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Convexly Totally Bounded and Strongly Totally Bounded Sets. Solution of a Problem of Idzik

E. DE PASCALE - G. TROMBETTA - H. WEBER

0. - Introduction

A long outstanding problem in fixed point theory is the following

PROBLEM OF SCHAUDER (Problem 54, [G]). *Does every continuous function $f : C \rightarrow C$ defined on a convex compact subset of a Hausdorff topological linear space have a fixed point?*

Schauder [S] gave a positive answer to this problem if the linear space is a Banach space. Thychonoff [T] generalized Schauder's theorem for locally convex spaces. Schauder's problem is still open even for metrizable (nonlocally convex) spaces. Idzik [I1] proved that the answer to Schauder's problem is "yes" if C is convexly totally bounded. This notion was introduced by Idzik [I1]: A subset K of a topological linear space E is called *convexly totally bounded* (ctb for short) if for every 0-neighbourhood U there are $x_1, \dots, x_n \in E$ and convex subsets $C_1, \dots, C_n$ of U such that $K \subset \bigcup_{i=1}^n (x_i + C_i)$. Idzik formulated – comparing his theorem with Schauder's problem – the following

PROBLEM (cf. Problem 4.7 of [I2]). *Is every convex compact subset of a Hausdorff topological linear space ctb?*

A positive answer to this problem would imply a positive answer to Schauder's problem.

In the first section of this paper we give a negative answer to Idzik's problem. In Section 2, we introduce the notion of strongly convexly totally bounded (sctb) sets and in Section 3 a parameter, which measures "the lack of strongly convexly total boundedness". This notion and this parameter is – in contrast to convexly total boundedness – invariant when one passes to the convex hull of a set. That admits the formulation of a fixed point theorem of

Darbo type [D] in nonlocally convex spaces. In Sections 4 and 5, we examine the mentioned parameter in the space l_p ($0 < p < 1$) and in the space L_0 of measurable functions.

The notion of scfb sets and the corresponding noncompactness measure is the main tool in [DP/T2] to get a best approximation result of Fan type in nonlocally convex linear spaces. In [W] strongly convexly total boundedness is linked with affine embeddability in locally convex spaces.

The main results of this paper were presented on the conference "Approximation Theory, Spline Functions and Applications" at Acquafredda di Maratea, Italy, April 29 – May 9, 1991.

In this paper we use the terminology of [J]. $\mathbb{N}$ and $\mathbb{R}$ stands for the sets of all natural and real numbers, respectively.

1. - A compact convex set not convexly totally bounded

In this section we construct a compact convex set, which is not ctb; this solves Problem 4.7 of Idzik [I2]. We obtain such a set by a modification of Robert's example for a compact convex set without extreme point, see [R], [Ro, Section 5.6]. Robert's example is based on his notion of *needle point*; that is a point $x_0 \neq 0$ of an (in the sense of [J]) F -normed linear space $(E, \|\cdot\|)$ with the following property: for every $\varepsilon > 0$, the ball $B_\varepsilon := \{x \in B : \|x\| \leq \varepsilon\}$ contains a finite set F such that $\text{co } F \subset \text{co}\{0, x_0\} + B_\varepsilon$ and $x_0 \in \text{co } F + B_\varepsilon$. (Hereby $\text{co } F$ denotes the convex hull of F). In our construction, a stronger property plays an important role:

DEFINITION 1.1. We call a point x_0 of an F -normed linear space $(E, \|\cdot\|)$ a *strong needle point*, if $x_0 \neq 0$ and, for every $\varepsilon > 0$, there is a natural number k and an infinite subset M of B_ε such that $\text{co } M \subset \text{co}\{0, x_0\} + B_\varepsilon$ and $x_0 \in \text{co } F + B_\varepsilon$ for every finite subset F of M , with $|F| \geq k$. (Hereby we denote by $|F|$ the cardinality of F).

LEMMA 1.2. For each $n \in \mathbb{N} \cup \{0\}$, let F_n be a finite subset of an F -normed linear space E and $\varepsilon_n > 0$ such that $\text{co } F_n \subset \text{co } F_0 + B_{\varepsilon_n}$. Assume that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. Then $C := \text{co} \bigcup_{n=0}^{\infty} F_n$ is totally bounded.

PROOF. Since $\bigcup_{i=0}^n F_i$ is a compact subset of a finite dimensional subspace of E , also $C_n := \text{co} \bigcup_{i=0}^n F_i$ is compact. Furthermore, we have $r_n := \sum_{i>n} \varepsilon_i \rightarrow 0$ ($n \rightarrow \infty$). Therefore it is enough to prove that $C \subset C_n + B_{r_n}$ for all $n \in \mathbb{N}$.

For $\alpha, \beta \in [0, 1]$, with $\alpha + \beta = 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha \cdot C_n + \beta \cdot \text{co } F_{n+1} &\subset \alpha \cdot C_n + \beta \cdot \text{co } F_0 + \beta \cdot B_{\varepsilon_{n+1}} \\ &\subset \alpha \cdot C_n + \beta \cdot C_n + B_{\varepsilon_{n+1}} \subset C_n + B_{\varepsilon_{n+1}}, \end{aligned}$$

hence $C_{n+1} = \bigcup_{\alpha \in [0,1]} \alpha \cdot C_n + (1 - \alpha) \cdot \text{co } F_{n+1} \subset C_n + B_{\varepsilon_{n+1}}$. It follows inductively

that $C_{n+k} \subset C_n + \sum_{i=n+1}^{n+k} B_{\varepsilon_i}$ for $n, k \in \mathbb{N}$; therefore $C_{n+k} \subset C_n + B_{r_n}$ and $C = \bigcup_{k=1}^{\infty} C_{n+k} \subset C_n + B_{r_n}$.

THEOREM 1.3. *Let E be a complete F -normed linear space, which contains a strong needle point. Then E contains a compact absolutely convex set, which is not *ctb*.*

PROOF. Let $\varepsilon_n > 0$, with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$, and x_0 be a strong needle point of E . Choose infinite subsets M_n of B_{ε_n} and $k_n \in \mathbb{N}$ such that $\text{co } M_n \subset \text{co}\{0, x_0\} + B_{\varepsilon_n}$, and $x_0 \in \text{co } F + B_{\varepsilon_n}$, for every finite subset F of M_n , with $|F| \geq k_n$. Let $F_0 := \{0, x_0\}$ and F_n be finite subsets of M_n with $|F_n| \geq n \cdot k_n$ for $n \in \mathbb{N}$. Since $C := \text{co} \bigcup_{n=0}^{\infty} F_n$ is totally bounded by Lemma 1.2, $\overline{C}$ and $\overline{C} - \overline{C}$ are compact. Therefore the closed absolutely convex hull $K := \overline{\text{aco}} \bigcup_{n=0}^{\infty} F_n$ of C is a compact subset of $\overline{C} - \overline{C}$ containing C . We now show that C is not *ctb*; then neither is K . Let $\varepsilon > 0$, with $x_0 \notin B_{4\varepsilon}$. Suppose that C is *ctb*. Then there are convex subsets C_i of B_{ε} and $y_i \in E$ such that $C \subset \bigcup_{i=1}^m (y_i + C_i)$. Choose $n \geq m$ with $\varepsilon_n \leq \varepsilon$. Since $F_n \subset C \subset \bigcup_{i=1}^m (y_i + C_i)$ and consequently

$$\sum_{i=1}^m |F_n \cap (y_i + C_i)| \geq |F_n| \geq n \cdot k_n \geq m \cdot k_n,$$

for some $j \in \{1, \dots, m\}$, the set $F := F_n \cap (y_j + C_j)$ has at least k_n elements. Therefore

$$x_0 \in \text{co } F + B_{\varepsilon_n} \subset y_j + C_j + B_{\varepsilon} \subset y_j + B_{2\varepsilon}.$$

Let $y \in F$. Then $y_j \in y - C_j \subset F_n - C_j \subset B_{\varepsilon_n} + B_{\varepsilon} \subset B_{2\varepsilon}$. It follows that $x_0 \in y_j + B_{2\varepsilon} \subset B_{2\varepsilon} + B_{2\varepsilon} \subset B_{4\varepsilon}$, a contradiction.

We now show the existence of strong needle points in the same way as Roberts has shown the existence of needle points; see [R], [Ro, Section 5.6].

In the following, let $\varphi : [0, +\infty[\rightarrow [0, +\infty[$ be a continuous increasing concave function such that

$\varphi(x) = 0$ iff $x = 0$ and

$\varphi(x)/x \rightarrow 0$ for $0 < x \rightarrow +\infty$.

If $\mathcal{U}$ is an algebra of sets and μ a $[0, 1]$ -valued measure ($:=$ finitely additive set function) on $\mathcal{U}$, we denote by $S(\mathcal{U})$ the space of real-valued $\mathcal{U}$ -simple functions, i.e. the linear hull of the system of all characteristic functions of sets of $\mathcal{U}$; furthermore we put

$$\|f\|_{\varphi} = \int \varphi \circ |f| d\mu \quad \text{and} \quad \|f\|_p := \left(\int |f|^p d\mu \right)^{1/p}$$

for $f \in S(\mathcal{U})$ and $1 \leq p < +\infty$. $\|\cdot\|_{\varphi}$ is then a Riesz pseudonorm in the sense of [A/B, p. 39] and $\|f\|_{\varphi} \leq \varphi(\|f\|_1)$ for $f \in S(\mathcal{U})$.

In the following let $\mathcal{U}_0$ be an algebra in the power set $\mathbb{P}(\Omega_0)$ of a non-empty set Ω_0 ,

$\mu_0 : \mathcal{U}_0 \rightarrow [0, 1]$ a measure with $\mu_0(\Omega_0) = 1$,

(A_n) a sequence in $\mathcal{U}_0$ with $0 < \mu_0(A_n) \rightarrow 0$ ($n \rightarrow \infty$),

$(\Omega, \mathcal{U}, \mu) = \bigotimes_{n \in \mathbb{N}} (\Omega_n, \mathcal{U}_n, \mu_n)$, where $(\Omega_n, \mathcal{U}_n, \mu_n) = (\Omega_0, \mathcal{U}_0, \mu_0)$ for $n \in \mathbb{N}$.

For $f : \Omega_0 \rightarrow \mathbb{R}$ and $i \in \mathbb{N}$, we define $S_i(f) : \Omega \rightarrow \mathbb{R}$ by $(S_i(f))(x_n) := f(x_i)$.

LEMMA 1.4. Let $f \in S(\mathcal{U}_0)$, with $\int f d\mu_0 = 1$, and $0 \leq t_i \leq t < +\infty$, with $\sum_{i=1}^n t_i \leq 1$. Then $g := \sum_{i=1}^n t_i (S_i(f) - 1) \in S(\mathcal{U})$, $\|g\|_1 \leq \sqrt{t} \cdot \|f - 1\|_2$, $\|g\|_{\varphi} \leq \varphi(\sqrt{t} \cdot \|f - 1\|_2)$.

For the proof, see [Ro, pp. 244-245] or [R, Lemma 3.8].

LEMMA 1.5. Let $0 < b \leq 1$ and $\delta > 0$. Then there are a non-negative function $f \in S(\mathcal{U}_0)$ and a number $a \in]0, b[$ with the following properties:

(i) $\int f d\mu_0 = 1$;

(ii) $\|f\|_{\varphi} \leq \delta$;

(iii) if $n \in \mathbb{N}$ and $t_i \geq b$, with $\sum_{i=1}^n t_i \leq 1$, then

$$\left\| \sum_{i=1}^n t_i S_i(f) \right\|_{\varphi} \leq \delta;$$

(iv) if $n \in \mathbb{N}$ and $0 \leq t_i \leq a$, with $\sum_{i=1}^n t_i \leq 1$, then

$$\left\| \sum_{i=1}^n t_i (S_i(f) - 1) \right\|_{\varphi} \leq \delta.$$

For the proof, see [Ro, Lemma 5.6.3] or [R, Lemma 3.10]. Hereby, 1.4 is used to prove (iv) of 1.5. With the aid of 1.5, one can prove the following proposition as [Ro, Proposition 5.6.4].

PROPOSITION 1.6. Denote by f_0 the constant function equal to 1 on Ω . Let $\varepsilon > 0$. Then there is a non-negative function $f \in S(\mathcal{U}_0)$ with the following properties:

- (i) $\int f \, d\mu_0 = 1$;
- (ii) $M := \{S_i(f) : i \in \mathbb{N}\}$ is contained in $B_{\varepsilon} := \{h \in S(\mathcal{U}) : \|h\|_{\varphi} \leq \varepsilon\}$;
- (iii) there is a $k \in \mathbb{N}$ such that, for every $I \subset \mathbb{N}$, with $|I| \geq k$, we have $f_0 \in \text{co}\{S_i(f) : i \in I\} + B_{\varepsilon}$;
- (iv) $\text{co } M \subset \text{co}\{0, f_0\} + B_{\varepsilon}$.

For small ε , a function f satisfying (i) and (ii) of 1.6 cannot be constant. The functions $S_i(f)$ are then different for different indices i ; therefore M is an infinite set. It follows:

THEOREM 1.7. Denote by N the space $N := \{f \in S(\mathcal{U}) : \|f\|_{\varphi} = 0\}$ of all $\mathcal{U}$ -simple null functions. Then $(S(\mathcal{U}), \|\cdot\|_{\varphi})/N$ contains strong needle points.

COROLLARY 1.8. Let λ be the Lebesgue measure on $[0, 1]$ and $\varphi : [0, +\infty[\rightarrow [0, +\infty[$ be a continuous increasing concave function such that $\varphi(x) = 0$ iff $x = 0$ and $\varphi(x)/x \rightarrow 0$ ($0 < x \rightarrow +\infty$). Then the Orlicz space $L_{\varphi}(\lambda)$ contains a compact absolutely convex set, which is not ctb.

PROOF. Since the measure algebras induced by λ and by $\lambda \otimes \lambda \otimes \lambda \otimes \dots$ are isomorphic, the space $L_{\varphi}(\lambda)$ is isomorphic to the completion of a space $(S(\mathcal{U}), \|\cdot\|_{\varphi})/N$ considered in 1.7. Since the last-named space contains strong needle points, so does $L_{\varphi}(\lambda)$. Now the assertion follows from 1.3.

COROLLARY 1.9. Let λ be the Lebesgue measure on $[0, 1]$ and $0 \leq p < 1$. Then $L_p(\lambda)$ contains a compact absolutely convex set, which is not ctb.

PROOF. Apply 1.8 with $\varphi(x) = x^p$ for $0 < p < 1$ and $\varphi(x) = x/(1+x)$ for $p = 0$.

2. - Strongly convexly totally bounded sets

In this section, let (E, τ) be a topological linear space. An important question in fixed point theory is: under which condition for a set $K \subset E$, the convex hull of any totally bounded subset of K is totally bounded, see [K, p. 10], [H1, p. 31], cf. also 3.1 (ii). Easy examples show that the convex hull of a ctb set does not need to be ctb, see 5.3 (b). Therefore we introduce in 2.1 a property stronger than convexly total boundedness, which is conserved passing to the convex hull, see 2.2 (b).

DEFINITION 2.1. A subset K of E is said to be *strongly convexly totally bounded* (sctb for short) if, for every 0-neighbourhood U , there is a convex subset C of U and a finite subset F of E such that $K \subset F + C$.

A subset of a *locally convex* linear space is sctb iff it is totally bounded.

PROPOSITION 2.2. (a) If K_1 and K_2 are sctb subsets of E , then the sets $K_1 \cup K_2$, $K_1 + K_2$ and $\alpha \cdot K_1$ for any $\alpha \in \mathbb{R}$ are sctb. (b) If K is a sctb subset of E , then the closed absolutely convex hull $\overline{\text{aco}} K$ of K is sctb.

PROOF. Let U be a circled closed 0-neighbourhood in E .

(i) There are finite sets $F_i \subset E$ and convex sets $C_i \subset U$ such that $K_i \subset F_i + C_i$ for $i = 1, 2$. Since $\text{co}(C_i \cup \{0\}) \subset U$, we may assume that $0 \in C_i$. Then the convex set $C := C_1 + C_2$ contains C_1 and C_2 . It follows that $K_1 \cup K_2 \subset (F_1 \cup F_2) + C$, $K_1 + K_2 \subset (F_1 + F_2) + C$, $C \subset U + U$. Hence $K_1 \cup K_2$ and $K_1 + K_2$ are sctb.

(ii) Since $\alpha \cdot K_1 \subset \alpha \cdot F_1 + \alpha \cdot C_1$, $\alpha \cdot C_1 \subset \alpha \cdot U$ and $\overline{K}_1 \subset F_1 + \overline{C}_1$, $\overline{C}_1 \subset \overline{U} = U$, the sets $\alpha \cdot K_1$ and $\overline{K}_1$ are sctb.

(iii) Let K be sctb; replacing K by $K \cup \{0\}$, we may assume that $0 \in K$. Let F be a finite subset of E and C a convex subset of U such that $K \subset F + C$, and V be a 0-neighbourhood in E . Since $E_0 := \text{span } F$ is finite dimensional, V contains a convex 0-neighbourhood V_0 in E_0 ; moreover $\text{co } F$ is a compact subset of E_0 . It follows that there is a finite subset F_0 of E_0 such that $\text{co } F \subset F_0 + V_0$. Since K is a subset of the convex set $\text{co } F + C$, we get $\text{co } K \subset \text{co } F + C \subset F_0 + (V_0 + C)$, where $V_0 + C$ is a convex subset of $V + U$. We have proved that $\text{co } K$ is sctb. Since $\text{aco } K \subset \text{co } K - \text{co } K$, the set $\text{aco } K$ is sctb by (i). It now follows from (ii) that the closure $\overline{\text{aco}} K$ is sctb.

Obviously every sctb set is ctb. In 5.3 (b), we give an example for a ctb set, the convex hull of which is not ctb; by 2.2 (b), such a set is ctb, but not sctb. But we do not know whether there are also *convex* ctb sets, which are not sctb.

In [W, Section 2] it is proved that a compact convex subset K of E is sctb iff there is a locally convex linear topology σ on E such that the

relative topologies $\sigma|K$ and $\tau|K$ coincide. Using the easy implication ($\Leftarrow$) of this equivalence for $\sigma = \sigma(E, E')$ one obtains:

PROPOSITION 2.3. *If the continuous dual E' of E separates the points, then every compact convex subset of E is sctb.*

Other examples for sctb sets are the compact convex order-bounded subsets of Orlicz function spaces, as mentioned at the end of [W, Section 2].

In 3.5 we will use the following fact:

PROPOSITION 2.4. *Let $(\tau_i)_{i \in I}$ be a family of linear topologies on E and $\tau = \sup \tau_i$. Then a set $K \subset E$ is sctb, ctb or totally bounded in (E, τ) iff K is sctb, ctb or totally bounded in (E, τ_i) , respectively, for every $i \in I$.*

PROOF (of the non-obvious implication ($\Leftarrow$) in the sctb case). Since every 0-neighbourhood in (E, τ) is a 0-neighbourhood in $(E, \sup \tau_i)$ for some finite subset J of I , we may assume that I is finite. Inductively ^{$i \in J$} we can reduce the assertion to the case that τ is the supremum of two linear topologies τ_1 and τ_2 .

Let U be a 0-neighbourhood in (E, τ) and V_i be a 0-neighbourhood in (E, τ_i) such that $(V_1 - V_1) \cap (V_2 - V_2) \subset U$. By assumption, there are convex sets $C_i \subset V_i$ and elements $x_r, y_s \in E$ such that $K \subset \bigcup_{r=1}^m (x_r + C_1)$ and $K \subset \bigcup_{s=1}^n (y_s + C_2)$. Choose $z_{rs} \in C_{rs} := (x_r + C_1) \cap (y_s + C_2)$ if $C_{rs} \neq \emptyset$ and z_{rs} arbitrary of E if $C_{rs} = \emptyset$. Then $C_{rs} \subset z_{rs} + C$, where $C := (C_1 - C_1) \cap (C_2 - C_2)$ is a convex subset of U . Moreover $K \subset \bigcup_{r,s} C_{rs} \subset \bigcup_{r,s} z_{rs} + C$.

3. - Noncompactness measures

One of the main tools in fixed point theory, after the pioneering work of Darbo [D], is the noncompactness measure. There are many axiomatic approaches to this concept (cf. [B/G], [B/R], [Rz], [H1] and references therein). All the approaches try to describe the minimal properties for a fixed point result. An axiomatic approach can be useful also in the frame of nonlocally convex spaces. Let I be a non-empty set and $V = [0, +\infty]^I$ the set of all functions from I to $[0, +\infty]$. V will be equipped with the usual algebraic operations, the usual order, and with the topology of pointwise convergence.

Let E be a Hausdorff topological linear space.

DEFINITION 3.1. We call a set function $\varphi : 2^E \rightarrow V$ a noncompactness measure in E , if φ has the following properties.

- (1) If A is a convex complete subset of E and $\varphi(A) = 0$, then A has the fixed point property.
- (2) $\varphi(A) = \varphi(\overline{\text{co}} A)$.

- (3) $A \subseteq B$ implies $\varphi(A) \leq \varphi(B)$.
- (4) If (A_n) is a decreasing sequence of complete non-empty subsets of E with $\varphi(A_n) \rightarrow 0$ ($n \rightarrow \infty$), then $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty.

Classical examples of noncompactness measures with $V = [0, +\infty]$ are the Hausdorff measure [G/G/M] and the Kuratowski measure [Ku].

DEFINITION 3.2. Let C be a non-empty subset of E and φ a noncompactness measure in E . A continuous function $f : C \rightarrow C$ is said to be a φ -contraction if there exists $0 \leq q < 1$ such that $\varphi(f(A)) \leq q \cdot \varphi(A)$ for every subset A of C .

THEOREM 3.3. Let C be a non-empty complete convex subset of E , φ a noncompactness measure and $f : C \rightarrow C$ a φ -contraction. If $\varphi(C) \in [0, +\infty[$, then f has a fixed point.

PROOF. It is classic. We define inductively a sequence of sets by $C_0 := C$ and $C_{n+1} := \overline{\text{co}} f(C_n)$. We have

$$\varphi(C_n) \leq q^n \cdot \varphi(C) \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{in } V.$$

By Axiom 4 in Definition 3.1, the set $C_\infty = \bigcap_{n \in \mathbb{N}} C_n$ is non-empty, moreover complete and convex. Since $f(C_\infty) \subseteq C_\infty$ and $\varphi(C_\infty) = 0$, the statement follows by Axiom 1 in Definition 3.1.

The crucial problem in the previous approach is that Axiom 2 in Definition 3.1 in general does not hold in the nonlocally convex case for the noncompactness measures usually used in locally convex spaces, cf. [H2], [DP/T1].

We will see in 3.8 that the set function $\overline{\gamma}_s$ introduced below, which measures the “nonstrongly convexly total boundedness”, is a noncompactness measure in the sense of Definition 3.1.

In the following, let $(\|\cdot\|_i)_{i \in I}$ be a family of F -seminorms inducing the topology in E .

NOTATION 3.4. For $i \in I$ and $A \subset E$ we put

$$B_{i,\varepsilon} := \{x \in E : \|x\|_i \leq \varepsilon\} \text{ for } \varepsilon > 0,$$

$$\gamma_i(A) := \inf\{\varepsilon \in]0, +\infty[: \text{there is a finite subset } F \subset E \text{ such that } A \subset F + B_{i,\varepsilon}\},$$

$$\overline{\gamma}_i(A) := \inf\{\varepsilon \in]0, +\infty[: \text{there are } x_1, \dots, x_n \in E \text{ and convex subsets } C_1, \dots, C_n \text{ of } B_{i,\varepsilon} \text{ such that } A \subset \bigcup_{i=1}^n x_i + C_i\},$$

$$\overline{\gamma}_{i,s}(A) := \inf\{\varepsilon \in]0, +\infty[: \text{there is a finite subset } F \subset E \text{ and a convex subset } C \text{ of } B_{i,\varepsilon} \text{ such that } A \subset F + C\}.$$

Furthermore, $\gamma := (\gamma_i)_{i \in I}$, $\overline{\gamma} := (\overline{\gamma}_i)_{i \in I}$, $\overline{\gamma}_s := (\overline{\gamma}_{i,s})_{i \in I}$.

γ is the noncompactness measure of Hausdorff. The set function $\bar{\gamma}$ is in some sense equivalent to the “measure of the lack of convex precompactness” introduced in the [DP/T1], Definition 2.1].

PROPOSITION 3.5. *Let $A \subset E$.*

- (a) $\gamma(A) \leq \bar{\gamma}(A) \leq \bar{\gamma}_s(A)$. *The three measures coincide if the F -seminorms $\|\cdot\|_i$ are even seminorms.*
- (b) *A is sctb, ctb or totally bounded iff $\bar{\gamma}_s(A) = 0$, $\bar{\gamma}(A) = 0$ or $\gamma(A) = 0$, respectively.*

PROOF. (a) of 3.5 is obvious; (b) follows from 2.4. Hereby observe that e.g. $\bar{\gamma}_{i,s}(A) = 0$ iff A is sctb in the $\|\cdot\|_i$ -topology.

LEMMA 3.6. γ , $\bar{\gamma}$ and $\bar{\gamma}_s$ have property (3) and (4) of 3.1.

PROOF. Obviously, γ , $\bar{\gamma}$ and $\bar{\gamma}_s$ are monotone. By 3.5 (a), it is enough to show that γ has property (4). Let (A_n) be a decreasing sequence of complete non-empty subsets of E with $\varphi(A_n) \rightarrow 0$ ($n \rightarrow \infty$) and $x_n \in A_n$ for $n \in \mathbb{N}$. Then

$$\gamma(\{x_i : i \in \mathbb{N}\}) = \gamma(\{x_i : i \geq n\}) \leq \gamma(A_n) \rightarrow 0 \quad (n \rightarrow +\infty),$$

hence $\gamma(\{x_i : i \in \mathbb{N}\}) = 0$ and $\{x_i : i \in \mathbb{N}\}$ is relatively compact. Therefore $\{x_i : i \in \mathbb{N}\}$ has cluster points, which obviously belong to the intersection

$$\bigcap_{n \in \mathbb{N}} A_n.$$

REMARK. 3.7. An obvious modification of a part of the proof of 2.2 shows that $\bar{\gamma}_s$ has property (2) of 3.1, whereas γ and $\bar{\gamma}$ do not as Example 5.3 (b) shows. $\bar{\gamma}$ and therefore $\bar{\gamma}_s$ satisfy (1) of 3.1 by Idzik’s fixed point theorem [11, Theorem 2.4]. Whether γ satisfies (1) of 3.1, is exactly the problem of Schauder mentioned in the introduction.

COROLLARY 3.8. $\bar{\gamma}_s$ is a noncompactness measure in the sense of 3.1.

An obvious generalization of the proof of 2.2 (a) yields:

PROPOSITION 3.9. *Let $A_1, A_2, A \subset E$ and $\alpha \in \mathbb{R}$. Then*

$$\bar{\gamma}_s(A_1 \cup A_2) \leq \bar{\gamma}_s(A_1) + \bar{\gamma}_s(A_2),$$

$$\bar{\gamma}_s(A_1 + A_2) \leq \bar{\gamma}_s(A_1) + \bar{\gamma}_s(A_2),$$

$$\bar{\gamma}_s(\alpha A) \leq n \cdot \bar{\gamma}(A) \text{ if } |\alpha| \leq n \in \mathbb{N}.$$

4. - Noncompactness measures in l_p , $0 < p < 1$

Let $0 < p < 1$. For any real sequence $x = (x_n)$, we put $\|x\|_p := \sum_{n \in \mathbb{N}} |x_n|^p$.

The space $(l_p, \|\cdot\|_p)$ is an example for an F -normed, nonlocally convex space, in which $\gamma(A) = \bar{\gamma}(A) = \bar{\gamma}_s(A)$ holds for every convex subset A .

THEOREM 4.1. (a) *If A is a pointwise bounded subset of l_p , then*

$$\gamma(A) = \inf_{n \in \mathbb{N}} \sup_{(x_k) \in A} \sum_{i=n}^{\infty} |x_i|^p.$$

(b) *If A is a convex subset of l_p , then $\gamma(A) = \bar{\gamma}(A) = \bar{\gamma}_s(A)$.*

PROOF. (a) For $n \in \mathbb{N}$ denote by $P_n : l_p \rightarrow l_p$ the projection $(x_i)_{i \in \mathbb{N}} \rightarrow (x_1, \dots, x_n, 0, 0, \dots)$. Let $Q_n = I - P_n$, where I is the identity. Put $\sigma(A) := \sup_{x \in A} \|x\|_p$ and

$$\tau(A) := \inf_{n \in \mathbb{N}} \sigma(Q_n(A)) = \inf_{n \in \mathbb{N}} \sup_{(x_k) \in A} \sum_{i=n}^{\infty} |x_i|^p \quad \text{for } A \subset l_p.$$

(i) $\tau(A) \leq \gamma(A)$: let $\alpha > \gamma(A)$ and F be a finite subset of l_p with $A \subset F + B_\alpha$. Then $\tau(A) \leq \tau(F) + \tau(B_\alpha) \leq 0 + \sigma(B_\alpha) = \alpha$.

(ii) Let A be pointwise bounded. Then $\gamma(A) \leq \tau(A)$. Let $\alpha > \tau(A)$.

Then $\alpha > \sigma(Q_n(A))$ for some $n \in \mathbb{N}$. Since $P_n(A)$ is a bounded subset of a finite dimensional space, $\gamma(P_n(A)) = 0$ and from $A \subset P_n(A) + Q_n(A)$ follows $\gamma(A) \leq \gamma(P_n(A)) + \gamma(Q_n(A)) \leq 0 + \sigma(Q_n(A)) < \alpha$.

(b) By (a) and 3.5 (a), it is enough to prove that $\bar{\gamma}_s(A) \leq \tau(A)$, if A is convex and pointwise bounded. Let $\alpha > \tau(A)$. As in (ii), we get $\bar{\gamma}_s(A) \leq \bar{\gamma}_s(P_n(A)) + \bar{\gamma}_s(Q_n(A))$ for some $n \in \mathbb{N}$. Moreover, $\bar{\gamma}_s(P_n(A)) = \gamma(P_n(A)) = 0$ since $P_n(A)$ is contained in a finite dimensional space and $\bar{\gamma}_s(Q_n(A)) \leq \sigma(Q_n(A)) < \alpha$, since $Q_n(A)$ is convex.

It follows from 4.1 (b) and 2.2 (b):

COROLLARY 4.2. *A subset of l_p is sctb iff its convex hull is totally bounded.*

By 4.1 (b) or by 2.3, a convex subset of l_p is totally bounded iff it is ctb iff it is sctb. In [W] it is proved that any subset of l_p is ctb iff it is sctb. On the other hand l_p contains totally bounded subsets, the convex hull of which is not totally bounded; such a set is totally bounded, but by 2.2 (b) not sctb.

A statement analogous to 4.1 (a) is given in [B/G] if $1 \leq p < \infty$.

5. - The noncompactness measure $\bar{\gamma}_s$ in L_0

In the following, let Ω be a non-empty set, $\mathcal{U}$ an algebra in the power set $\mathbb{P}(\Omega)$ of Ω and $\eta : \mathbb{P}(\Omega) \rightarrow [0, +\infty[$ a submeasure, i.e. a monotone, subadditive function with $\eta(\emptyset) = 0$. Then $\|f\| := \inf\{a > 0 : \eta(\{|f| \geq a\}) \leq a\}$ defines a group seminorm on $\mathbb{R}^\Omega$. Let L_0 be the closure of the space $S := \text{span}\{\chi_A : A \in \mathcal{U}\}$ of $\mathcal{U}$ -simple functions in $(\mathbb{R}^\Omega, \|\cdot\|)$. We will identify functions $f, g \in \mathbb{R}^\Omega$, for which $\|f - g\| = 0$. Then the space $(L_0, \|\cdot\|)$ of “measurable functions” becomes an F -normed linear space.

In [A/DP], [T/W], the following two parameters λ and ω are used to estimate the noncompactness measure γ in L_0 :

$$\lambda(M) := \inf\{\varepsilon > 0 : \text{there is an } a \in [0, +\infty[\text{ such that } \eta(\{|f| \geq a\}) \leq \varepsilon \text{ for all } f \in M\},$$

$$\omega(M) := \inf\{\varepsilon > 0 : \text{there is a partition } A_1, \dots, A_n \in \mathcal{U} \text{ of } \Omega \text{ such that for every } f \in M \text{ there is a set } D \subset \Omega \text{ with } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(s) - f(t)| : s, t \in A_i \setminus D\} \leq \varepsilon \text{ for } i = 1, \dots, n\}.$$

By [T/W, 2.2.9] we have $\max\{\lambda, \omega/2\} \leq \gamma \leq \lambda + \omega$. To estimate $\bar{\gamma}_s$ we use the following parameters:

$$\bar{\lambda}(M) := \inf\{\varepsilon > 0 : \text{there is a set } D \subset \Omega \text{ such that } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(x)| : f \in M, x \in \Omega \setminus D\} < +\infty\},$$

$$\bar{\omega}(M) := \inf\{\varepsilon > 0 : \text{there is a partition } A_1, \dots, A_n \in \mathcal{U} \text{ of } \Omega \text{ and a set } D \subset \Omega \text{ such that } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(s) - f(t)| : s, t \in A_i \setminus D\} \leq \varepsilon, \text{ for } i = 1, \dots, n \text{ and } f \in M\}.$$

Obviously, $\lambda \leq \bar{\lambda}$ and $\omega \leq \bar{\omega}$.

THEOREM 5.1. $\bar{\gamma}_s(M) \leq \bar{\gamma}(M) + \bar{\omega}(M)$ holds for $M \subset L_0$.

PROOF. It is similar to that of the known inequality $\gamma \leq \lambda + \omega$. Let $\alpha > \bar{\lambda}(M)$ and $\beta > \bar{\omega}(M)$. There are sets $D_1, D_2 \subset \Omega$ and a partition $A_1, \dots, A_n \in \mathcal{U}$ of Ω such that

$$\eta(D_1) < \alpha \text{ and } s := \sup\{|f(x)| : f \in M, x \in \Omega \setminus D_1\} < +\infty,$$

$$\eta(D_2) < \beta \text{ and } \sup\{|f(s) - f(t)| : s, t \in A_i \setminus D_2\} \leq \beta \text{ for } i = 1, \dots, n.$$

Let $k, m \in \mathbb{N}$ such that $1/m < \alpha$ and $-s + k/m > s$. We put $Y := \{-s + i/m : i = 0, \dots, k\}$ and $F := \left\{ \sum_{i=1}^n y_i \cdot \chi_{A_i} : y_i \in Y \right\}$. For $f \in M$, there is a $g \in F$ such that $|f(x) - g(x)| \leq 1/m + \beta/2 \leq \alpha + \beta$ for every $x \in \Omega \setminus (D_1 \cup D_2)$; therefore $f - g \in C := \{h \in L_0 : \sup\{|h(x)| : x \in \Omega \setminus (D_1 \cup D_2)\} \leq \alpha + \beta\}$. Since F is finite and C a convex subset of $B_{\alpha+\beta}$, it follows that $\bar{\gamma}_s(M) \leq \alpha + \beta$.

One immediately sees that $\bar{\lambda}(\text{co } M) = \bar{\lambda}(M)$ and $\bar{\omega}(\text{co } M) = \bar{\omega}(M)$ for $M \subset L_0$ and that $\bar{\lambda}$ and $\bar{\omega}$ are monotone. Therefore it follows from 5.1 and 3.8:

COROLLARY 5.2. $\bar{\lambda} + \bar{\omega}$ is a noncompactness measure in L_0 (in the sense of 3.1).

We see in 5.3 (a) that in the inequality $\max\{\lambda, \omega/2\} \leq \gamma$ one cannot replace λ, ω, γ by $\bar{\lambda}, \bar{\omega}, \bar{\gamma}_s$.

EXAMPLE 5.3. Let $\mathcal{U}$ be the Borel algebra of $\Omega = [0, 1]$ and $\mu = \eta|_{\mathcal{U}}$ the Lebesgue measure. Let $A_1, A_2, \dots$ be an enumeration of the intervals $[(i-2)/2^n, i/2^n[$ ($i, n \in \mathbb{N}; i \leq 2^n$), (a_n) a sequence of positive numbers such that $a_n \rightarrow +\infty$ ($n \rightarrow \infty$), $f_n = a_n \chi_{A_n}$ for $n \in \mathbb{N}$ and $M = \{f_n : n \in \mathbb{N}\}$.

- (a) $\bar{\lambda}(M) = \bar{\omega}(M) = 1$. But $\bar{\gamma}_s(M) = 0$ if $a_n \mu(A_n) \rightarrow 0$ ($n \rightarrow \infty$).
 (b) M is ctb. But $\text{co } M$ is not bounded and therefore not ctb if $a_n \mu(A_n) \rightarrow +\infty$ ($n \rightarrow \infty$).

(a) We prove the last assertion. If $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and $a_n \mu(A_n) \leq \varepsilon^2$ for $n \geq n_0$, then $f_n \in C := \left\{ f \in S : \int |f| d\mu \leq \varepsilon^2 \right\}$ for $n \geq n_0$ and C is a convex subset of B_ε . Hence M is scbt.

(b) Let $n \in \mathbb{N}$, $\varepsilon = 2^{-n}$ and $B_i = [(i-1)/2^n, i/2^n[$ for $i \leq 2^n$. Then $C_i := \{f \chi_{B_i} : f \in L_0\}$ are convex subsets of B_ε and $M \setminus \bigcup_{1 \leq i \leq 2^n} C_i$ is finite. Hence M is ctb.

Let $b_i := a_k$ if $B_i = A_k$. Then

$$\text{co } M \ni \sum_{i \leq 2^n} 2^{-n} b_i \chi_{B_i} \geq \min_{1 \leq i \leq 2^n} b_i \mu(B_i) \cdot \chi_{[0,1]}.$$

Therefore $\text{co } M$ is not bounded (in the linear topological sense), if $a_n \mu(A_n) \rightarrow +\infty$ ($n \rightarrow \infty$).

EXAMPLE 5.4. Let $\mathcal{U}, \Omega, \mu, \eta$ be chosen as in 5.3 and $n \in \mathbb{N}$.

- (a) For $A_{i,n} := \{f \chi_{[(i-1)/n, i/n[} : 0 \leq f \leq 2n\}$, $A_n := \bigcup_{i=1}^n A_{i,n}$, we have $\bar{\lambda}(A_n) = 0$, $\omega(A_n) \leq 1/n$, but $\bar{\gamma}_s(A_n) \geq 1/2$ since $A := \{f \in L_0 : 0 \leq f \leq 2\} \subset \text{co } A_n$ and therefore

$$1/2 = \gamma(A) \leq \bar{\gamma}_s(\text{co } A_n) = \bar{\gamma}_s(A_n).$$

- (b) For $B_{i,n} := \{c \chi_{[(i-1)/n, i/n[} : c \in \mathbb{R}\}$, $B_n := \bigcup_{i=1}^n B_{i,n}$ we have $\lambda(B_n) = 1/n$, $\bar{\omega}(B_n) = 0$, but $\bar{\gamma}_s(B_n) = 1$ since $B := \{c \chi_{[0,1[} : c \in \mathbb{R}\} \subset \text{co } B_n$ and therefore

$$1 = \lambda(B) \leq \gamma(B) \leq \bar{\gamma}_s(\text{co } B_n) = \bar{\gamma}_s(B_n) \leq 1.$$

In contrast to 5.4, we will see that, under the assumptions of 5.4, $\bar{\lambda}(M) = \omega(M) = 0$ or $\lambda(M) = \bar{\omega}(M) = 0$ implies $\bar{\gamma}_s(M) = 0$.

PROPOSITION 5.5. *Let $M \subset L_0$ and $\lambda(M) = 0$. Then $\bar{\lambda}(M) \leq \bar{\omega}(M)$ and therefore $\bar{\gamma}_s(M) \leq 2 \cdot \bar{\omega}(M)$.*

PROOF. Let $\lambda(M) = 0$. By 5.1, it is enough to show that $\bar{\lambda}(M) \leq \bar{\omega}(M)$. Let $\alpha > \bar{\omega}(M)$. Then there is a set $D \subset \Omega$ with $\eta(D) < \alpha$ and a partition $A_1, \dots, A_n \in \mathcal{U}$ of Ω such that

$$\sup\{|f(s) - f(t)| : s, t \in A_i \setminus D\} < \alpha \text{ for } i = 1, \dots, n.$$

We may assume that $\eta(A_i \setminus D) > 0$ for $i < m$ and $A_i \subset D$ for $i \geq m$, for some $m \in \mathbb{N}$. Since $\lambda(M) = 0$, there is a $b \in [0, +\infty[$ and, for every $f \in M$, a set $D(f) \subset \Omega$ such that $\eta(D(f)) < \min_{i < m} \eta(A_i \setminus D)$ and $|f(x)| \leq b$ for $x \in \Omega \setminus D(f)$. Let $f \in M$ and $x \in \Omega \setminus D$. We show that $|f(x)| \leq \alpha + b$. In fact, if $i < m$ with $x \in A_i$, then $\eta(D(f)) < \eta(A_i \setminus D)$; therefore there is an $y \in (A_i \setminus D) \setminus D(f)$ and $|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq \alpha + b$. It follows that $\bar{\lambda}(M) \leq \eta(D) < \alpha$.

PROPOSITION 5.6. *Assume that $\eta(B) = \inf\{\eta(A) : B \subset A \in \mathcal{U}\}$ for any $B \subset \Omega$ and that $\mu := \eta|_{\mathcal{U}}$ is additive. Let $M \subset L_0$ and $\omega(M) = 0$. Then $\bar{\gamma}_s(M) \leq \bar{\lambda}(M)$.*

PROOF. Let $\omega(M) = 0$ and $\alpha > \bar{\lambda}(M)$. By assumption, there is a set $D \in \mathcal{U}$, with $\eta(D) < \alpha$, and a number $c > 0$ such that $|f(x)| \leq c$ for $f \in M$ and $x \in \Omega \setminus D$. $M_1 := L_0 \cdot \chi_D$ is a convex subset of B_α , hence $\bar{\gamma}_s(M_1) \leq \alpha$. The set $M_2 := M \cdot \chi_{\Omega \setminus D}$ is totally bounded, since $\omega(M_2) = \lambda(M_2) = 0$. On $\{f \in L_0 : |f| \leq c\}$ the $\|\cdot\|$ -topology coincides with the $\|\cdot\|_1$ -topology, where $\|f\|_1 := \int |f| d\mu$. Therefore M_2 is also totally bounded with respect to the (semi-)norm $\|\cdot\|_1$ and therefore scrb, i.e. $\bar{\gamma}_s(M_2) = 0$. Since $M \subset M_1 + M_2$, it follows $\bar{\gamma}_s(M) \leq \bar{\gamma}_s(M_1) + \bar{\gamma}_s(M_2) \leq \alpha$.

Under the assumption of 5.6, a set $M \subset L_0$ is scrb if $\omega(M) = \bar{\gamma}(M) = 0$, in particular, if M is totally bounded and $\bar{\lambda}(M) = 0$. The next proposition clarifies the meaning of $\bar{\lambda}(M) = 0$.

PROPOSITION 5.7. *Assume that $\eta(B) = \inf\{\eta(A) : B \subset A \in \mathcal{U}\}$ for $B \subset \Omega$. Then for $M \subset L_0$, $\bar{\lambda}(M) = 0$ iff $M \subset [-\varphi, \varphi]$ for some $\varphi \in L_0$, ≥ 0 .*

PROOF. $\Rightarrow$: Let $\bar{\lambda}(M) = 0$. By assumption, there are $A_n \in \mathcal{U}$ and $a_n \in [0, +\infty[$ such that $\eta(\Omega \setminus A_n) \leq 1/n$ and $|f(x)| \leq a_n$ for $f \in M$ and $x \in A_n$. Define $B_n := A_n \setminus \bigcup_{i < n} A_i$, $\varphi_n = \sum_{i=1}^n a_i \chi_{B_i}$, $\varphi = \sum_{i=1}^{\infty} a_i \chi_{B_i}$. Then $\varphi_n \in S$, $\|\varphi - \varphi_n\| \leq \eta(\Omega - A_n) \leq 1/n$, hence $\varphi \in L_0$. Moreover, $|f(x)| \leq \varphi(x)$ for $f \in M$ and $x \in \Omega \setminus D$, where $D = \Omega \setminus \bigcup_{n=1}^{\infty} A_n$ and $\eta(D) = 0$.

$\Leftarrow$: Let φ be a positive function of L_0 , $M \subset [-\varphi, \varphi]$ and $\varepsilon > 0$. Then there is a positive number c such that $\eta(\{\varphi \geq c\}) \leq \varepsilon$, hence $\bar{\lambda}(M) \leq \eta(\{\varphi \geq c\}) \leq \varepsilon$.

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