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A New Approach to the Ricci Flow on S^2

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1. - Introduction

In this paper we study the Ricci flow of R. Hamilton on the 2-dimensional sphere S^2,

$$\frac{\partial g}{\partial t} = (r - R)g, \tag{1}$$

where g denotes the evolving metric, R the scalar curvature and r the average of R. The following result has been obtained by Hamilton [H1] and B. Chow [Ch].

THEOREM 1.1. *For any smooth initial metric, the solution of* (1) *exists for all times and converges exponentially to a metric of constant curvature as* $t \to \infty$.

(Hamilton assumed the additional condition $R > 0$, which was eventually removed by Chow.)

The situation of the Ricci flow on S^2 is surprisingly delicate and differs very much from the 3-dimensional case in [H2] and the 4-dimensional case in [H3]. The proof of Theorem 1.1 given in [H1] is very intricate, it involves the Harnack inequality for the scalar curvature, monotonicity of a delicate new geometric quantity called "entropy" and analysis of soliton solutions of the Ricci flow. Our purpose here is to provide an elementary proof of Theorem 1.1, first under the assumption that the initial metric is conformal to the standard metric on S^2. By virtue of the uniformization theorem this proof then extends to all cases. (Of course, it should be noted that the proof of Hamilton-Chow does not use the uniformization theorem. Instead, the uniformization theorem is recovered.) We remark that the Ricci flow on S^2 is not only significant for its own sake, it also sheds lights on neck pinching along the Ricci deformation of

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3-manifolds. From this point of view, an elementary proof of Theorem 1.1 is particularly desirable.

Let g_{S^2} denote the standard metric on S^2 of volume 4π and write $g = e^u g_{S^2}$ for a smooth function u. Then the equation (1) is equivalent to the following

$$\frac{\partial e^u}{\partial t} = \Delta_{S^2} u - 2 + re^u, \tag{2}$$

where Δ_{S^2} denotes the standard Laplacian on S^2. The average scalar curvature r can be written as

$$r = \frac{8\pi}{\mathrm{vol}(e^u g_{S^2})}.$$

It remains a constant in time because the volume is preserved along the Ricci flow. On the other hand, we note that the metric $e^u g_{S^2}$ has scalar curvature r if and only if u satisfies

$$\Delta_{S^2} u - 2 + re^u = 0. \tag{3}$$

We have:

THEOREM 1.2. *For any smooth initial function u_0, the solution u of* (2) *exists for all times and converges exponentially as $t \to \infty$ to a solution of* (3) *with $r = 8\pi/\mathrm{vol}(e^{u_0} g_{S^2})$. Moreover the following estimate holds*

$$|\nabla_{S^2} u| \leq C \tag{4}$$

with C depending only on u_0.

Indeed, the key to long time existence and convergence is the gradient estimate (4). The proof of this estimate is along the lines of the argument in [Y] for the Harnack inequality for solutions of the Yamabe flow. A delicate difference between the Ricci flow on S^2 and the Yamabe flow is that the latter is a (negative) gradient flow whereas the former is not. In [Y], L. Simon's result in [S] is applied to derive uniqueness of the asymptotic limit. The gradient flow property of the Yamabe flow is essential for this application. In the present paper we obtain uniqueness of the asymptotic limit (indeed exponential convergence) by a more elementary argument.

2. - Proof of Theorem 1.2

We consider S^2 as the standard unit sphere in $\mathbb{R}^3$ and let $F : S^2 \to \mathbb{R}^2$ be the stereographic projection, whose inverse is given by

$$F^{-1}(x) = \left(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right), \quad x \in \mathbb{R}^2.$$

We introduce the following coordinates around the north pole $p_0 = (0, 0, 1)$:

$$G(x) = \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right), \quad |x| \leq 1.$$

For a given smooth function f on S^2, we set $(F^{-1})^*(f g_{S^2}) = \overline{f} g_{\mathbb{R}^2}$, where $g_{\mathbb{R}^2}$ denotes the euclidean metric. Then $\overline{f}$ is given by

$$\overline{f}(x) = f(F^{-1}(x)) \frac{4}{(1+|x|^2)^2}. \tag{5}$$

A simple computation leads to:

LEMMA 2.1. *Define* $a_0 = f(p_0) = (f \circ G)(0)$, $a_1 = \dfrac{\partial (f \circ G)}{\partial x^i}(0)$ *and* $a_{ij} = \dfrac{\partial^2 (f \circ G)}{\partial x^i \partial x^j}(0)$. *Then we have the following expansions near* ∞*:*

$$\begin{aligned} \overline{f}(x) &= \frac{4}{|x|^4} \left(a_0 + \frac{a_i x_i}{|x|^2} + \left(\frac{1}{2} a_{ij} - 2 a_0 \delta_{ij} \right) \frac{x_i x_j}{|x|^4} + O\left(\frac{1}{|x|^3} \right) \right) \\ \frac{\partial \overline{f}}{\partial x_i} &= \frac{16}{|x|^6} x_i \left(a_0 + \frac{a_j x_j}{|x|^2} \right) + \frac{4a_i}{|x|^6} - \frac{8x_i}{|x|^8} a_j x_j + O\left(\frac{1}{|x|^7} \right). \end{aligned} \tag{6}$$

Now let u_0 be a smooth function on S^2 and u the unique smooth solution of (2) with initial value u_0 on a maximal time interval $[0, T^*)$. We set $f = e^u$, define w, $\overline{f}$ in terms of

$$(F^{-1})^*(f g_{S^2}) = \overline{f} g_{\mathbb{R}^2}, \quad \overline{f} = e^w,$$

and set

$$a_0(t) = f(p_0, t), \quad a_i(t) = \frac{\partial (f(\cdot, t) \circ G)}{\partial x^i}(0), \quad a_{ij}(t) = \frac{\partial^2 (f(\cdot, t) \circ G)}{\partial x^i \partial x^j}(0).$$

Note that

$$w(x, t) = u(F^{-1}(x), t) + \log \frac{4}{(1+|x|^2)^2} \tag{7}$$

and that w satisfies the following flow equation

$$e^w \frac{\partial w}{\partial t} = \Delta w + r e^w, \tag{8}$$

where r is the average scalar curvature of the metric $e^u g_{S^2}$. By Lemma 2.1, the expansion (6) holds for $\overline{f}(\cdot, t)$ with $a_0 = a_0(t)$, $a_i = a_i(t)$ and $a_{ij}(t) = a_{ij}(t)$. We notice that the expansion is uniform for all $t \in [0, T]$, where T is any given number in $[0, T^*)$.

Our purpose is to estimate ∇u. We introduce the *center* $y(t)$ of $w(\cdot, t)$ as $y(t) = (y_1(t), y_2(t))$, where

$$y_i(t) = \frac{a_i(t)}{4a_0(t)}.$$

PROPOSITION 2.2. *There is a constant* $C > 0$ *depending only on* $\|u_0\|_{C^4}$ *such that*

$$\|y(t)\| \leq C$$

for all $t \in [0, T^*)$.

PROOF. Consider a given $T \in (0, T^*)$. Performing a rotation of coordinates and the transformation $x_2 \mapsto -x_2$ if necessary, we may assume $y_2(T) = \max_i |y_i(T)|$. By the expansion (6) and the arguments for Lemma 4.2 in [Gi-Ni-Nir] we derive that for some $\lambda_0 \geq 1$ depending only on $\|u_0\|_{C^4}$ the following holds: For each $\lambda \geq \lambda_0$, $\overline{f}(x, 0) > \overline{f}(x^\lambda, 0)$ whenever $x_2 < \lambda$, where $x^\lambda = (x_1, 2\lambda - x_2)$ for $x = (x_1, x_2)$. (Thus x^λ is the reflection of x about the plane $x_2 = \lambda$.) Consequently

$$w(x, 0) > w(x^\lambda, 0) \tag{9}$$

whenever $x_2 < \lambda$, $\lambda \geq \lambda_0$.

By the same argument and the fact that the expansion (6) for $\overline{f}(\cdot, t)$ is uniform for all $t \in [0, T]$, there is some $\lambda_1 \geq \lambda_0$ such that for each $\lambda \geq \lambda_1$

$$w(x, t) > w(x^\lambda, t) \text{ whenever } t \in [0, T] \text{ and } x_2 < \lambda. \tag{10}$$

We are going to show $y_2(T) \leq \lambda_0$. For this purpose we consider the function $w^\lambda(x, t) \equiv w(x^\lambda, t)$ on the region $x_2 \leq \lambda$, $0 \leq t \leq T$ and define

$$I = \{\lambda : \lambda > \lambda_0,\ \lambda > \max_{0 \leq t \leq T} y_2(t),\ w^\lambda \leq w\}.$$

Note that w^λ solves (8) and coincides with w along the plane $x_2 = \lambda$. By (10), I is nonempty. I is also open. Indeed, $w^\lambda \equiv w$ can never happen for $\lambda \geq \lambda_0$ because of (9). Hence for a given $\lambda \in I$, the maximum principle implies

$$w^\lambda < w \text{ for } x_2 < \lambda, \tag{11}$$

and the standard proof of the parabolic version of the Hopf boundary point lemma implies

$$\frac{\partial w}{\partial x_2} < 0 \tag{12}$$

along the plane $x_2 = \lambda$.

Consequently,

$$\overline{f}^\lambda < \overline{f} \text{ for } x_2 < \lambda \tag{13}$$

and

$$\frac{\partial \overline{f}}{\partial x_2} < 0 \tag{14}$$

along the plane $x_2 = \lambda$, where $\overline{f}^\lambda(x,t) \equiv \overline{f}(x^\lambda, t)$ is defined on $x_2 \leq \lambda$. For each fixed $t \in [0,T]$, we shift the origin to $y(t)$ to obtain the new expansion for $\overline{f}(\cdot,t)$ in the new coordinates

$$\begin{aligned} \overline{f}(\cdot,t) &= \frac{4}{|x|^4}\left(a_0 + \frac{a_{ij}x_ix_j}{|x|^4} + O\left(\frac{1}{|x|^3}\right)\right), \\ \frac{\partial \overline{f}}{\partial x_i} &= \frac{16}{|x|^6}a_0x_i + O\left(\frac{1}{|x|^7}\right), \end{aligned} \tag{15}$$

with different coefficients a_{ij}. The plane $x_2 = \lambda$ becomes the plane $x_2 = \lambda - y_2(t)$ in the new coordinates. Because $\lambda \in I$, we have $\lambda - y_2(t) > 0$. Hence we can argue as in [Gi-Ni-Nir] to show that there is an $\varepsilon(t) > 0$ with the following property:

$$\text{If } \lambda' \in (\lambda - \varepsilon(t), \lambda + \varepsilon(t)), \text{ then } \overline{f}^{\lambda'}(\cdot,t) \leq \overline{f}(\cdot,t).$$

Since $\lambda > \max_{0\leq t\leq T} y_2(t)$ and the expansion (15) is uniform for all $t \in [0,T]$, we can choose $\varepsilon(t)$ uniformly for all $t \in [0,T]$. Since $\overline{f}^\lambda \leq \overline{f}$ is equivalent to $w^\lambda \leq w$, it follows that $(\lambda - \varepsilon, \lambda + \varepsilon) \subset I$ for some $\varepsilon > 0$. Thus the openness of I has been shown.

Next we prove that I is closed in (λ_0, ∞). Let $\lambda > \lambda_0$ be in the closure of I. By continuity, we have $w^\lambda \leq w$ and $\lambda \geq \max_{0\leq t\leq T} y_2(t)$. If $\lambda = \max_{0\leq t\leq T} y_2(t)$, then $\lambda = y_2(t_0)$ for some $t_0 \in [0,T]$. Now we choose $y_0(t_0)$ as the new origin and consider the stereographic projection $F : S^2 \to \mathbb{R}^2$. Define z and z^λ in the following way

$$F^*(e^w g_{\mathbb{R}^2}) = e^z g_{S^2}, \quad F^*(e^{w^\lambda} g_{\mathbb{R}^2}) = e^{z^\lambda} g_{S^2}.$$

Then, z, z^λ are defined on $S^2_+ \times [0,T]$ for a hemisphere S^2_+. The functions z and z^λ satisfy the equation (2). We also know that $z^\lambda \leq z$ and z^λ coincides with z along ∂S^2_+. Moreover, Lemma 2.1 and the expansion (15) (for $t = t_0$) imply that

$$\frac{\partial e^z(\cdot,t_0)}{\partial \nu}(\text{north pole}) = \frac{\partial e^{z^\lambda}(\cdot,t_0)}{\partial \nu}(\text{north pole}) = 0.$$

where ν denotes the inward unit normal of ∂S^2_+. Hence

$$\frac{\partial z(\cdot,t_0)}{\partial \nu}(\text{north pole}) = \frac{\partial z^\lambda(\cdot,t_0)}{\partial \nu}(\text{north pole}) = 0.$$

By the Hopf boundary point lemma we then deduce that $z \equiv z^{\lambda}$. This implies that $w \equiv w^{\lambda}$, which is impossible because of (9). We conclude that $\lambda > \max_{0 \le t \le T} y_2(t)$, whence $\lambda \in I$. This shows that I is closed. We infer that $I = (\lambda_0, \infty)$. This proves $y_2(T) \le \lambda_0$. Hence $|y(T)| \le C$. Since T is arbitrary, the proposition is proven. □

PROOF OF THEOREM 1.2. Proposition 2.2 readily implies $|\nabla u(p_0, t)| \le C$ for $t \in [0, T^*)$ and a positive constant C depending only on $\|u_0\|_{C^4}$. By a rotation we can bring any point of S^2 to the north pole p_0, whence the gradient estimate (4) follows.

Integrating the estimate (4) from a minimal point to a maximal point of u along a great circle yields the following Harnack inequality for the conformal factor $f = e^u$:

$$\inf_t e^u \ge c \sup_t e^u \tag{16}$$

for a positive constant c. Next we compute the rate of change of volume along (2), or equivalently (1) with $g = e^u g_{S^2}$,

$$\frac{d\,\mathrm{vol}(g)}{dt} = \int_{S^2} \frac{\partial}{\partial t}(dv) = \frac{1}{2} \int_{S^2} \mathrm{trace}\left(\frac{\partial g}{\partial t}\right) dx = 0.$$

Hence the volume remains a constant. But

$$\mathrm{vol}(g) = \int_{S^2} e^u dv_{S^2}.$$

Thus the Harnack inequality (16) implies that $|u|$ is uniformly bounded. The linear theory and bootstrapping then yield uniform smooth estimates for u on $S^2 \times [0, T^*)$. It follows that $T^* = \infty$, since otherwise we would be able to extend u beyond T^*. Moreover, u subconverges smoothly as $t \to \infty$.

To analyze the limits, we multiply the equation

$$\frac{\partial e^u}{\partial t} = (r - R)e^u$$

(this is a direct consequence of (1)) by $\dfrac{\partial u}{\partial t}$ and then integrate with respect to dv_{S^2}. Then we obtain

$$\int_{S^2} \frac{\partial e^u}{\partial t} \frac{\partial u}{\partial t} dv_{S^2} = \int_{S^2} (r - R)^2 dv_g. \tag{17}$$

But

$$\int_{S^2} \frac{\partial e^u}{\partial t} \frac{\partial u}{\partial t} \, dv_{S^2} = \int_{S^2} (\Delta_{S^2} u - 2 + re^u) \frac{\partial u}{\partial t} \, dv_{S^2}$$

$$= -\frac{d}{dt} \int_{S^2} \left(\frac{|\nabla_{S^2} u|^2}{2} + 2u \right) dv_{S^2}.$$

where constancy of the volume has been used. We set

$$E(u) = \int_{S^2} \left(\frac{|\nabla_{S^2} u|^2}{2} + 2u \right) dv_{S^2}.$$

Since u is uniformly bounded, there exists a constant C_0 such that $E(u(t)) \geq C_0$ for all t.

Integrating (17) along with (18) then yields

$$\int_0^\infty \int_{S^2} (r - R)^2 dv_g dt \leq E(u_0) - C_0 < \infty. \tag{19}$$

On the other hand, from [H1], p. 239, we have the following simple equation for the scalar curvature R,

$$\frac{\partial R}{\partial t} = \Delta R + R^2 - rR,$$

where $\Delta = \Delta_g$. This equation and the estimates for u imply uniform smooth estimates for $\frac{\partial R}{\partial t}$, which along with (19) then imply that the limit metrics all have the same constant scalar curvature r. Consequently, the scalar curvature R of g converges smoothly to r as $t \to \infty$. Since $r > 0$, we deduce that for large time R is bounded from below by a positive constant.

Our final goal is to prove unique convergence of g. Following [H1], p. 241, we consider the potential ϕ which is defined to be the solution of the equation

$$\Delta \phi = R - r$$

with mean value (in the metric g) zero. Let M_{ij} denote the trace-free part of the second covariant derivative of ϕ, i.e.

$$M_{ij} = \nabla_i \nabla_j \phi - \frac{1}{2} \Delta \phi \cdot g_{ij}.$$

Then we have the following evolution equation ([H1], p. 253)

$$\frac{\partial}{\partial t} |M_{ij}|^2 = \Delta |M_{ij}|^2 - 2|\nabla_k M_{ij}|^2 - 2R|M_{ij}|^2.$$

Since the scalar curvature R is bounded from below by a positive constant for large time, it follows from the maximum principle that

$$|M_{ij}| \leq Ce^{-ct} \tag{20}$$

for some positive constants C and c. The estimates for u imply uniform smooth estimates for M_{ij}. By a simple interpolation we then deduce from (20) that all the derivatives of M_{ij} converge to zero exponentially as $t \to \infty$.

Next we consider the following modified Ricci flow as in [H1]

$$\frac{\partial}{\partial t} g_{ij} = 2M_{ij} = (r - R)g_{ij} - 2\nabla_i \nabla_j \phi, \tag{21}$$

with the same initial data $e^{u_0} g_{S^2}$ as before. This equation differs from the Ricci flow only by transport along a one-parameter family of diffeomorphisms generated by the gradient vector field of the potential ϕ. The exponential decay of M_{ij} and its derivatives implies that the solution $\tilde{g}$ converges exponentially to a unique limit $\tilde{g}_\infty$. But $\tilde{g}(t)$ has the same scalar curvature as $g(t)$, consequently $\tilde{g}_\infty$ has constant scalar curvature r. It follows that the scalar curvature of $\tilde{g}$, and hence g, converges exponentially. By the equation (1), the metric g and hence the function u converge exponentially. $\square$

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