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Degenerate Variational Inequalities with Gradient Constraints

HI JUN CHOE - YONG SUN SHIM *

1. - Introduction

In this paper we consider degenerate variational inequalities for certain convex function classes which involve gradient constraints. We show the existence and $C^{1,\alpha}$ regularity for the solutions of degenerate variational inequalities with some general constraints on gradients. These variational inequalities arise in elasto-plasticity and optimal control problems. A typical example is the minimization problem

$$\min \int_{\Omega} (|\nabla u|^p - fu)dx, \ \ 1 < p < \infty$$

with respect to a function class $K = \{u \in W_0^{1,p}(\Omega) + u_0 : G(\nabla u) \leq 0\}$, where G is a convex function.

Without any restriction on the test function class, the regularity questions of the solutions of degenerate elliptic equations of p-Laplacian type have been considered by many people. Uhlenbeck [Uhl] proved $C^{1,\alpha}$ regularity for p-Laplacian system when $p > 2$. Lewis [Lew], Di Benedetto [DiB] and Tolksdorff [To1] proved $C^{1,\alpha}$ regularity for p-Laplacian equation for all $1 < p < \infty$. Employing a comparison argument, Toklsdorff [To2] proved $C^{1,\alpha}$ regularity for p-Laplacian system for all $1 < p < \infty$.

There have also been many studies for obstacle problems of degenerate elliptic equations (see Ziemer and Michael [Mic], Choe [Ch1], Lieberman [Li1], Mu [Mu1], Lindquist [Lind] etc.).

On the other hand, many people have studied strongly elliptic gradient constraint problems. In particular, Brezis and Stampacchia [Br2] considered the test function class $\{|\nabla u| \leq 1\}$ and proved $W^{2,p}$ regularity for solutions. We

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also recall that Gerhardt proved $W^{2,p}$ regularity for a quasilinear operator with strong ellipticity. In [Eva], Evans studied the problem of solving a linear second order elliptic variational inequality with a function class $K = \{|\nabla u| \leq g\}$ for some smooth function g. He proved $W^{2,p}$ regularity for solutions and $W^{2,\infty}$ regularity for restricted cases. His result for $W^{2,\infty}$ regularity was extended by Wiegner [Wi]. Ishii and Koike [IK] also studied the existence and uniqueness of the solutions of variational inequalities of the forms which are considered by Evans. Caffarelli and Riviere [Caf] proved $W^{2,\infty}$ regularity for elasto-plastic problems such as the example given above using a priori estimates on the free boundary. Finally Choe and Shim [Ch2] showed the existence and $C^{1,\alpha}$ regularity for a quasilinear operator under some general setting on the constraint.

Now we state the problem.

Suppose that

$$G(A) : \mathbb{R}^n \to \mathbb{R}$$

is a C^2 convex function and strictly convex on A such that

$$G_{A_iA_j}(A)\xi_i\xi_j \geq c|\xi|^2 \tag{1}$$

for all A, $\xi \in \mathbb{R}^n$ and for some positive constant c. This convexity is necessary for regularity of viscosity solutions of a certain Hamilton-Jacobi equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^3 boundary. Let u_0 be a $C^2(\overline{\Omega})$ function and assume

$$G(\nabla u_0(x)) \leq 0$$

for all $x \in \overline{\Omega}$. Let K be the closed convex function class defined by

$$K = \{v \in W_0^{1,p}(\Omega) + u_0 : G(\nabla v) \leq 0\}$$

that is nonempty since $u_0 \in K$.

Suppose that $\{a_i : i = 1, \ldots, n\}$ are functions

$$a_i(x, u, A) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \ \ i = 1, \ldots, n$$

satisfying:

i) $a_i(x, u, A)$ are C^1 function in $A \in \mathbb{R}^n$ for all $u \in \mathbb{R}$ and for all $x \in \Omega$ with the degenerate ellipticity condition

$$a_{i,A_j}(x, u, A)\xi_i\xi_j \geq \lambda|A|^{p-2}|\xi|^2, \ \ 1 < p < \infty$$

for some positive constant λ and for all $x \in \Omega$, for all $u \in \mathbb{R}$ and for all A, $\xi \in \mathbb{R}^n$.

ii) $a_i(x, u, A)$ are Hölder continuous in x for all (u, A), that is,

$$|a_i(x, u, A) - a_i(y, u, A)| \leq c|x - y|^\alpha$$

for all x, $y \in \Omega$, for some $\alpha > 0$, c and for all $(u, A) \in \mathbb{R} \times \mathbb{R}^n$.

iii) $a_i(x, u, A)$ are Hölder continuous in u for all A and for all $x \in \Omega$, that is,

$$|a_i(x, u, A) - a_i(x, v, A)| \leq c|u - v|^\alpha$$

for some $\alpha > 0$, for some c, for all $A \in \mathbb{R}^n$, and for all u, $v \in \mathbb{R}$ and for all $x \in \Omega$.

Suppose that

$$b(x, u, A) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

is bounded if $|(u, A)|$ is bounded, that is, if $|(u, A)| \leq M$, then there is a constant $c(M)$ such that

$$|b(x, u, A)| \leq c(M)$$

for almost all $x \in \Omega$.

Suppose that

$$g_i(x, u) : \Omega \times \mathbb{R} \to \mathbb{R}, \quad i = 1, 2, \ldots, n$$

are Hölder continuous such that

$$|g_i(x, u) - g_i(y, v)| \leq c(|x - y|^\alpha + |u - v|^\alpha), \quad i = 1, \ldots, n.$$

for some c and for some $\alpha > 0$.

Note that we do not assume any growth condition on a_i as $|(u, A)|$ goes to ∞.

We say that $u \in K$ is a weak solution to

$$-(a_i(x, u, \nabla u))_{x_i} - (g_i(x, u))_{x_i} + b(x, u, \nabla u) \geq 0 \tag{2}$$

if u satisfies

$$\int_\Omega a_i(x, u, \nabla u)(v - u)_{x_i} + g_i(x)(v - u)_{x_i} + b(x, u, \nabla u)(v - u)dx \geq 0$$

for all $v \in K$.

The following theorem is our main result in this paper.

THEOREM. *There exists a weak solution $u \in K$ to* (2). *Furthermore,*

$$u \in C^{1,\alpha}(\overline{\Omega})$$

for some $\alpha > 0$.

For the proof, we follow the idea of [Ch2]. The interior $C^{1,\alpha}$ regularity in Theorem 1 is proved employing a comparison method suitable for using Campanato space techniques. Indeed, a nice integral estimate for p-Laplacian function was proved by Lieberman [Li2] and used for regularity problems of degenerate obstacle problems (see [Ch1], [Li1]). We consider a comparison function which is a solution to a homogeneous differential equation of the p-Laplacian form with the same type of constraint. Then a suitable perturbation technique such a [Gia] is used for interior $C^{1,\alpha}$ regularity. In fact, a $C^{1,\alpha}$ comparison function is constructed by considering a bilateral obstacle problem. The existence and uniqueness property of the solution of bilateral obstacle problem follows from the penalization and the theory of monotone operators. Among others things, we first show that a solution of the bilateral obstacle problem is $C^{1,\alpha}$, where the obstacles are defined by the solutions of the vanishing viscosity equations. Sending the viscosity term to zero, we conclude that the bilateral obstacles converge uniformly to the viscosity solutions to certain Hamilton-Jacobi equations. In fact, the Perron process for the viscosity solutions of Hamilton-Jacobi equations, discovered by Ishii [Is2], characterizes the upper and lower envelopes for the function class K. Furthermore, the semiconcavity and semiconvexity regularity for the viscosity solutions to Hamilton-Jacobi equations is translated to $C^{1,\alpha}$ regularity in the interior to the solutions of the bilateral obstacle problems. We then use a maximum principle to show that the solution to the bilateral obstacle problem, where obstacles are characterized by the viscosity solutions to certain Hamilton-Jacobi equations, is the solution to the variational inequality with a nice differential operator. A comparison argument then shows that the solution to (2) is $C^{1,\alpha}$ in the interior.

Near the boundary, similarly, we follow a comparison argument in which comparison functions come from the variational inequalities with a nice differential operator. We show by the maximum principle that the solution to the variational inequality with respect to K is the solution to the bilateral obstacle problem. It is important to recall that the boundary regularity result of Krylov [Kry] for non-divergent equation is exploited to prove boundary regularity of the solutions of degenerate elliptic obstacle problems (see [Li3] and [Lin]). For the regularity of the comparison functions, we use the fact that, near the boundary, the viscosity solution to Hamilton-Jacobi equation can be characterized using the characteristic method if the boundary and the boundary data are smooth enough, that is C^3. Hence C^2 regularity for the viscosity solutions near the boundary follows immediately. Combining Krylov's boundary regularity result and smoothness of viscosity solutions near the boundary, we show again that the solution to the bilateral obstacle problem is a $C^{1,\alpha}$ function near the boundary. Hence we can proceed to show that the solution u has a Campanato type growth condition near the boundary by the usual comparison argument.

Once we have a priori $C^{1,\alpha}(\overline{\Omega})$ regularity, the existence result follows from Leray-Schauder's fixed point theorem.

The following symbols will be used.

x_0: a generic point,

$|E|$: the Lebesgue measure of E,

$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$,

$$(w)_{R,x_0} = \frac{1}{|B_R(x_0) \cap \Omega|} \int_{B_R(x_0)\cap\Omega} w dx,$$

$W^{1,p}(\Omega)$: the Sobolev space with L^p norm,

$W_0^{1,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$,

$||w||_L$: the L-norm of w in Ω,

w_η: the directional derivative of w along η.

We drop out the generic point x_0 in various expressions when there is no confusion. As usual double indices mean summation up to n.

2. - Interior $C^{1,\alpha}$ regularity for simple cases

We transform the gradient constraint problem to a bilateral obstacle problem. To achieve this, the solutions to vanishing viscosity equations are used to approximate the bilateral obstacle problems where obstacles are defined using the viscosity solutions to vanishing viscosity equations. Indeed, sending the viscosity term to zero, we prove the local $C^{1,\alpha}$ regularity when obstacles are solutions to certain Hamilton-Jacobi equations.

Suppose that $B_R \subset \Omega$ and $w_0 \in W^{1,\infty}(B_R)$ with $G(\nabla w_0) \leq 0$. Fix μ and ε as small positive numbers. Then there exists a unique solution $w = w^{+,\mu,\varepsilon}$ to the vanishing viscosity equation

$$L^{+,\mu,\varepsilon}(w) = -\varepsilon\Delta w + G(\nabla w) = 0 \tag{3}$$

with $w = w_0 + \mu$ on ∂B_R. The existence and uniqueness of such solutions to vanishing viscosity equations are known by a result of Lions (see [Lio2]).

LEMMA 1. *There exists a unique Lipschitz solution w to* (3) *with*

$$||w||_{W^{1,\infty}} \leq c$$

for some c independent of μ and ε. Furthermore for all $x \in B_{(1-\delta)R}$, w is semiconcave, that is,

$$\frac{\partial^2 w}{\partial \varsigma^2}(x) \leq c, \text{ for all } \varsigma \in \mathbb{R}^n \text{ with } |\varsigma| \leq 1 \tag{4}$$

in the sense of distribution for some c independent of μ and ε.

One can find the proof of Lemma 1 in Theorem 2.2 of [Lio1]. Observe that the strict convexity condition

$$G_{A_iA_j}(A)\xi_i\xi_j \geq c|\xi|^2$$

for some $c > 0$ and for all $\xi \in \mathbb{R}^n$ is needed to assure the semiconcavity result (4) for the viscosity solutions. Let $w^{+,\mu,0}$ be the viscosity solution to

$$G(\nabla w^{+,\mu,0}) = 0$$

with the boundary condition $w^{+,\mu,0} = w_0 + \mu$ on ∂B_R.

The existence of such a viscosity solution $w^{+,\mu,0}$ can be proved by the Perron process [Is1] or sending ε to zero in (3). When $\min G < 0$, we have a comparison principle proved by Ishii (see Theorem 1 in [Is2]) and the uniqueness follows immediately. If $\min G = 0$, then the uniqueness follows from the fact that the minimizer of G is unique. The following lemma is in [Lio1].

LEMMA 2. *As ε goes to zero, $w^{+,\mu,\varepsilon}$ converges to $w^{+,\mu,0}$ uniformly in $C(\overline{B_{(1-\delta)R}})$ for each $\delta > 0$ and $w^{+,\mu,0} \geq v$ for all viscosity subsolutions v such that*

$$G(\nabla v) \leq 0$$

with $v = w_0 + \mu$ on ∂B_R.

Similarly, we can find $w^{-,\mu,\varepsilon}$ as the solution to the vanishing viscosity equation

$$-\varepsilon \Delta w - G(\nabla w) = 0$$

where $w = w_0 - \mu$ on ∂B_R for each positive μ and ε. Hence we get the following lemma.

LEMMA 3. *As ε goes to zero, $w^{-,\mu,\varepsilon}$ converges uniformly in $C(\overline{B_{(1-\delta)R}})$ for each $\delta > 0$ to the viscosity solution $w^{-,\mu,0}$ of*

$$-G(\nabla w^{-,\mu,0}) = 0$$

with $w^{-,\mu,0} = w_0 - \mu$ on ∂B_R and

$$\|w^{-,\mu,\varepsilon}\|_{W^{1,\infty}} \leq c$$

for some c independent of ε and μ. Furthermore, for all $x \in B_{(1-\delta)R} w$ is semiconvex

$$\frac{\partial^2 w^{-,\mu,\varepsilon}}{\partial \varsigma^2}(x) \geq c, \text{ for all } |\varsigma| \leq 1$$

in the sense of distribution for some c independent of μ and ε.

Since $w^{-,\mu,0}$ is Lipschitz and $-G(\nabla w^{-,\mu,0}) = 0$ in the viscosity sense, we see that $G(\nabla w^{-,\mu,0}) = 0$ a.e. and hence by the convexity of G that $G(\nabla w^{-,\mu,0}) \leq 0$ in the viscosity sense. By the comparison argument as in lemma 2 we see that each given $\mu > 0$,

$$w^{-,\mu,\varepsilon} < w^{+,\mu,\varepsilon} \text{ and } w^{-,\mu,0} < w^{+,\mu,0}$$

if ε is sufficiently small compared to μ.

We define the function class $J_R^{\mu,\varepsilon}$ by

$$J_R^{\mu,\varepsilon} = \{v \in W_0^{1,2}(B_R) + w_0 : w^{-,\mu,\varepsilon} \leq v \leq w^{+,\mu,\varepsilon}\}$$

for each given positive μ and ε. Note that $w_0 \in J_R^{\mu,\varepsilon}$. Since $w^{-,\mu,\varepsilon}$ and $w^{+,\mu,\varepsilon}$ converge uniformly to $w^{-,\mu,0}$ and $w^{+,\mu,0}$ respectively, we see that for given $\mu > 0$

$$w^{-,\mu,\varepsilon} < w^{+,\mu,\varepsilon}$$

for sufficiently small $\varepsilon > 0$ compared to μ.

Now we consider a bilateral obstacle problem. We freeze coefficients. For notational convenience, we write

$$-(a_i(\nabla w))_{x_i} = -(a_i(x_0, w_0(x_0), \nabla w))_{x_i}$$

and assume that $w = w^{\mu,\varepsilon} \in J_R^{\mu,\varepsilon}$ is the solution to the bilateral obstacle problem

$$-(a_i(\nabla w))_{x_i} \geq 0$$

with respect to the function class $J_R^{\mu,\varepsilon}$. The following lemma is essential in our comparison argument.

LEMMA 4. *If $B_R \subset \Omega$ and $\rho \leq \frac{R}{4}$, then the estimate*

$$\int_{B_\rho} |\nabla w - (\nabla w)_\rho|^p dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w - (\nabla w)_R|^p dx + cR^{n+\sigma} \tag{5}$$

holds for some c independent of μ and ε and for some $\sigma \in (0, p)$. In particular, ∇w is locally Hölder continuous in B_R with

$$\|w\|_{C^{1,\alpha}_{\text{loc}}} \leq c$$

for some c independent of μ and ε.

PROOF. We follow a penalization method. Let β_1 be a nondecreasing smooth function such that

$$\begin{aligned} &0 < \beta_1(t) < 5t, \text{ if } t > 0 \\ &\beta_1(t) = 0, \text{ if } t \leq 0 \\ &\beta_1'(t) \geq 0 \\ &\beta_1''(t) \geq 0. \end{aligned}$$

Similarly we define β_2 by a nonincreasing smooth function satisfying

$$5t < \beta_2(t) < 0, \text{ if } t < 0$$
$$\beta_2(t) = 0, \text{ if } t \geq 0$$
$$\beta_2'(t) \geq 0$$
$$\beta_2''(t) \leq 0.$$

We approximate our problem. Fix $p \in \mathbb{R}^n$ so that $G(p) \leq 0$ and let $0 < \theta < 1$. Define $w_\theta(x) = \theta p \cdot x + w_0((1-\theta)x)$. Then w_θ is defined in $B_{\frac{R}{1-\theta}}$ and satisfies $G(\nabla w_\theta) \leq 0$ a.e. Also, $w_\theta \to w_0$ uniformly in B_R as $\theta \to 0$. Moreover we approximate w_θ by a smooth function using the mollification technique and we denote the smooth function again w_θ. Let $v = v_\theta^{\mu,\varepsilon,\tau} \in W_0^{1,p}(B_R) + w_\theta$ be the solution to the penalized equation

$$L^{\mu,\varepsilon,\tau}(v) = -(a_i(\nabla v))_{x_i} + \frac{\beta_1(v - w^{+,\mu,\varepsilon})}{\tau} + \frac{\beta_2(v - w^{-,\mu,\varepsilon})}{\tau} = 0 \tag{6}$$

for a given small positive number τ. The existence and uniqueness can be proved from the monotone operator theory (see [Har]). Since all the following estimate is independent of θ, we omit θ in the various expressions from now on. Taking some large number c so that

$$\|w^{+,\mu,\varepsilon}\|_{W^{1,\infty}(B_R)} + \|w^{-,\mu,\varepsilon}\|_{W^{1,\infty}(B_R)} \leq c$$

as a supersolution to the operator $L^{\mu,\varepsilon,\tau}$, we can prove that v is bounded from above. In a similar way v can be shown to be bounded from below. Hence we conclude that

$$\|v\|_{L^\infty} \leq c \tag{7}$$

for some c independent of μ, ε and τ.

Next we estimate the Lipschitz norm of v. Let $\rho = \operatorname{dist}(x, \partial B_R)$ and $\phi^+ = w_0 + \nu\rho - \nu\rho^{\frac{3}{2}}$ for some large ν. We know already that

$$|\nabla\rho| = 1 \text{ and } |\nabla^2\rho| \leq c$$

near the boundary of B_R. Therefore we see that

$$\phi^+ = v \text{ on } \partial B_R \text{ and } \phi^+ \geq v \text{ on } \partial B_{(1-\delta)R}$$

for some small δ when ν is sufficiently large. Since for some large ν,

$$\phi^+(x) \geq w^{+,\mu,\varepsilon},$$

$$\frac{\beta_1(v - w^{+,\mu,\varepsilon})}{\tau} + \frac{\beta_2(v - w^{-,\mu,\varepsilon})}{\tau} \geq 0$$

and

$$-a(\nabla\phi^+)_{i,A_j}\phi^+_{x_ix_j} \geq 0,$$

we see that

$$L^{\mu,\varepsilon,\tau}(\phi^+) \geq 0$$

near the boundary of B_R, that is, ϕ^+ is a supersolution to $L^{\mu,\varepsilon,\tau}$ and we conclude that

$$v \leq \phi^+$$

near the boundary of B_R. Similarly we have

$$v \geq \phi^-$$

for some ϕ^- near the boundary of B_R and we see that

$$\|\nabla v\|_{L^\infty(\partial B_R)} \leq c$$

for some c independent of μ, ε and τ. Since $L^{\mu,\varepsilon,\tau}$ is a monotone operator, then $|\nabla v|^2$ satisfies the weak maximum principle. Hence it follows that

$$\|\nabla v\|_{L^\infty(B_R)} \leq c \tag{8}$$

for some c independent of μ, ε and τ.

Now we apply the L^q-theory for degenerate elliptic equation. We first prove that

$$\left\|\frac{\beta_1}{\tau}\right\|_{L^q} + \left\|\frac{\beta_2}{\tau}\right\|_{L^q} \leq c \tag{9}$$

for all $q \in (1,\infty)$ and for some c independent of μ, ε, τ and q. With this L^q estimate on $\dfrac{\beta_1}{\tau}$ and $\dfrac{\beta_2}{\tau}$, we conclude that v is in $C^{1,\alpha}_{\text{loc}}(B_R)$ and ∇v is in C^α for some $\alpha \in (0,1)$ with Hölder norm independent of μ, ε and τ. To see (7), let us choose a nonnegative smooth cutoff function η so that

$$\eta = 1 \text{ in } B_{(1-\delta)R}, \quad \eta = 0 \text{ in } \partial B_{(1-\delta/2)R}$$

and

$$|\nabla\eta| \leq \frac{c}{\delta R}$$

for some c and appropriate δ.

Applying $\left(\dfrac{\beta_1}{\tau}\right)^{q-1}\eta^q$ as a test function to (6), we get

$$\begin{aligned}\int a_i(\nabla v)\left[\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^p\right]_{x_i} dx + \int\left(\frac{\beta_1}{\tau}\right)^q \eta^q dx \\ + \int\left(\frac{\beta_2}{\tau}\right)\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q dx = 0.\end{aligned} \tag{10}$$

Note that

$$\left(\frac{\beta_1}{\tau}(t)\right)\left(\frac{\beta_2}{\tau}(t)\right)=0$$

for all t. Subtracting

$$\int a_i(\nabla w^{+,\mu,\varepsilon})\left[\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q\right]_{x_i}dx$$

from the both sides of (10), we have

$$\begin{aligned}(11)\qquad &(q-1)\int[a_i(\nabla v)-a_i(\nabla w^{+,\mu,\varepsilon})](v-w^{+,\mu,\varepsilon})_{x_i}\left(\frac{\beta_1}{\tau}\right)^{q-2}\frac{\beta_1'}{\tau}\eta^q dx\\ &+q\int[a_i(\nabla v)-a_i(\nabla w^{+,\mu,\varepsilon})]\eta_{x_i}\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^{q-1}dx+\int\left(\frac{\beta_1}{\tau}\right)^q\eta^q dx\\ &=\int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon})w^{+,\mu,\varepsilon}_{x_ix_j}\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q dx.\end{aligned}$$

From the ellipticity condition for a_i we see that

$$\int[a_i(\nabla v)-a_i(\nabla w^{+,\mu,\varepsilon})](v-w^{+,\mu,\varepsilon})_{x_i}\left(\frac{\beta_1}{\tau}\right)^{q-2}\frac{\beta_1'}{\tau}\eta^q dx\geq 0.$$

Therefore we have

$$\begin{aligned}(12)\qquad \int\left(\frac{\beta_1}{\tau}\right)^q\eta^q dx\leq \frac{c}{\delta R}&\int(\|\nabla v\|_{L^\infty}+\|\nabla w^{+,\mu,\varepsilon}\|_{L^\infty})^{p-1}\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^{q-1}dx\\ &+\int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon})w^{+,\mu,\varepsilon}_{x_ix_j}\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q dx.\end{aligned}$$

Since $w^{+,\mu,\varepsilon}$ is semiconcave (see Lemma 1), we have

$$w^{+,\mu,\varepsilon}_{x_ix_j}\xi_i\xi_j\leq c|\xi|^2$$

for some c independent of μ and ε. Moreover, a_{i,A_j} is a positive semidefinite matrix. Therefore,

$$a_{i,A_j}(\nabla w^{+,\mu,\varepsilon})w^{+,\mu,\varepsilon}_{x_ix_j}\leq c_1$$

for some c_1 independent of μ and ε. It follows that

$$\begin{aligned}(13)\qquad &\int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon})w^{+,\mu,\varepsilon}_{x_ix_j}\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q dx\\ &\leq\int c_1\left(\frac{\beta_1}{\tau}\right)^{q-1}\eta^q dx\leq\frac{1}{4}\int\left(\frac{\beta_1}{\tau}\right)^q\eta^q dx+cR^n\end{aligned}$$

for some c. Using Young's inequality on the first term of the right hand side of (12) and the estimate (13) we have

$$\left\| \frac{\beta_1}{\tau} \right\|_{L^q(B_{(1-\delta)R})} \le c$$

for some c independent of μ, ε and τ. Similarly we also have

$$\left\| \frac{\beta_2}{\tau} \right\|_{L^q(B_{(1-\delta)R})} \le c$$

for some c independent of μ, ε and τ. This proves (7).

Now we prove (5). Suppose $u^{\mu,\varepsilon,\tau} \in W_0^{1,p}(B_{\frac{R}{2}}) + v^{\mu,\varepsilon,\tau}$ is the solution to

$$-(a_i(\nabla u))_{x_i} = 0.$$

Then we have the following integral estimate

$$\int_{B_\rho} |\nabla u^{\mu,\varepsilon,\tau} - (\nabla u^{\mu,\varepsilon,\tau})_\rho|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla u^{\mu,\varepsilon,\tau} - (\nabla u^{\mu,\varepsilon,\tau})_R|^2 dx + cR^{n+\sigma}$$

for some σ and all $0 < \rho < \dfrac{R}{4}$ (see [Ch1] or [Li2]). From ellipticity condition we have

$$\begin{aligned} \int_{B_{\frac{R}{2}}} (|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v|^2 dx &\le c \int_{B_{\frac{R}{2}}} (a_i(\nabla u) - a_i(\nabla v))(\nabla u - \nabla v) dx \\ &\le \int_{B_{\frac{R}{2}}} \left(\left|\frac{\beta_1}{\tau}\right| + \left|\frac{\beta_1}{\tau}\right| \right) |u - v| dx. \end{aligned}$$

Hence from Poincaré inequality and Hölder inequality we obtain

$$\int_{B_{\frac{R}{2}}} |\nabla u - \nabla v|^p dx \le cR^{n+\nu}$$

for some c depending on $\left\| \frac{\beta_1}{\tau} \right\|_{L^q(B_{\frac{R}{2}})}$ and $\left\| \frac{\beta_1}{\tau} \right\|_{L^q(B_{\frac{R}{2}})}$, where

$$\nu = n + \frac{p}{p-1}\left(1 - \frac{n}{q}\right) \text{ when } p \ge 2$$

and

$$\nu = n + p\left(1 - \frac{n}{q}\right) \text{ when } 1 < p < 2.$$

From usual comparison argument we have immediately the following Campanato growth condition for v

$$\int_{B_\rho} |\nabla v^{\mu,\varepsilon,\tau} - (\nabla v^{\mu,\varepsilon,\tau})_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla v^{\mu,\varepsilon,\tau} - (\nabla v^{\mu,\varepsilon,\tau})_R|^p dx + cR^{n+\sigma}$$

for some $\sigma \in (0,p)$, for all $\rho \le \frac{R}{2}$ and for some c independent of μ, ε and τ.

Sending τ to zero and using Minty's lemma ([Chi]), we conclude that the unique solution to (6) is in $C^{1,\alpha}_{\mathrm{loc}}$ and satisfies the Campanato type growth condition

$$\int_{B_\rho} |\nabla w^{\mu,\varepsilon} - (\nabla w^{\mu,\varepsilon})_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w^{\mu,\varepsilon} - (\nabla w^{\mu,\varepsilon})_R|^p dx + cR^{n+\sigma}$$

for some $\sigma \in (0,p)$, for all $\rho \le \frac{R}{2}$ and for some c independent of μ and ε. □

Sending ε to zero we have that the unique solution $w^{\mu,0}$ to the variational inequality

$$-(a_i(\nabla w^{\mu,0}))_{x_i} \ge 0 \tag{14}$$

with respect to $J_R^{\mu,0}$ is in $C^{1,\alpha}_{\mathrm{loc}}$ and satisfies the Campanato type growth condition (5).

Set w^+ as the viscosity solutions to Hamilton-Jacobi equation

$$G(\nabla w^+) = 0, \;\; w^+ = w_0 \text{ on } \partial B_R$$

and w^- as the viscosity solution to

$$-G(\nabla w^-) = 0, \;\; w^- = w_0 \text{ on } \partial B_R.$$

Also set $J_R = \{v \in W_0^{1,\infty}(B_R) + w_0 : w^- \le v \le w^+\}$. Suppose w is the unique solution to the variational inequality

$$-(a_i(\nabla w))_{x_i} \ge 0 \tag{15}$$

with respect to J_R. Then it is easy to see that

$$w^{\mu,0} \to w \text{ uniformly in } C^{1,\alpha}_{\mathrm{loc}}$$

as μ goes to zero. Hence we have the following lemma.

LEMMA 5. *If w is the unique solution to* (15) *with respect to J_R, then w satisfies*

$$\int_{B_\rho} |\nabla w - (\nabla w)_\rho|^2 dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w - (\nabla w)_R|^2 dx + cR^{n+\sigma} \tag{16}$$

for some $\sigma \in (0,2)$, for all $\rho \leq \dfrac{R}{2}$ and for some c independent of ρ and R.

Now we want to show that the solution w to the variational inequality (15) with respect to J_R is indeed the unique solution to the variational inequality (15) with respect to a function class K_R, where K_R is defined by

$$K_R = \{v \in W_0^{1,\infty}(B_R) + w_0 : G(\nabla v) \leq 0\}.$$

We note that $K_R \subset J_R$. Hence if we show that $w \in K_R$, that is, $G(\nabla w) \leq 0$, then w is the solution to the variational inequality (15) with respect to K_R.

We define contact sets I_R^- and I_R^+ by

$$I_R^- = \{x \in B_R : w(x) = w^-(x)\}$$

and

$$I_R^+ = \{x \in B_R : w(x) = w^+(x)\}.$$

We also define I_R by

$$I_R = I_R^- \cup I_R^+.$$

Since G is a C^2 convex function, we have the following maximum principle.

LEMMA 6. *We have that*

$$\max_{B_R \backslash I_R} G(\nabla w) \leq 0. \tag{17}$$

Note that w satisfies the degenerate elliptic equation

$$-(a_i(\nabla w))_{x_i} = 0$$

in $B_R \backslash I_R$. Hence we see that that $G(\nabla w)$ is a subsolution to

$$-(a_{i,A_j}(\nabla w) G_{x_j})_{x_i} \leq 0$$

in $B_R \backslash I_R$. We omit this rather direct computation.

As in the proof of Lemma 4 we regularize w_0 by w_θ and the using the regularity result in section 3 we can assume that w_θ is differentiable on $\partial B_{\frac{R}{1-\theta}}$ and $G(\nabla w_\theta)$ is continuous in $\overline{B}_{\frac{R}{1-\theta}}$. Since all the estimate is independent of θ, we omit θ in various expressions. Since $w^+(x) = w^-(x) = w(x)$ for all $x \in \partial B_R$ and $w^-(x) \le w(x) \le w^+(x)$ for all $x \in B_R$, we have that

$$\frac{\partial w^+(x)}{\partial \eta} \le \frac{\partial w(x)}{\partial \eta} \le \frac{\partial w^-(x)}{\partial \eta}$$

for all $x \in \partial B_R$, where η is the outward normal vector at $x \in \partial B_R$. Since

$$\frac{\partial w^+(x)}{\partial \tau} = \frac{\partial w^-(x)}{\partial \tau} = \frac{\partial w(x)}{\partial \tau}$$

for all tangent vector τ at $x \in \partial B_R$, we have for each $x \in \partial B_R$

$$\nabla w(x) = t\nabla w^+(x) + (1-t)\nabla w^-(x)$$

for some $t \in [0,1]$. Since G is convex, we obtain

$$G(\nabla w(x)) \le tG(\nabla w^+(x)) + (1-t)G(\nabla w^-(x)) \le 0$$

for all $x \in \partial B_R$.

Now we show that

$$G(\nabla w(x)) \le 0$$

on ∂I_R. Recall that w^+ (resp. w^-) is semiconcave (resp. semiconvex). Then we find that for each $x \in I_R^+ \cap B_R$ (resp. $x \in I_R^- \cap B_R$) w^+ (resp. w^-) is differentiable. For instance, if $x \in I_R^+ \cap B_R$, then w^+ is superdifferentiable since it is semiconcave, and also subdifferentiable since w is in C^1 and $w^+ - w$ attains a minimum. Once we know the differentiablity we see that $G(\nabla w) \le 0$ on $I_R \cap B_R$, which is enough to apply the maximum principle. Indeed, if $x \in I_R^+ \cap B_R$, then $G(\nabla w^+(x)) = G(\nabla w(x)) = 0$ since w^+ is a viscosity solution. Similarly we see that $G(\nabla w^-(x)) = G(\nabla w(x)) = 0$ for $x \in I_R^- \cap B_R$.

3. - Boundary regularity for simple case

In this section we show that a Campanato type growth condition holds for solutions near the boundary for simple case.

Let $x_0 \in \partial\Omega$ and $\partial\Omega$ be C^3. Also let w_0 be a Lipschitz function in $B_R \cap \Omega$ and a C^2 function on $\partial\Omega \cap B_R$. From the Perron process we know that the viscosity solution w^+ to Hamilton-Jacobi equation

$$G(\nabla w^+) = 0, \ w^+ = w_0 \text{ on } \partial(\Omega \cap B_R) \tag{18}$$

can be characterized by

$$w^+(x) = \sup\{v(x) : v = w_0 \text{ on } \partial(\Omega \cap B_R),$$

$$v \text{ is a viscosity subsolution of } G(\nabla v) = 0\}.$$

For all subsolutions v of $G(\nabla v) = 0$ with $v \leq w_0$ on $\partial(B_R \cap \Omega)$, it holds that

$$w^+ \geq v.$$

Similarly we find a viscosity solution w^- to

$$-G(\nabla w^-) = 0, \;\; w^- = w_0 \text{ on } \partial(B_R \cap \Omega)$$

and for all supersolutions v of $-G(\nabla v) = 0$ with $v \geq w_0$ on $\partial(B_R \cap \Omega)$, it holds that

$$w^- \leq v.$$

When $\min G = 0$ the C^2 regularity of w^+ and w^- near $\partial\Omega$ is trivial. When $\min G < 0$, near $\partial\Omega$ we can compute w^+ by the method of characteristics and hence w^+ is $C^2(\overline{\Omega_\delta})$ for some small δ (see the appendix and Lemma 2.2 in [Fle]), where Ω_δ is defined by $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$.

Now we need a Krylov type boundary estimate for solutions of non-divergent elliptic equations (see [Li3] and Lemma 4.1 in [Lin]). Let $x_0 \in \partial\Omega$ and $\Delta_{x_0,r} = \partial\Omega \cap B_r(x_0)$.

LEMMA 7. *Let w be a solution of the following equation*

$$a_{ij}(x)w_{x_ix_j} = 0 \text{ in } B_R \cap \Omega, \;\; w = \phi \text{ on } \Delta_{x_0,R},$$

where $\phi \in C^{1,\alpha}$ and $[a_{ij}]$ is stricly positive matrix with bounded measurable coefficients. Then there exist $\mu = \mu(n, \lambda, \alpha, R)$, $c = c(n, \lambda, \alpha, R)$ and $A(y)$ defined on $B_{\frac{R}{2}}(x_0)$ such that

$$|w(x) - \phi(x) - A(y)\nu(y)(x - y)| \leq c(\|\nabla\phi\|_{C^\alpha})|x - y|^{1+\mu} \tag{19}$$

for any pair $(x, y) \in (B_R(y_0) \cap \Omega) \times \Delta_{x_0,R}$, where $\nu(y)$ is the unit inner normal of $\partial\Omega$ at y, and

$$|A(y)| \leq c(\|w - \phi\|_{L^\infty} + \|\nabla\phi\|_{C^\alpha}) \text{ for } y \in \Delta_{x_0,R}.$$

We prove a lemma which describes the size of the oscillation of the solution w near the boundary to the variational inequality

$$-(a_i(x_0, w_0(x_0), \nabla w))_{x_i} \geq 0 \tag{20}$$

with respect to $K_{R,x_0} = \{v \in W_0^{1,p}(B_R(x_0) \cap \Omega) + w_0 : G(\nabla v) \leq 0\}$.

LEMMA 8. *If $\rho \leq \dfrac{R}{2}$, then w satisfies*

$$\int_{B_\rho \cap \Omega} |\nabla w - (\nabla w)_\rho|^p dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla w - (\nabla w)_R|^p dx + cR^{n+\sigma} \tag{21}$$

for some c and some $\sigma \in (0, 1)$.

PROOF. We consider a bilateral obstacle problem as in section 2. We drop out the generic point x_0 in a_i as follows

$$a_i(A) = a_i(x_0, w_0(x_0), A).$$

Define a function class J_R by

$$J_R = \{v \in W_0^{1,2}(B_R \cap \Omega) + w_0 : w^- \leq v \leq w^+\}.$$

Let $v \in J_R$ be the unique solution to the variational inequality

$$-(a_i(\nabla v))_{x_i} \geq 0 \tag{22}$$

with respect to J_R. Again the existence and uniqueness for solution to (22) follow from the monotone operator theory.

Let v^- be the solution to

$$-(a_i(\nabla v^-))_{x_i} = -(a_i(\nabla w^-))_{x_i} \tag{23}$$

and suppose that $v - w_0 \in W_0^{1,p}(B_\delta \cap \Omega)$. Note that w_0 is C^2 near $B_\delta \cap \partial\Omega$ and w^- is $C^2(\overline{B_\delta \cap \Omega})$ for some small δ which we determine later. Since $v^- = w_0 \geq w^-$ on $\partial(B_\delta \cap \Omega)$, it follows from the maximum principle that

$$v^- \geq w^-$$

in $B_\delta \cap \Omega$.

With a typical regularization and the boundary estimate (20) in Lemma 7 an integral estimate of v^- near boundary can be proved. Note that a similar estimate appears in ([Li3]). Since $w^- \in C^2(\overline{B_\delta \cap \Omega})$ and $A(y)$ in Lemma 7 is Hölder continuous, we see that for small $\rho \leq \frac{\delta}{2}$,

$$\int_{B_\rho \cap \Omega} |\nabla v^- - (\nabla v^-)_\rho|^p dx \leq c \left(\frac{\rho}{\delta}\right)^{n+\sigma} \int_{B_\delta} |\nabla v^- - (\nabla v^-)_\delta|^p dx + c\delta^{n+\sigma}$$

for some c and for some $\sigma \in (0,1)$. Now it is evident that $\overline{v} := v^- \wedge w^+ \in J_\delta$ and is an admissible competing function to (22).

Now we assume $p \in [2, \infty)$. In this case we have

$$\begin{aligned}
\int_{B_\delta \cap \Omega} |\nabla v - \nabla v^-|^p dx &\le c \int_{B_\delta \cap \Omega} [a_i(\nabla v) - a_i(\nabla v^-)](v - v^-)_{x_i} dx \\
&= c \int_{B_\delta \cap \Omega} a_i(\nabla v)(v - v^- \wedge w^+)_{x_i} dx + c \int_{B_\delta \cap \Omega} a_i(\nabla v)(v^- \wedge w^+ - v^-)_{x_i} dx \\
&\quad - \int_{B_\delta \cap \Omega} a_i(\nabla v^-)(v - v^-)_{x_i} dx \\
&=: I + II + III.
\end{aligned} \tag{24}$$

We note that since $v^- \wedge w^+$ is an admissible competing function in J_δ,

$$I \le 0. \tag{25}$$

As in Lemma 3, it can be shown that

$$\|\nabla v\|_{L^\infty(B_\delta \cap \Omega)} \le c$$

for some c. Thus we see that

$$\begin{aligned}
II &\le c \int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+) - \nabla v^-| dx \\
&\le c\delta^{n\left(1-\frac{1}{p}\right)} \left(\int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^p dx \right)^{\frac{1}{p}}
\end{aligned} \tag{26}$$

for some c. We use the equation (23). Since $v^- \wedge w^+ - v^- \in W_0^{1,p}(B_\delta \cap \Omega)$, we have

$$\begin{aligned}
&\int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^p dx \\
&\le c \int_{B_\delta \cap \Omega} [a_i(\nabla v^-) - a_i(\nabla w^+)](v^- - v^- \wedge w^+)_{x_i} dx \\
&= c \int_{B_\delta \cap \Omega} [a_i(\nabla w^-) - a_i(\nabla w^-(x_0))](v^- - v^- \wedge w^+)_{x_i} dx \\
&\quad - c \int_{B_\delta \cap \Omega} [a_i(\nabla w^+) - a_i(\nabla w^+(x_0))](v^- - v^- \wedge w^+)_{x_i} dx
\end{aligned}$$

for some c. We note that since w^- and w^+ are C^2 near $\partial\Omega$,

$$|a_i(\nabla w^-) - a_i(\nabla w^-(x_0))|,\ |a_i(\nabla w^-) - a_i(\nabla w^+(x_0))| \le c\delta$$

for all $x \in B_\delta \cap \Omega$. Hence using Young's inequality we have

$$\int_{B_\delta\cap\Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^p dx \le c\delta^{n+\frac{p}{p-1}}$$

and

$$II \le c\delta^{n+\frac{1}{p-1}} \tag{27}$$

for some c. Similarly,

$$\begin{aligned} III &= -c \int_{B_\delta\cap\Omega} a_i(\nabla v^-)(v - v^-)_{x_i} dx \\ &= -c \int_{B_\delta\cap\Omega} [a_i(\nabla w^-) - a_i(\nabla w^-(x_0))](v - v^-)_{x_i} dx \\ &\le \frac{1}{4} \int_{B_\delta\cap\Omega} |\nabla v - \nabla v^-|^p dx + c\delta^{n+\frac{p}{p-1}} \end{aligned} \tag{28}$$

for some c.

Combining (24) through (28), we conclude that

$$\int_{B_\delta\cap\Omega} |\nabla v - \nabla v^-|^p dx \le c\delta^{n+\frac{p}{p-1}} \tag{29}$$

for some c.

Now we assume $p \in (1,2)$. In this case we have

$$\begin{aligned} &\int_{B_\delta\cap\Omega} (|\nabla v| + |\nabla v^-|)^{p-2} |\nabla v - \nabla v^-|^2 dx \\ &\le c \int_{B_\delta\cap\Omega} [a_i(\nabla v) - a_i(\nabla v^-)](v - v^-)_{x_i} dx \\ &= c \int_{B_\delta\cap\Omega} a_i(\nabla v)(v - v^- \wedge w^+)_{x_i} dx \\ &\quad + c \int_{B_\delta\cap\Omega} a_i(\nabla v)(v^- \wedge \qquad\qquad)_{x_i} dx \\ &\quad - \int_{B_\delta\cap\Omega} a_i(\nabla v^-)(v - v^-)_{x_i} dx =: I + II + III. \end{aligned} \tag{30}$$

As in the case of $p \in [2, \infty)$,

$$I \leq 0. \tag{31}$$

We also have

$$\begin{aligned} II &\leq c \int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+) - \nabla v^-| dx \\ &\leq c \left(\int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+)|^p + |\nabla v^-|^p dx \right)^{\frac{2-p}{2}} \\ &\quad \cdot \left(\int_{B_\delta \cap \Omega} (|\nabla(v^- \wedge w^+)| + |\nabla v^-|)^{p-2} |\nabla(v^- \wedge w^+) - \nabla v^-|^2 dx \right)^{\frac{p}{2}} \end{aligned} \tag{32}$$

for some c. Since

$$\int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+)|^p + |\nabla v^-|^p dx \leq c\delta^n$$

we have

$$II \leq c\delta^{\frac{n(2-p)}{2}} \left(\int_{B_\delta \cap \Omega} (|\nabla(v^- \wedge w^+)| + |\nabla v^-|)^{p-2} |\nabla v^- \wedge w^+ - \nabla v^-|^2 dx \right)^{\frac{p}{2}}.$$

From the structure of the equation and $v^- \wedge w^+ - v^- \in W_0^{1,p}(B_\delta \cap \Omega)$, we have

$$\begin{aligned} &\int_{B_\delta \cap \Omega} (|\nabla v^- \wedge w^+| + |\nabla v^-|)^{p-2} |\nabla(v^- \wedge w^+) - \nabla v^-|^2 dx \\ &\leq c \int_{B_\delta \cap \Omega} [a_i(\nabla v^-) - a_i(\nabla w^+)](v^- - v^- \wedge w^+)_{x_i} dx \\ &= c \int_{B_\delta \cap \Omega} [a_i(\nabla w^-) - a_i(\nabla w^-(x_0))](v^- - v^- \wedge w^+)_{x_i} dx \\ &\quad - c \int_{B_\delta \cap \Omega} [a_i(\nabla w^+) - a_i(\nabla w^+(x_0))](v^- - v^- \wedge w^+)_{x_i} dx \end{aligned}$$

for some c. Since w^- and w^+ are C^2 near $\partial\Omega$, we see that

$$|a_i(\nabla w^-) - a_i(\nabla w^-(x_0))|, \ |a_i(\nabla w^-) - a_i(\nabla w^+(x_0))| \leq c\delta$$

for all $x \in B_\delta \cap \Omega$ and hence from Hölder inequality

$$\int_{B_\delta \cap \Omega} (|\nabla v^- \wedge w^+| + |\nabla v^-|)^{p-2} |\nabla(v^- \wedge w^+) - \nabla v^-|^2 dx \leq c\delta^{n+2}.$$

Consequently

$$II \leq c\delta^{n+p} \tag{33}$$

for some c. Similarly,

$$\begin{aligned} III &= -c \int_{B_\delta \cap \Omega} a_i(\nabla v^-)(v - v^-)_{x_i} dx \\ &= -c \int_{B_\delta \cap \Omega} [a_i(\nabla w^-) - a_i(\nabla w^-(x_0))](v - v^-)_{x_i} dx \\ &\leq \frac{1}{4} \int_{B_\delta \cap \Omega} (|\nabla v| + |\nabla v^-|)^{p-2} |\nabla v - \nabla v^-|^2 dx + c\delta^{n+2} \end{aligned} \tag{34}$$

for some c. Combining (30) through (34), we conclude that

$$\int_{B_\delta \cap \Omega} (|\nabla v| + |\nabla v^-|)^{p-2} |\nabla v - \nabla v^-|^2 dx \leq c\delta^{n+p} \tag{35}$$

for some c. Hence we obtain

$$\begin{aligned} \int_{B_\delta \cap \Omega} |\nabla v - \nabla v^-|^p dx &\leq c \left(\int_{B_\delta \cap \Omega} |\nabla v|^p + |\nabla v^-|^p dx \right)^{\frac{2-p}{2}} \\ &\cdot \left(\int_{B_\delta \cap \Omega} (|\nabla v| + |\nabla v^-|)^{p-2} |\nabla v - \nabla v^-|^2 dx \right)^{\frac{p}{2}} \leq c\delta^{n+\frac{p^2}{2}} \end{aligned} \tag{36}$$

for some c.

Now we apply a comparison argument to estimate the oscillation of ∇v. For each small $\rho < \frac{\delta}{2}$, we have

$$\begin{aligned} &\int_{B_\rho \cap \Omega} |\nabla v - (\nabla v)_\rho|^p dx \\ &\leq c \int_{B_\rho \cap \Omega} |\nabla v^- - (\nabla v^-)_\rho|^p dx + c \int_{B_\rho \cap \Omega} |\nabla v - \nabla v^-|^p dx \end{aligned} \tag{37}$$

for some c. Since v^- is a solution to (23) and satisfies a Campanato type growth condition for ∇v^-, then we estimate the first term of the right hand side of (37) as follows:

$$\begin{aligned}\int_{B_\rho\cap\Omega} |\nabla v^- - (\nabla v^-)_\rho|^p dx \le c\left(\frac{\rho}{\delta}\right)^{n+\sigma} \int_{B_\delta\cap\Omega} |\nabla v - (\nabla v)_\delta|^p dx \\ + c\int_{B_\delta\cap\Omega} |\nabla v - \nabla v^-|^2 dx + c\delta^{n+\sigma}\end{aligned} \tag{38}$$

for some c and σ. Furthermore using the estimate (29) and (36) we have

$$\int_{B_\delta\cap\Omega} |\nabla v - \nabla v^-|^2 dx \le c\delta^{n+\sigma} \tag{39}$$

for some c. Therefore combining (37), (38) and (39), it follows that ∇v is Hölder continuous in $\overline{B_{\frac{\delta}{2}} \cap \Omega}$ and satisfies

$$\int_{B_\delta\cap\Omega} |\nabla v - (\nabla v)_\rho|^p dx \le c\left(\frac{\rho}{\delta}\right)^{n+\sigma} \int_{B_\delta\cap\Omega} |\nabla v - (\nabla v)_\delta|^p dx + c\delta^{n+\sigma} \tag{40}$$

for all $\rho \le \dfrac{\delta}{2}$ and for some c. □

Since the viscosity solution w^+ and w^- can be derived from the method of characteristics if $\partial\Omega$ is smooth enough, e.g. C^3, we can prove that the viscosity solutions $w^\pm$ to Hamilton-Jacobi equation

$$\pm G(\nabla w^\pm) = 0$$

with $w^\pm = w_0$ on $\partial\Omega$ are $C^2(\overline{B_R \cap \Omega})$ for $R < \delta$.

Note that $K_R \subset J_R$. So if we show that

$$v \in K_R,$$

that is,

$$G(\nabla v) \le 0$$

for all $x \in B_\delta \cap \Omega$, then v is the unique solution to the variational inequality (20) with respect to K_R. Thus $w = v$ and we conclude that ∇w is Hölder continuous up to $B_{\frac{\delta}{2}} \cap \partial\Omega$ and satisfies the Campanato growth condition (21).

Here we use the maximum principle again for $G(\nabla v)$. From a direct computation we see that $G(\nabla v)$ is a subsolution to a strictly elliptic equation and

$$-(a_{i,A_j}(\nabla u) G_{x_j})_{x_i} \le 0$$

and G satisfies the maximum principle. Let I_R^- and I_R^+ be the contact set such that

$$I_R^- = \{x \in B_R \cap \Omega : v(x) = w^-(x)\} \text{ and } I_R^+ = \{x \in B_R \cap \Omega : v(x) = w^+(x)\}.$$

Consequently from the maximum principle it follows that max $G(\nabla v)$ is attained on $\partial(B_R \cap \Omega \backslash (I_R^- \cup I_R^+))$ and this in turn gives

$$G(\nabla v) \leq 0.$$

Therefore we conclude that

$$v \in K_R.$$

4. - Regularity

We prove that the solution u to the variational inequality (2) under the full generality in Section 1 with respect to K is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Here we employ the perturbation techniques using the interior and boundary regularity results for the simple cases from sections 2 and 3.

We approximate our differential operator. Since the function class is bounded in $W^{1,\infty}$, there exists some large number M such that

$$|v(x)| + |\nabla v(x)| \leq M$$

for all $x \in \Omega$ and $v \in K$. Hence we have

$$(v(x), \nabla v(x)) \in B_M(0) \subset \mathbb{R}^{n+1}$$

for all $x \in \Omega$. We can find functions

$$\overline{a}_i(x, v, A) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, n$$

and

$$\overline{b}(x, v, A) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

such that

$$\overline{a}_i(x, v, A) = a_i(x, v, A), \ i = 1, \ldots, n$$

and

$$\overline{b}(x, v, A) = b(x, v, A)$$

for all $(v, A) \in B_{2M} \subset \mathbb{R}^{n+1}$, and $\overline{a}_i$ and $\overline{b}$ satisfy

$$c_1(M)|A|^{p-2}|\xi|^2 \leq \overline{a}_{i,A_j}(x, v, A)\xi_i\xi_j \leq c_2(M)|A|^{p-2}|\xi|^2$$

and

$$\bar{b}(x, v, A) \leq c(M)$$

for some $c_1(M) > 0$ and for all $(x, v, A) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. For notational simplicity we write a_i and b instead of $\bar{a}_i$ and $\bar{b}$.

Let $u \in K$ be the solution to

$$-(a_i(x, u, \nabla u))_{x_i} - (g_i(x, u))_{x_i} + b(x, u, \nabla u) \geq 0 \tag{41}$$

with respect to K. The following lemma is our main result in this section.

LEMMA 9. *Fix α minimum of $\frac{1}{2}$ and the α's in the assumptions* i) *and* ii) *in the introduction. Suppose $x_0 \in \overline{\Omega}$ and $\rho \leq \frac{R}{2}$. Then for some $\sigma \in (0, 2\alpha)$, ∇u satisfies*

$$\int_{B_\rho \cap \Omega} |\nabla u - (\nabla u)_\rho|^p dx \leq c \left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla u - (\nabla u)_R|^p dx + cR^{n+\sigma} \tag{42}$$

for some c depending on u only through M. Consequently, $u \in C^{1,\alpha}(\overline{\Omega})$.

PROOF. Let $x_0 \in \overline{\Omega}$ and K_R be the function class with domain in $B_R \cap \Omega$ such that $K_R = \{v \in W_0^{1,\infty}(B_R \cap \Omega + u : G(\nabla v) \leq 0\}$. Let $\overline{u} \in K_R$ be the solution to the frozen coefficient variational inequality

$$-(a_i(x_0, u(x_0), \nabla \overline{u}))_{x_i} \geq 0$$

with respect to K_R. From sections 2 and 3, $\overline{u}$ satisfies a Campanato type growth condition

$$\int_{B_\rho \cap \Omega} |\nabla \overline{u} - (\nabla \overline{u})_\rho|^p dx \leq c \left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla \overline{u} - (\nabla \overline{u})_R|^p dx + cR^{n+\sigma} \tag{43}$$

for some c, for some $\sigma \in (0, 2\alpha)$ and for all $\rho \leq \frac{R}{2}$.

Therefore we have

$$\begin{aligned} \int_{B_\rho \cap \Omega} |\nabla u - (\nabla u)_\rho|^p dx &\leq c \int_{B_\rho \cap \Omega} |\nabla \overline{u} - (\nabla \overline{u})_\rho|^p dx + 8 \int_{B_\rho \cap \Omega} |\nabla \overline{u} - \nabla u|^p dx \\ &\leq c \left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla u - (\nabla u)_R|^p dx + cR^{n+\sigma} + c \int_{B_R \cap \Omega} |\nabla u - \nabla \overline{u}|^p dx \end{aligned} \tag{44}$$

for some c and for all $\sigma \in (0, 2\alpha)$. Now we see that $\overline{u} \in K_R$ and is an admissible competing function to (41). Hence we have

$$\int_{B_R\cap\Omega} a_i(x,u,\nabla u)(\overline{u}-u)_{x_i}dx + \int_{B_R\cap\Omega} [g_i(x,u)-g_i(x_0,u(x_0))](\overline{u}-u)_{x_i}dx + \int_{B_R\cap\Omega} b(x,u,\nabla u)(\overline{u}-u)dx \geq 0 \tag{45}$$

and

$$\int_{B_R\cap\Omega} a_i(x_0,u(x_0),\nabla\overline{u})(u-\overline{u})_{x_i}dx \geq 0. \tag{46}$$

Subtracting (46) from (45) we have

$$\begin{aligned} &\int_{B_R\cap\Omega} [a_i(x_0,u(x_0),\nabla\overline{u}) - a_i(x,u,\nabla u)](\overline{u}-u)_{x_i}dx \\ &\leq \int_{B_R\cap\Omega} [g_i(x,u)-g_i(x_0,u(x_0))](\overline{u}-u)_{x_i}dx \\ &+ \int_{B_R\cap\Omega} b(x,u,\nabla u)(\overline{u}-u)dx. \end{aligned} \tag{47}$$

The left-hand side of (47) can be written as

$$\begin{aligned} &\int_{B_R\cap\Omega} [a_i(x_0,u(x_0),\nabla\overline{u}) - a_i(x,u,\nabla u)](\overline{u}-u)_{x_i}dx \\ &= \int_{B_R\cap\Omega} [a_i(x_0,u(x_0),\nabla\overline{u}) - a_i(x_0,u(x_0),\nabla u)](\overline{u}-u)_{x_i}dx \\ &+ \int_{B_R\cap\Omega} [a_i(x_0,u(x_0),\nabla u) - a_i(x_0,u,\nabla u)](\overline{u}-u)_{x_i}dx \\ &+ \int_{B_R\cap\Omega} [a_i(x_0,u,\nabla u) - a_i(x,u,\nabla u)](\overline{u}-u)_{x_i}dx =: I + II + III. \end{aligned}$$

Considering the ellipticity condition i) in Section 1 we estimate

$$\int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2}|\nabla\overline{u}-\nabla u|^2 dx \leq cI \tag{48}$$

for some c. From Hölder continuity of a_i with respect to x and Lipschitz continuity of $\overline{u}$ and u, we have

$$\begin{aligned} |III| &\leq c \int_{B_R\cap\Omega} R^\alpha |\nabla\overline{u} - \nabla u| dx \\ &\leq \varepsilon \int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2} |\nabla\overline{u} - \nabla u|^2 dx + cR^{n+2\alpha} \end{aligned} \tag{49}$$

for some c and small ε. From Young's inequality we also have

$$\begin{aligned} II &= \int_{B_R\cap\Omega} [a_i(x_0, u(x_0), \nabla u) - a_i(x, u, \nabla u)](\overline{u} - u)_{x_i} dx \\ &\leq c \int_{B_R\cap\Omega} |u(x) - u(x_0)|^\alpha |\nabla\overline{u} - \nabla u| dx \\ &\leq c \int_{B_R\cap\Omega} R^\alpha |\nabla\overline{u} - \nabla u| dx \\ &\leq \varepsilon \int_{B_R\cap\Omega} (|\nabla\overline{u} + |\nabla u|)^{p-2} |\nabla\overline{u} - \nabla u|^2 dx + cR^{n+2\alpha} \end{aligned} \tag{50}$$

for some small ε, where we used the fact that u is Lipschitz continuous and

$$|u(x) - u(x_0)| \leq c|x - x_0|.$$

Finally using Poincaré's inequality we estimate the right-hand side of (47) as follows:

$$\begin{aligned} &\int_{B_R\cap\Omega} [g_i(x, u) - g_i(x_0, u(x_0))](\overline{u} - u) dx \\ &\leq \varepsilon \int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2} |\nabla\overline{u} - \nabla u|^2 dx + cR^{n+2\alpha} \end{aligned}$$

and

$$\int_{B_R\cap\Omega} b(x, u, \nabla u)(\overline{u} - u) dx \leq \varepsilon \int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2} |\nabla\overline{u} - \nabla u|^2 dx + cR^{n+2\alpha}.$$

Thus combining all these together we have that for sufficiently small ε

$$\int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2} |\nabla\overline{u} - \nabla u|^2 dx \leq cR^{n+2\alpha} \tag{51}$$

for some c. We know that

$$\int_{B_R\cap\Omega} |\nabla\overline{u} - \nabla u|^p dx \le c \int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2}|\nabla\overline{u} - \nabla u|^2 dx \tag{52}$$

for some c when $p \in [2,\infty)$ and

$$\begin{aligned}\int_{B_R\cap\Omega} |\nabla\overline{u} - \nabla u|^p dx \le c &\left(\int_{B_R\cap\Omega} |\nabla\overline{u}|^p + |\nabla u|^p dx\right)^{\frac{2-p}{2}} \\ &\cdot \left(\int_{B_R\cap\Omega} (|\nabla\overline{u}| + |\nabla u|)^{p-2}|\nabla\overline{u} - \nabla u|^2 dx\right)^{\frac{p}{2}}\end{aligned} \tag{53}$$

for some c when $p \in (1,2)$.

Therefore using the estimate (51) on (53) we conclude that

$$\int_{B_\rho\cap\Omega} |\nabla u - (\nabla u)_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R\cap\Omega} |\nabla u - (\nabla u)_R|^p dx + cR^{n+\sigma}$$

for some c and this completes the proof. □

5. - Existence

We employ Leray-Schauder's fixed point theorem to show the existence of the solution to

$$L(u) = -(a_i(x,u,\nabla u))_{x_i} - (g_i(x,u))_{x_i} + b(x,u,\nabla u) \ge 0 \tag{54}$$

with respect to $K = \{v \in W_0^{1,\infty}(\Omega) + u_0 : G(\nabla u) \le 0\}$.

We define a compact map $T : K \to K$. Let $v \in K$ and $u = T(v)$ be the solution to the variational inequality

$$L(v,u) = -(a_i(x,v,\nabla u))_{x_i} - (g_i(x,u))_{x_i} + b(x,u,\nabla u) \ge 0 \tag{54}$$

with respect to $K = \{v \in W_0^{1,\infty}(\Omega) + u_0 : G(\nabla u) \le 0\}$.

We define a compact map $T : K \to K$. Let $v \in K$ and $u = T(v)$ be the solution to the variational inequality

$$L(v,u) = -(a_i(x,v,\nabla u))_{x_i} - (g_i(x,v))_{x_i} + b(x,v,\nabla v) \ge 0 \tag{55}$$

with respect to K. We note that K is a bounded closed convex subset of $W_0^{1,2}(\Omega)+u_0$. Moreover for each fixed $v \in K$, $L(v,u)$ is strictly monotone as an operator of u. Therefore from the Theorem 1.1 in [Har] we see that there is a unique solution $u = T(v) \in K$ to (55) and hence T is well defined.

From the $C^{1,\alpha}(\overline{\Omega})$ regularity result in section 4 we have that for each $v \in K$

$$u = T(v) \in C^{1,\alpha}(\overline{\Omega}).$$

for some fixed $\alpha > 0$. Moreover the $C^{1,\alpha}$ norm of u is bounded by some fixed number M independent of $v \in K$, $C^{1,\alpha}$ norm depends on v only through the upper bound of the Lipschitz norm of v. We note that the space $C^{1,\alpha}(\overline{\Omega})$ is compactly imbedded in the space of Lipschitz functions $\mathrm{Lip}(\overline{\Omega})$. Hence the image of K under the map T is a precompact subset of K. Therefore from Leray-Schauder's fixed point theorem we conclude that there is a fixed point u for T such that

$$u = T(u)$$

and u is a $C^{1,\alpha}(\overline{\Omega})$ solution to (54) with respect to K.

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