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On the Stationary Motion of Compressible Viscous Fluids

PAOLO SECCHI

1. - Introduction

In this paper we continue our study, see [5], about the stationary motion of a compressible, viscous and heat-conductive fluid in a bounded domain Ω of \mathbb{R}^3 , in the presence of self-gravitation, with the velocity field satisfying a slip boundary condition instead of the usual adherence condition. The corresponding Navier-Stokes equations for the unknown velocity field $u(x) = (u_1(x), u_2(x), u_3(x))$, density $\rho(x)$ and absolute temperature $\Theta(x)$ are

$$(1.1) \quad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla p(\rho, \Theta) = \rho[f - (u \cdot \nabla)u - \nabla U], \\ \operatorname{div}(\rho u) = 0, \\ -\chi\Delta\Theta + c_v\rho u \cdot \nabla\Theta + \Theta p'_\Theta \operatorname{div} u = \rho g + \alpha(u) \quad \text{in } \Omega. \end{cases}$$

Here the pressure $p = p(\rho, \Theta)$ is a known smooth function of ρ and Θ ; U is the Newtonian gravitational potential given by

$$(1.2) \quad U(x) = -\gamma \int_{\Omega} \frac{\rho(y)}{|x-y|} dy,$$

where γ is the constant of gravitation; μ is the shear viscosity and $\nu = \mu + \mu'$, where μ' is the bulk viscosity; χ is the coefficient of heat conductivity and c_v is the specific heat at constant volume. In order to avoid technicalities we will assume that the coefficients μ, ν, χ, c_v are constant. In general, μ and μ' must satisfy the physical constraints $\mu \geq 0$, $2\mu + 3\mu' \geq 0$; the latter implies $\nu \geq \mu/3$. Since the fluid is viscous we will assume $\mu > 0$ and also $\nu > \frac{\mu}{3}$, $\chi > 0$, $c_v > 0$

(see Remark (iii) at the end of Section 3). Finally, f denotes the given external force field, g the given heat supply and $\alpha = \alpha(u)$ the dissipation function

$$\alpha(u) = 2\mu T(u) : T(u) + (\nu - \mu)(\operatorname{div} u)^2$$

where $T(u) = \frac{1}{2} (D_i u_j + D_j u_i)_{1 \leq i, j \leq 3}$ is the deformation tensor and

$$T(u) : T(v) = \frac{1}{4} \sum_{i, j=1}^3 (D_i u_j + D_j u_i)(D_i v_j + D_j v_i)$$

with $D_i = \frac{\partial}{\partial x_i}$. Since the total mass of the fluid is given, we impose the condition

$$(1.3) \quad \frac{1}{|\Omega|} \int_{\Omega} \rho(x) dx = m_0$$

where $m_0 > 0$ is given. On the boundary $\Gamma \equiv \partial\Omega$, instead of the usual adherence condition $u = 0$, we impose for u the slip boundary condition

$$(1.4) \quad \begin{aligned} u \cdot n &= 0, \\ t_i \cdot T(u) \cdot n &= 0 \quad \text{on } \Gamma, \quad i = 1, 2, \end{aligned}$$

where n is the unit outward normal vector to Γ and t_1, t_2 span the tangent plane. For Θ we impose the Dirichlet condition

$$(1.5) \quad \Theta = \Theta_e \quad \text{on } \Gamma$$

(for other boundary conditions for Θ see Remark (ii) at the end of Section 3).

In our previous paper [5] we proved the existence of a unique solution (u, ρ, Θ) in the Sobolev spaces $W^{j+2,p} \times W^{j+1,p} \times W^{j+2,p}$, for any integer $j \geq 1$ and real $p > 3$, provided that the data $(f, g, \Theta_e) \in W^{j,p} \times W^{j,p} \times W^{j+2-\frac{1}{p},p}(\Gamma)$ belong to a suitable neighbourhood of $(0, 0, \Theta_0)$, $\Theta_0 = \text{const} > 0$, and that γ is sufficiently small. The purpose of the present paper is to cover also the case $j = 0$, that is we prove the existence of a solution (u, ρ, Θ) in $W^{2,p} \times W^{1,p} \times W^{2,p}$ for small enough data $(f, g, \Theta_e - \Theta_0) \in L^p \times L^p \times W^{2-\frac{1}{p},p}(\Gamma)$ and small enough γ (for a different regularity of the temperature Θ see Remark (i) at the end of Section 3).

As in [5] the core of the paper is the study of the linearized system (2.1) for (u, σ) , $\sigma = \rho - \rho_0$, where ρ_0 is the equilibrium state. In order to solve it we introduced an equivalent formulation of (2.1). Such a formulation, in the present context of a solution (u, σ) in $W^{2,p} \times W^{1,p}$, loses its meaning because of the lower regularity (see in particular (2.24), (2.25) in [5]). We overcome this difficulty by introducing a different approach which gives us the density as

solution of a linear transport equation, obtained in turn as solution of a Neumann problem. The result is obtained without resorting to weak formulations of the equations for σ .

Moreover, this new approach applies as well to the case $j \geq 1$ already considered in [5], without additional difficulties (see the Remark at the end of Section 2).

Before stating our main result let us introduce some notation. By $c, C, C_i, k_i, i \geq 0$, we denote positive constants depending at most on Ω, j, p , unless explicitly stated otherwise.

We denote by $W^{j,p}$, j a positive integer, $1 < p \leq +\infty$, the Sobolev space $W^{j,p}(\Omega)$, endowed with the usual norm $\|\cdot\|_{j,p}$. For real $s > 0$, $W^{s,p}$ denotes the Sobolev space $W^{s,p}(\Omega)$ of fractional order s with norm $\|\cdot\|_{s,p}$ (for the definition see [1]). The norm in $L^p = L^p(\Omega)$ is denoted by $|\cdot|_p$, $1 \leq p \leq +\infty$. If $p = 2$ we write $W^{j,2} = H^j$ whose norm is simply denoted by $\|\cdot\|_j$; the norm of $L^2 = H^0$ is denoted by $\|\cdot\|$ ($= |\cdot|_2$). On the boundary we use trace spaces $W^{j-\frac{1}{p},p}(\Gamma)$ with norm $\|\cdot\|_{j-\frac{1}{p},p,\Gamma}$. For convenience we use the same symbols for spaces of vector-valued functions. We denote by $\overline{W}^{j,p}$ the space of scalar functions $\{\sigma \in W^{j,p} : \bar{\sigma} = 0\}$ where $\bar{\sigma}$ is the mean value of σ over Ω . We denote by $W_b^{j,p}$ the space of vector-valued functions u in $W^{j,p}$ such that $u \cdot n = 0$, $t_i \cdot T(u) \cdot n = 0$ on $\Gamma, i = 1, 2$ (here $j \geq 2$). Let us introduce the space $H = \{u \in H^1 : u \cdot n = 0$ on $\Gamma\}$ endowed by the norm $\|u\|_H^2 = \|\nabla u\|^2 = \sum_{i,j=1}^3 \|D_i u_j\|^2$. In H this norm is equivalent to the H^1 -norm since

$$\|u\| \leq k_0 \|u\|_H \quad \text{for all } u \in H,$$

see [8]. Let us denote by H' its dual space with norm $\|\cdot\|_{H'}$. Associated to the linear problem

$$(1.6) \quad \begin{cases} \mu \Delta u - \nu \nabla \operatorname{div} u = f & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \end{cases}$$

let us consider the following variational problem: find $u \in H$ such that

$$(1.7) \quad a(u, v) := 2\mu \int_{\Omega} T(u) : T(v) + (\nu - \mu) \int_{\Omega} \operatorname{div} u \operatorname{div} v = \int_{\Omega} f \cdot v$$

for any $v \in H$. Observe that the bilinear form $a(u, v)$ is bicontinuous in H .

To obtain the coerciveness in H of $a(u, v)$ we must exclude the rigid body motions

$$S = \{u \in H : T(u) = 0\} = \{u = b \wedge (x - x_0) : u \cdot n = 0 \text{ on } \Gamma\}$$

from H provided that $S \neq \emptyset$, i.e. Ω is a body of revolution around its axis of symmetry $b \in \mathbb{R}^3$. If $S \neq \emptyset$ we let \mathbb{H} denote the subspace of vectors in H which are orthogonal to rigid motions. If $S = \emptyset$, then $\mathbb{H} = H$. For each $u \in \mathbb{H}$ we have Korn's inequality

$$(1.8) \quad \|u\|_H^2 \leq k_1 \int_{\Omega} T(u) : T(u),$$

see [8]. For the sake of simplicity we assume in our main Theorem 1 that $S = \emptyset$. Partial results in the case of domains Ω with symmetry can be obtained as in Section 4 of [5].

Let us now introduce the equilibrium solutions. By an equilibrium solution we mean a regular solution (u, ρ, Θ) of (1.1)-(1.5) in the case $f \equiv 0$, $g \equiv 0$ in Ω , such that $u \equiv 0$ in Ω , $\Theta \equiv \Theta_0 = \text{const} > 0$ in Ω and $\rho > 0$ in $\bar{\Omega}$. Hence ρ solves

$$(1.9) \quad \begin{cases} \rho \nabla U + \nabla p(\rho, \Theta_0) = 0 & \text{in } \Omega, \\ U(x) = -\gamma \int_{\Omega} \frac{\rho(y)}{|x-y|} dy & x \in \Omega. \end{cases}$$

From [5] we have:

PROPOSITION 1. *Let $p > 3$ and assume that $\Gamma \in C^2$, $p'_\rho(\rho, \Theta_0) > 0$ for $\rho > 0$. Then, given $\varepsilon > 0$ there exists $\gamma_0 > 0$ such that for any $0 \leq \gamma \leq \gamma_0$ there exists a solution $\rho_0 \in W^{2,p}$ of (1.9) such that $\rho_0 > 0$ in $\bar{\Omega}$ and*

$$(1.10) \quad \|\nabla \rho_0\|_{1,p} \leq \varepsilon.$$

Let us state now our main result.

THEOREM 1. *Let $p > 3$. Let us assume that $\Gamma \in W^{4-\frac{1}{p},p}$ and that Ω has no axis of symmetry, i.e. $S = \emptyset$. Let $p \in C^3$ with $p'_\rho(\rho, \Theta_0) > 0$ for $\rho > 0$. Let $(f, g, \Theta_e) \in L^p \times L^p \times W^{2-\frac{1}{p},p}(\Gamma)$. There exist positive constants c_0, γ_0 such that if*

$$0 \leq \gamma \leq \gamma_0, \quad |f|_p + |g|_p + \|\Theta_e - \Theta_0\|_{2-\frac{1}{p},p,\Gamma} \leq c_0,$$

then there exists a unique solution $(u, \rho, \Theta) \in W^{2,p} \times W^{1,p} \times W^{2,p}$ of problem (1.1)-(1.5).

Let (ρ_0, Θ_0) be an equilibrium solution with $\bar{\rho}_0 = m_0$ and let U_0 denote the gravitational potential corresponding to ρ_0 ; define $\sigma = \rho - \rho_0$, $\theta = \Theta - \Theta_0$. Let us write

$$p(\rho, \Theta) = p(\rho_0 + \sigma, \Theta_0 + \theta) = p(\rho_0, \Theta_0) + \pi\sigma + \pi_0\theta + \omega(\sigma, \theta)$$

where $\pi \equiv p'_\rho(\rho_0, \Theta_0) > 0$, $\pi_0 \equiv p'_\Theta(\rho_0, \Theta_0)$, $\omega(0, 0) = 0$, $\omega(\sigma, \theta) = 0(|\sigma|^2 + |\theta|^2)$ as

$|\sigma| + |\theta| \rightarrow 0$. Problem (1.1)-(1.5) can be written as

$$(1.11) \quad \begin{cases} \mu \Delta u - \nu \nabla \operatorname{div} u + \nabla(\pi \sigma) = F(u, \sigma, \theta) & \text{in } \Omega, \\ \operatorname{div}(m_0 + \sigma)u = E(u) & \text{in } \Omega, \\ \chi \Delta \theta = G(u, \sigma, \theta) & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \\ \theta = \theta_e & \text{on } \Gamma, \\ \bar{\sigma} = 0, \end{cases}$$

where, by definition

$$(1.12) \quad \begin{cases} F(u, \sigma, \theta) = (\rho_0 + \sigma)[f - (u \cdot \nabla)u - \nabla U] + \rho_0 \nabla U_0 \\ \quad - \nabla[\pi_0 \theta + \omega(\sigma, \theta)], \\ U(x) = -\gamma \int_{\Omega} \frac{\rho_0(y) + \sigma(y)}{|x - y|} dy, \\ E(u) = \operatorname{div}(m_0 - \rho_0)u, \\ G(u, \sigma, \theta) = -c_v(\rho_0 + \sigma)u \cdot \nabla \theta \\ \quad + \frac{\Theta_0 + \theta}{\rho_0 + \sigma} p'_{\Theta}(\rho_0 + \sigma, \Theta_0 + \theta)u \cdot \nabla(\rho_0 + \sigma) \\ \quad + (\rho_0 + \sigma)g + \alpha(u), \\ \theta_e = \Theta_e - \Theta_0. \end{cases}$$

Observe that (1.11)₂ is used to deduce the expression of G .

The plan of the paper is the following: in Section 2 we study the linearized system (2.1) while in Section 3 we consider the nonlinear problem (1.11) and prove Theorem 1.

2. - The linearized system

In this section we study the linear system

$$(2.1) \quad \begin{cases} \mu \Delta u - \nu \nabla \operatorname{div} u + \nabla(\pi \sigma) = F & \text{in } \Omega, \\ \operatorname{div}(m_0 u + \sigma v) = E & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \\ \bar{\sigma} = 0. \end{cases}$$

Here we assume that the given vector field v satisfies

$$(2.2) \quad v \cdot n = 0 \quad \text{on } \Gamma,$$

and that the given function E satisfies the necessary compatibility condition $\overline{E} = 0$.

THEOREM 2. *Let $p > 3$, $\Gamma \in W^{4-\frac{1}{p}, p} \subset C^3$, $S = \emptyset$, $p(\rho, \Theta) \in C^2$. Assume that $\rho_0 \in W^{2,p}$ with $\rho_0 > 0$ in $\overline{\Omega}$, $F \in L^p$, $E \in \overline{W}^{1,p}$ and let $v \in W^{2,p}$ satisfy (2.2). There exist positive constants k_2, k_3 such that if*

$$(2.3) \quad \|\nabla \rho_0\|_{1,p} \leq k_2, \quad \|v\|_{2,p} \leq k_3,$$

then there exists a unique solution $(u, \sigma) \in W_b^{2,p} \times \overline{W}^{1,p}$ of problem (2.1). Moreover

$$(2.4) \quad \|u\|_{2,p} + \|\sigma\|_{1,p} \leq C_0 (\|F\|_p + \|E\|_{1,p})$$

where C_0 depends on $\Omega, p, \mu, \nu, \pi, \|\rho_0\|_{2,p}$.

PROOF. We prove this result by the continuity method. The first step consists in proving an a priori estimate for a solution (u, σ) in $H^1 \times L^2$.

LEMMA 2.1. *If v is sufficiently small, see (2.11), then a solution (u, σ) in $H^2 \times H^1$ of (2.1) satisfies*

$$(2.5) \quad \|u\|_H \leq A (\|F\|_{H'} + \|E\|),$$

$$(2.6) \quad \|\sigma\| \leq C_1 \left| \frac{1}{\pi} \right|_{\infty} (\|F\|_{H'} + \|u\|_H),$$

where

$$A = \frac{2}{\mu_*} \max \left\{ \frac{2}{\mu_*} + \frac{\mu_*}{4}, \frac{4}{m_0^2 \mu_*} C_1^2 |\pi|_{\infty}^2 \left| \frac{1}{\pi} \right|_{\infty}^2 \right\} + \frac{C_1}{m_0 \mu_*} |\pi|_{\infty} \left| \frac{1}{\pi} \right|_{\infty},$$

$$\mu_* = \frac{1}{k_1} \min \{2\mu, 3\nu - \mu\} > 0.$$

PROOF. We first multiply (2.1)₁ by u and integrate over Ω . Integrating by parts and using Korn's inequality give

$$(2.7) \quad \mu_* \|u\|_H^2 \leq \int F \cdot u - \int \nabla(\pi\sigma) \cdot u$$

where \int denotes integration over Ω . Using (2.1)₂ gives

$$(2.8) \quad \int \nabla(\pi\sigma) \cdot u = \frac{1}{2} \int (\pi\sigma)^2 \operatorname{div} \left(\frac{v}{\pi m_0} \right) - \int \frac{\pi\sigma}{m_0} E;$$

hence from (2.7), (2.8) we obtain

$$(2.9) \quad \frac{\mu_*}{2} \|u\|_H^2 \leq \frac{1}{\mu_*} \|F\|_{H'}^2 + \frac{1}{2} \left| \operatorname{div} \left(\frac{v}{\pi m_0} \right) \right|_\infty \|\pi\sigma\|^2 + \frac{1}{m_0} \|E\| \|\pi\sigma\|.$$

Since from (2.1)₁ we have

$$(2.10) \quad \begin{aligned} \|\sigma\| &= \sup_{\psi \in H} \frac{\left| \int \sigma \operatorname{div} \psi \right|}{\|\operatorname{div} \psi\|} \\ &\leq \left| \frac{1}{\pi} \right|_\infty \sup_{\psi \in H} \frac{\left| \int F \cdot \psi - a(\psi, u) \right|}{\|\operatorname{div} \psi\|} \leq C_1 \left| \frac{1}{\pi} \right|_\infty (\|F\|_{H'} + \|u\|_H), \end{aligned}$$

which gives (2.6), from (2.9), (2.10) we obtain (2.5) if

$$(2.11) \quad 2C_1^2 |\pi|_\infty^2 \left| \frac{1}{\pi} \right|_\infty^2 \left| \operatorname{div} \left(\frac{v}{\pi m_0} \right) \right|_\infty \leq \frac{\mu_*}{4}$$

(see [5] for details). \square

The next step consists in proving an a priori estimate of a solution in $W^{2,p} \times W^{1,p}$.

LEMMA 2.2. *If v is small enough, see (2.23), then a solution $(u, \sigma) \in W^{2,p} \times W^{1,p}$ of (2.1) satisfies (2.4).*

PROOF. Since for the below computations at least one more derivative is needed, we approximate u, σ, F, E by more regular functions. First of all we observe that $W_b^{3,p}$ is dense in $W_b^{2,p}$. Indeed, for $u \in W_b^{2,p}$ let $w_m \in W^{3,p}$ be such that $w_m \rightarrow u$ in the topology of $W^{2,p}$. We solve the following trace problem: find $z_m \in W^{3,p}$ such that

$$\begin{aligned} z_m \cdot n &= w_m \cdot n \\ t_i \cdot T(z_m) \cdot n &= t_i \cdot T(w_m) \cdot n, \quad i = 1, 2, \end{aligned}$$

on Γ . We have

$$\|z_m\|_{j,p} \leq C(\|w_m \cdot n\|_{j-1/p,p,\Gamma} + \sum_{i=1,2} \|t_i \cdot T(w_m) \cdot n\|_{j-1-\frac{1}{p},p,\Gamma}), \quad j = 2, 3,$$

which implies $z_m \rightarrow 0$ in $W^{2,p}$. Hence $u_m = w_m - z_m \in W_b^{3,p}$, $u_m \rightarrow u$ in $W^{2,p}$.

Moreover, $\overline{W}^{2,p}$ is dense in $\overline{W}^{1,p}$. Indeed, for $\sigma \in \overline{W}^{1,p}$ let $\tau_m \in W^{2,p}$ be such that $\tau_m \rightarrow \sigma$ in $W^{1,p}$. Then $\sigma_m = \tau_m - \bar{\tau}_m \in \overline{W}^{2,p}$ and $\sigma_m \rightarrow \sigma$ in $W^{1,p}$.

Given F, E, v as in Theorem 2 and a solution $(u, \sigma) \in W_b^{2,p} \times \overline{W}^{1,p}$ let us consider $u_m \in W_b^{3,p}$ with $u_m \rightarrow u$ in $W^{2,p}$, $\sigma_m \in \overline{W}^{2,p}$ with $\sigma_m \rightarrow \sigma$ in $W^{1,p}$,

$F_m \in W^{1,p}$ with $F_m \rightarrow F$ in L^p , $E_m \in \overline{W}^{2,p}$ with $E_m \rightarrow E$ in $W^{1,p}$ as $m \rightarrow +\infty$. For F, F_m let us consider the decompositions $F = \varphi + \nabla\psi$, $\varphi \in L^p$ with $\operatorname{div} \varphi = 0$ in Ω , $\varphi \cdot n = 0$ on Γ , $\psi \in \overline{W}^{1,p}$, $F_m = \varphi_m + \nabla\psi_m$, $\varphi_m \in W^{1,p}$ with $\operatorname{div} \varphi_m = 0$ in Ω , $\varphi_m \cdot n = 0$ on Γ , $\psi_m \in \overline{W}^{2,p}$; we have $\psi_m \rightarrow \psi$ in $W^{1,p}$ as $m \rightarrow +\infty$. From (2.1)₂ we deduce that $\operatorname{div}(\sigma v) \in \overline{W}^{1,p}$; let $a_m \in \overline{W}^{2,p}$ be such that $a_m \rightarrow \operatorname{div}(\sigma v)$ in $W^{1,p}$. For these approximations let us introduce the differences δ_m, ε_m defined by

$$(2.12) \quad \mu \Delta u_m - \nu \nabla \operatorname{div} u_m + \nabla(\pi \sigma_m) = F_m + \delta_m,$$

$$(2.13) \quad \operatorname{div}(m_0 u_m) + a_m = E_m + \varepsilon_m,$$

$\delta_m \in W^{1,p}$, $\varepsilon_m \in \overline{W}^{2,p}$, $\delta_m \rightarrow 0$ and $\varepsilon_m \rightarrow 0$ in L^p and $W^{1,p}$ respectively as $m \rightarrow +\infty$. Applying the div operator to (2.12) and the laplacian to (2.13) give respectively

$$(2.14) \quad \begin{aligned} (\nu + \mu) \Delta \operatorname{div} u_m + \Delta(\pi \sigma_m) &= \Delta \psi_m + \operatorname{div} \delta_m, \\ m_0 \Delta \operatorname{div} u_m + \Delta a_m &= \Delta(E_m + \varepsilon_m). \end{aligned}$$

We eliminate $\Delta \operatorname{div} u_m$ from (2.14) and obtain

$$(2.15) \quad \Delta W_m = \operatorname{div} \left(\delta_m + \frac{\nu + \mu}{m_0} \nabla \varepsilon_m \right) \quad \text{in } \Omega,$$

where $W_m = \pi \sigma_m + \frac{\nu + \mu}{m_0} (a_m - E_m) - \psi_m$. Taking the scalar product on Γ of (2.12) times n and applying the normal derivate $\partial/\partial n$ to (2.13) give

$$(2.16) \quad \begin{aligned} \mu \Delta u_m \cdot n - \nu \frac{\partial}{\partial n} \operatorname{div} u_m + \frac{\partial}{\partial n} (\pi \sigma_m) &= \frac{\partial \psi_m}{\partial n} + \delta_m \cdot n, \\ m_0 \frac{\partial}{\partial n} \operatorname{div} u_m + \frac{\partial a_m}{\partial n} &= \frac{\partial}{\partial n} (E_m + \varepsilon_m). \end{aligned}$$

Now we observe that the boundary conditions (2.1)_{3,4} imply that $\Delta u_m \cdot n - \frac{\partial}{\partial n} \operatorname{div} u_m$ does not contain second order derivatives of u_m ; indeed if the boundary is flat this difference is equal to zero, and in the general case this fact can be proved with a long but straightforward computation. Hence we can introduce a vector function h_1 and a matrix function h_2 such that

$$(2.17) \quad \mu \Delta u_m \cdot n = \mu \frac{\partial}{\partial n} \operatorname{div} u_m + h_1 \cdot u_m + h_2 : \nabla u_m \quad \text{on } \Gamma.$$

h_1 contains at most second order derivatives of n, t_1, t_2 , hence $h_1 \in W^{1-\frac{1}{p},p}(\Gamma)$; h_2 contains at most first order derivatives, hence $h_2 \in W^{2-\frac{1}{p},p}(\Gamma)$.

From (2.16), (2.17) we obtain

$$(2.18) \quad \frac{\partial W_m}{\partial n} = h_1 \cdot u_m + h_2 : \nabla u_m + \left(\delta_m + \frac{\nu + \mu}{m_0} \nabla \varepsilon_m \right) \cdot n \quad \text{on } \Gamma.$$

We multiply (2.15) by $\phi \in W^{1,q}$, $\frac{1}{q} + \frac{1}{p} = 1$, integrate over Ω by parts and use (2.18). Passing to the limit as $m \rightarrow +\infty$ gives

$$(2.19) \quad \int_{\Gamma} \nabla W \cdot \nabla \phi = \int_{\Gamma} (h_1 \cdot u + h_2 : \nabla u) \phi \quad \text{for any } \phi \in W^{1,q},$$

where $W = \pi \sigma + \frac{\nu + \mu}{m_0} (\operatorname{div}(\sigma v) E) - \psi \in W^{1,p}$ and \int_{Γ} denotes integration over Γ . Both sides of (2.19) define a linear continuous functional on $W^{1,q}$. The norm of the functional $\phi \mapsto \int_{\Gamma} \nabla W \cdot \nabla \phi$ is $|\nabla W|_p$. Given $s, \frac{1}{p} < s < 1$, $u \in W^{2,p} \subset W^{1+s,p}$ gives $h_1 \cdot u + h_2 : \nabla u \in W^{s-\frac{1}{p},p}(\Gamma) \subset L^p(\Gamma)$. Hence the norm of the functional $\phi \mapsto \int_{\Gamma} (h_1 \cdot u + h_2 : \nabla u) \phi$ can be estimated by $c\|u\|_{1+s,p}$. Then equality (2.19) implies

$$(2.20) \quad |\nabla W|_p \leq c\|u\|_{1+s,p}.$$

From the Poincaré inequality and the fact that $\operatorname{div}(\sigma v), E, \psi$ have mean value zero we deduce

$$|W|_p \leq |W - \overline{W}|_p + |\overline{W}|_p \leq c(|\nabla W|_p + |\pi|_{\infty} \|\sigma\|).$$

Then from the above inequality and (2.20) we obtain

$$(2.21) \quad \|W\|_{1,p} \leq c(\|u\|_{1+s,p} + |\pi|_{\infty} \|\sigma\|).$$

Consider now the linear transport equation

$$(2.22) \quad \pi \sigma + \frac{\nu + \mu}{m_0} \operatorname{div}(\sigma v) = \psi + \frac{\nu + \mu}{m_0} E + W \equiv \Lambda \quad \text{in } \Omega.$$

From [3], see in particular Theorem 2.3 and part (i) of the proof of Theorem 1.1, we have that, since $p > 3$, if

$$(2.23) \quad \frac{\nu + \mu}{m_0} C_2 \left\| \frac{v}{\pi} \right\|_{2,p} < \frac{1}{2},$$

where C_2 is a suitable constant depending only on Ω, p , then for any $\Lambda \in W^{1,p}$, there exists a unique solution $\pi\sigma \in W^{1,p}$ of (2.22) and

$$\|\pi\sigma\|_{1,p} \leq 2\|\Lambda\|_{1,p}.$$

Using (2.21) we obtain

$$(2.24) \quad \|\pi\sigma\|_{1,p} \leq c \left(|F|_p + \frac{\nu + \mu}{m_0} \|E\|_{1,p} + \|u\|_{1+s,p} + |\pi|_\infty \|\sigma\| \right).$$

Consider now the elliptic system

$$(2.25) \quad \begin{cases} \mu\Delta u - \nu\nabla \operatorname{div} u = F - \nabla(\pi\sigma) & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2. \end{cases}$$

The weak formulation of (2.25) is (1.7) with $f = F - \nabla(\pi\sigma)$, where $a(u, v)$ is a bilinear form, bicontinuous and coercive in H . The boundary conditions are complementing in the sense of Agmon, Douglis and Nirenberg [2]. Hence the solution u belongs to $W^{2,p}$ if $F - \nabla(\pi\sigma) \in L^p$ and moreover

$$(2.26) \quad \|u\|_{2,p} \leq c(|F|_p + \|\pi\sigma\|_{1,p})$$

holds. From (2.24), (2.26) we obtain that $\|u\|_{2,p}$ can be estimated by the right-hand side of (2.24). Now, from $W^{2,p} \subset W^{1+s,p} \subset H$, since in particular the first imbedding is compact, we deduce that for any positive ε there exists a constant $c(\varepsilon)$ such that

$$\|u\|_{1+s,p} \leq \varepsilon \|u\|_{2,p} + c(\varepsilon) \|u\|_H.$$

For ε small enough, taking account of (2.5), (2.6), we then obtain

$$\|u\|_{2,p} \leq c \left[|F|_p + \frac{\nu + \mu}{m_0} \|E\|_{1,p} + \left(A + |\pi|_\infty \left| \frac{1}{\pi} \right|_\infty (1 + A) \right) (\|F\|_{H'} + \|E\|) \right]$$

and from (2.24)

$$\|\sigma\|_{1,p} \leq c \left\| \frac{1}{\pi} \right\|_{1,p} \left[|F|_p + \frac{\nu + \mu}{m_0} \|E\|_{1,p} + \left(A + |\pi|_\infty \left| \frac{1}{\pi} \right|_\infty (1 + A) \right) (\|F\|_{H'} + \|E\|) \right]$$

which gives the thesis. \square

The rest of the proof is as in [5]; we briefly recall the main steps, see [5] for details.

(i) We first prove the existence of a solution of (2.1) in the particular case of μ/ν sufficiently small. We define $q = \frac{1}{\mu}(\pi_1\sigma - \nu \operatorname{div} u)$ where $\pi_1 = p'_\rho(m_0, \Theta_0) > 0$. Then (2.1) is transformed in the following Stokes problem and linear transport equation

$$(2.27) \quad \begin{cases} \Delta u + \nabla q = \frac{1}{\mu} [F + \nabla(\pi_1 - \pi)\sigma] & \text{in } \Omega, \\ \operatorname{div} u = \frac{\mu}{\nu} \left(\frac{\pi_1}{\mu} \sigma - q \right) & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \\ \bar{q} = 0, \end{cases}$$

$$(2.28) \quad \begin{cases} \frac{\pi_1}{\nu} \sigma + \operatorname{div} \left(\frac{1}{m_0} \sigma v \right) = \frac{\mu}{\nu} q + \frac{1}{m_0} E & \text{in } \Omega, \\ \bar{\sigma} = 0. \end{cases}$$

We solve (2.27), (2.28) by finding a fixed point of the map $\Psi_0 : (\sigma^*, q^*) \mapsto (\sigma, q)$ in the square $\Sigma_0 = \{(\sigma^*, q^*) \in \overline{W}^{1,p} \times \overline{W}^{1,p} : \|\sigma^*\|_{1,p} \leq B, \|q^*\|_{1,p} \leq B\}$ where (σ, q) is the solution of (2.27), (2.28) for $(\sigma^*, q^*) \in \Sigma_0$ inserted in the right-hand side and B is chosen large enough. If $|\nabla \rho_0|_p \geq c\|\pi - \pi_1\|_{1,p}$, $\frac{\nu}{m_0\pi_1} \|\psi\|_{2,p}$ and $\frac{\mu}{\nu}$ are small enough the map Ψ_0 is a contraction in Σ_0 . Then, there exists a unique fixed point, that is a solution of (2.27), (2.28), with u solution of (2.27) corresponding to the fixed point $\sigma^* = \sigma$, $q^* = q$.

(ii) Secondly we consider the general case, with no restriction on the viscosity coefficients μ and ν ; we prove the existence of a solution of (2.1) by the continuity method. Choose μ_0, ν_0 such that μ_0/ν_0 is so small that the result proved in (i) holds. For $\tau \in [0, 1]$ define

$$\begin{aligned} \mu_\tau &= (1 - \tau)\mu_0 + \tau\mu, & \nu_\tau &= (1 - \tau)\nu_0 + \tau\nu, \\ L_\tau(u, \sigma) &= (-\mu_\tau \Delta u - \nu_\tau \nabla \operatorname{div} u + \nabla(\pi\sigma), \operatorname{div}(m_0 u + \sigma v)), \\ X &= W_b^{2,p} \times \overline{W}^{1,p} & Y &= L^p \times \overline{W}^{1,p}. \end{aligned}$$

Consider the set

$$T = \{\tau \in [0, 1] : \text{for each } (F, E) \in Y \text{ there exists a unique solution } (u, \sigma) \in X \text{ of } L_\tau(u, \sigma) = (F, E)\}.$$

Since $0 \in T$, T is not empty. Using (2.4) we prove that T is open and closed, i.e. $T \equiv [0, 1]$. Then for each $(F, E) \in Y$ there exists a solution $(u, \sigma) \in X$ of (2.1). From the linearity of the problem and (2.4) the uniqueness of the solution follows. This complete the proof of Theorem 2. \square

REMARK. The same approach can be followed also for obtaining solutions $(u, \sigma) \in W^{j+2,p} \times \overline{W}^{j+1,p}$, $j \geq 1$. In that case the proof is simplified since it is not necessary, due to the higher regularity, to introduce the approximations u_m, σ_m, \dots and in (2.20) (and below) it is sufficient to consider $\|u\|_{j+1,p}$ instead of a norm of fractional order. In particular, (2.21) can be substituted by $\|W\|_{j+1,p} \leq c(\|u\|_{j+1,p} + |\pi|_\infty \|\sigma\|)$ (see also [5]).

3. - Proof of Theorem 1

Since the proof is essentially the same as in [5] we give just a sketch of it. We solve (1.11) by finding a fixed point of the map $\Psi : (v, \sigma^*, \theta^*) \mapsto (u, \sigma, \theta)$, where (u, σ, θ) is the solution of (1.11) with $F(v, \sigma^*, \theta^*), G(v, \sigma^*, \theta^*)$ in the right-hand side and the equation $\operatorname{div}(m_0 u + \sigma v) = E(u^*)$ instead of $\operatorname{div}(m_0 + \sigma)u = E(u)$.

We consider the set

$$\Sigma = \left\{ (u, \sigma, \theta) \in W_b^{2,p} \times \overline{W}^{1,p} \times W^{2,p} : \|u\|_{2,p} + \|\sigma\|_{1,p} + \|\theta\|_{2,p} \leq k_4 \right\},$$

where $k_4 \leq k_3$ is such that $|\sigma|_\infty \leq c\|\sigma\|_{1,p} \leq ck_4 \leq \frac{1}{2} \min_{\overline{\Omega}} \rho_0(x)$. The first step consists in proving that $\Psi(\Sigma) \subseteq \Sigma$. This follows using Theorem 2, estimate (2.4) and a well-known estimate for the Dirichlet problem (1.11)_{3,6} under the requirement that $\gamma_0, |f|_p, |g|_p, \|\Theta_e - \Theta_0\|_{2-\frac{1}{p},p,\Gamma}, k_4$ are sufficiently small. Observe that the requirement that γ_0 is small implies, by Proposition 1, that $\nabla \rho_0$ is small. The second step consists in estimating the difference $(u_1 - u_2, \sigma_1 - \sigma_2, \theta_1 - \theta_2)$ for $(u_i, \sigma_i, \theta_i) = \Psi(v_i, \sigma_i^*, \theta_i^*), (v_i, \sigma_i^*, \theta_i^*) \in \Sigma$. For such differences we consider the norms $\|u_1 - u_2\|_H, \|\sigma_1 - \sigma_2\|, \|\theta_1 - \theta_2\|_1$ which we estimate using (2.5), (2.6) and $\|\theta_1 - \theta_2\|_1 \leq c\|G_1 - G_2\|_{-1}$ where $G_i = G(v_i, \sigma_i^*, \theta_i^*)$ and $\|\cdot\|_{-1}$ denotes the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$. Again, provided that $\gamma_0, |f|_p, |g|_p, k_4$ are sufficiently small, we prove that Ψ is a contraction in Σ with respect to a suitable norm in $H \times L^2 \times H^1$. Hence there exists a unique fixed point in Σ of the map Ψ , i.e. a solution of (1.11). This completes the proof. \square

REMARKS. (i) If in theorem 1 we assume $(g, \Theta_e) \in W^{1,p} \times W^{3-\frac{1}{p},p}(\Gamma)$ (instead of $(g, \Theta_e) \in L^p \times W^{2-\frac{1}{p},p}(\Gamma)$) we obtain $\Theta \in W^{3,p}$.

(ii) Results similar to Theorem 1 can be obtained if we consider for the temperature Θ , instead of a Dirichlet boundary condition, either a Neumann b.c. $\frac{\partial \Theta}{\partial n} = \Theta_e$ on Γ , or an oblique b.c. $\frac{\partial \Theta}{\partial n} = h(\Theta_e - \Theta)$ on Γ , $h > 0$, where in each case $\Theta_e \in W^{1-\frac{1}{p},p}(\Gamma)$. In the case of the Neumann b.c. the total amount of temperature is also assigned. A different regularity, as in (i), can be obtained also with such boundary conditions.

(iii) If $\mu > 0$, $\nu = \mu/3$ the operator $-\mu\Delta - \nu\nabla \operatorname{div}$ is elliptic, but the coerciveness of the associated bilinear form, under the boundary conditions (1.4), fails. For this reason our method does not apply. The same difficulty was met in [4] for the stationary problem and in [7] (see also [9]) for the evolutionary problem with free boundary.

The fluid is viscous even if we assume that the shear viscosity μ vanishes and the bulk viscosity μ' is strictly positive, namely if $\mu = 0$, $\nu > 0$. In this case the correct boundary condition is $u \cdot n = 0$ on Γ . The motion of a viscous flow under these assumptions on the viscosity coefficients has been studied only in the evolutionary case in [6].

REFERENCES

- [1] R.A. ADAMS, *Sobolev Spaces*. Academic Press, New York, 1975.
- [2] S. AGMON - A. DOUGLIS - L. NIRENBERG, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, Part II*. **17** (1964), 35-92.
- [3] H. BEIRÃO DA VEIGA, *Existence Results in Sobolev Spaces for a Stationary Transport Equation*. *Ricerche Mat.*, **36** (1987), 173-184.
- [4] R. FARWIG, *Stationary Solutions of Compressible Navier-Stokes Equations with Slip Boundary Condition*. *Comm. Partial Differential Equations*, **14** (1989), 1579-1606.
- [5] P. SECCHI, *On a Stationary Problem for the Compressible Navier-Stokes Equations*. *Differential Integral Equations*, to appear.
- [6] P. SECCHI, *Existence Theorems for Compressible Viscous Fluids Having Zero Shear Viscosity*. *Rend. Sem. Mat. Univ. Padova*, **70** (1983), 73-102.
- [7] P. SECCHI - A. VALLI, *A Free Boundary Problem for Compressible Viscous Fluids*. *J. Reine Angew. Math.*, **341** (1983), 1-31.
- [8] V.A. SOLONNIKOV - V.E. ŠČADILOV, *On a Boundary Value Problem for a Stationary System of Navier-Stokes Equations*. *Proc. Steklov Inst. Math.*, **125** (1973), 186-199.
- [9] V.A. SOLONNIKOV - A. TANI, *Free Boundary Problem for a Viscous Compressible Flow with a Surface Tension*. In: *C. Carathéodory: An International Tribute*, ed. T.M. Rassias, World Scientific Publ. Co., 1991, 1270-1303.

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