

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

LUZ M. FERNÁNDEZ-CABRERA

JOSÉ L. TORREA

The two weight problem for operators in the upper half-plane

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 20, n° 4 (1993), p. 545-574

<http://www.numdam.org/item?id=ASNSP_1993_4_20_4_545_0>

© Scuola Normale Superiore, Pisa, 1993, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

The Two Weight Problem for Operators in the Upper Half-Plane

LUZ M. FERNÁNDEZ-CABRERA - JOSÉ L. TORREA

0. - Introduction

Let $f \mapsto \tilde{f}$ denote the conjugate function operator on the torus $\mathbf{T} \simeq [-1/2, 1/2)$, and consider the weighted L^2 -inequality

$$(0.1) \quad \int_{\mathbf{T}} |\tilde{f}(x)|^2 u(x) dx \leq C \int_{\mathbf{T}} |f(x)|^2 v(x) dx.$$

The question, raised for the first time by Muckenhoupt, is the following:
Find all $v(x)$ (resp. all $u(x)$) such that (0.1) holds for some $u(x)$ (resp. all $v(x)$).

Using complex-variable methods the following simple answer was given by Koosis: (0.1) holds for some non-trivial $u(x)$ if and only if $v^{-1} \in L^1(\mathbf{T})$, and it holds for some non-trivial $v(x)$ if and only if $u \in L^1(\mathbf{T})$; see [K].

A more systematic study of this kind of problem was made later on by different authors. The general setting is the following.

Let $\left(\sum_1, \mathcal{A}_1, m_1\right), \left(\sum_2, \mathcal{A}_2, m_2\right)$ be two measure spaces and T be a linear operator. Find conditions on $v(x)$ (resp. $u(x)$) such that

$$(0.2) \quad \left(\int_{\sum_2} |Tf(x)|^q u(x) dm_2(x) \right)^{1/q} \leq C \left(\int_{\sum_1} |f(x)|^p v(x) dm_1(x) \right)^{1/p}$$

is satisfied for some $u(x)$ (resp. $v(x)$) where u and v are positive measurable functions).

Essentially two methods are used to deal with this problem. The first one is a constructive method, which means the following: given a weight v a new weight u is constructed such that (0.2) is satisfied. The second one uses non-constructive techniques of factorization of operators and then the existence of a certain u satisfying (0.2) is proved. The first method together with the A_p -weights theory was used in [C-J] in order to prove the following results.

0.3 THEOREM. *Let M be the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ and assume $1 < p < \infty$. The following conditions are equivalent:*

- (i) *There exists $u \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx.$$

- (ii) *v belongs to the class D_p^* , i.e.*

$$\sup_{r \geq 1} \frac{1}{r^{np'}} \int_{|x| \leq r} v^{-p'/p}(x) dx < +\infty.$$

0.4 THEOREM. *Let $1 < p < \infty$; on $\mathbb{R}^n$ the following conditions are equivalent:*

- (i) *There exists $u \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Sf(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx$$

for all singular integral operators S .

- (ii) *v belongs to the class D_p , i.e.*

$$\int_{\mathbb{R}^n} \frac{v^{-p'/p}(x) dx}{(1 + |x|)^{np'}} \leq C < +\infty.$$

The following theorems were also proved using the constructive method.

0.5 THEOREM. ([G-G]). *Let $1 < p < \infty$; on $\mathbb{R}^n$ the following conditions are equivalent:*

- (i) *There exists $v \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx.$$

- (ii) *u belongs to the class Z_p , i.e.*

$$\int_{\mathbb{R}^n} \frac{u(x)}{(1 + |x|)^{np}} dx < +\infty.$$

0.6 THEOREM. ([H-M-S]). Let I_γ denote the fractional integral operator

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(y) |x - y|^{\gamma-n} dy, \quad 0 < \gamma < n;$$

and assume $1 < p, q < \infty$, $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$. Then the following conditions are equivalent:

(i) There exists $u \not\equiv 0$ such that

$$\left(\int_{\mathbb{R}^n} |I_\gamma f(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}.$$

(ii) v belongs to the class D_p^γ , i.e.

$$\int_{\mathbb{R}^n} \frac{v(x)^{-p'/p}}{(1 + |x|)^{(n-\gamma)p'}} dx \leq C < +\infty.$$

0.7 THEOREM. Let M_γ denote the fractional maximal operator

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{r^{n-\gamma}} \int_{|x-y|<r} |f(y)| dy, \quad 0 < \gamma < n$$

and assume $1 < p, q < \infty$, $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$. Then the following conditions are equivalent:

(i) There exists $u \not\equiv 0$ such that

$$\left(\int_{\mathbb{R}^n} |M_\gamma f(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}.$$

(ii) v belongs to the class $D_p^{*\gamma}$, i.e.

$$\sup_{r \geq 1} \frac{1}{r^{(n-\gamma)p'}} \int_{|x| \leq r} v(x)^{-p'/p} dx \leq C < +\infty.$$

The non-constructive method was developed by Rubio de Francia and with this method he proved Theorems 0.3, 0.4, 0.5, 0.6 and 0.7 for $p = q$ (see [GC-R de F], Chapt. VI] for a complete description of the method). The method of Rubio cannot be applied in the cases $p \neq q$ but, being non-constructive, it can be applied to a huge class of operators and measure spaces; in particular the A_p -theory is not needed.

This paper grew out of an effort to understand better the two methods mentioned above, and, as a consequence, to solve the general problem (0.2)

for operators acting on the upper half-plane (see Section 1), in which case the A_p -theory is not available. We also look at both methods from the point of view of vector-valued function theory: then maximal operators can be handled as ℓ^∞ -valued linear operators.

The organization of the paper is as follows. In Section 1 we introduce the operators in the upper half-plane for which we want to solve problem (0.2); we also prove some estimates (see Proposition 1.7) that have interest in themselves and that we shall use later. Section 2 contains the basic lemmas that are needed for the methods mentioned above; both methods have a common part of strategy: given a function f , decompose it as $f_1 + f_2$, where $f_1 = f\chi_B$ and B is a certain ball, while Tf_2 is estimated in both methods with a local L^∞ -bound. In this estimate, when working in Banach lattices X , it is natural to consider a new class $D_{p,X}^\gamma$ of weights (see (2.15)); the class $D_{p,X}^\gamma$ solves problem (0.2) for fractional maximal operators acting on lattice-valued functions; the classes D_p^* , D_p^γ , $0 < \gamma < n$, solve the problem for the Poisson integral and the fractional integral operator in the upper half-plane (even with Banach-valued functions), whereas the class D_p solves the problem for the generalization of the Riesz transform with U.M.D.-valued functions. All these results are proved in Section 3 (Theorems 3.5, 3.9, 3.1 and 3.7). In Section 4 we see that the class $D_{p,X}^\gamma$ is an intermediate class between D_p^γ and $D_p^{*,\gamma}$.

1. - Notation and background

We shall consider the following operators:

$$M_\gamma f(x, t) = \sup_{r \geq t} \frac{1}{r^{n-\gamma}} \int_{B(x, r)} |f(y)| dy, \quad (x, t) \in \mathbb{R}_+^{n+1}, \quad 0 \leq \gamma < n,$$

$$T_\gamma f(x, t) = \int_{\mathbb{R}^n} \frac{f(y)}{(|x - y| + t)^{n-\gamma}} dy, \quad (x, t) \in \mathbb{R}_+^{n+1}, \quad 0 < \gamma < n,$$

$$Q_i f(x, t) = c_n \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}} f(y) dy, \quad (x, t) \in \mathbb{R}_+^{n+1}, \quad i = 1, 2, \dots, n,$$

$$Pf(x, t) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}} f(y) dy, \quad (x, t) \in \mathbb{R}_+^{n+1}$$

where $\mathbb{R}_+^{n+1}$ denotes the upper half-plane $\mathbb{R}^n \times [0, \infty)$, c_n is a constant depending on the dimension and $B(x, r)$ is the ball $\{y \in \mathbb{R}^n : |x - y| < r\}$.

The operator P is the Poisson-Integral, M_0 was introduced in [F-S] and M_γ and T_γ were studied in [R-T].

Given a measure $d\mu$ on $\mathbb{R}_+^{n+1}$ we shall say, as usual, that $d\mu$ is a Carleson measure if there exists a constant C such that for any ball $B = B(x, r)$ in $\mathbb{R}^n$

we have $\mu(\hat{B}) \leq C|B|$ where $\hat{B} = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| + t < r\}$ and $|B|$ stands for the Lebesgue measure of B .

1.1 REMARK. It is known (see [R-T]) that if μ is a Carleson measure then M_γ and T_γ map $L^1(\mathbb{R}^n, dx)$ into weak- $L^{\frac{n}{n-\gamma}}(\mathbb{R}_+^{n+1}, d\mu)$ and P and Q_i map $L^1(\mathbb{R}^n, dx)$ into weak- $L^1(\mathbb{R}_+^{n+1}, d\mu)$.

Moreover if a weight v belongs to D_p then there exists a weight u such that M_0 maps $L^p(v)$ into $L^p(u, d\mu)$ (see [F-T]).

On the other hand it is known that if for an open set $\theta \subset \mathbb{R}^n$ one defines

$$T(\theta) = \{(x, t) \in \mathbb{R}_+^{n+1} : B(x, t) \subset \theta\}$$

then $d\mu$ is a Carleson measure if and only if $\mu(T(\theta)) \leq C|\theta|$ (see [A-B] and [J]).

1.2 REMARK. The operators defined above generalize several known operators. In particular:

- the fractional maximal operator of order γ

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)| dy = M_\gamma f(x, 0),$$

(observe that $M_0 = M$ is the Hardy-Littlewood maximal operator);

- the fractional integral

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\gamma}} dy = T_\gamma f(x, 0);$$

- the Riesz transforms

$$R_i f(x) = p.v. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x - y|^{n+1}} f(y) dy = Q_i f(x, 0).$$

Now we shall consider vector-valued extensions of the above operators.

We observe that T_γ and P are positive ($f \geq 0 \Rightarrow T_\gamma f \geq 0$ and $Pf \geq 0$) and linear operators. Therefore given any Banach space E we can consider the vector-valued extensions $\tilde{T}_\gamma$ and $\tilde{P}$ such that

$$(1.3) \quad \begin{aligned} \langle \tilde{T}_\gamma f(x, t), e^* \rangle &= T_\gamma(\langle f, e^* \rangle)(x, t) \\ \langle \tilde{P} f(x, t), e^* \rangle &= P(\langle f, e^* \rangle)(x, t) \end{aligned}$$

where f is an E -valued strongly measurable function and $e^* \in E^*$. It is well known (see [L-T]) that the extensions $\tilde{T}_\gamma$ and $\tilde{P}$ have the same boundedness properties as T_γ and P . In particular, if $d\mu$ is a Carleson measure, $\tilde{T}_\gamma$ maps $L_E^1(\mathbb{R}^n, dx)$ into weak- $L_E^{\frac{n}{n-\gamma}}(\mathbb{R}_+^{n+1}, d\mu)$ and $\tilde{P}$ maps $L_E^1(\mathbb{R}^n, dx)$ into weak- $L_E^1(\mathbb{R}_+^{n+1}, d\mu)$ for any Banach space E .

We recall that the class of the Banach spaces E such that the Riesz transforms $\tilde{R}_i$ are bounded from $L^2_E(\mathbb{R}^n, dx)$ into $L^2_E(\mathbb{R}^n, dx)$ has been characterized by Burkholder and Bourgain (see [Bk] and [B]) and it is denoted by U.M.D. Therefore choosing the Carleson measure $d\mu(x, t) = dx \otimes \delta_0(t)$, where δ_0 is Dirac's delta, it is clear that if the $\tilde{Q}_i$'s are bounded from $L^1_E(\mathbb{R}^n, dx)$ into weak- $L^1_E(\mathbb{R}^{n+1}, d\mu)$ then E must be in the U.M.D. class. On the other hand

$$\{(x, t) : \|\tilde{Q}_i f(x, t)\|_E > \lambda\} \subset T(\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}} \lambda\})$$

and therefore if E is U.M.D., using Remark 1.1, for any Carleson measure $d\mu$ we have

$$\begin{aligned} \mu(\{(x, t) : \|\tilde{Q}_i f(x, t)\|_E > \lambda\}) &\leq \mu(T(\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}} \lambda\})) \\ &\leq C \{\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}} \lambda\}\} \leq \frac{C}{\lambda} \|f\|_{L^1_E}. \end{aligned}$$

1.4 DEFINITION. Let $(\Omega, \mathcal{A}, \nu)$ be a complete σ -finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally-integrable real-valued functions on Ω is called a Köthe function space if the following two conditions hold:

- (1) If $|f(\omega)| \leq |g(\omega)|$ a.e. on Ω with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.
- (2) For every $E \in \mathcal{A}$ with $\nu(E) < +\infty$ the characteristic function χ_E of E belongs to X .

Every Köthe function space is a Banach lattice with the obvious order ($f \geq 0$ if $f(\omega) \geq 0$ a.e.).

Given a measurable function g on Ω such that $gf \in L^1(\nu)$ for every $f \in X$, one defines an element x_g^* in X^* by

$$x_g^*(f) = \int_{\Omega} f(\omega)g(\omega)d\nu(\omega).$$

The linear space of these x_g^* is denoted by X' . It is known (see [L-T]) that X' is a norming subspace of X^* if and only if whenever $\{f_n\}_{n=1}^{\infty}$ and f are non-negative elements of X such that $f_n(\omega) \uparrow f(\omega)$ a.e. we have $\|f_n\| \rightarrow \|f\|$.

Let X be a Banach lattice and let J be a finite subset of the set $\mathbb{Q}_+$ of positive rational numbers. Given a locally-integrable function $f : \mathbb{R}^n \rightarrow X$ (this means, of course, that f is strongly measurable and that the $\mathbb{R}$ -valued function $y \mapsto \|f(y)\|_X$ is locally-integrable) we define

$$(1.5) \quad \tilde{M}_{\gamma, J} f(x, t) = \sup_{\substack{r \in J \\ r \geq t}} \frac{1}{r^{n-\gamma}} \int_{B(x, r)} |f(y)| dy,$$

where the sup is taken in the lattice X .

When X is a Köthe function space, as it will be the case in the sequel, it is clear that $\tilde{M}_{\gamma,J}f(x,t)$ is a function of ω given by

$$\tilde{M}_{\gamma,J}f(x,t)(\omega) = \sup_{\substack{r \in J \\ r \geq J}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y,\omega)| dy,$$

where the sup is now taken with respect to the order of $\mathbb{R}$.

In this situation we can consider f and $\tilde{M}_{\gamma,J}f$ as functions on $\mathbb{R}^n \times \Omega$ and $\mathbb{R}_+^{n+1} \times \Omega$ respectively. If $\gamma > 0$ we have

$$M_{\gamma}f(x,t) \leq T_{\gamma}f(x,t), \quad (x,t) \in \mathbb{R}_+^{n+1}$$

and therefore for any Banach lattice X and any finite subset J of $\mathbb{Q}_+$, we have

$$\tilde{M}_{\gamma,J}f(x,t) \leq \tilde{T}_{\gamma}f(x,t), \quad (x,t) \in \mathbb{R}_+^{n+1}$$

where now the inequality holds with respect to the order of the lattice X . By the above results about $\tilde{T}_{\gamma}$ we have that given a Banach lattice X , a Carleson measure $d\mu$ and $\gamma > 0$, the inequality

$$\mu(\{(x,t) : \|\tilde{M}_{\gamma,J}f(x,t)\|_X > \lambda\}) \leq C \left(\frac{1}{\lambda} \int \|f(x)\|_X dx \right)^{\frac{n}{n-\gamma}}$$

holds with the constant C depending on $d\mu$ and γ but not on J .

1.6 DEFINITION. *We shall say that a Banach lattice X satisfies the Hardy-Littlewood (H.L.) property if the inequality*

$$|\{x \in \mathbb{R}^n : \|\tilde{M}_J f(x)\|_X > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_X dx$$

holds, with C independent of J (see [GC-M-T]); $\tilde{M}_J$ is defined by

$$\tilde{M}_J f(x) = \sup_{r \in J} \frac{1}{r^n} \int_{B(x,r)} |f(y)| dy,$$

the sup being taken with respect to the lattice order.

It is easy to check that for every for any finite subset J of $\mathbb{Q}_+$ we have

$$\{(x,t) : \|\tilde{M}_{0,J}f(x,t)\|_X > \lambda\} \subset T(\{x : \|\tilde{M}_J f(x)\|_X > \lambda\}),$$

therefore if X has the H.L. property, using Remark 1.1, we have for any Carleson measure $d\mu$

$$\begin{aligned} & \mu(\{(x, t) : \|\tilde{M}_{0,J}f(x, t)\|_X > \lambda\}) \\ & \leq \mu(T(\{x : \|\tilde{M}_Jf(x)\|_X > \lambda\})) \leq C|\{x : \|\tilde{M}_Jf(x)\|_X > \lambda\}| \\ & \leq \frac{C}{\lambda} \|f\|_{L_X^1}. \end{aligned}$$

On the other hand, choosing $d\mu(x, t) = dx \otimes \delta_0(t)$, it is clear that if for a Banach lattice X we have the inequality

$$\mu(\{(x, t) : \|\tilde{M}_{0,J}f(x, t)\|_X > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_X dx$$

with C independent of J , then, Definition 1.6, X must have the H.L. property. Now we collect the above results for further reference.

1.7 PROPOSITION.

- (1.8) *Let E be an arbitrary Banach space and $\tilde{T}_\gamma, \tilde{P}$ be the E -valued extensions defined in (1.3); then for any Carleson measure $d\mu$ in $\mathbb{R}_+^{n+1}$, we have*

$$\mu(\{(x, t) : \|\tilde{T}_\gamma f(x, t)\|_E > \lambda\}) \leq C \left(\frac{1}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_E dx \right)^{\frac{n}{n-\gamma}}$$

and

$$\mu(\{(x, t) : \|\tilde{P}f(x, t)\|_E > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_E dx.$$

- (1.9) *Let X be an arbitrary Banach lattice and $\tilde{M}_{\gamma,J}$ be the operator defined in (1.5); then for any Carleson measure $d\mu$ in $\mathbb{R}_+^{n+1}$ and any $\gamma > 0$ we have*

$$\mu(\{(x, t) : \|\tilde{M}_{\gamma,J}f(x, t)\|_X > \lambda\}) \leq C \left(\frac{1}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_X dx \right)^{\frac{n}{n-\gamma}}$$

with C independent of J .

- (1.10) *Let E be a Banach space and $\tilde{Q}_i$ be the vector-valued extension of Q_i . For any Carleson measure $d\mu$ in $\mathbb{R}_+^{n+1}$ we have*

$$\mu(\{(x, t) : \|\tilde{Q}_i f(x, t)\|_E > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_E dx \quad i = 1, 2, \dots, n$$

if and only if E is U.M.D.

(1.11) Let X be a Banach lattice and $\tilde{M}_{0,J}$ be the operator defined in (1.5). For any Carleson measure $d\mu$ in $\mathbb{R}_+^{n+1}$ we have

$$\mu(\{(x, t) : \|\tilde{M}_{0,J}f(x, t)\|_X > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_X dx$$

with C independent of J , if and only if X has the H.L. property.

2. - Technical lemmas

2.1 LEMMA. Assume that $d\mu$ is a Carleson measure in $\mathbb{R}_+^{n+1}$. Let $1 < p < \infty$, $B_1 = \{x : |x| \leq 1\}$ and let v be a non-negative measurable function in $\mathbb{R}^n$ such that $M_0(v^{-p'/p}\chi_{B_1})(x, t) < +\infty$ a.e. in (x, t) and $\int_{B_1} v^{-p'/p} dx > 0$. Under these conditions the weight $w(x, t) = [M_0(v^{-p'/p}\chi_{B_1})(x, t)]^{-\beta}$ with $\beta > p - 1$ satisfies

(2.2) For any $\gamma \geq 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} \geq 0$

$$\left(\int_{\hat{B}_1} M_\gamma(f\chi_{B_1})(x, t)^q w(x, t)^{q/p} d\mu(x, t) \right)^{1/q} \leq C_{\gamma,p,q} \left(\int_{B_1} |f(x)|^p v(x) dx \right)^{1/p}.$$

(2.3) For any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} \geq 0$

$$\left(\int_{\hat{B}_1} T_\gamma(f\chi_{B_1})(x, t)^q w(x, t)^{q/p} d\mu(x, t) \right)^{1/q} \leq C_{\gamma,p,q} \left(\int_{B_1} |f(x)|^p v(x) dx \right)^{1/p}.$$

PROOF. If $\frac{1}{r} > \frac{1}{p} - \frac{\gamma}{n} = \frac{1}{q}$, by Hölder's inequality, we have

$$\begin{aligned} & \left(\int_{\hat{B}_1} M_\gamma f(x, t)^r w^{r/p}(x, t) d\mu(x, t) \right)^{1/r} \\ & \leq \mu(\hat{B}_1)^{1/(q/r)'} \left(\int_{\hat{B}_1} M_\gamma f(x, t)^q w^{q/p}(x, t) d\mu(x, t) \right)^{1/q} \end{aligned}$$

Then we need to consider only the case $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Let

$$E_k = \{(x, t) \in \mathbb{R}_+^{n+1} : M_0(v^{-p'/p} \chi_{B_1})(x, t) \leq 2^k\},$$

$k = 0, 1, 2, \dots$. We define

$$T_k f(x, t) = M_\gamma(f v^{-p'/p} \chi_{B_1})(x, t) \chi_{E_k}(x, t);$$

as the operator M_γ maps $L^1(\mathbb{R}^n, dx)$ into weak- $L^{\frac{n}{n-\gamma}}(\mathbb{R}_+^{n+1}, d\mu)$, see Remark 1.1, we have

$$(2.4) \quad \mu(\{(x, t) : T_k f(x, t) > \lambda\}) \leq C \left(\frac{1}{\lambda} \int_B f(x) v^{-p'/p}(x) dx \right)^{\frac{n}{n-\gamma}}.$$

On the other hand, for any $r > 0$, Hölder's inequality gives

$$\begin{aligned} & \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)| \chi_{B_1}(y) v^{-p'/p}(y) dy \\ & \leq \left(\frac{1}{r^{n-\gamma}} \int_{B(x,r)} \chi_{B_1} v^{-p'/p} dx \right)^{1-\gamma/n} \left(\int_{B_1} |f|^{n/\gamma} v^{-p'/p} dx \right)^{\gamma/n}; \end{aligned}$$

therefore, by the definition of E_k , we obtain

$$(2.5) \quad \|T_k f\|_\infty \leq C 2^{k(1-\gamma/n)} \left(\int_{B_1} |f|^{n/\gamma} v^{-p'/p} dx \right)^{\gamma/n}.$$

Applying Marcinkiewicz's interpolation theorem to (2.4) and (2.5) we obtain

$$\left(\int_{\mathbb{R}_+^{n+1}} |T_k f|^q d\mu \right)^{1/q} \leq C_{\alpha,p} 2^{k/p'} \left(\int_{\mathbb{R}^n} |f|^p v^{-p'/p} dx \right)^{1/p},$$

for $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$.

Replacing, in the last inequality, $f v^{-p'/p}$ by f it follows that

$$(2.6) \quad \left(\int_{E_k} M_\gamma(f \chi_{B_1})^q d\mu \right)^{1/q} \leq C 2^{k/p'} \left(\int_{B_1} |f|^p v dx \right)^{1/p}$$

The assumption that $M_0(v^{-p'/p}\chi_{B_1})(x, t) < +\infty$ a.e. in (x, t) implies that $\mu(\mathbb{R}_+^{n+1} \setminus \cup E_k) = 0$; therefore:

$$(2.7) \quad \int_{\hat{B}_1} M_\gamma(f\chi_{B_1})^q w^{q/p} d\mu = \left(\int_{\hat{B}_1 \cap E_0} + \sum_{k=1}^{\infty} \int_{\hat{B}_1 \cap (E_k \setminus E_{k-1})} \right) M_\gamma(f\chi_{B_1})^q w^{q/p} d\mu.$$

On the other hand if $(x, t) \in \hat{B}_1$ then

$$\int_{B_1} v^{-p'/p}(y) dy \leq C \sup_{r \geq t} \frac{1}{r^n} \int_{B(x, r)} v^{-p'/p}(y) \chi_{B_1}(y) dy;$$

therefore, as $\int_{B_1} v^{-p'/p} > 0$, we have that $\omega(x, t)$ is bounded for $(x, t) \in \hat{B}_1 \cap E_0$.

Then, using (2.6) and (2.7), we get

$$\begin{aligned} \int_{\hat{B}_1} M_\gamma(f\chi_{B_1})^q w^{q/p} d\mu &\leq \sum_{k=0}^{\infty} 2^{-k\beta q/p} \int_{E_k} M_\gamma(f\chi_{B_1})^q d\mu \\ &\leq C_{\gamma, p} \left(\int_{B_1} |f|^p v dx \right)^{q/p} \sum_{k=0}^{\infty} 2^{-k\beta q/p/2^{kq/p'}}; \end{aligned}$$

since $\beta > p - 1$, the above geometric series is convergent and this completes the proof of (2.2).

In order to estimate $T_\gamma(f\chi_{B_1})$ we shall use the following inequality, see [R-T], valid for $\varepsilon > 0$ small enough

$$(2.8) \quad |T_\gamma f(x, t)| \leq C [M_{\gamma-\varepsilon} f(x, t) M_{\gamma+\varepsilon} f(x, t)]^{1/2}$$

where C depends on ε and γ .

Let $\varepsilon > 0$, $\gamma_1 = \gamma - \varepsilon$ and $\gamma_2 = \gamma + \varepsilon$; if ε is small enough we get $0 < \gamma_i < \frac{n}{p}$. Let q_1 and q_2 be such that $\frac{1}{q_1} = \frac{1}{q} + \frac{\varepsilon}{n}$ and $\frac{1}{q_2} = \frac{1}{q} - \frac{\varepsilon}{n}$; in particular $\frac{1}{q_1} = \frac{1}{p} - \frac{\gamma_1}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\gamma_2}{n}$.

By (2.2) we have

$$\left(\int_{\hat{B}_1} M_{\gamma_i}(f\chi_{B_1})^{q_i} w^{q_i/p} d\mu \right)^{1/q_i} \leq C \left(\int_{B_1} |f|^p v dx \right)^{1/p}, \quad i = 1, 2.$$

Then applying (2.8) and Hölder's inequality with exponents $\frac{2q_i}{p}$ we obtain

$$\begin{aligned} & \left(\int_{\tilde{B}_1} T_\gamma(f\chi_{B_1})^q w^{q/p} d\mu \right)^{1/q} \\ & \leq C \left(\int_{\tilde{B}_1} \mathcal{M}_{\gamma_1}(f\chi_{B_1})^{q_1} w^{q_1/p} d\mu \right)^{1/2q_1} \left(\int_{\tilde{B}_1} \mathcal{M}_{\gamma_2}(f\chi_{B_1})^{q_2} w^{q_2/p} d\mu \right)^{1/2q_2} \\ & \leq C \left(\int_{B_1} |f|^p v dx \right)^{1/p}. \end{aligned} \quad \square$$

Now we state the vector-valued version of Lemma 2.1.

2.9 LEMMA. Assume that $d\mu$ is a Carleson measure in $\mathbb{R}_+^{n+1}$. Let $1 < p < \infty$ and let v be a non-negative measurable function in $\mathbb{R}^n$ such that $\mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x, t) < \infty$ a.e. in (x, t) and $\int_{B_1} v^{-p'/p} dx > 0$. Under these conditions the weight w defined in Lemma 2.1 satisfies the following properties:

(2.10) For any Köthe function space X , any finite subset J of $\mathbb{Q}_+$, any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$

$$\left(\int_{\tilde{B}_1} \|\tilde{\mathcal{M}}_{\gamma, J} f(x, t)\|_X^q w(x, t)^{q/p} d\mu(x, t) \right)^{1/q} \leq C_{\gamma, p, q} \left(\int_{B_1} \|f(x)\|_X^p v(x) dx \right)^{1/p}$$

with $C_{\gamma, p, q}$ independent of J and X ;

(2.11) For any Banach space E , any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$

$$\left(\int_{\tilde{B}_1} \|\tilde{T}_\gamma f(x, t)\|_E^q w(x, t)^{q/p} d\mu(x, t) \right)^{1/q} \leq C_{\gamma, p, q} \left(\int_{B_1} \|f(x)\|_E^p v(x) dx \right)^{1/p}$$

with $C_{\gamma, p, q}$ independent of E .

PROOF. Since $\|\tilde{T}_\gamma f(x, t)\|_E \leq T_\gamma(\|f\|_E)(x, t)$, (2.11) is a direct consequence of (2.3). On the other hand if X is a Banach space of classes of equivalence of measurable functions on $(\Omega, \mathcal{A}, \nu)$ we have for any $\omega \in \Omega$ and $\gamma > 0$

$$\tilde{\mathcal{M}}_\gamma f(x, t)(\omega) = \sup_{r \geq t} \frac{1}{r^{n-\gamma}} \int_{B(x, r)} |f(y, \omega)| dy \leq C_\gamma T_\gamma(|f(\cdot, \omega)|)(x, t)$$

and therefore, by definition (1.4), we have

$$\|\tilde{M}_\gamma f(x, t)\|_X \leq \|\tilde{T}_\gamma(|f|)(x, t)\|_X \leq T_\gamma(\|f\|_X)(x, t) = T_\gamma(\|f\|_X)(x, t);$$

now (2.10) follows from (2.11). $\square$

The following Lemma can be found in [F-T], Theorem 1. We state it here for further reference.

2.12 LEMMA. *Let $(Y, d\nu)$ be a measure space, F and G be Banach spaces and $\{A_k\}_{k=0}^\infty$ be a sequence of disjoint sets in Y such that $\bigcup_{k=0}^\infty A_k = Y$. Assume that $0 < s < p < \infty$ and that $(T_j)_{j \in I}$ is a family of sublinear operators which satisfies*

$$\left\| \left(\sum_j \|T_j f_j\|_F^p \right)^{1/p} \right\|_{L^s(A_k, d\nu)} \leq C_k \left(\sum_j \|f_j\|_G^p \right)^{1/p}, \quad k \in \mathbb{N}$$

where, for each $k \in \mathbb{N}$, C_k is a constant depending on G , F , p and s . Then there exists a positive function $u(x)$ on Y such that

$$\left(\int_Y \|T_j f(x)\|_F^p u(x) d\nu(x) \right)^{1/p} \leq C \|f\|_G \quad (j \in I)$$

where C is a constant depending on G , F , p and s .

2.13 DEFINITION. *Given $0 \leq \gamma < n$ and a finite subset L of $\mathbb{Q}_+ \cap [1, \infty)$ we define the function $\varphi_{\gamma, L}$ on $\mathbb{R}^n \times \Omega$ as*

$$\varphi_{\gamma, L}(x, \omega) = \frac{\chi_{B_\omega}(x)}{|B_\omega|^{1-\frac{\gamma}{n}}}$$

where B_ω is a ball centered at the origin with radius $r_\omega \in L$.

2.14 DEFINITION. *Let v be a weight in $\mathbb{R}^n$ (i.e. a real-valued, locally-integrable function with $v(x) \geq 0$ a.e. in $\mathbb{R}^n$). Let X be a Köthe function space with X' norming and $1 < p < \infty$. We shall say that v belongs to the class $D_{p, X}^\gamma$ if the inequality*

$$\int_{\mathbb{R}^n} \|a\varphi_{\gamma, L}(x)\|_{X'}^{p'} v(x)^{-p'/p} dx \leq C$$

holds for any $a \in X'$ with $\|a\|_{X'} \leq 1$ and with the constant C independent of $\varphi_{\gamma, L}$.

2.15 LEMMA. *Assume $0 \leq \gamma < n$. Let X be a Köthe function space with X' norming and let $1 < p < \infty$. For a weight v in $\mathbb{R}^n$ the following properties are equivalent:*

- (i) $v \in D_{p,X}^\gamma$.
- (ii) Given a ball $B = B(0, R) = \{x : |x| \leq R\}$, $R \geq 1$, and a X -valued locally-integrable function f with support in the complement of $2B = B(0, 2R)$, inequality

$$\sup_{(x,t) \in \hat{B}} \|\tilde{M}_{\gamma,J} f(x,t)\|_X \leq C \|f\|_{L_X^p(v)}$$

holds with the constant C independent of J .

PROOF. (i) $\Rightarrow$ (ii) Let f be a locally-integrable function with support in $\mathbb{R}^n \setminus B(0, 2R)$. Given $(x, t) \in \hat{B}$ and a rational number r , the integral $\int_{B(x,r)} |f(y)| dy$ is equal to zero unless $B(x, r) \cap \mathbb{R}^n \setminus B(0, 2R) \neq \emptyset$; in this case $r > R$ and then $B(x, r) \subset B(0, 2r)$. Hence if $J \subset \mathbb{Q}_+$ we have

$$\begin{aligned} \mathcal{M}_{\gamma,J} f(x,t)(\omega) &= \sup_{\substack{r \in J \\ r \geq t}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y, \omega)| dy \\ &= C \sup_{\substack{r \in J \\ r \geq \max(t, R)}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y, \omega)| dy \\ &\leq C \sup_{\substack{r \in J \\ r \geq \max(t, R)}} \frac{1}{r^{n-\gamma}} \int_{B(0, 2r)} |f(y, \omega)| dy \\ &\leq C \sup_{\substack{r \in J \\ r \geq \max(1, t)}} \frac{1}{r^{n-\gamma}} \int_{B(0, 2r)} |f(y, \omega)| dy \\ &= C \int_{\mathbb{R}^n} |f(y, \omega)| \varphi_{\gamma, L_t}(y, \omega) dy, \end{aligned}$$

where

$$L_t = \{2r : r \in J, r \geq \max(1, t)\}, \quad \varphi_{\gamma, L_t}(y, \omega) = \frac{\chi_{B_\omega}(y)}{r_\omega^{n-\gamma}},$$

and for each ω , $r_\omega = s \in L_t$ if

$$\sup_{\substack{r \in J \\ r \geq \max(1, t)}} \frac{1}{(2r)^{n-\gamma}} \int_{B(0, 2R)} |f(y, \omega)| dy = \frac{1}{s^{n-\gamma}} \int_{B(0, s)} |f(y, \omega)| dy.$$

In the last equality we have used the hypothesis that J is finite and the consequence that for each (ω, t) the sup is in fact a maximum.

As X' is norming, we have

$$\|\mathcal{M}_{\gamma,J}f(x,t)\|_X \leq C \sup_{\substack{a \in X' \\ \|a\| \leq 1}} \int_{\Omega} \int_{\mathbb{R}^n} |f(y,\omega)| \varphi_{\gamma,L_t}(y,\omega) a(\omega) dy d\omega.$$

Now we observe that for each y , the function $\omega \mapsto \varphi_{\gamma,L_t}(y,\omega)$ is a step function and therefore the function $\omega \mapsto a(\omega)\varphi_{\gamma,L_t}(y,\omega)$ is an element of X' ; therefore by duality and Hölder's inequality, we have

$$\begin{aligned} \|\mathcal{M}_{\gamma,J}f(x,t)\|_X &\leq C \sup_{\substack{a \in X' \\ \|a\| \leq 1}} \left(\int_{\mathbb{R}^n} \|f(y)\|_X^p v(y) dy \right)^{1/p} \\ &\quad \times \left(\int_{\mathbb{R}^n} \|a\varphi_{\gamma,L_t}(y)\|_{X'}^{p'} v(y)^{-p'/p} dy \right)^{1/p'} \leq C \|f\|_{L_X^p(v)}. \end{aligned}$$

(ii) $\Rightarrow$ (i) Let $B_1 = \{x : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. By duality we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \|a\varphi_{\gamma,L}(y)\|_{X'}^{p'} v(y)^{-p'/p} dy \\ &= \sup_{\substack{g \in L_X^p \\ \|g\| \leq 1}} \left(\int_{\mathbb{R}^n} \int_{\Omega} a(\omega)\varphi_{\gamma,L}(y,\omega) g(y,\omega) v(y)^{-1/p} d\omega dy \right)^{p'} \\ &\leq \sup_{\substack{g \in L_X^p \\ \|g\| \leq 1}} \left(\int_{\mathbb{R}^n} \int_{\Omega} a(\omega)\varphi_{\gamma,L}(y,\omega) g(y,\omega) \chi_{\mathbb{R}^n \setminus 2B_1}(y) v(y)^{-1/p} d\omega dy \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \int_{\Omega} a(\omega)\varphi_{\gamma,L}(y,\omega) g(y,\omega) \chi_{2B_1}(y) v(y)^{-1/p} d\omega dy \right)^{p'} \\ &\leq \sup_{\substack{g \in L_X^p \\ \|g\| \leq 1}} \left(\int_{\Omega} a(\omega) \mathcal{M}_{\gamma,L}(g\chi_{\mathbb{R}^n \setminus 2B_1} v^{1/p})(0,t,\omega) d\omega \right. \\ &\quad \left. + \int_{2B_1} \int_{\Omega} a(\omega)\varphi_{\gamma,L}(y,\omega) g(y,\omega) v(y)^{-1/p} d\omega dy \right)^{p'} \end{aligned}$$

where $0 \leq t < 1$.

To estimate the first summand we observe that $(0, t) \in \hat{B}_1$ and therefore we have

$$\|\mathcal{M}_{\gamma,L}(g\chi_{\mathbb{R}^n \setminus 2B_1} v^{-1/p})(0, t)\|_X \leq C \|g\chi_{\mathbb{R}^n \setminus 2B_1} v^{-1/p}\|_{L_X^p(v)} \leq C$$

On the other hand since $\varphi_{\gamma,L}(y, \omega) \leq 1$ we have

$$\begin{aligned} & \left| \int_{2B_1} \int_{\Omega} a(\omega) \varphi_{\gamma,L} \varphi_{\gamma,L}(y, \omega) g(y, \omega) v(y)^{-1/p} d\omega dy \right| \\ & \leq \int_{2B_1} \int_{\Omega} |a(\omega)| |g(y, \omega)| v(y)^{-1/p} d\omega dy \\ & \leq \|a\|_{X^*} \|g\|_{L_X^p} \left(\int_{2B_1} v(y)^{-p'/p} dy \right)^{1/p'} \leq C. \end{aligned}$$

This completes the proof. $\square$

2.16 LEMMA. *Let $0 \leq \gamma < n$, E and F be Banach spaces and v be a weight in $\mathbb{R}^n$ such that $v \in D_p^\gamma$ (see 0.6). Assume that τ_γ is an operator from $L_E^p(\mathbb{R}^n, dx)$ into $L_F^q(\mathbb{R}_+^{n+1}, d\mu)$ such that*

$$\|\tau_\gamma f(x, t)\|_F \leq C \int \frac{\|f(y)\|_E}{(|x - y| + t)^{n-\gamma}} dy = CT_\gamma(\|f\|_E)(x, t).$$

Then given a ball $B = B(0, R)$, $R \geq 1$, and a E -valued locally-integrable function f with support in the complement of $2B$, inequality

$$\sup_{(x,t) \in \hat{B}} \|\tau_\gamma f(x, t)\|_F \leq C_\gamma \|f\|_{L_E^p(v)}$$

holds with C independent of τ_γ .

PROOF. It is sufficient to prove that

$$\sup_{(x,t) \in \hat{B}} T_\gamma(\|f\|_E)(x, t) \leq C_\gamma \|f\|_{L_E^p(v)}.$$

If $(x, t) \in \hat{B}$, we have

$$\begin{aligned}
 T_\gamma(\|f\|_E)(x, t) &= \int_{|x|+t < R < 2|y|} \frac{\|f(y)\|_E}{(|x-y|+t)^{n-\gamma}} dy \\
 &\leq \int_{|x| < R < 2|y|} \frac{\|f(y)\|_E}{|x-y|^{n-\gamma}} dy \leq \int_{\mathbb{R}^n} \frac{\|f(y)\|_E}{(|y|+1)^{n-\gamma}} dy \\
 &\leq \left(\int_{\mathbb{R}^n} \|f(y)\|_E^p v(y) dy \right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{v(y)^{-p'/p}}{(1+|y|)^{n-\gamma}} dy \right)^{1/p'} \\
 &\leq C \|f\|_{L_E^p(v)}.
 \end{aligned}$$

□

3. - Main results

In this section we shall state and prove the main results of the paper. We recall that in the Introduction (0.3 and 0.6) we have defined the classes D_p^* and D_p^γ .

3.1 THEOREM. *Let E be a Banach space, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}^n$.*

(3.2) *If $v \in D_p^*$, $1 < p < \infty$, then there exists a weight u in $\mathbb{R}_+^{n+1}$ such that*

$$\int_{\mathbb{R}_+^{n+1}} \|\tilde{P}f(x, t)\|_E^p u(x, t) d\mu(x, t) \leq C \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx.$$

(3.3) *v belongs to D_p^γ with $1 < p < \infty$ and $0 < \gamma < n$, if and only if there exists a weight u in $\mathbb{R}_+^{n+1}$ such that*

$$\left(\int_{\mathbb{R}_+^{n+1}} \|\tilde{T}_\gamma f(x, t)\|_E^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx \right)^{1/p}$$

with $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$ and $1 < q < \infty$.

3.4 COROLLARY. *Let $0 < \gamma < n$, E and F be Banach spaces, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}^n$ such that $v \in D_p^\gamma$. Assume that τ_γ is a bounded sublinear operator from $L_E^p(\mathbb{R}^n, dx)$ into $L_F^q(\mathbb{R}_+^{n+1}, d\mu)$ such that*

$$\|\tau_\gamma f(x, t)\|_F \leq CT_\gamma(\|f\|_E)(x, t).$$

Then there exists a weight u in $\mathbb{R}_+^{n+1}$ such that

$$\left(\int_{\mathbb{R}^{n+1}} \|\tau_\gamma f(x, t)\|_F^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx \right)^{1/p}$$

with $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$ and $1 < q < \infty$.

3.5 THEOREM. Let X be a Banach lattice, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}^n$; then v belongs to $D_{p,X}^\gamma$ (see 2.14) $0 < \gamma < n$ and $1 < p < \infty$, if and only if there exists a weight u in $\mathbb{R}_+^{n+1}$ such that

$$(3.6) \quad \left(\int_{\mathbb{R}_+^{n+1}} \|\tilde{M}_{\gamma,J} f(x, t)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p v(x) dx \right)^{1/p}$$

where the constant C_γ is independent of the finite set J , $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$ and $1 < q < \infty$.

3.7 THEOREM. Let E be a Banach space with the U.M.D. property, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}^n$. Then v belongs to D_p with $1 < p < \infty$, if and only if there exists u in $\mathbb{R}_+^{n+1}$ such that

$$(3.8) \quad \int_{\mathbb{R}_+^{n+1}} \|\tilde{Q}_i f(x, t)\|_E^p u(x, t) \leq C \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx, \quad i = 1, \dots, n.$$

3.9 THEOREM. Let X be a Banach lattice with the H.L. property, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}^n$. Then v belongs to $D_{p,X}$ (see 2.14) with $1 < p < \infty$, if and only if there exists u in $\mathbb{R}_+^{n+1}$ such that

$$(3.10) \quad \int_{\mathbb{R}_+^{n+1}} \|\tilde{M}_J f(x, t)\|_X^p u(x, t) d\mu(x, t) \leq C \int_{\mathbb{R}^n} \|f(x)\|_X^p v(x) dx$$

where the constant C is independent of the set J .

PROOF OF 3.1. We first observe that for every $(x, t) \in \mathbb{R}_+^{n+1}$ we have

$$\|\tilde{P}f(x, t)\|_E \leq P(\|f\|_E)(x, t)$$

and

$$\|\tilde{T}_\gamma f(x, t)\|_E \leq T_\gamma(\|f\|_E)(x, t).$$

Therefore it is enough to prove the result for the special case $E = \mathbb{R}$ and $f \geq 0$.

In order to see that condition D_p^* is sufficient we observe that, by Remark 1.1 there exists a weight u satisfying

$$\int_{\mathbb{R}^{n+1}} \mathcal{M}_0 f(x, t)^p u(x, t) d\mu(x, t) \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx$$

and then (3.2) follows since

$$Pf(x, t) \leq \mathcal{M}_0 f(x, t), \quad (x, t) \in \mathbb{R}_+^{n+1}.$$

Now we shall show that condition D_p^γ is sufficient in (3.3). We assume first that $\frac{1}{p} - \frac{\gamma}{n} > 0$. Without loss of generality we may assume that $v^{-p'/p}$ is not equal to zero a.e. on $B(0, 1)$. Given a function f we write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$. To deal with f_1 , remark that

$$\int_{B(0,1)} v(y)^{-p'/p} dy \leq C \int_{\mathbb{R}^n} \frac{v(y)^{-p'/p}}{(1+|y|)^{(n-\gamma)p'}} \leq C.$$

Therefore as $\mathcal{M}_0$ maps $L^1(\mathbb{R}^n, dx)$ into weak- $L^1(\mathbb{R}_+^{n+1}, d\mu)$, see [F-S], we have that

$$\mathcal{M}_0(v^{-p'/p}\chi_{B(0,1)})(x, t) < +\infty \text{ a.e. in } (x, t)$$

and then we are in the hypothesis of Lemma 2.1. Therefore, if u is defined as

$$u^{p/q} = \mathcal{M}_0(v^{-p'/p}\chi_{B_1})^{-\beta'} \chi_{\hat{B}_1}$$

we get by (2.3)

$$\begin{aligned} \left(\int_{\hat{B}_1} |T_\gamma f_1(x, t)|^q u(x, t) d\mu(x, t) \right)^{1/q} &\leq C_{\gamma, p, q} \left(\int_{B_1} |f_1(x)|^p v(x) dx \right)^{1/p} \\ &\leq C_{\gamma, p, q} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}. \end{aligned}$$

In order to consider f_2 we apply Lemma 2.16 and get

$$\begin{aligned} & \left(\int_{\hat{B}_1} |T_\gamma f_2(x, t)|^q u(x, t) d\mu(x, t) \right)^{1/q} \\ & \leq C_{\gamma, p, q} \left(\int_{\mathbb{R}^n} |f_2(x)|^p v(x) dx \right)^{1/p} \left(\int_{\hat{B}_1} u(x, t) d\mu(x, t) \right)^{1/q} \\ & \leq C_{\gamma, p, q} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p} \end{aligned}$$

where for the last inequality we have used the hypotheses that u is a bounded function and that μ is a Carleson measure.

Suppose now that $\frac{1}{p} - \frac{\gamma}{n} \leq 0$. As before we write $f = f_1 + f_2$ where $f_1 = f\chi_{B(0,2)}$.

To deal with f_1 we first remark that there always exists δ with $0 < \delta < n$, such that $\frac{1}{q} \geq \frac{1}{p} - \frac{\delta}{n} > 0$. It turns out that $\delta < \gamma$; then $D_p^\gamma \subset D_p^\delta$ and hence we can apply the previous case obtaining a weight u supported in $\hat{B}_1 = \hat{B}(0, 1)$ and satisfying

$$\left(\int_{\hat{B}_1} |T_\delta f_1|^q(x, t) u(x, t) d\mu(x, t) \right)^{1/q} \leq C \left(\int |f_1(x)|^p v(x) dx \right)^{1/p}.$$

Then the desired estimate for $T_\gamma f_1$ follows from inequality

$$\chi_{\hat{B}_1}(x, t) T_\gamma f_1(x, t) \leq C T_\delta f_1(x, t).$$

The estimate for f_2 is proved using the same method as in the previous case.

We finally show that condition D_p^γ is necessary in (3.3).

Given $(z, t) \in \hat{B}(0, 1)$ we have

$$T_\gamma f(z, t) = \int_{\mathbb{R}^n} \frac{f(y)}{(|z - y| + t)^{n-\gamma}} dy \geq \int_{\mathbb{R}^n} \frac{f(y)}{(|y| + 1)^{n-\gamma}} dy.$$

Therefore, using the hypothesis, we have

$$\int_{\mathbb{R}^n} \frac{f(y)}{(|y| + 1)^{n-\gamma}} dy \left(\int_{\hat{B}(0,1)} u(z, t) d\mu(z, t) \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(y)|^p v(y) dy \right)^{1/p}.$$

Now, if we take $f(y) = g(y)v^{-1/p}(y)$, we can conclude that

$$\int_{\mathbb{R}^n} g(y) \frac{v^{-1/p}(y)}{(|y|+1)^{n-\gamma}} dy \leq C \left(\int_{\mathbb{R}^n} g^p(y) dy \right)^{1/p}$$

and this implies that

$$\int_{\mathbb{R}^n} \frac{v^{-p'/p}(y)}{(|y|+1)^{(n-\gamma)p'}} dy \leq C.$$

This concludes the proof of Theorem 3.1. $\square$

PROOF OF 3.5. Assume that $v \in D_{p,X}^\gamma$ and suppose that $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} > 0$. Given a function f , write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$, and remark that

$$\int_{B(0,1)} v^{-p'/p}(x) dx = \int_{\mathbb{R}^n} \|a\varphi_{\gamma,L}(x)\|_{X^*}^{p'} v(x)^{-p'/p} dx \leq C$$

where $\varphi_{\gamma,L}(x, \omega) = \frac{\chi_{B(0,1)}(x)}{|B(0,1)|^{1-\gamma/n}}$ and $\|a\|_{X^*} = 1$. Therefore, as in the proof of (3.3), since $\mathcal{M}_0$ maps $L^1(\mathbb{R}^n, dx)$ into weak- $L^1(\mathbb{R}_+^{n+1}, d\mu)$, we have that $\mathcal{M}_0(v^{-p'/p}\chi_{B(0,1)})(x, t) < +\infty$ a.e. in (x, t) and then we are in the hypothesis of Lemma 2.9. Therefore, if u is defined as

$$u^{p/q} = \mathcal{M}_0(v^{-p'/p}\chi_{B_1})^{-\beta'} \chi_{\hat{B}_1},$$

we get

$$\begin{aligned} \left(\int_{\hat{B}_1} \|\tilde{\mathcal{M}}_{\gamma,J} f_1(x, t)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} &\leq C_{\gamma,p,q} \left(\int_{B_1} \|f_1(x)\|_X^p v(x) dx \right)^{1/p} \\ &\leq C_{\gamma,p,q} \left(\int \|f(x)\|_X^p v(x) dx \right)^{1/p}. \end{aligned}$$

In order to consider f_2 we apply Lemma 2.15 and get

$$\begin{aligned} &\left(\int_{\hat{B}_1} \|\tilde{\mathcal{M}}_{\gamma,J} f_2(x, t)(x, t)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} \\ &\leq C_{\gamma,p,q} \|f\|_{L_X^p(v)} \cdot \left(\int_{\hat{B}_1} u(x, t) d\mu(x, t) \right)^{1/q} \\ &\leq C_{\gamma,p,q} \|f\|_{L_X^p(v)}; \end{aligned}$$

for the last inequality we have used the hypotheses that u is a bounded function and that μ is a Carleson measure.

If $\frac{1}{p} - \frac{\gamma}{n} \leq 0$ we continue as in the corresponding proof of (3.3).

In order to see that condition $D_{p,X}^\gamma$ is necessary we shall prove that (3.6) implies (ii) of 2.15. We first observe that if f is a X -valued locally-integrable function with support in the complement of a ball $2B = B(0, 2R)$, then for any $(x, t), (z, t) \in \hat{B} = \hat{B}(0, R)$, we have

$$\tilde{M}_{\gamma,J} f(x, t) \leq C \tilde{M}_{\gamma,L} f(0, R) \leq C' M_{\gamma,K} f(z, t)$$

where the sets L and K are defined by

$$L = \{2r : r \in J, r \geq R\}, \quad K = \{3r : r \in J, r \geq R\}.$$

Then if u is the weight whose existence is given by the hypothesis we have

$$\begin{aligned} \sup_{(x,t) \in \hat{B}} \|\tilde{M}_{\gamma,J} f(x, t)\|_X &\leq C \|\tilde{M}_{\gamma,L} f(0, R)\|_X \\ &= C \left(\int_{\hat{B}} u(x, t) d\mu(x, t) \right)^{-1} \left(\int_{\hat{B}} \|\tilde{M}_{\gamma,L} f(0, R)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} \\ &\leq C \left(\int_{\hat{B}(0,1)} u(x, t) d\mu(x, t) \right)^{-1} \left(\int_{\mathbb{R}^n} \|\tilde{M}_{\gamma,K} f(x, t)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} \\ &\leq C \|f\|_{L_X^q(v)}. \end{aligned} \quad \square$$

PROOF OF 3.9. The necessity of condition $D_{p,X}$ can be proved as in Theorem 3.5. In order to see that condition $D_{p,X}$ is sufficient, we shall apply Lemma 2.12. We consider the following sequence of subsets of $\mathbb{R}_+^{n+1}$:

$$\begin{aligned} S_0 &= \{(x, t) \in \mathbb{R}_+^{n+1} : |x| + t \leq 1\}, \\ S_k &= \{(x, t) \in \mathbb{R}_+^{n+1} : 2^{k-1} \leq |x| + t < 2^k\} \quad k = 1, 2, \dots \end{aligned}$$

Given $k \geq 0$ we decompose each function f as $f' + f''$, where $f' = f \chi_{B_{k+1}}$ and $B_k = \{x : |x| < 2^k\}$.

Using Lemma 2.15 we have

$$\sup_{(x,t) \in S_k} \|\tilde{M}_J f''(x, t)\| \leq C \|f''\|_{L_X^p(v)}$$

with the constant C independent of J . Thus

$$\sup_{(x,t) \in S_k} \left(\sum_j \|\tilde{\mathcal{M}}_J f_j''(x,t)\|_X^p \right)^{1/p} \leq C \left(\sum_j \|f_j''\|_{L_X^p(v)}^p \right)^{1/p}.$$

Therefore, the vector-valued inequality of the hypothesis of Lemma 2.12 for the functions (f_j'') follows immediately with $G = L_X^p(v)$, $F = X$ and $A_k = S_k$.

On the other hand we use Cotlar's inequality (see V.2.8 in [GC-R de F]), and (1.11) to obtain for $0 < s < 1$

$$\begin{aligned} & \left\| \left(\sum_j \|\tilde{\mathcal{M}}_J f_j'\|_X^p \right)^{1/p} \right\|_{L^s(S_k, d\mu)} \\ & \leq C_s \mu(S_k)^{1/s-1} \sup_{\lambda > 0} \lambda \mu \left(\left\{ (x,t) : \sum_j \|\tilde{\mathcal{M}}_J f_k'(x,t)\|_X^p > \lambda^p \right\} \right) \\ & \leq C_s |S_k|^{1/s-1} \int_{|x| < 2^{k+1}} \left(\sum_j \|f_j(x)\|_X^p \right)^{1/p} dx \\ & \leq C_s |S_k|^{1/s} \left(\int_{\mathbb{R}^n} \sum_j \|f_j(x)\|_X^p v(x) dx \right)^{1/p} \left(\frac{1}{|B_k|^{p'}} \int_{|x| < 2^{k+1}} v(x)^{-p'/p} dx \right)^{1/p'} \\ & \leq C_s |S_k|^{1/s} \left(\sum_j \|f_j\|_G^p \right)^{1/p}; \end{aligned}$$

for the last inequality we have used the fact that

$$\frac{1}{|B_k|^{p'}} \int_{B_k} v^{-p'/p} = \int_{\mathbb{R}^n} \|a\varphi_L(x)\|_{X^*}^{p'} v(x)^{-p'/p} dx \leq C$$

with $\varphi_L(x, \omega) = \frac{\chi_{B_k}(x)}{|B_k|}$ and $\|a\|_{X^*} = 1$.

Now a direct application of Lemma 2.12 shows the existence of a function u satisfying (3.10). $\square$

PROOF OF 3.7. Sufficiency of condition D_p can be established as in Theorem 3.7.

For necessity, we remark that if $d\mu(x, t) = dx \otimes \delta_0(t)$ then inequality (3.8) becomes

$$\int_{\mathbb{R}^n} \|R_i f(x)\|_E^p u(x) dx \leq C \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx$$

and in this case it is known (see [GC-R de F], p. 561 for a proof) that $v \in D_p$. For a general Carleson measure μ the mentioned proof can be easily adapted. $\square$

3.11 REMARK. Given a Carleson measure $d\mu$ in $\mathbb{R}_+^{n+1}$ we define for $1 < p < \infty$ and $0 \leq \gamma < n$ the following classes of weights in $\mathbb{R}_+^{n+1}$:

$$D_p^{*\gamma}(d\mu) = \left\{ v : \sup_{R \geq 1} R^{(\gamma-n)p'} \int_{|x|+t \leq R} v(x, t)^{-p'/p} d\mu(x, t) < +\infty \right\},$$

$$D_p^\gamma(d\mu) = \left\{ v : \int_{\mathbb{R}_+^{n+1}} v(x, t)^{-p'/p} (|x| + t + 1)^{(\gamma-n)p'} d\mu(x, t) < +\infty \right\}.$$

Then we have the following result.

3.12 THEOREM. Let $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and v be a weight in $\mathbb{R}_+^{n+1}$.

(3.13) v belongs to $D_p^*(d\mu)$ with $1 < p < \infty$ if and only if there exists a weight u in $\mathbb{R}^n$ such that for any Banach space E

$$\int_{\mathbb{R}_+^{n+1}} \|\tilde{P}^* g(x)\|_E^p u(x) dx \leq C \int_{\mathbb{R}_+^{n+1}} \|g(x, t)\|_E^p v(x, t) d\mu(x, t)$$

where $\tilde{P}^*$ is the vector-valued extension of the positive operator

$$P^* g(x) = \int_{\mathbb{R}_+^{n+1}} \frac{g(y, t) d\mu(y, t)}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}}.$$

(3.14) v belongs to $D_p^\gamma(d\mu)$ with $1 < p < \infty$ and $0 < \gamma < n$ if and only if there exists a weight u in $\mathbb{R}^n$ such that for any Banach E

$$\left(\int_{\mathbb{R}^n} \|\tilde{T}_\gamma^* g(x)\|_E^q u(x) dx \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}_+^{n+1}} \|g(x, t)\|_E^p v(x, t) d\mu(x, t) \right)^{1/p},$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}, \quad 1 < q < \infty.$$

(3.15) Let $0 \leq \gamma < n$, E and F be Banach spaces and suppose that $v \in D_p^\gamma(d\mu)$. Assume that τ_γ^* is a bounded sublinear operator from $L_E^p(\mathbb{R}_+^{n+1}, d\mu)$ into

$L_F^q(\mathbb{R}^n, dx)$ such that

$$\|\tau_\gamma^* g(x)\|_F \leq C \int_{\mathbb{R}_+^{n+1}} \frac{\|g(y, t)\|_E d\mu(y, t)}{(|x - y| + t)^{n-\gamma}}.$$

Then there exists a weight u in $\mathbb{R}^n$ such that

$$\left(\int_{\mathbb{R}^n} \|\tau_\gamma^* g(x)\|_F^q u(x) dx \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}_+^{n+1}} \|g(x, t)\|_E^p v(x, t) d\mu(x, t) \right)^{1/p}.$$

(3.16) Let E be a Banach space with the U.M.D. property. The weight v belongs to $D_p(d\mu)$ with $1 < p < \infty$ if and only if there exists u in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \|\tilde{Q}_i^* g(x)\|_E^p u(x) dx \leq C \int_{\mathbb{R}_+^{n+1}} \|g(x, t)\|_E^p v(x, t) d\mu(x, t)$$

where $\tilde{Q}_i^*$ is the vector-valued extension of the operator

$$\tilde{Q}_i^* g(x) = \int_{\mathbb{R}_+^{n+1}} \frac{y_i - x_i}{(|y - x|^2 + t^2)^{\frac{n+1}{2}}} g(y, t) d\mu(y, t).$$

PROOF. Technical results analogous to Proposition 1.7 and Lemmas 2.1, 2.9 and 2.16 hold also for the operators $\tilde{P}^*$, $\tilde{Q}_i^*$, $\tilde{T}_\gamma^*$ and τ_γ^* , and this theorem is therefore established following the lines of the proofs of Theorems 3.1, 3.4 and 3.7. $\square$

4. - Duality of the two-weight problem

Let $(Y, d\mu)$ and $(Z, d\nu)$ be measure spaces, E and F be Banach spaces and assume that T is a linear operator such that for some pair of exponents (p, q) , $1 < p, q < \infty$, T is bounded from $L_E^p(d\nu)$ into $L_F^q(d\mu)$ and there exists a pair of weights (u, v) such that

$$(4.1) \quad \left(\int_Y \|Tf(y)\|_F^q u(y) d\mu(y) \right)^{1/q} \leq C \left(\int_Z \|f(z)\|_E^p v(z) d\nu(z) \right)^{1/p}.$$

Then can consider the transpose operator T^* from $L_{F^*}^q(d\mu)$ into $(L_E^p(d\nu))^*$ defined by

$$\langle T^* g, f \rangle = \int_Y \langle Tf(y), g(y) \rangle d\mu(y).$$

If we know that T^*g is a E^* -valued function then we actually have that

T^* maps $L_{F^*}^{q'}(d\mu)$ into $L_{E^*}^{p'}(d\nu)$ and moreover the following inequality holds:

$$(4.2) \quad \left(\int_Z \|T^*g(z)\|_{E^*}^{p'} v^{1-p'}(z) d\nu(z) \right)^{1/p'} \leq C \left(\int_Y \|g(y)\|_{F^*}^{q'} u(y)^{1-q'} d\mu(y) \right)^{1/q'}.$$

In this case we also have that the pair (u, v) satisfies (4.1) if and only if the pair $(v^{1-p'}, u^{1-q'})$ satisfies (4.2).

In other words finding necessary and sufficient conditions for a weight u to satisfy inequality (4.1) is equivalent to finding necessary and sufficient conditions for a weight $u^{1-q'}$ to satisfy (4.2).

4.3 DEFINITION. Let $d\mu$ be a Carleson measure on $\mathbb{R}_+^{n+1}$. We shall say that a weight u in $\mathbb{R}_+^{n+1}$ satisfies condition $Z_p^\gamma(d\mu)$ with $1 < p < \infty$ and $0 \leq \gamma < n$, if and only if

$$\int_{\mathbb{R}_+^{n+1}} \frac{u(x, t)}{(|x| + t + 1)^{(n-\gamma)p}} d\mu(x, t) \leq C < +\infty.$$

4.4 THEOREM. Let E be a Banach space, $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and u be a weight in $\mathbb{R}_+^{n+1}$. The following conditions are equivalent for $1 < p < \infty$:

- (i) $u^{1-p'} \in D_p^*(d\mu)$;
- (ii) There exists a weight v in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}_+^{n+1}} \|\tilde{P}f(x, t)\|_E^p u(x, t) \leq C \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx.$$

4.5 THEOREM. Let $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and u be a weight in $\mathbb{R}_+^{n+1}$. Assume $1 < p < \infty$. The following conditions are equivalent:

- (i) $u \in Z_p(d\mu)$;
- (ii) For any U.M.D. Banach space E there exists a weight v such that

$$\int_{\mathbb{R}_+^{n+1}} \|\tilde{Q}_i f(x, t)\|_E^p u(x, t) d\mu(x, t) \leq C_p \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx, \quad i = 1, 2, \dots, n;$$

- (iii) For any Banach lattice X with the H.L. property there exists a weight v such that

$$\int_{\mathbb{R}_+^{n+1}} \|\tilde{M}_{0,L} f(x, t)\|^p u(x, t) d\mu(x, t) \leq C \int_{\mathbb{R}^n} \|f(x)\|^p v(x) dx$$

with the constant C independent of the finite set L .

4.6 THEOREM. Let $d\mu$ be a Carleson measure in $\mathbb{R}_+^{n+1}$ and u be a weight in $\mathbb{R}_+^{n+1}$. Assume $0 < \gamma < n$, $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$ and $1 < q < \infty$. The following conditions

are equivalent:

- (i) $u \in Z_p^\gamma(d\mu)$;
(ii) For any Banach space E there exists a weight v such that

$$\left(\int_{\mathbb{R}_+^{n+1}} \|\tilde{T}_\gamma f(x, t)\|_E^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx \right)^{1/p};$$

- (iii) For any Banach lattice X there exist a weight v such that

$$\left(\int_{\mathbb{R}_+^{n+1}} \|\tilde{M}_{\gamma, L} f(x, t)\|_X^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p v(x) dx \right)^{1/p}$$

with the constant C independent of the finite set L .

PROOF OF 4.4, 4.5 AND 4.6. The results for the operators $\tilde{P}$, $\tilde{T}_\gamma$ and $\tilde{Q}_i$ follow directly using duality and (3.13), (3.14) and (3.16).

For the operator $\tilde{M}_{\gamma, L}$ we consider the $\ell^\infty(X)$ -valued linear operator

$$\tilde{T}_{\gamma, L} f(x, t) = \left\{ \frac{1}{r^{n-\gamma}} \int_{B(x, r)} f(y) dy \right\}_{r \in L}.$$

We have

$$\tilde{M}_{\gamma, L} f(x, t) = |\tilde{T}_{\gamma, L} f(x, t)|,$$

where $|\cdot|$ denotes the absolute value in the lattice

$$\ell^\infty(X) = \{(x_n) \in X : \|(x_n)\| = \|\sup |x_n|\|_X\},$$

and

$$\|\tilde{M}_{\gamma, L} f(x, t)\|_X = \|\tilde{T}_{\gamma, L} f(x, t)\|_{\ell^\infty(X)}.$$

Therefore statement (iii) is equivalent to the following:

There exists a weight v such that

$$\left(\int_{\mathbb{R}_+^{n+1}} \|\tilde{T}_{\gamma, L} f(x, t)\|_{\ell^\infty(X)}^q u(x, t) d\mu(x, t) \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p v(x) dx \right)^{1/p}.$$

But this fact can be obtained in a standard way using duality and Theorem 3.5. This completes the proof of Theorems 4.4, 4.5 and 4.6. $\square$

5. - Properties of the classes $D_{p,X}^\gamma$

5.1 PROPOSITION. *Let X be a Köthe function space with X' norming and $1 < p < \infty$. The following relations hold:*

- (i) $D_p^\gamma \subset D_p^\delta \subset D_p^0$ for $0 < \delta < \gamma$.
- (ii) $D_p^\gamma \subset D_{p,X}^\gamma \subset D_{p,X}^{*,\gamma}$ for $0 < \gamma < n$.

PROOF. (i) is obvious.

In order to show (ii) we remark that for every ball $B = B(0, r)$ (with $r \geq 1$)

$$\frac{\chi_B(x)}{|B|^{1-\gamma/n}} \leq \frac{C}{(1+|x|)^{n-\gamma}};$$

then if $a \in X'$ and $\varphi_{\gamma,L}$ is a function of the type defined in 2.13 we have

$$a(\omega)\varphi_{\gamma,L}(x, \omega) \leq \frac{Ca(\omega)}{(1+|x|)^{n-\gamma}}.$$

Therefore $\|a\varphi_{\gamma,L}(x)\|_{X^*} \leq C\|a\|_{X^*} \frac{1}{(1+|x|)^{n-\gamma}}$ which implies that $D_p^\gamma \subset D_{p,X}^\gamma$.

Now we shall see that $D_{p,X}^\gamma \subset D_{p,X}^{*,\gamma}$. Given a ball $B(0, R)$ with $R \geq 1$ we consider

$$\varphi_{\gamma,L}(x, \omega) = \frac{\chi_B(x)}{|B|^{1-\gamma/n}}$$

where $L = \{R\}$; then for $a \in X'$ with $\|a\|_{X^*} = 1$, we have

$$\|a\varphi(x)\|_{X^*} = \frac{\chi_B(x)}{|B|^{1-\gamma/n}}.$$

Therefore

$$\begin{aligned} \frac{1}{R^{(n-\gamma)p'}} \int_{|x| \leq R} v(x)^{-p'/p} dx &= C \frac{1}{|B|^{(1-\gamma/n)p'}} \int_B v(x)^{-p'/p} dx \\ &= C \int_{\mathbb{R}^n} \|a\varphi(x)\|_{X^*}^{p'} v(x)^{-p'/p} dx \leq C'. \end{aligned} \quad \square$$

In case $X = \ell^r$ with $1 \leq r \leq \infty$ we denote D_{p,ℓ^r}^γ by $D_{p,r}^\gamma$. For this case we have the following result.

5.2 COROLLARY. *Assume $1 < p < \infty$ and $0 \leq \gamma < n$. Then $D_{p,1}^\gamma = D_p^\gamma$ and $D_{p,\infty}^\gamma = D_p^{*,\gamma}$.*

PROOF. It is clear that $v \in D_{p,r}^\gamma$ with $1 < r \leq \infty$ if and only if

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left| a_j \frac{\chi_{B_{r_j}}(x)}{r_j^{n-\gamma}} \right|^{r'} \right)^{p'/r'} v(y)^{-p'/p} dy \leq C$$

for any sequence $(a_j) \in \ell^{p'}$ with $\|(a_j)\|_{\ell^{p'}} \leq 1$ and any sequence of balls (B_{r_j}) centered at the origin with rational radii r_j bigger than one.

If $v \in D_p^*$ using Minkowski's inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_j \left| a_j \frac{\chi_{B_{r_j}}(x)}{r_j^{n-\gamma}} \right| \right)^{p'} v(x)^{-p'/p} dx \\ & \leq \left(\sum_j \left(\int_{\mathbb{R}^n} \frac{\chi_{B_{r_j}}(x)}{|r_j|^{(n-\gamma)p'}} v(x)^{-p'/p} dx \right)^{1/p'} a_j \right)^{p'} \\ & \leq C \left(\sum_j a_j \right)^{p'} \leq C'. \end{aligned}$$

Therefore $D_p^{*,\gamma} \subset D_{p,\infty}^\gamma$.

On the other hand $v \in D_{p,1}^\gamma$ if and only if

$$\int_{\mathbb{R}^n} \sup_j \left| a_j \frac{\chi_{B_{r_j}}(y)}{r_j^{n-\gamma}} \right|^{p'} v(y)^{-p'/p} dy \leq C$$

for any $(a_j) \in \ell^\infty$ with $\|(a_j)\|_{\ell^\infty} \leq 1$ and any sequence of balls (B_{r_j}) centered at the origin with rational radii (r_j) bigger than one.

Now if $v \in D_{p,1}^\gamma$ and if we denote by B_k the ball centered at the origin with radius 2^k for $k = 0, 1, 2, \dots$ then

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}} dx = \int_{B_0} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}} dx + \sum_{k=1}^{\infty} \int_{B_k \setminus B_{k-1}} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}} dx \\ & \leq C \left(\int_{B_0} v(x)^{-p'/p} dx + \sum_{k=1}^{\infty} \int_{B_k} \frac{v(x)^{-p'/p}}{2^{k(n-\gamma)p'}} dx \right) \\ & \leq C \int_{\mathbb{R}^n} \sup_k \frac{\chi_{B_k}(x)}{2^{k(n-\gamma)p'}} v(x)^{-p'/p} dx \leq C'. \end{aligned}$$

Therefore $D_{p,1}^\gamma \subset D_p^\gamma$. □

REFERENCES

- [A-B] E. AMAR - A. BONAMI, *Measures de Carleson d'ordre α et solutions au bord de l'équation $\bar{\partial}$* . Bull. Soc. Math. France **107** (1979), 23-48.
- [B] J. BOURGAIN, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*. Ark. Mat. **22** (1983), 163-168.
- [Bk] D. BURKHOLDER, *A geometrical characterization of Banach spaces in which martingale differences are unconditional*. Ann. Probab. **9** (1981), 997-1011.
- [C-J] L. CARLESON - P.W. JONES, *Weighted norm inequalities and a theorem of Koosis*. Inst. Mittag-Leffler, Report n. 2, 1981.
- [F-S] C. FEFFERMAN - E.M. STEIN, *Some maximal inequalities*. Amer. J. Math. **1** (1971), 107-115.
- [F-T] L.M. FERNANDEZ-CABRERA - J.L. TORREA, *Vector-valued inequalities with weights*. Publ. Mat. **37** (1993) 177-208.
- [GC-M-T] J. GARCIA-CUERVA - R.A. MACIAS - J.L. TORREA, *The Hardy-Littlewood property of Banach Lattices*. To appear in Israel J. Math.
- [GC-R de F] J. GARCIA-CUERVA - J.L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*. North-Holland, Math. Studies 116, 1985.
- [G-G] A.E. GATTO - C.E. GUTIERREZ, *On weighted norm inequalities for the maximal function*. Studia Math. **76** (1983), 59-62.
- [H-M-S] E. HARBOURE - R.A. MACIAS - C. SEGOVIA, *Boundedness of fractional operators on L^p spaces with different weights*. Trans. Amer. Math. Soc. **285** (1984), 629-647.
- [J] R. JOHNSON, *Application of Carleson measures to partial differential equations and Fourier multiplier problems*. Lectures Notes in Mathematics, n. 992, Springer-Verlag 1983, 16-72.
- [K] P. KOOSIS, *Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques*. C.R. Acad. Sci. Paris Sér. I Math. **291** (1980), 255-259.
- [L-T] J. LINDENSTRAUSS - L. TZAFRIRI, *Classical Banach spaces II. Function spaces*. Springer-Verlag, Berlin, 1979.
- [R-T] F.J. RUIZ - J.L. TORREA, *Weighted and vector-valued inequalities for potential operators*. Trans. Amer. Math. Soc. **295** (1986), 213-232.

Escuela Universitaria de Estadística
 Universidad Complutense de Madrid
 Madrid
 Spain

Department of Mathematics
 Universidad Autónoma de Madrid
 28049 Madrid
 Spain