t-r)(-A)^{1/2-\Delta}\tilde{g}\big(X(r)\big)\big|^2_{\mathcal{L}_2(H)}\to 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|S(t-r)\big(R^2(\lambda)-I\big)Bk\big(X(r)\big)\big|^2_{\mathcal{L}_2(U,H)} \\
&=\big|\big(R^2(\lambda)-I\big)S(t-r)(-A)^{1/2-\gamma}\tilde{k}\big(X(r)\big)\big|^2_{\mathcal{L}_2(U,H)}\to 0
\end{aligned}
$$

as $\lambda\to\infty$ since $R(\lambda)\to I$ in the strong operator topology and $|R(\lambda)|_{\mathcal{L}(H)}$ is bounded. Therefore

$$
\begin{aligned}
\beta_\lambda(t) := |x - x_\lambda|^p &+ \left[\int_0^t \left|S(t-r)\big(R(\lambda) - I\big)f\big(X(r)\big)\right| \mathrm{d}r\right]^p \\
&+ \left[\int_0^t \left|S(t-r)\big(R^2(\lambda) - I\big)Bh\big(X(r)\big)\right| \mathrm{d}r\right]^p \\
&+ \left[\int_0^t \left|S(t-r)\big(R^2(\lambda) - I\big)g\big(X(r)\big)\right|^2_{\mathcal{L}_2(H)} \mathrm{d}r\right]^{p/2} \\
&+ \left[\int_0^t \left|S(t-r)\big(R^2(\lambda) - I\big)Bk\big(X(r)\big)\right|^2_{\mathcal{L}_2(U,H)} \mathrm{d}r\right]^{p/2} \to 0 \quad \text{a.s.}
\end{aligned}
\tag{1.32}
$$

as $\lambda \to \infty$ by the dominated convergence theorem. Moreover, since

$$\beta_\lambda(t) \leq k_7 \left(1 + |x|^p + \int_0^T |X(r)|^p \mathrm{d}r\right), \quad t \in [0, T],$$

and $\mathrm{E}\beta_\lambda(t) = \kappa_\lambda(t)$, $t \in [0, T]$, we can use again the dominated convergence theorem to obtain $\kappa_\lambda(t) \to 0$ as $\lambda \to \infty$ for $t \in [0, T]$, and

$$\sup_{t\in[0,T],\lambda\in\mathbb{R}_+} \kappa_\lambda(t) < \infty. \tag{1.33}$$

Now (1.24), (1.25) follow from (1.31) by the Gronwall lemma. □

Let $V \in \mathcal{L}(H)$, $V = V^*$, be an operator satisfying $|\langle Vx, Ax\rangle| \leq k|x|^2$, $x \in D_A^1$, for some $k > 0$. It is easy to see that the function $x \mapsto \langle Vx, Ax\rangle$, $x \in D_A^1$, is extendable to a continuous function $\Phi: H \to \mathbb{R}$. Indeed, for (y_n), $y \in D_A^1$, $y_n \to y$ in H, we can write

$$
\begin{aligned}
|\langle Ay_n, Vy_n\rangle - \langle Ay, Vy\rangle| &\leq |\langle A(y_n - y), V(y_n - y)\rangle| \\
&+ |2\langle Vy, Ay\rangle - \langle Vy_n, Ay\rangle - \langle Vy, Ay_n\rangle| = S_1 + S_2 \to 0, \quad n \to \infty,
\end{aligned}
$$

since $S_1 \leq k|y_n - y|^2 \to 0$, $\langle Vy_n, Ay\rangle \to \langle Vy, Ay\rangle$ by the boundedness of V, and $Ay_n \to Ay$ weakly on H.

PROPOSITION 1.5. *Let $V \in \mathcal{L}(H)$, $V = V^*$, be such that*

$$V \in \mathcal{L}(D_A^{-\alpha}, D_{A^*}^{\alpha}) \tag{1.34}$$

for some $\alpha > \max(1/2 - \Delta, 1/2 - \gamma, 1 - \varepsilon)$ *and*

$$\langle Vx, Ax\rangle \leq k|x|^2, \quad x \in D_A^1, \tag{1.35}$$

is fulfilled for some $k > 0$. *Then*

$$\begin{aligned}
&\mathrm{E}\langle VX(t), X(t)\rangle - \mathrm{E}\langle Vx, x\rangle \\
&= \mathrm{E}\int_0^t \big(2\Phi(X(s)) + 2\langle f(X(s)), VX(s)\rangle \\
&\quad + 2\langle h(X(s)), B^*VX(s)\rangle_U \\
&\quad + \mathrm{Tr}\{(-A^*)^{1/2-\Delta}Vg(X(s))g^*(X(s))(-A^*)^{\Delta-1/2}\} \\
&\quad + \mathrm{Tr}\{(-A^*)^{1/2-\gamma}VBk(X(s))k^*(X(s))B^*(-A^*)^{\gamma-1/2}\}\big)\,\mathrm{d}s,
\end{aligned} \tag{1.36}$$

holds for $t \geq 0$, $\mathcal{F}_0$*-measurable* x, $\mathrm{E}|x|^p < \infty$ *with* $p > \max(1/\Delta, 1/\gamma, 1-\varepsilon)$, *where* $\Phi: H \to \mathbb{R}$ *is the continuous extension of* $\langle V\cdot, A\cdot\rangle$.

PROOF. Set

$$\begin{aligned}
L_\lambda V(y) = {}&2\Phi(y) + 2\langle R(\lambda)f(y), Vy\rangle + 2\langle h(y), B^*\big(R^*(\lambda)\big)^2Vy\rangle_U \\
&+ \mathrm{Tr}\{VR^2(\lambda)g(y)g^*(y)\big(R^*(\lambda)\big)^2\} \\
&+ \mathrm{Tr}\{VR^2(\lambda)Bk(y)k^*(y)B^*\big(R^*(\lambda)\big)^2\},
\end{aligned}$$

$$\begin{aligned}
LVy = {}&2\Phi(y) + 2\langle f(y), Vy\rangle + 2\langle h(y), B^*Vy\rangle_U \\
&+ \mathrm{Tr}\{(-A^*)^{1/2-\Delta}Vg(y)g^*(y)(-A^*)^{\Delta-1/2}\} \\
&+ \mathrm{Tr}\{(-A^*)^{1/2-\gamma}VBk(y)k^*(y)B^*(-A^*)^{\gamma-1/2}\}
\end{aligned}$$

for $\lambda > 0$, $y \in H$. The Itô lemma applied to the strong solutions X_λ of the equations (1.23) yields

$$\mathrm{E}\langle VX_\lambda(t), X_\lambda(t)\rangle - \mathrm{E}\langle Vx_\lambda, x_\lambda\rangle = \mathrm{E}\int_0^t L_\lambda V\big(X_\lambda(s)\big)\,\mathrm{d}s, \quad t \geq 0, \tag{1.37}$$

(cf. Theorem 4.17 in [8]). We have to justify the limit passage in (1.37) for $\lambda \to \infty$. We have that

$$\begin{aligned}
&\big|L_\lambda V\big(X_\lambda(s)\big) - LV\big(X(s)\big)\big| \\
&\leq \big|L_\lambda V\big(X_\lambda(s)\big) - L_\lambda V\big(X(s)\big)\big| + \big|L_\lambda V\big(X(s)\big) - LV\big(X(s)\big)\big|, \quad s \geq 0.
\end{aligned} \tag{1.38}$$

Since $(-A^*)^{1/2-\Delta}V \in \mathcal{L}(H)$, $(-A^*)^{1/2-\gamma}V \in \mathcal{L}(H)$ by (1.34) and

$$R^2(\lambda)g(y)g^*(y)\big(R^*(\lambda)\big)^2 \in \mathcal{L}_1(H),$$
$$R^2(\lambda)Bk(y)k^*(y)B^*\big(R^*(\lambda)\big)^2 \in \mathcal{L}_1(H)$$

by (A2), (A3), respectively, we have

$$\begin{aligned}
&\operatorname{Tr}\big\{VR^2(\lambda)g(y)g^*(y)\big(R^*(\lambda)\big)^2\big\}\\
&= \operatorname{Tr}\big\{(-A^*)^{1/2-\Delta}VR^2(\lambda)g(y)g^*(y)\big(R^*(\lambda)\big)^2(-A^*)^{\Delta-1/2}\big\},\\
&\operatorname{Tr}\big\{VR^2(\lambda)Bk(y)k^*(y)B^*\big(R^*(\lambda)\big)^2\big\}\\
&= \operatorname{Tr}\big\{(-A^*)^{1/2-\gamma}VR^2(\lambda)Bk(y)k^*(y)B^*\big(R^*(\lambda)\big)^2(-A^*)^{\gamma-1/2}\big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big|L_\lambda\big(X(s)\big) - LV\big(X(s)\big)\big|\\
&\le 2\big|\big(R(\lambda)-I\big)f\big(X(s)\big)\big|\,|VX(s)|\\
&\quad + 2\big|h\big(X(s)\big)\big|_U \cdot \big|B^*\big(\big(R^*(\lambda)\big)^2 - I\big)VX(s)\big|_U\\
&\quad + \big|\operatorname{Tr}\big\{(-A^*)^{1/2-\Delta}VR^2(\lambda)g\big(X(s)\big)g^*\big(X(s)\big)\big(R^*(\lambda)\big)^2(-A^*)^{\Delta-1/2}\\
&\quad - (-A^*)^{1/2-\Delta}Vg\big(X(s)\big)g^*\big(X(s)\big)(-A^*)^{\Delta-1/2}\big\}\big|\\
&\quad + \big|\operatorname{Tr}\big\{(-A^*)^{1/2-\gamma}VR^2(\lambda)Bk\big(X(s)\big)k^*\big(X(s)\big)B^*\big(R^*(\lambda)\big)^2(-A^*)^{\gamma-1/2}\\
&\quad - (-A^*)^{1/2-\gamma}VBk\big(X(s)\big)k^*\big(X(s)\big)B^*(-A^*)^{\gamma-1/2}\big\}\big|.
\end{aligned} \tag{1.39}$$

Since $R^2(\lambda)y \to y$ and $R(\lambda)y \to y$ for $y \in H$ and $|R(\lambda)|_{\mathcal{L}(H)}$, $|R^2(\lambda)|_{\mathcal{L}(H)} = \big|\big(R^*(\lambda)\big)^2\big|_{\mathcal{L}(H)}$ are bounded the first two summands on the r.h.s. of (1.39) converge to zero a.s. Moreover, we have that $\Psi_1 := (-A^*)^{1/2-\Delta}V(-A)^{1/2-\Delta} \in \mathcal{L}(H)$ and $\Psi_2 := (-A^*)^{1/2-\gamma}V(-A)^{1/2-\gamma} \in \mathcal{L}(H)$, therefore

$$\begin{aligned}
&\big|\operatorname{Tr}\big\{\Psi_1(-A)^{\Delta-1/2}g\big(X(s)\big)g^*\big(X(s)\big)(-A^*)^{\Delta-1/2}\\
&- \Psi_1R^2(\lambda)(-A)^{\Delta-1/2}g\big(X(s)\big)g^*\big(X(s)\big)(-A^*)^{\Delta-1/2}\big(R^*(\lambda)\big)^2\big\}\big| \to 0 \quad \text{a.s.},\\
&\big|\operatorname{Tr}\big\{\Psi_2(-A)^{\gamma-1/2}Bk\big(X(s)\big)k^*\big(X(s)\big)B^*(-A^*)^{\gamma-1/2}\\
&- \Psi_2R^2(\lambda)Bk\big(X(s)\big)k^*\big(X(s)\big)B^*(-A^*)^{\gamma-1/2}\big(R^*(\lambda)\big)^2\big\}\big| \to 0 \quad \text{a.s.},
\end{aligned}$$

for all $s \ge 0$ by (A2) and (A3), respectively. By the dominated convergence theorem we obtain

$$\mathrm{E}\int_0^t \big|L_\lambda V\big(X(s)\big) - LV\big(X(s)\big)\big|\,ds \to 0, \quad \lambda \to \infty. \tag{1.40}$$

Furthermore, by (A2), (A3), (1.34) and the Lipschitz continuity of the coefficients f, h, $\tilde{g}$ and $\tilde{k}$ we obtain

$$
\begin{aligned}
&\big|L_\lambda V\big(X_\lambda(s)\big) - L_\lambda V\big(X(s)\big)\big| \le 2\big|\Phi\big(X_\lambda(s)\big) - \Phi\big(X(s)\big)\big| \\
&\quad + 2|R(\lambda)|_{\mathcal{L}(H)}\big|f\big(X_\lambda(s)\big) - f\big(X(s)\big)\big|\,|VX(s)| \\
&\quad + 2|R(\lambda)|_{\mathcal{L}(H)}\big|f\big(X_\lambda(s)\big)\big| \cdot \big|V\big(X_\lambda(s) - X(s)\big)\big| \\
&\quad + 2|B^*|_{\mathcal{L}(D_{A^*}^{1-\varepsilon},U)}|R^*(\lambda)^2|_{\mathcal{L}(D_{A^*}^{1-\varepsilon})}|V|_{\mathcal{L}(H,D_{A^*}^{1-\varepsilon})} \\
&\quad \cdot \Big(|X(s) - X_\lambda(s)| \cdot \big|h\big(X_\lambda(s)\big)\big|_U + |X(s)| \cdot \big|h\big(X(s)\big) - h\big(X_\lambda(s)\big)\big|_U\Big) \\
(1.41)\qquad &\quad + \big|\Psi_1 R^2(\lambda)(-A)^{\Delta-1/2}\big(g\big(X(s)\big)g^*\big(X(s)\big) \\
&\quad - g\big(X_\lambda(s)\big)g^*\big(X_\lambda(s)\big)\big)(-A^*)^{\Delta-1/2}\big(R^*(\lambda)\big)^2\big|_{\mathcal{L}_1(H)} \\
&\quad + \big|\Psi_2 R^2(\lambda)(-A)^{\gamma-1/2}B\big(k\big(X(s)\big)k^*\big(X(s)\big) \\
&\quad - k\big(X_\lambda(s)\big)k^*\big(X_\lambda(s)\big)\big)\big)B^*(-A^*)^{\gamma-1/2}\big(R^*(\lambda)\big)^2\big|_{\mathcal{L}_1(H)} \\
&\le 2\big|\Phi\big(X_\lambda(s)\big) - \Phi\big(X(s)\big)\big| + K|X(s) - X_\lambda(s)|\big(1 + |X(s)| + |X_\lambda(s)|\big), \\
&\quad s \in [0,t],
\end{aligned}
$$

for some $K > 0$ independent of $\lambda \in \mathbb{R}_+$. From Lemma 1.4 it follows that

$$
(1.42)\qquad \int_0^t \mathrm{E}|X(s) - X_\lambda(s)|\big(1 + |X(s)| + |X_\lambda(s)|\big)\,\mathrm{d}s \to 0
$$

and there exists a sequence $\lambda_n \to \infty$ such that $\Phi\big(X_{\lambda_n}(s)\big) \to \Phi\big(X(s)\big)$ almost everywhere on $(0,t) \times \Omega$. By (1.35) and Lemma 1.4 the sequence $\big|\Phi\big(X(s)\big) - \Phi\big(X_{\lambda_n}(s)\big)\big|$ is equi-integrable on $(0,t) \times \Omega$, therefore

$$
(1.43)\qquad \mathrm{E}\int_0^t \big|\Phi\big(X(s)\big) - \Phi\big(X_{\lambda_n}(s)\big)\big|\,\mathrm{d}s \to 0, \quad \lambda_n \to \infty.
$$

Now (1.41), (1.42) and (1.43) yield

$$
\mathrm{E}\int_0^t \big|L_{\lambda_n}V\big(X_{\lambda_n}(s)\big) - L_{\lambda_n}V\big(X(s)\big)\big|\,\mathrm{d}s \to 0, \quad \lambda_n \to \infty,
$$

which together with (1.40) implies

$$
(1.44)\qquad \mathrm{E}\int_0^t L_{\lambda_n}V\big(X_{\lambda_n}(s)\big)\,\mathrm{d}s \to \mathrm{E}\int_0^t LV\big(X(s)\big)\,\mathrm{d}s
$$

and the limit passage in (1.37) is verified. □

REMARK. From (A1) it follows that if for $y \in H$ the operator $Vg(y)g^*(y)$ is nuclear then

$$\operatorname{Tr}\{(-A^*)^{1/2-\Delta}Vg(y)g^*(y)(-A^*)^{\Delta-1/2}\} = \operatorname{Tr}\{Vg(y)g^*(y)\}$$

(cf. [16], Theorem III.8.2) and an analogous statement holds true for the last term on the right-hand side of (1.36). The formula (1.36) then has the more usual form known from the Itô lemma.

COROLLARY 1.6. *Let $V \in \mathcal{L}(H)$ satisfy the assumptions of Proposition* 1.5 *and let x, y be arbitrary H-valued $\mathcal{F}_0$-measurable random variables satisfying* $\mathrm{E}|x|^p + \mathrm{E}|y|^p < \infty$ *for some* $p > \max(1/\Delta, 1/\gamma, 1/\varepsilon)$. *Then*

$$\begin{aligned}
&\mathrm{E}\langle V\big(X^x(t) - X^y(t)\big), X^x(t) - X^y(t)\rangle \\
&\quad - \mathrm{E}\langle V(x-y), x-y\rangle \\
&= \mathrm{E}\int_0^t \Big\{2\Phi\big(X^x(s) - X^y(s)\big) + 2\langle f\big(X^x(s)\big) \\
&\quad - f\big(X^y(s)\big), V\big(X^x(s) - X^y(s)\big)\rangle \\
&\quad + 2\langle h\big(X^x(s)\big) - h\big(X^y(x)\big), B^*V\big(X^x(s) - X^y(s)\big)\rangle_U \\
&\quad + \operatorname{Tr}\big\{(-A^*)^{1/2-\Delta}V\big(g\big(X^x(s)\big) - g\big(X^y(s)\big)\big) \\
&\quad \times \big(g^*\big(X^x(s)\big) - g^*\big(X^y(s)\big)\big)(-A^*)^{\Delta-1/2}\big\} \\
&\quad + \operatorname{Tr}\big\{(-A)^{1/2-\gamma}VB\big(k\big(X^x(s)\big) - k\big(X^y(s)\big)\big) \\
&\quad \times \big(k^*\big(X^x(s)\big) - k^*\big(X^y(s)\big)\big)B^*(-A^*)^{\gamma-1/2}\big\}\Big\}\,\mathrm{d}s
\end{aligned} \tag{1.45}$$

holds for $t \geq 0$.

2. - Invariant measures and stability

For $x \in H$, $T > 0$, denote by μ_T^x the measure on $\mathcal{B}(H)$ defined by

$$\mu_T^x(A) = \frac{1}{T}\int_0^T P(t,x,A)\,\mathrm{d}t, \quad A \in \mathcal{B}(H),$$

where $P = P(t,x,A)$ is the transition probability function of the Markov process corresponding to the solution of (1.1). This process is Feller by Proposition 1.2 and, therefore, if there exists an $x \in H$ such that the family $(\mu_T^x)_{T\geq 1}$ is relatively compact in the sense of weak convergence of measures, then there exists an invariant measure $\mu^* \in \mathcal{P}(H)$ for the system (1.1), that is, a measure satisfying

$$\mu^*(A) = \int_H P(t,y,A)\,\mathrm{d}\mu^*(y), \quad t \geq 0, \quad A \in \mathcal{B}(H).$$

The measure μ^* can be found as a weak limit of a subsequence $(\mu^x_{T_n})$, $T_n \to \infty$. The weak compactness is usually proved by means of the Prokhorov theorem by which (μ^x_T) is weakly compact provided it is tight, i.e., there exists a sequence $K_n \subset H$ of compact sets such that

$$\sup_T \mu^x_T(H \setminus K_n) \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

If the dimension of the state space is finite then it is usual to take $K_n = B_n = \{x \in H; |x| \leq n\}$, $n \in \mathbb{N}$, and verify (2.1) by means of a suitable Lyapunov type theorem. However, in the infinite-dimensional case the condition

$$\sup_T \mu^x_T(H \setminus B_n) \to 0 \quad \text{as } n \to \infty \tag{2.2}$$

is, in general, insufficient for the compactness of (μ^x_T) as well as for the existence of an invariant measure as shown in a counterexample by Vrkoč [32]. Nevertheless, as we show below in case of the problem (1.1) with (A1)–(A3) fulfilled the "average boundedness in probability" (2.2) already guarantees the existence of the invariant measure.

THEOREM 2.1. *Assume* (A1)–(A3) *and let there exist an $x \in H$ such that*

$$\frac{1}{T}\int_0^T P(t,x,H \setminus B_n)\,\mathrm{d}t \to 0 \tag{2.3}$$

holds uniformly in $T \geq T_0$ as $n \to \infty$, where $T_0 \geq 0$ and $B_R = \{y \in H, |y| \leq R\}$, $R \geq 0$. Then there exists an invariant measure $\mu^* \in \mathcal{P}$ for the system (1.1).

REMARK. Note that even in the particular case $h = 0$, $k = 0$, Theorem 2.1 presents a strengthening of [18], Theorem 4.3, and [26], Theorem 3.3, in which it was essential that the operator A had to be self–adjoint. A similar result for $h = 0$, $k = 0$, but with $S(\cdot)$ non-analytic, had been obtained in [10] (Theorems 4 and 6). The method of proof of Theorem 2.1 is different from the above mentioned cases.

At first we shall prove the following result:

PROPOSITION 2.2. *Take $\delta \in (0, \min(\varepsilon, \Delta, \gamma))$ and $p > \max\big((\Delta-\delta)^{-1}, (\gamma-\delta)^{-1}, (\varepsilon-\delta)^{-1}\big)$. For any $T > 0$ there exists a constant $C = C(T,p,\delta)$ such that for every H-valued $\mathcal{F}_0$-measurable random variable satisfying* $\mathrm{E}|x|^p < \infty$ *we have $X^x(T) \in D^\delta_A$ and*

$$\mathrm{E}|(-A)^\delta X^x(T)|^p \leq C(1 + \mathrm{E}|x|^p). \tag{2.4}$$

PROOF. The constants k_i, $i = 1, 2, \dots$ in the below proof can be chosen independently of $t \in [0, T]$ and x. By (A2), (A3) and the at most linear growth of f, h, $\tilde{g}$ and $\tilde{k}$ we have

$$
\begin{aligned}
&\mathrm{E}|X^x(t)|^p \\
&\le k_1 \left(\mathrm{E}|x|^p + \mathrm{E}\left| \int_0^t S(t-r)\big(f(X^x(r)) + Bh(X^x(r))\big)\,\mathrm{d}t \right|^p \right. \\
&\quad + \mathrm{E}\left| \int_0^t S(t-r) g(X^x(r))\,\mathrm{d}W_r \right|^p \\
&\quad \left. + \mathrm{E}\left| \int_0^t S(t-r) Bk(X^x(r))\,\mathrm{d}V_r \right|^p \right) \\
&\le k_2 \left(1 + \mathrm{E}|x|^p + \mathrm{E}\left(\int_0^t (t-r)^{\varepsilon-1} |X^x(r)|\,\mathrm{d}r \right)^p \right. \\
&\quad + \mathrm{E}\left(\int_0^t (t-r)^{2\Delta-1} |X^x(r)|^2 \mathrm{d}r \right)^{p/2} \\
&\quad \left. + \mathrm{E}\left(\int_0^t (t-r)^{2\gamma-1} |X^x(r)|^2 \mathrm{d}r \right)^{p/2} \right), \quad t \in [0, T],
\end{aligned}
\tag{2.5}
$$

for some k_1, $k_2 < \infty$. The Hölder inequality yields

$$
\mathrm{E}|X^x(t)|^p \le k_3 \left(1 + \mathrm{E}|x|^p + \int_0^t \mathrm{E}|X^x(r)|^p \mathrm{d}r \right), \quad t \in [0, T], \tag{2.6}
$$

which implies

$$
\mathrm{E}|X^x(t)|^p \le k_4 (1 + \mathrm{E}|x|^p), \quad t \in [0, T], \tag{2.7}
$$

by the Gronwall lemma. It follows that

$$
\begin{aligned}
\mathrm{E}|(-A)^{\delta} X^{x}(T)|^{p} \leq k_5 \Bigg(& \mathrm{E}|(-A)^{\delta} S(T)x|^{p} \\
&+ \mathrm{E}\left|\int_0^T (-A)^{\delta} S(T-r)\big(f(X^x(r)) + Bh(X^x(r))\big)\,\mathrm{d}r\right|^p \\
&+ \mathrm{E}\left|\int_0^T (-A)^{\delta} S(T-r) g(X^x(r))\,\mathrm{d}W_r\right|^p \\
&+ \mathrm{E}\left|\int_0^T (-A)^{\delta} S(T-r) Bk(X^x(r))\,\mathrm{d}V_r\right|^p \Bigg) \\
\leq k_6 \Bigg(& 1 + \frac{\mathrm{E}|x|^p}{T^{p\delta}} + \mathrm{E}\left(\int_0^T \frac{|h(X^x(r))|}{(T-r)^{1-\varepsilon+\delta}}\,\mathrm{d}r\right)^p \\
&+ \mathrm{E}\left(\int_0^T (T-r)^{2\Delta-2\delta-1} |\tilde{g}(X^x(r))|_{\mathcal{L}_2(H)} \mathrm{d}r\right)^{p/2} \\
&+ \mathrm{E}\left(\int_0^T (T-r)^{2\gamma-2\delta-1} |\tilde{k}(X^x(r))|_{\mathcal{L}_2(U,H)} \mathrm{d}r\right)^{p/2} \Bigg) \\
\leq k_7 \Bigg[& 1 + \frac{\mathrm{E}|x|^p}{T^{p\delta}} + \Bigg(\left(\int_0^T r^{(\varepsilon-\delta-1)q} \mathrm{d}r\right)^{p/q} \\
&+ \left(\int_0^T r^{(2\Delta-2\delta-1)\eta} \mathrm{d}r\right)^{p/2\eta} + \left(\int_0^T r^{(2\gamma-2\delta-1)\eta} \mathrm{d}r\right)^{p/2\eta} \Bigg) \\
&\times \int_0^T \mathrm{E}|X^x(r)|^p \mathrm{d}r \Bigg],
\end{aligned}
$$

where $q = p(p-1)^{-1}$, $\eta = p(p-2)^{-1}$, which together with (2.7) concludes the proof. □

PROOF OF THEOREM 2.1. Take $0 < \delta < \min(\varepsilon, \Delta, \gamma)$ and $p > \max\big((\Delta -$

$\delta)^{-1},(\gamma-\delta)^{-1},(\varepsilon-\delta)^{-1}\big)$. For $n \in \mathbb{N}$ set

$$K_n = \mathrm{Cl}_H(-A)^{-\delta}B_n.$$

By (A1) the sets K_n are compact in H and for every $R>0$ we have

$$\tag{2.9} \begin{aligned} &\sup_{|y|\le R} P(1,y,H\setminus K_n) \le \sup_{|y|\le R} \mathrm{P}[|(-A)^{\delta}X^{y}(1)| \ge n] \\ &\le \sup_{|y|\le R} n^{-p}\mathrm{E}|(-A)^{\delta}X^{y}(1)|^{p} \le n^{-p}C(1,p,\delta)(1+R^{p}) \to 0, \quad n\to\infty. \end{aligned}$$

Now for $T>1$ the Chapman–Kolmogorov inequality yields

$$\begin{aligned} &\frac{1}{T}\int_1^T P(t,x,H\setminus K_n)\,\mathrm{d}t \\ &= \frac{1}{T}\int_1^T\int_H P(1,y,H\setminus K_n)P(t-1,x,\,\mathrm{d}y)\,\mathrm{d}t \\ &= \frac{T-1}{T}\int_H P(1,y,H\setminus K_n)\,\mathrm{d}\mu^{x}_{T-1}(y) \\ &\le \mu^{x}_{T-1}(H\setminus B_R) + \mu^{x}_{T-1}(B_R)\sup_{|y|\le R} P(1,y,H\setminus K_n), \end{aligned}$$

for any $R>0$. By (2.3) and (2.9) this implies

$$\tag{2.10} \mu^{x}_{T}(H\setminus K_n) \to 0, \quad n\to\infty,$$

uniformly in $T\ge 1$. Consequently, the family $(\mu^{x}_{T})_{T\ge 1}$ is weakly compact by the Prokhorov theorem. Since the process X^{x} is Feller by Proposition 1.2 it is standard to show that the weak limit μ^{*} of a subsequence $\mu^{x}_{T_n}$, $T_n\to\infty$, is an invariant measure. □

The condition (2.3) is usually verified by means of a Lyapunov method and we shall also carry out this procedure. However, let us notice that, for example, the Lyapunov function $V(x)=|x|^{p}$, $p>0$, which is often used in similar situations, cannot be used here as the formal stochastic differential of $V\big(X^{x}(t)\big)$ contains terms which are either infinite or do not make sense, like $\langle X^{x}(t),Bh\big(X^{x}(t)\big)\rangle$, $\mathrm{Tr}\,g\big(X^{x}(t)\big)g^{*}\big(X^{x}(t)\big)$, etc. We must restrict ourselves to functionals allowed by the above proved Itô type lemma (Proposition 1.5). Set

$$\tag{2.11} V := 2\int_0^{+\infty} S^{*}(t)S(t)\,\mathrm{d}t.$$

The integral in (2.11) converges in the operator norm of $\mathcal{L}(H)$, clearly $V = V^*$ and $V \geq 0$. We have the following

LEMMA 2.3. *For any* $\beta \geq 0$, $\lambda \geq 0$, $\lambda + \beta < 1$, *we have that* $V \in \mathcal{L}(D_A^{-\beta}, D_{A^*}^{\lambda})$ *and*

$$|V|_{\mathcal{L}(D_A^{-\beta}, D_{A^*}^{\lambda})} \leq 2M^2(2\omega)^{\beta+\lambda-1}\Gamma(1-\lambda-\beta),$$

where Γ *is the Gamma-function and* M, ω *are the constants from the estimate* (1.3).

The proof of Lemma 2.3 is obtained by a straightforward application of (1.3). In the sequel denote by K_f, K_h, $K_{\tilde{g}}$ and $K_{\tilde{k}}$ the Lipschitz constants of the functions f, h, $\tilde{g}$ and $\tilde{k}$, respectively.

PROPOSITION 2.4. *Assume that*

$$\begin{aligned}(2.12)\qquad & K_f|V|_{\mathcal{L}(H)} + K_h|B^*V|_{\mathcal{L}(H,U)} + \frac{1}{2}K_{\tilde{g}}^2|V|_{\mathcal{L}(D_A^{\Delta-1/2}, D_{A^*}^{1/2-\Delta})} \\ & + \frac{1}{2}K_{\tilde{k}}^2|V|_{\mathcal{L}(D_A^{\gamma-1/2}, D_{A^*}^{1/2-\gamma})} < 1.\end{aligned}$$

Then for every H*-valued,* $\mathcal{F}_0$*-measurable* x *such that* $\mathrm{E}|x|^p < \infty$ *for some* $p > \max(1/\varepsilon, 1/\gamma, 1/\Delta)$ *we have*

$$\int_0^T \mathrm{E}|X^x(t)|^2 \mathrm{d}t \leq C_1(1+T), \quad T \geq 0, \tag{2.13}$$

where $C_1 < \infty$ *is a constant independent of* $T \geq 0$.

PROOF. It is standard to verify that

$$\langle Vx, Ax\rangle = -|x|^2, \quad x \in \mathcal{D}(A), \tag{2.14}$$

hence by Lemma 2.3 the operator V satisfies all assumptions of Proposition 1.5 which yields

$$\begin{aligned}\mathrm{E}\langle VX(t), X(t)\rangle - \mathrm{E}\langle Vx, x\rangle = \mathrm{E}\int_0^t \Big(& -2|X(s)|^2 \\ & + 2\langle VX(s), f\big(X(s)\big)\rangle + 2\langle h\big(X(s)\big), B^*VX(s)\rangle_U \\ & + \mathrm{Tr}\big\{(-A^*)^{1/2-\Delta}Vg\big(X(s)\big)g^*\big(X(s)\big)(-A^*)^{\Delta-1/2}\big\} \\ & + \mathrm{Tr}\big\{(-A^*)^{1/2-\gamma}VBk\big(X(s)\big)k^*\big(X(s)\big)B^*(-A^*)^{\gamma-1/2}\big\}\Big)\,\mathrm{d}s\end{aligned}$$

for $X(s) = X^x(s)$. Denote by r the left-hand side of the inequality (2.12). By the nonnegativity of V, (A2) and (A3), we get

$$0 \le \mathrm{E}\langle Vx, x\rangle + \mathrm{E}\int_0^T \left(k_1 + k_2|X(s)| + (r-1)|X(s)|^2\right) \mathrm{d}s, \quad T \ge 0,$$

for some k_1, k_2 independent of T and taking $\kappa > 0$ such that $r + \kappa - 1 < 0$ we obtain

$$(1 - r - \kappa)\int_0^T \mathrm{E}|X(s)|^2 \mathrm{d}s \le \mathrm{E}\langle Vx, x\rangle + \mathrm{E}\int_0^T \left(k_1 + k_2|X(s)| - \kappa|X(s)|^2\right) \mathrm{d}s, \quad T \ge 0,$$

and (2.13) follows. □

COROLLARY 2.5. *Assume that* (2.12) *holds. Then there exists an invariant measure* μ^* *for the system* (1.1).

PROOF. For $x \in H$ we have by the Chebyshev inequality

$$\frac{1}{T}\int_0^T P(t, x, H \setminus B_n)\,\mathrm{d}t \le \frac{1}{Tn^2}\int_0^T \mathrm{E}|X^x(t)|^2 \mathrm{d}t \to 0, \quad n \to \infty,$$

uniformly in $T \ge 0$ by Proposition 2.4. Thus Theorem 2.1 can be applied. □

REMARK. By Lemma 2.3 it is easily seen that (2.12) is satisfied if

$$2M^2\left(K_f(2\omega)^{-1} + K_h|B|_{\mathcal{L}(U, D_A^{\varepsilon-1})}(2\omega)^{-\varepsilon}\Gamma(\varepsilon) + \frac{1}{2}K_g^2(2\omega)^{-2\Delta}\Gamma(2\Delta) + \frac{1}{2}K_k^2(2\omega)^{-2\gamma}\Gamma(2\gamma)\right) < 1 \tag{2.15}$$

holds, which is a condition formulated in terms of the coefficients of the system (1.1).

PROPOSITION 2.6. *Assume* (2.12). *Then for every* $p > \max(1/\varepsilon, 1/\Delta, 1/\gamma)$ *there exists a constant* $C_2 > 0$ *such that*

$$\int_0^{+\infty} \mathrm{E}|X^x(t) - X^y(t)|^2 \mathrm{d}t \le C_2 \mathrm{E}|x - y|^2 \tag{2.16}$$

holds for every H*-valued,* $\mathcal{F}_0$*-measurable random variables* x, y, *satisfying* $\mathrm{E}|x|^p + \mathrm{E}|y|^p < \infty$.

PROOF. As noticed in the proof of Proposition 2.4 the operator V defined by (2.11) satisfies the assumptions of Proposition 1.5. Thus we can use Corollary 1.6 to obtain

$$\mathrm{E}\langle VX^x(t) - VX^y(t), X^x(t) - X^y(t)\rangle - \mathrm{E}\langle V(x-y), x-y\rangle$$

$$= \mathrm{E}\int_0^t \Big(-2|X^x(s) - X^y(s)|^2 + 2\langle V\big(X^x(s) - X^y(s)\big), f\big(X^x(s)\big) - f\big(X^y(s)\big)\rangle$$

$$+ 2\langle h\big(X^x(s)\big) - h\big(X^y(s)\big), B^*V\big(X^x(s) - X^y(s)\big)\rangle_U$$

$$+ \mathrm{Tr}\{(-A^*)^{1/2-\Delta}V\big(g\big(X^x(s)\big) - g\big(X^y(s)\big)\big)$$

$$\times \big(g^*\big(X^x(s)\big) - g^*\big(X^y(s)\big)\big)(-A^*)^{\Delta-1/2}\}$$

$$+ \mathrm{Tr}\{(-A^*)^{1/2-\gamma}VB\big(k\big(X^x(s)\big) - k\big(X^y(s)\big)\big)$$

$$\times \big(k^*\big(X^x(s)\big) - k^*\big(X^y(s)\big)\big)B^*(-A^*)^{\gamma-1/2}\}\Big)\,\mathrm{d}s$$

$$\le \mathrm{E}\int_0^t (r-1)|X^x(s) - X^y(s)|^2 \mathrm{d}s, \quad t \ge 0,$$

where r stands for the left-hand side of (2.12). Hence

$$\int_0^t \mathrm{E}|X^x(s) - X^y(s)|^2 \mathrm{d}s \le c\mathrm{E}\langle V(x-y), (x-y)\rangle$$

for some $c > 0$ independent of x, y and t, which yields (2.16). □

In Theorem 2.8 and Corollary 2.10 below we prove global attractiveness of the invariant measure corresponding to (1.1).

LEMMA 2.7. *Let Z be the linear space of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$ with values in a Banach space $(\Xi, \|\cdot\|)$ and let $T(t,s)$, $t \ge s \ge 0$, be a family of nonlinear operators with domains $Y_s \subset Z$ with properties:*

(i) $T(t,s)Y_s \subset Y_t$, $t \ge s$,

(ii) $T(t,t)y = y$, $t \ge 0$, $y \in Y_t$,

(iii) $T(t,u)T(u,s) = T(t,s)$ *on* Y_s, $s \le u \le t$,

(iv) $T(\cdot, s)y$ *is measurable on* $[s, +\infty) \times \Omega$ *for each* $s \ge 0$, $y \in Y_s$,

(v) $\mathrm{E}\|T(t,s)x - T(t,s)y\|^p \le g(t-s)\mathrm{E}\|x-y\|^p$, $t \ge s$,

$$x, y \in Y_s(p) := \{z \in Y_s; \mathrm{E}\|z\|^p < \infty\},$$

for a positive continuous function $g\colon [0,\infty) \to \mathbb{R}_+$ *and some* $p \ge 2$.

Then the two conditions (2.17), (2.18) *below are equivalent:*

$$\exists K > 0, \quad \int_0^\infty \mathrm{E}\|T(t,s)x - T(t,s)y\|^p \mathrm{d}t \le K\mathrm{E}\|x-y\|^p, \quad x, y \in Y_s(p), \tag{2.17}$$

$$\begin{gathered} \exists M < \infty, \quad \exists a > 0, \\ \mathrm{E}\|T(t,s)x - T(t,s)y\|^p \le Me^{-a(t-s)}\mathrm{E}\|x-y\|^p, \\ x, y \in Y_s(p). \end{gathered} \tag{2.18}$$

The idea of the proof of Lemma 2.7 basically belongs to Datko [7]. In Ichikawa [19], Theorem 2.2, an analogous result has been established for the nonlinear stochastic case. The proof of Lemma 2.7 is a straightforward modification of the Ichikawa's proof (for differences of solutions) and can be omitted.

THEOREM 2.8. *Assume* (A2), (A3) *and* (2.12). *Then for any* $t_0 > 0$, $0 < \delta < \min(\varepsilon, \Delta, \gamma)$ *and* $p > \max\left((\Delta-\delta)^{-1}, (\gamma-\delta)^{-1}, (\varepsilon-\delta)^{-1}\right)$ *there exist constants* $M < \infty$, $a > 0$, *such that*

$$\mathrm{E}\left|(-A)^\delta\left(X^x(t) - X^y(t)\right)\right|^2 \le Me^{-at}\mathrm{E}|x-y|^2, \quad t \ge t_0, \tag{2.19}$$

holds for all H*-valued,* $\mathcal{F}_0$*-measurable* x, y, *satisfying* $\mathrm{E}|x|^p + \mathrm{E}|y|^p < \infty$.

PROOF. *Step I.*

At first we show that for every $T > 0$ there exists a constant $M_T < \infty$ independent of x, y, such that

$$\mathrm{E}\left|(-A)^\delta\left(X^x(T) - X^y(T)\right)\right|^p \le M_T\mathrm{E}|x-y|^p. \tag{2.20}$$

We have

$$
\begin{aligned}
&\mathrm{E}\big|(-A)^{\delta}\big(X^{x}(T)-X^{y}(T)\big)\big|^{p}\\
&\le 5^{p-1}\Bigg(\mathrm{E}|(-A)^{\delta}S(T)(x-y)|^{p}\\
&\quad+\mathrm{E}\left[\int_0^T \big|(-A)^{\delta}S(T-r)\big(f\big(X^{x}(r)\big)-f\big(X^{y}(r)\big)\big)\big|\,\mathrm{d}r\right]^{p}\\
&\quad+\mathrm{E}\left[\int_0^T \big|(-A)^{\delta}S(T-r)B\big(h\big(X^{x}(r)\big)-h\big(X^{y}(r)\big)\big)\big|\,\mathrm{d}r\right]^{p}\\
&\quad+\mathrm{E}\left[\int_0^T \big|(-A)^{\delta}S(T-r)\big(g\big(X^{x}(r)\big)-g\big(X^{y}(r)\big)\big)\big|^{2}_{\mathcal{L}_2(H)}\mathrm{d}r\right]^{p/2}\\
&+\mathrm{E}\left[\int_0^T \big|(-A)^{\delta}S(T-r)B\big(k\big(X^{x}(r)\big)-k\big(X^{y}(r)\big)\big)\big|^{2}_{\mathcal{L}_2(U,H)}\mathrm{d}r\right]^{p/2}\Bigg)
\end{aligned}
\tag{2.21}
$$

and similarly as in the proof of (2.4) we get

$$
\mathrm{E}\big|(-A)^{\delta}\big(X^{x}(T)-X^{y}(T)\big)\big|^{p}\le c\left(\mathrm{E}|x-y|^{p}+\int_0^T \mathrm{E}|X^{x}_s-X^{y}_s|^{p}\mathrm{d}s\right)
\tag{2.22}
$$

for some $c>0$ which together with Proposition 1.2 yields (2.20).

Step II.

We prove that there exists a constant $K>0$ such that

$$
\mathrm{E}\int_0^{+\infty}\big|(-A)^{\delta}\big(X^{x}(t)-X^{y}(t)\big)\big|^{2}\mathrm{d}t\le K\mathrm{E}|(-A)^{\delta}(x-y)|^{2},
\tag{2.23}
$$

for all D^{δ}_{A}-valued, $\mathcal{F}_0$-measurable x, y, satisfying $\mathrm{E}|(-A)^{\delta}x|^{p}+\mathrm{E}|(-A)^{\delta}y|^{p}<\infty$.

From (1.3) and the Lipschitz continuity of f, $\tilde{g}$, h and $\tilde{k}$ we obtain

$$\begin{aligned}
&\mathrm{E}\left|(-A)^{\delta}\left(X^{x}(t)-X^{y}(t)\right)\right|^{2} \\
&\leq c_1\left\{e^{-\omega t}\mathrm{E}|(-A)^{\delta}(x-y)|^{2}\right. \\
&\quad+\mathrm{E}\left(\int_0^t e^{-\omega(t-s)}|X^x(s)-X^y(s)|\,\mathrm{d}s\right)^2 \\
&\quad+\mathrm{E}\left(\int_0^t e^{-\omega(t-s)}(t-s)^{\varepsilon-\delta-1}|X^x(s)-X^y(s)|\,\mathrm{d}s\right)^2 \\
&\quad+\mathrm{E}\int_0^t e^{-2\omega(t-s)}(t-s)^{2\Delta-2\delta-1}|X^x(s)-X^y(s)|^2\mathrm{d}s \\
&\quad\left.+\mathrm{E}\int_0^t e^{-2\omega(t-s)}(t-s)^{2\gamma-2\delta-1}|X^x(s)-X^y(s)|^2\mathrm{d}s\right\}, \\
&t\geq 0
\end{aligned} \tag{2.24}$$

for a constant c_1 independent of x, y and $t \geq 0$. By the Young inequality we have

$$\begin{aligned}
&\mathrm{E}\int_0^{+\infty}\left(\int_0^t e^{-\omega(t-s)}|X^x(s)-X^y(s)|\,\mathrm{d}s\right)^2\mathrm{d}t \\
&\leq\left(\int_0^{+\infty}e^{-\omega t}\mathrm{d}t\right)^2\int_0^{+\infty}\mathrm{E}|X^x(s)-X^y(s)|^2\mathrm{d}s,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&\mathrm{E}\int_0^{+\infty}\left(\int_0^t e^{-\omega(t-s)}(t-s)^{\varepsilon-\delta-1}|X^x(s)-X^y(s)|\,\mathrm{d}s\right)^2\mathrm{d}t \\
&\leq\left(\int_0^{+\infty}\frac{e^{-\omega t}}{t^{1-\varepsilon+\delta}}\,\mathrm{d}t\right)^2\int_0^{+\infty}\mathrm{E}|X^x(s)-X^y(s)|^2\mathrm{d}s,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\mathrm{E}\int_0^{+\infty}\int_0^t \frac{e^{-2\omega(t-s)}}{(t-s)^{1-2\Delta+2\delta}}\,|X^x(s)-X^y(s)|^2\mathrm{d}s\mathrm{d}t\\ &\le \int_0^{+\infty}\frac{e^{-2\omega t}}{t^{1-2\Delta+2\delta}}\,\mathrm{d}t\int_0^{+\infty}\mathrm{E}|X^x(s)-X^y(s)|^2\mathrm{d}s,\end{aligned} \tag{2.27}$$

$$\begin{aligned}&\mathrm{E}\int_0^{+\infty}\int_0^t \frac{e^{-2\omega(t-s)}}{(t-s)^{1-2\gamma+2\delta}}\,|X^x(s)-X^y(s)|^2\mathrm{d}s\mathrm{d}t\\ &\le \int_0^{+\infty}\frac{e^{-2\omega t}}{t^{1-2\gamma+2\delta}}\,\mathrm{d}t\int_0^{+\infty}\mathrm{E}|X^x(s)-X^y(s)|^2\mathrm{d}s,\end{aligned} \tag{2.28}$$

Taking into account (2.24)–(2.28) and Proposition 2.6 we obtain (2.23).

Step III.

Denote by $|\cdot|_\delta$ the norm of the space D_A^δ, let Z be the space of random variables defined on $(\Omega,\mathcal{F},\mathrm{P})$ with values in D_A^δ,

$$Y_s = \{x \in Z, x \text{ is } \mathcal{F}_s\text{-measurable, } \mathrm{E}|x|_\delta^p < \infty\}$$

and let us define $T(t,s)y$, $t \ge s \ge 0$, $y \in Y_s$, as the solution $X(t)$ of (1.1) satisfying the initial condition $T(s,s)y = X(s) = y$. From the basic properties of solutions of (1.1) it follows that (i)–(iv) from Lemma 2.7 holds true. Since the system (1.1) is autonomous we have by (2.21)

$$\begin{aligned}&\mathrm{E}|T(t,s)x - T(t,s)y|_\delta^2 \le [\mathrm{E}|T(t,s)x - T(t,s)y|_\delta^p]^{2/p}\\ &\le \tilde{M}_{t-s}^{2/p}|x-y|_\delta^2\end{aligned} \tag{2.29}$$

for x, $y \in D_A^\delta$ deterministic, where $\tilde{M} = \tilde{M}_t$ is bounded for t in compact intervals, which verifies (v) from Lemma 2.7 by the Markov property. The condition (2.17) has been verified by Step II of the present proof. Therefore by Lemma 2.7 we obtain

$$\mathrm{E}\big|(-A)^\delta\big(X^x(t) - X^y(t)\big)\big|^2 \le \hat{M}e^{-at}\mathrm{E}|(-A)^\delta(x-y)|^2, \quad t \ge 0, \quad y \in Y_0,$$

for some $\hat{M} < \infty$, $a > 0$. Using (2.20) with $T = t_0$ and again the Markov property we have

$$\mathrm{E}\big|(-A)^\delta\big(X^x(t)-X^y(t)\big)\big|^2 \le \hat{M}e^{-a(t-t_0)}\mathrm{E}\big|(-A)^\delta\big(X^x(t_0)-X^y(t_0)\big)\big|^2$$
$$\le \hat{M}M_{t_0}^{2/p}e^{at_0}e^{-at}\mathrm{E}|x-y|^2$$

for $t \ge t_0$, x, y H-valued, $\mathcal{F}_0$-measurable, $\mathrm{E}|x|^p + \mathrm{E}|y|^p < \infty$. □

Theorem 2.8 tells us that solutions of (1.1) are exponentially stable in the quadratic mean in the norm $|\cdot|_\delta$. As a consequence we establish a subsequent result on limit behavior of probability distributions of solutions to (1.1).

Let $\{S_t\}_{t\ge0}$ be the family of operators on the space of probability measures $\mathcal{P}(H)$ defined by

$$S_t\nu(A) = \int_H P(t,y,A)\mathrm{d}\nu(x), \quad t\ge 0, \quad A\in\mathcal{B}(H), \quad \nu\in\mathcal{P}(H).$$

Clearly, a measure $\mu^*\in\mathcal{P}(H)$ is invariant for (1.1) if and only if $S_t\mu^*=\mu^*$ for all $t\ge 0$.

DEFINITION 2.9. Let $\delta\ge 0$ be given. We say that a sequence $(\nu_n)\subset\mathcal{P}(H)$ converges D_A^δ-weakly to $\nu\in\mathcal{P}(H)$ if ν_n are concentrated on D_A^δ for n large and

$$\int_{D_A^\delta}\varphi\,\mathrm{d}\nu_n \to \int_{D_A^\delta}\varphi\,\mathrm{d}\nu, \quad n\to\infty, \tag{2.30}$$

holds for any bounded and continuous function $\varphi\colon D_A^\delta\to\mathbb{R}$.

It is obvious that D_A^0-weak convergence coincides with the usual weak convergence of measures on H and $D_A^{\delta_1}$-weak convergence implies $D_A^{\delta_2}$-weak convergence for $\delta_1\ge\delta_2$. The converse implication does not hold for unbounded A.

COROLLARY 2.10. *Assume* (A1)–(A3) *and* (2.12). *Then there exists a unique invariant measure* $\mu^*\in\mathcal{P}(H)$ *for the system* (1.1). *Moreover, for every* $\nu\in\mathcal{P}(H)$ *and* $0\le\delta<\min(\varepsilon,\Delta,\gamma)$ *the system* $(S_t\nu)$ *converges* D_A^δ*-weakly to* μ^* *as* $t\to\infty$.

PROOF. Let $\varphi\colon D_A^\delta\to\mathbb{R}$ be bounded and Lipschitz continuous. For all deterministic x, $y\in H$ by Theorem 2.8 we have

$$\big|\mathrm{E}\varphi\big(X^x(t)\big)-\mathrm{E}\varphi\big(X^y(t)\big)\big| \le \mathrm{const}\,\mathrm{E}|X^x(t)-X^y(t)|_\delta \to 0, \quad t\to\infty.$$

By the dominated convergence theorem it follows that

$$\left| \int_{D_A^\delta} \varphi \,\mathrm{d}S_t\nu - \int_{D_A^\delta} \varphi \,\mathrm{d}\mu^* \right| \leq \int_{H\times H} \left| \int_{D_A^\delta} \varphi(z)P(t,y,\,\mathrm{d}z) \right.$$

$$\left. - \int_{D_A^\delta} \varphi(z)P(t,x,\,\mathrm{d}z) \right| \nu \times \mu^*(\mathrm{d}y,\,\mathrm{d}x) \to 0, \quad t \to \infty.$$

□

3. - Examples

EXAMPLE 3.1 (stochastic parabolic equation of $2m$-th order). Let $A(x,D)$ be a $2m$-th order differential operator of the form

$$A(x,D)y = \sum_{|p|,|q|\leq m} (-1)^{|p|} D^p\big(a_{pq}(x)D^q y\big), \quad x \in G, \tag{3.1}$$

where G is a bounded domain in $\mathbb{R}^n$, the boundary ∂G of G being infinitely smooth with $x \in G$ locally on one side of the boundary. The coefficients a_{pq} are in $\mathbb{C}^\infty(\overline{G})$ for all values of multiindices p, q, $|p| \leq m$, $|q| \leq m$, where $|\cdot|$ stands for the length of a multiindex. Assume that $A(x,D)$ is uniformly elliptic, i.e.,

$$\sum_{|p|=|q|=m} (-1)^m a_{pq}(x)\xi^{p+q} \geq \hat{\nu}|\xi|^{2m}, \quad x \in \overline{G}, \quad \xi \in \mathbb{R}^n, \tag{3.2}$$

for some $\hat{\nu} > 0$, where $\xi^{p+q} = \xi_1^{p_1+q_1} \ldots \xi_n^{p_n+q_n}$, $p = (p_1,\ldots,p_n)$, $q = (q_1,\ldots,q_n)$. Furthermore, let $\mathcal{B} = (\mathcal{B}_0,\ldots,\mathcal{B}_{m-1})$ be a system of boundary operators defined by

$$\mathcal{B}_j\varphi = \sum_{|h|\leq m_j} b_{jh}(x)D^h\varphi, \quad x \in \partial G, \tag{3.3}$$

for $j = 0,1,\ldots,m-1$ and $0 \leq m_0 < m_1 < \ldots < m_{m-1} \leq 2m-1$, b_{jh}, $\varphi \in \mathbb{C}^\infty(\partial G)$. We assume that the system $(\mathcal{B}_j)$ is normal and covers $A(x,D)$ and there exists a Green function of the problem $\dot{y} = A(x,D)y$, $\mathcal{B}y = 0$ (cf. [25], [4]). For example, we can consider the Dirichlet boundary problem in which case $\mathcal{B}_j = \dfrac{\partial^j}{\partial\nu^j}$, $j = 0,\ldots,m-1$, is the j-th normal derivative.

Under the above assumptions there exists a $\beta \geq 0$ such that the operator $-A(x,D) - \beta I$ defined on $\{y \in \mathbb{C}^\infty(\overline{G}), \mathcal{B}y = 0\}$ has an extension A, $\mathcal{D}(A) = \{y \in H^{2m}(G); \mathcal{B}y = 0\}$ which is a generator of an exponentially stable analytic semigroup on $H = L_2(G)$. Furthermore, set $U = H^{\sigma_0}(\partial G) \times H^{\sigma_1}(\partial G) \times \ldots \times H^{\sigma_{m-1}}(\partial G)$, $\sigma_j \in \mathbb{R}$, $\sigma_j > -(m_j+1/2)$, $j = 0, 1, \ldots, m-1$, (see, for instance, [25] for the definition of $H^{\sigma_j}(\partial G)$) and take

$$s \in \big(0, \min_j(\sigma_j + m_j + 1/2, m_0 + 1/2)\big), \quad \varepsilon = \frac{s}{2m}.$$

It is well known that for the elliptic problem

$$A(x,D)y + \beta y = 0 \text{ on } G, \quad \mathcal{B}y = g \text{ on } \partial G, \tag{3.4}$$

there exists the Dirichlet mapping $D \in \mathcal{L}(U, D_A^\varepsilon)$, $D\colon g \mapsto -y$, (cf. [25]). In particular, in the second-order case with the Dirichlet boundary conditions we have $m = 1$, $U = H^{\sigma_0}(\partial G)$, $\sigma_0 > -1/2$, and we can take $0 < \varepsilon < \left(\frac{\sigma_0}{2} + \frac{1}{4}\right), \varepsilon < 1/4$. For the Neumann boundary conditions we have $U = H^{\sigma_0}(\partial G)$ with $\sigma_0 > -3/2$ and

$$0 < \varepsilon < \frac{\sigma_0}{2} + \frac{3}{4}, \varepsilon < \frac{3}{4}.$$

The problem which is dealt with is given heuristically by the equation

$$\begin{aligned} \frac{\partial y}{\partial t}(t,x) &= -A(x,D)y(t,x) + F\big(y(t,x)\big) \\ &+ \Gamma\big(y(t,x)\big)\eta_1(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \end{aligned} \tag{3.5}$$

with the initial and boundary conditions

$$y(0,x) = y_0(x), \quad x \in G, \tag{3.6}$$

$$\mathcal{B}y(t,x) = h\big(y(t,\cdot)\big)(x) + K\big(y(t,\cdot)\big)(x)\eta_2(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial G, \tag{3.7}$$

where $F\colon \mathbb{R} \to \mathbb{R}$, $\Gamma\colon \mathbb{R} \to \mathbb{R}$, $h\colon H \to U$ and $K\colon H \to U$ are Lipschitz continuous, Γ and K are bounded, η_1 and η_2 stand for mutually stochastically independent, space-dependent Gaussian noises on G and ∂G, respectively.

In order to give an exact mathematical sense to (3.5)–(3.7) we proceed similarly to analogous cases [20], [21], [13], [11]. Treating the right-hand side of (3.7) as a "nice" sufficiently smooth function and introducing appropriate infinite-dimensional Wiener processes we derive an equation of the form (1.2) which is the mild form of the equation of the type (1.1) with suitably defined coefficients. Denote by $g_1 = g_1(t)$ the right hand side of (3.7). For $v(t) = y(t) + Dg_1(t)$ we have formally

$$\dot{v}(t) = -A(x, D)v(t) + D\dot{g}_1(t) + F\big(y(t)\big) + \Gamma\big(y(t)\big)\eta_1(t) \tag{3.8}$$

on $\mathbb{R}_+ \times G$ and

$$v(0) = y_0 + Dg_1(0), \quad \mathcal{B}v = 0 \quad \text{on } \mathbb{R}_+ \times \partial G. \tag{3.9}$$

The mild solution for the homogeneous problem is defined by

$$\begin{aligned} v(t) = S(t)\big(y_0 + Dg_1(0)\big) &+ \int_0^t S(t-r)D\dot{g}_1(r)\,\mathrm{d}r \\ &+ \int_0^t S(t-r)f\big(y(r)\big)\,\mathrm{d}r \\ &+ \int_0^t S(t-r)g\big(y(r)\big)\,\mathrm{d}W_r, \quad t \geq 0, \end{aligned} \tag{3.10}$$

where $f: H \to H$, $f(x)(\theta) := F\big(x(\theta)\big)$, $x \in H$, $\theta \in G$, W_r is an H-valued cylindrical Wiener process and $g: H \to \mathcal{L}(H)$,

$$[g(x)l](\theta) = \Gamma\big(x(\theta)\big) \cdot [Q_1^{1/2}(l)](\theta), \quad l \in H, \quad \theta \in G,$$

where $Q_1 \in \mathcal{L}(H)$ is the incremental covariance of the Wiener process corresponding to η_1. Integrating formally by parts in (3.10) and substituting the right-hand side of (3.8) for g_1 we arrive at

$$\begin{aligned} y(t) = S(t)y_0 &+ \int_0^t AS(t-r)Dh\big(y(r)\big)\,\mathrm{d}r \\ &+ \int_0^t S(t-r)f\big(y(r)\big)\,\mathrm{d}r \\ &+ \int_0^t S(t-r)g\big(y(r)\big)\,\mathrm{d}W_r \\ &+ \int_0^t AS(t-r)Dk\big(y(r)\big)\,\mathrm{d}V_r, \quad t \geq 0, \end{aligned} \tag{3.11}$$

where V_r is a cylindrical Wiener process on U and

$$k: H \to \mathcal{L}(U), \quad \big(k(x)l\big)(\theta) = K(x)(\theta) \cdot [Q_2^{1/2} l](\theta), \quad l \in U, \quad \theta \in \partial G,$$

where $Q_2 \in \mathcal{L}(U)$ is the incremental covariance operator of the Wiener process corresponding to η_2. Setting $B = [D^* A^*]^* \in \mathcal{L}(U, D_A^{\varepsilon-1})$ we obtain the equation of the form (1.1) whose mild solution has been defined as a solution to (1.2).

It is well known that the assumption (A1) is satisfied since the domain G of the linear elliptic problem (3.4) is bounded. However, the assumptions (A2) and (A3) need not be fulfilled in general. They can be viewed as conditions on the noise in the domain G and the boundary ∂G, respectively. For example, if W_t and V_t are "genuine" H and U-valued Wiener processes (that is, Q_1 and Q_2 are nuclear) and $\varepsilon > 1/2$ then (A2) and (A3), respectively, are always satisfied (see the note following (A3)).

Another important particular case when (A2) is satisfied is the case

$$(-A)^{\Delta - 1/2} \in \mathcal{L}_2(H)$$

for some $\Delta > 0$. This is true for $\dfrac{n}{4m} < \dfrac{1}{2}$ when we can take $0 < \Delta < \dfrac{1}{2} - \dfrac{n}{4m}$.

Similarly, (A3) is always fulfilled if $(-A)^{\gamma + 1/2 - \varepsilon} \in \mathcal{L}_2(H)$ for some $\gamma > 0$ which is true for $\varepsilon > \dfrac{n}{4m} + \dfrac{1}{2}$ in which case we take

$$0 < \gamma < \varepsilon - \frac{n}{4m} - \frac{1}{2},$$

(cf. [12], Theorem 3.2.2).

EXAMPLE 3.2 (structurally damped plates with point and distributed random loading). The model is formally described by the equation

$$\begin{aligned} &u_{tt}(t,x) + \Delta^2 u(t,x) - \rho \Delta u_t(t,x) = F\big(u(t,x), u_t(t,x)\big) \\ &\quad + \Gamma\big(u(t,x))\big) N(t,x) + \big(h\big(u(t,\cdot), u_t(t,\cdot)\big) \\ &\quad + K\big(u(t,\cdot), u_t(t,\cdot)\big)\eta(t)\big)\delta(x - x_0), \quad (t,x) \in \mathbb{R}_+ \times G, \end{aligned} \tag{3.12}$$

with the initial and boundary conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in G, \tag{3.13}$$

$$u\Big|_{\mathbb{R}_+ \times \partial G} = \Delta u\Big|_{\mathbb{R}_+ \times \partial G} = 0, \tag{3.14}$$

where $G \subset \mathbb{R}^n$ is a bounded open domain with a smooth boundary ∂G, the functions $F: \mathbb{R}^2 \to \mathbb{R}$, $\Gamma: \mathbb{R}^2 \to \mathbb{R}$,

$$h: \big(H^2(G) \cap H_0^1(G)\big) \times L_2(G) \to \mathbb{R},$$

$$K: \big(H^2(G) \cap H_0^1(G)\big) \times L_2(G) \to \mathbb{R}$$

are Lipschitz continuous, Γ is bounded, $N(t,x)$ and η_t are stochastically independent, Gaussian noises on G and in $\mathbb{R}$, respectively, $\delta(x-x_0)$ stands for the Dirac function at $x_0 \in G$ and $\rho > 0$ is a constant. In order to give an exact sense to (3.12)–(3.14) we rewrite the second order equation (3.12) as a first order system for $X(t) = \big(x(t), y(t)\big) = \big(u(t,\cdot), u_t(t,\cdot)\big)$ in a suitable state space. Set

$$\mathcal{A}h = \Delta^2 h, \quad \mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(G); h\Big|_{\partial G} = \Delta h\Big|_{\partial G} = 0 \right\}$$

and for the basic Hilbert space H take

$$H = \mathcal{D}(\mathcal{A}^{1/2}) \times L_2(G) = \big(H^2(G) \cap H_0^1(G)\big) \times L_2(G).$$

The operator A defined by

$$A = \begin{pmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{1/2} \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}),$$

is a generator of a stable analytic semigroup on H (cf. [5], Theorem 1.2). Proceeding similarly to the deterministic case (cf. [5], [24], and the references therein) we can rewrite the system (3.12)–(3.14) in the form

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + f\big(X(t)\big) + Bh\big(X(t)\big)\,\mathrm{d}t + g\big(X(t)\big)\,\mathrm{d}W_t \\ &\quad + Bk\big(X(t)\big)\,\mathrm{d}V_t, \quad t \geq 0, \quad X(0) = (u_0, u_1) \in H, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} &f: H \to H, \quad f(x,y)(\theta) = \begin{pmatrix} 0 \\ F\big(x(\theta), y(\theta)\big) \end{pmatrix}, \\ &(x,y) \in H, \quad \theta \in G, \end{aligned} \tag{3.15}$$

$$B\alpha = \alpha \begin{pmatrix} 0 \\ \delta(x - x_0) \end{pmatrix}, \quad \alpha \in \mathbb{R} = U, \tag{3.16}$$

$$g: H \to \mathcal{L}(H), \quad g(x,y)[h_1,h_2](\theta) = \begin{pmatrix} 0 \\ \Gamma\big(x(\theta)\big) \cdot Q^{1/2}(h_1,h_2)(\theta) \end{pmatrix}, \tag{3.17}$$

and $Q \in \mathcal{L}\big(L_2(G)\big)$ is the incremental covariance operator of an $L_2(G)$-valued Wiener process corresponding to $N(t,x)$. Finally, we put

$$k: H \to \mathcal{L}(\mathbb{R}); \quad k(x,y)\alpha = K\big(x(\cdot), y(\cdot)\big)\alpha, \quad \alpha \in \mathbb{R},$$

and let W_t and V_t be cylindrical Wiener processes on H and $\mathbb{R}$, respectively.

It has been proved in [6], Theorem 1.1, that $B \in \mathcal{L}(U, D_A^{\varepsilon-1})$ for $\varepsilon < 1-n/4$, thus we can satisfy our assumptions with an $\varepsilon > 0$ provided $n \leq 3$. Compactness of the resolvent of A (the assumption (A1)) has been proved in [5], Lemma A1. As in the previous Example the assumptions (A2) and (A3) can be viewed as conditions on the corresponding noise terms. For example, as follows from the note following (A3), if Q is nuclear then (A2) is always satisfied. Also, from the characterization

$$D_A^{\alpha} = D_{\mathcal{A}}^{1/2+\alpha/2} \times D_{\mathcal{A}}^{\alpha/2}, \quad \alpha \in \mathbb{R},$$

(cf. [6]) we obtain that if the space dimension n is one then the operator $(A)^{\Delta-1/2}$ is Hilbert–Schmidt on H with any $0 < \Delta < 1/4$ and, consequently, (A2) holds with any covariance $Q \in \mathcal{L}\big(L_2(G)\big)$, Q symmetric and nonnegative.

The condition (A3), for example, is fulfilled with no further restriction on K for $\varepsilon > 1/2$ which however may be satisfied together with the above condition $\varepsilon < 1 - n/4$ only for $n = 1$.

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