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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 3 (1994), p. 377-386

<http://www.numdam.org/item?id=ASNSP_1994_4_21_3_377_0>

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An important direction in real algebraic geometry during the past decade is the construction of real algebraic hypersurfaces with prescribed topology [4], [7], [8], [9]. Central to these developments is a combinatorial construction due to O.Ya. Viro, which is based on regular triangulations of Newton polytopes. Using this technique, significant progress has been made in the study of low degree curves in the real projective plane (Hilbert's 16th problem). The objective of this note is to extend Viro's Theorem to the case of complete intersections. Our construction uses mixed decompositions of the Newton polytopes (see e.g. [6]). It generalizes both Viro's theorem for hypersurfaces and the observations on zero-dimensional complete intersections in [5].

Acknowledgements

I am grateful to Askold Khovanskii for introducing me to Viro's Theorem, and to Jean-Jaques Risler for suggesting the problem of extending it to complete intersections. Work on this paper has been supported by the David and Lucile Packard Foundation and the National Science Foundation (DMS-9201453 and DMS-9258547).

1. - Asymptotic analysis of hypersurfaces

We recall Viro's theorem for hypersurfaces, following the exposition given by Gel'fand, Kapranov and Zelevinsky in [1]. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of lattice points, and let $Q = \mathrm{conv}(\mathcal{A})$. Let $\omega : \mathcal{A} \to \mathbb{Z}$ be any function such that the coherent polyhedral subdivision Δ_ω of $(\mathcal{A}, Q)$ is a triangulation (cf. [1], [2], [3], [5]). Fix nonzero real numbers $c_{\mathbf{a}}, \mathbf{a} \in \mathcal{A}$. For each positive real number t

Pervenuto alla Redazione il 6 Maggio 1993 e in forma definitiva l'11 Aprile 1994.

we consider the Laurent polynomial

$$f_t(x_1, \dots, x_n) = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} t^{\omega(\mathbf{a})} \mathbf{x}^{\mathbf{a}}. \tag{1}$$

We wish to describe the zero set of f_t for t very close to the origin. Here the zeros are not to be taken in $\mathbb{R}^n$. Instead, we will first study the zero sets of f_t in each orthant, and afterwards in their natural toric compactification.

Let $Z_+(f_t)$ denote the zero set of f_t in the positive orthant $(\mathbb{R}_+)^n$. Let $\mathrm{Bar}(\Delta_\omega)$ denote the first barycentric subdivision of the regular triangulation Δ_ω. Each facet σ of $\mathrm{Bar}(\Delta_\omega)$ is incident to a unique point $\mathbf{a} \in \mathcal{A}$. (Facet means maximal cell). We define the sign of a facet σ to be the sign of the real number $c_{\mathbf{a}}$. The sign of any lower dimensional cell $\tau \in \mathrm{Bar}(\Delta_\omega)$ is defined as follows:

$$\mathrm{sign}(\tau) := \begin{cases} + & \text{if } \mathrm{sign}(\sigma) = + \text{ for all facets } \sigma \text{ containing } \tau, \\ - & \text{if } \mathrm{sign}(\sigma) = - \text{ for all facets } \sigma \text{ containing } \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let $Z_+(\Delta_\omega, f)$ denote the subcomplex of $\mathrm{Bar}(\Delta_\omega)$ consisting of all cells τ with $\mathrm{sign}(\tau) = 0$.

THEOREM 1. (Viro [7], see also [1, Thm. XI.5.6]) *For sufficiently small $t > 0$, the real algebraic set $Z_+(f_t) \subset (\mathbb{R}_+)^n$ is homeomorphic to the simplicial complex $Z_+(\Delta_\omega, f) \subset \Delta_\omega$.*

REMARK. Theorem 1 and all subsequent assertions are understood in the embedded sense, that is, there exists a homeomorphism between the orthant $(\mathbb{R}_+)^n$ and the interior of Q which maps $Z_+(f_t)$ into $Z_+(\Delta_\omega, f) \cap \mathrm{int}(Q)$.

Naturally, a signed version of Theorem 1 holds in each of the 2^n orthants

$$(\mathbb{R}_+)^\epsilon := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \mathrm{sign}(x_i) = \epsilon_i \text{ for } i = 1, \dots, n\},$$

where $\epsilon \in \{-, +\}^n$. Let $Z_\epsilon(f_t)$ denote the zero set of f_t in $(\mathbb{R}_+)^\epsilon$. It corresponds to the zero set of $f^\epsilon(x_1, \dots, x_n) := f(\epsilon_1 x_1, \dots, \epsilon_n x_n)$ in $(\mathbb{R}_+)^n$. Theorem 1 implies that the real algebraic set $Z_\epsilon(f_t)$ is homeomorphic to the simplicial complex $Z^\epsilon(\Delta_\omega, f) := Z_+(\Delta_\omega, f^\epsilon)$.

Let $X_{\mathcal{A}}$ denote the projective toric variety in $P(\mathbb{C}^{\mathcal{A}})$ associated with the configuration $\mathcal{A}$. We will consider the *real toric variety* $X_{\mathcal{A}}(\mathbb{R}) := X_{\mathcal{A}} \cap P(\mathbb{R}^{\mathcal{A}})$ and its positive part $X_{\mathcal{A}}(\mathbb{R}_+) := X_{\mathcal{A}} \cap P(\mathbb{R}_+^{\mathcal{A}})$. There is a well-known surjection, called the *moment map*, which takes the toric variety $X_{\mathcal{A}}$ onto the polytope $Q = \mathrm{conv}(\mathcal{A})$. The restriction of the moment map defines a homeomorphism between $X_{\mathcal{A}}(\mathbb{R}_+)$ and the interior of Q. Our use of the moment map will be entirely analogous to that of Section 5.D in [1, Chapter XI].

By restricting the moment map to each orthant, we obtain the following recipe for gluing the real toric variety $X_{\mathcal{A}}(\mathbb{R})$. Let F be a face of Q. Two sign vectors $\delta, \epsilon \in \{+, -\}^n$ are said to *agree on* F if either $\epsilon^{\mathbf{a}} = \delta^{\mathbf{a}}$ for all $\mathbf{a} \in F \cap \mathcal{A}$, or $\epsilon^{\mathbf{a}} \neq \delta^{\mathbf{a}}$ for all $\mathbf{a} \in F \cap \mathcal{A}$. (Here we abbreviate, as usual, $\epsilon^{\mathbf{a}} := \epsilon_1^{a_1} \cdots \epsilon_n^{a_n}$). For each $\epsilon \in \{+,-\}^n$ we take a copy Q_ϵ of the polytope Q. If $F \subset Q$ is a face, then F_ϵ denotes the corresponding face of Q_ϵ.

PROPOSITION 2. [1, Thm. XI.5.4] *The real toric variety $X_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to the space obtained by gluing the polytopes Q_ϵ, $\epsilon \in \{+,-\}^n$, according to the following identifications: For any face $F \subset Q$ we identify F_ϵ and F_δ whenever ϵ and δ agree on F.*

The regular triangulation Δ_ω for each polytope Q_ϵ gives rise to a triangulation Δ'_ω of $X_{\mathcal{A}}(\mathbb{R})$. Each facet of its first barycentric subdivision $\mathrm{Bar}(\Delta'_\omega)$ lies in a unique Q_ϵ and is incident to a unique $\mathbf{a} \in \mathcal{A}$. The sign of this facet is defined to be the sign of the real number $c_{\mathbf{a}}\epsilon^{\mathbf{a}}$. The sign of each lower-dimensional cell is defined by the rule (2), applied separately in each orthant. Let $Z(\Delta_\omega, f)$ denote the subcomplex of $\mathrm{Bar}(\Delta'_\omega)$ consisting of all cells τ with $\mathrm{sign}(\tau) = 0$. This subcomplex is glued from the 2^n complexes $Z^\epsilon(\Delta_\omega, f)$ via the rule in Proposition 2. Our input polynomial (1) is identified with a linear form

$$f_t = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} t^{\omega(\mathbf{a})} \cdot z_{\mathbf{a}}, \tag{3}$$

where the $z_{\mathbf{a}}$ are the coordinate functions on $P(\mathbb{C}^{\mathcal{A}})$. Let $Z(f_t)$ denote the set of zeros of f_t in the real toric variety $X_{\mathcal{A}}(\mathbb{R})$. The real algebraic variety $Z(f_t)$ is the natural toric compactification of the zero set of (1) in $(\mathbb{R}\backslash\{0\})^n$. Note that the positive part $Z(f_t) \cap P(\mathbb{R}_+^{\mathcal{A}})$ is identified with $Z_+(f_t) \subset (\mathbb{R}_+)^n$ via the parametrization $\mathbf{x} = (x_1, \ldots, x_n) \mapsto (\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A})$ of the toric variety. The same holds for all other orthants.

THEOREM 3. (Viro [7], see also [1, Thm. XI.5.6]) *For sufficiently small $t > 0$, the real algebraic set $Z(f_t) \subset X_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to the simplicial complex $Z(\Delta_\omega, f) \subset \Delta'_\omega$.*

The most important instance of this construction concerns the set $\mathcal{A}$ of all non-negative integer vectors $(j_1, \ldots, j_n)$ with $j_1 + \ldots + j_n \leq d$. In this case f_t is a dense polynomial of degree d in n variables. The toric variety $X_{\mathcal{A}}(\mathbb{R})$ equals real projective n-space $P^n(\mathbb{R})$. Proposition 2 gives a recipe for gluing $P^n(\mathbb{R})$ from 2^n copies for the simplex $Q = \mathrm{conv}(\mathcal{A})$. Theorem 3 gives a purely combinatorial construction for the real projective hypersurface $\{f_t = 0\}$. Viro and collaborators have applied this construction with great success in the case of curves ($n = 2$). An extensive list of examples can be found in [8].

2. - Asymptotic analysis of complete intersections

We replace the single input equation (1) by a system of k equations

$$f_{i,t}(x_1,\ldots,x_n) := \sum_{\mathbf{a}\in\mathcal{A}_i} c_{i,\mathbf{a}} t^{\omega_i(\mathbf{a})}\mathbf{x}^{\mathbf{a}} \qquad (i=1,\ldots,k). \tag{4}$$

Here the $c_{i,\mathbf{a}}$ are non-zero real numbers, and $\mathcal{A}_1,\ldots,\mathcal{A}_k \subset \mathbb{Z}^n$ are (generally distinct) finite sets of lattice points. So, we have k distinct Newton polytopes $Q_i = \mathrm{conv}(\mathcal{A}_i)$. We assume that the pointwise sum $\mathcal{A} := \mathcal{A}_1 + \cdots + \mathcal{A}_k$ affinely generates the lattice $\mathbb{Z}^n$. In what follows we consider $\mathcal{A}$ as a multiset of cardinality equal to the product of the cardinalities of the $\mathcal{A}_i$. Let $Q := \mathrm{conv}(\mathcal{A}) = Q_1 + \cdots + Q_k \subset \mathbb{R}^n$ denote the Minkowski sum of the given Newton polytopes.

The functions $\omega_i : \mathcal{A}_i \to \mathbb{Z}$ are assumed to be *sufficiently generic* in the following precisely defined sense. We extend the ω_i to a unique function

$$\omega : \mathcal{A} \to \mathbb{Z}, \quad \mathbf{a}^{(1)} + \ldots + \mathbf{a}^{(k)} \mapsto \omega_1(\mathbf{a}^{(1)}) + \ldots + \omega_k(\mathbf{a}^{(k)}). \tag{5}$$

This is well-defined because $\mathcal{A}$ is a multiset. Let Δ_ω denote the coherent polyhedral subdivision of $(Q,\mathcal{A})$ defined by ω. In precise technical terms Δ_ω is a collection of subsets of the multiset $\mathcal{A}$, see e.g. [1], [2], [3]. The subdivisions Δ_ω were introduced in [6], where we called them *tight coherent mixed decompositions*, or TCMD's for short. Each facet F of Δ_ω has a unique representation

$$F = F_1 + F_2 + \cdots + F_k, \tag{6}$$

where F_i is a subset of $\mathcal{A}_i$. By *sufficiently generic* we mean that each of the sums (6) is direct, i.e., for each facet F of Δ_ω we have

$$\dim(F) = \dim(F_1) + \dim(F_2) + \cdots + \dim(F_k). \tag{7}$$

Let $\mathrm{Bar}(\Delta_\omega)$ denote the first barycentric subdivision of the mixed decomposition Δ_ω. Each facet σ of $\mathrm{Bar}(\Delta_\omega)$ is incident to a unique point $\mathbf{a} = \mathbf{a}^{(1)} + \ldots + \mathbf{a}^{(k)}$ in $\mathcal{A}$. We define the sign of σ to be the sign vector

$$\mathrm{sign}(\sigma) := (\mathrm{sign}(c_{1,\mathbf{a}^{(1)}}),\ldots,\mathrm{sign}(c_{k,\mathbf{a}^{(k)}})) \in \{-,+\}^k. \tag{8}$$

The set $\{-,0,+\}$ is partially ordered by $0 < -$ and $0 < +$. Let $\{-,0,+\}^k$ denote the product poset. We define the sign of a cell τ of $\mathrm{Bar}(\Delta_\omega)$ to be the infimum in $\{-,0,+\}^k$ of the signs of all facets σ containing τ. Note that this is consistent with (2) for $k=1$. Let $Z_+(\Delta_\omega, f_1,\ldots,f_k)$ denote the subcomplex of $\mathrm{Bar}(\Delta_\omega)$ consisting of all cells τ with $\mathrm{sign}(\tau) = (0,0,\ldots,0)$. Let $Z_+(f_{1,t},\ldots,f_{k,t})$ denote the common zero set of (4) in $(\mathbb{R}_+)^n$. The following result generalizes Theorem 1.

THEOREM 4. *For sufficiently small $t > 0$, the real algebraic set $Z_+(f_{1,t}, \ldots, f_{k,t}) \subset (\mathbb{R}_+)^n$ is homeomorphic to the simplicial complex $Z_+(\Delta_\omega, f_1, \ldots, f_k) \subset \Delta_\omega$.*

PROOF. For each $i = 1, \ldots, k$ there is a surjective morphism of toric varieties

$$\gamma_i : X_{\mathcal{A}} \to X_{\mathcal{A}_i} \tag{9}$$

The morphism γ_i maps the real part $X_{\mathcal{A}}(\mathbb{R})$ onto the real part $X_{\mathcal{A}_i}(\mathbb{R})$, and it maps $X_{\mathcal{A}}(\mathbb{R}_+)$ onto $X_{\mathcal{A}_i}(\mathbb{R}_+)$. The polynomial $f_{i,t}$ is identified with a linear form on $X_{\mathcal{A}_i}$ as in (3). We define $Z_+(f_{i,t})$ to be the zero set in $X_{\mathcal{A}}(\mathbb{R}_+)$ of the composition $f_{i,t} \circ \gamma_i$. Their intersection $\cap_{i=1}^k Z_+(f_{i,t})$ coincides with $Z_+(f_{1,t}, \ldots, f_{k,t})$.

Let $Z_+(\Delta_\omega, f_i)$ denote the subcomplex of $\mathrm{Bar}(\Delta_\omega)$ consisting of those cells τ for which $\mathrm{sign}(\tau)$ is zero in coordinate i. We apply Theorem 1 to any regular triangulation which refines the mixed decomposition Δ_ω. The moment map induces a homeomorphism between $X_{\mathcal{A}}(\mathbb{R}_+)$ and Q. This homeomorphism identifies $Z_+(f_{i,t})$ and $Z_+(\Delta_\omega, f_i)$. Theorem 4 follows by taking the intersection over all $i = 1, \ldots, k$. □

We next state the generalization of Theorem 3 to complete intersections. We define $Z(f_{1,t}, \ldots, f_{k,t})$ to be the set of common zeros of $f_{1,t} \circ \gamma_1, \ldots, f_{k,t} \circ \gamma_k$ in the real toric variety $X_{\mathcal{A}}(\mathbb{R})$. We glue $X_{\mathcal{A}}(\mathbb{R})$ from 2^n disjoint copies Q_ϵ of the polytope Q, using Proposition 2. Each polytope Q_ϵ comes with its own mixed decomposition Δ_ω. By gluing these together we get a cell decomposition Δ'_ω, which we call the *mixed decomposition* of the toric variety $X_{\mathcal{A}}(\mathbb{R})$ induced by ω.

Let $\mathrm{Bar}(\Delta'_\omega)$ denote the first barycentric subdivision of the mixed decomposition. Each facet σ of the simplicial complex $\mathrm{Bar}(\Delta'_\omega)$ lies in a unique Q_ϵ and is incident to a unique point $\mathbf{a} = \mathbf{a}^{(1)} + \ldots + \mathbf{a}^{(k)}$ in $\mathcal{A}$. We define

$$\mathrm{sign}(\sigma) := (\mathrm{sign}(c_{1,\mathbf{a}^{(1)}}\epsilon^{\mathbf{a}^{(1)}}), \ldots, \mathrm{sign}(c_{k,\mathbf{a}^{(k)}}\epsilon^{\mathbf{a}^{(k)}})) \in \{-,+\}^k.$$

For each lower-dimensional cell $\tau \in \mathrm{Bar}(\Delta'_\omega)$ we define $\mathrm{sign}(\tau)$ to be the infimum in $\{-,0,+\}^k$ of the signs of all facets σ containing τ. Let $Z(\Delta_\omega, f_1, \ldots, f_k)$ denote the subcomplex of $\mathrm{Bar}(\Delta'_\omega)$ consisting of all cells τ with $\mathrm{sign}(\tau) = (0, \ldots, 0)$. The next theorem is the main result in this paper. Its proof follows from Theorems 3 and 4.

THEOREM 5. *For sufficiently small $t > 0$, the real algebraic set $Z(f_{1,t}, \ldots, f_{k,t}) \subset X_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to the simplicial complex $Z(\Delta_\omega, f_1, \ldots, f_k) \subset \Delta'_\omega$.*

Theorem 5 applies in particular to complete intersections of hypersurfaces in real projective n-space. Let $\mathcal{A}_i$ of all non-negative integer vectors $(j_1, \ldots, j_n)$ with $j_1 + \ldots + j_n \leq d_i$, where d_i is some positive integer. Hence $f_{i,t}$ is a dense polynomial of degree d_i. The toric variety $X_{\mathcal{A}}(\mathcal{R})$ and each of the toric varieties

$X_{\mathcal{A}_i}(\mathbb{R})$ is isomorphic to $P^n(\mathbb{R})$ via the Veronese embedding. The surjection γ_i in (9) is an isomorphism. Theorem 5 gives a purely combinatorial construction for the real projective $(n-k)$-fold $\{f_{1,t} = \ldots = f_{k,t} = 0\}$.

AN EXAMPLE IN THE PLANE. We illustrate Theorem 5 for the intersection of two curves in the real projective plane. Consider the equations

$$f_t(x,y) := y^3 - txy^2 - t^5x^2y + t^{12}x^3 - ty^2 + t^4xyt^9x^2 - t^5y - t^9x + t^{12}$$

$$g_t(x,y) := t^8y^2 - t^6xy + t^6x^2 - t^3y - t^2x + 1$$

for some very small parameter value $t > 0$. The cubic curve $Z(f_t)$ consists of two ovals. Its intersection $Z_+(f_t)$ with the positive quadrant has four connected components, three of which are unbounded. In each of the other three quadrants $Z_\epsilon(f_t)$ has two unbounded connected components. This information can be read off from the Viro diagram in Figure 1.

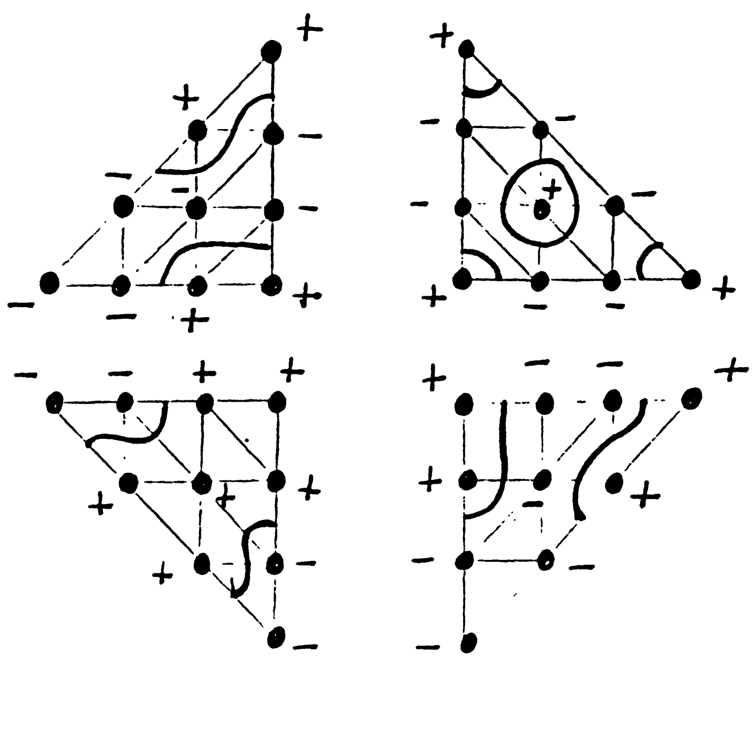

Figure 1. – *A cubic curve in the projective plane*

Similarly, we have a Viro diagram for the quadratic curve $Z(g_t)$:

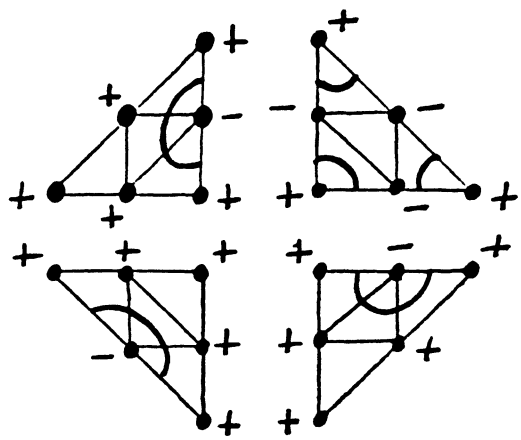

Figure 2. – *A quadratic curve in the projective plane*

Here is the construction of their intersection in the mixed decomposition Δ_ω:

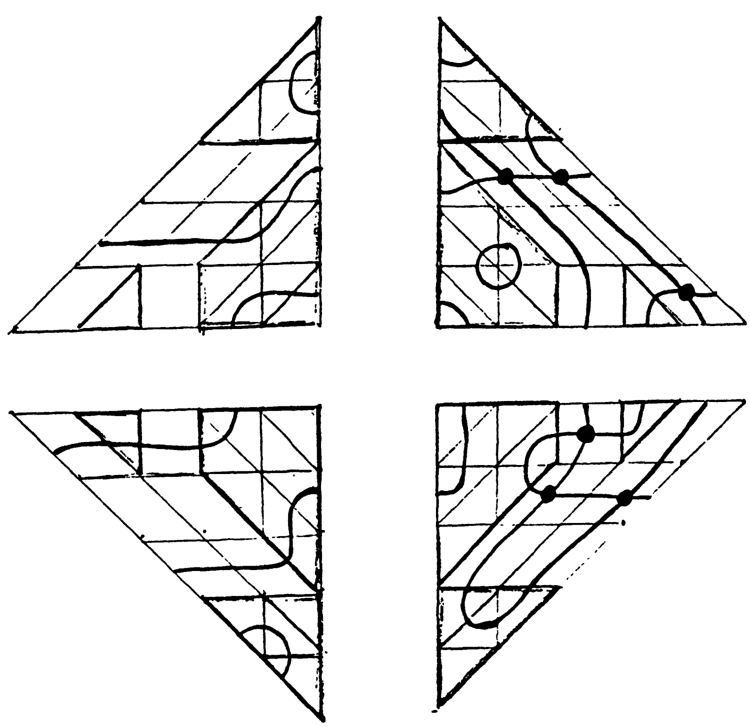

Figure 3. – *Intersection of two curves in the projective plane*

This shows that all six points in $Z(f_t, g_t)$ are real. Three of the points lie in the positive quadrant, while the three others lie in the quadrant indexed by $\epsilon = (+, -)$.

REMARK. An asymptotic analysis of complete intersections in complex projective toric varieties was carried out already by Danilov and Khovanskii in [10, § 6]. Our constructions in Section 2 make this analysis more effective by using mixed decompositions of the Minkowski sum of Newton polytopes. While the emphasis in the present note lies on real varieties, the underlying techniques can be extended to complex varieties as well.

3. - Curves in projective 3-space

We now specialize to the case $k = 2$, $n = 3$ of complete intersection curves in real projective 3-space $P^3(\mathbb{R})$. Consider two equations of degree r and s respectively:

$$(10)\qquad \begin{aligned} f_t(x,y,z) &= \sum_{0\le i+j+k\le r} c_{ijk} t^{\alpha_{ijk}} x^i y^j z^k \\ g_t(x,y,z) &= \sum_{0\le i+j+k\le s} d_{ijk} t^{\beta_{ijk}} x^i y^j z^k \end{aligned}$$

where the c_{ijk}, d_{ijk} are non-zero real numbers, and $\alpha_{ijk}, \beta_{ijk}$ are sufficiently generic integers. For t fixed, let $\mathcal{C}_t$ denote the common zero set of f_t and g_t in $P^3(\mathbb{R})$. For all $t \gg 0$, $\mathcal{C}_t$ is a curve of the same topological type in $P^3(\mathbb{R})$.

Let $\mathcal{A}^{(r)}$ denote the set of lattice points (i, j, k) with $0 \le i + j + k \le r$, and consider the tetrahedron $Q^{(r)} = \mathrm{conv}(\mathcal{A}^{(r)})$. The integers α_{ijk} define a regular triangulation $\Delta^{(r)}$ of $(Q^{(r)}, \mathcal{A}^{(r)})$, and the integers β_{ijk} define a regular triangulation $\Delta^{(s)}$ of $(Q^{(s)}, \mathcal{A}^{(s)})$. Together they define a mixed decomposition Δ of $(Q^{(r+s)}, \mathcal{A}^{(r)} + \mathcal{A}^{(s)})$. By our genericity assumption, each 3-cell of Δ has the form $F_1 + F_2$, where either:

(i) F_1 is a vertex (0-cell) in $\Delta^{(r)}$ and F_2 is a tetrahedron (3-cell) in $\Delta^{(s)}$, or vice versa;

(ii) or F_1 is an edge (1-cell) in $\Delta^{(r)}$ and F_2 is a triangle (2-cell) in $\Delta^{(s)}$, or vice versa.

A cell of type (ii) is a *prism*; it has five 2-faces, two triangles and three *parallelograms*, the latter being 2-cells $E_1 + E_2$ where E_1 is an edge in $\Delta^{(r)}$ and E_2 is an edge in $\Delta^{(s)}$.

For each $\sigma \in \{-, +\}^3$ we place a copy Δ_σ of the subdivided tetrahedron Δ into the orthant indexed σ. The union of the eight tetrahedra Δ_σ is a regular octahedron. By identifying antipodal boundary points of the octahedron, we obtain a polyhedral complex Δ' homeomorphic to $P^3(\mathbb{R})$. We call Δ_σ an *orthant* in Δ'.

Let Γ denote the graph on the set of all prisms in Δ', where two prisms are connected by an edge if an only if they share a parallelogram face. The graph Γ is embedded as a 1-dimensional subcomplex in $\mathrm{Bar}(\Delta')$, the first barycentric subdivision of Δ'. Let Γ_σ denote the restriction of Γ to the orthant Δ_σ. Note that Γ depends only on the integer exponents α_{ijk} and β_{ijk}, but not on the coefficients of the equations (10). The main task in computing the graph Γ is to find the mixed decomposition Δ. This can be done using any convex hull algorithm for points in four dimensions.

We next define a subgraph G of Γ which depends on the signs of the coefficients in (10). Each vertex in Δ is the sum of a unique pair of points (i,j,k) in $\mathcal{A}^{(r)}$ and (i',j',k') in $\mathcal{A}^{(s)}$. The *label* of that vertex is the vector $(\mathrm{sign}(c_{ijk}), \mathrm{sign}(d_{i'j'k'}))$ in $\{-,+\}^2$. The corresponding vertex in $\Delta_\sigma = \Delta_{(\sigma_1,\sigma_2,\sigma_3)}$ inherits the *label*

$$(\mathrm{sign}(c_{ijk})\sigma_1^i\sigma_2^j\sigma_3^k,\ \ \mathrm{sign}(d_{i'j'k'})\sigma_1^i\sigma_2^j\sigma_3^k) \in \{(-,-),(-,+),(+,-),(+,+)\}.$$

A parallelogram in Δ or Δ' is said to be *good* if the labels of its four vertices are distinct, i.e., if the set of labels equals $\{(-,-),(-,+),(+,-),(+,+)\}$. We define G to be the subgraph of Γ consisting of all edges whose parallelograms are good. Hence G is a one-dimensional subcomplex of the first barycentric subdivision of Δ'. We abbreviate $G_\sigma := G \cap \Gamma_\sigma$ for each orthant. Theorem 5 implies the following result.

COROLLARY 6. *For $t \gg 0$, the embedded curve $\mathcal{C}_t \subset P^3(\mathbb{R})$ is homeomorphic to the embedded graph $G \subset \Delta'$. This homeomorphism respects orthants in $P^3(\mathbb{R})$ and in Δ'.*

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