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Regularity of Free Boundaries in Two Dimensions

MAKOTO SAKAI

1. - Introduction

In this paper we discuss regularity of free boundaries in two dimensions which appear in an obstacle problem. Let u be a non-negative function defined in the unit disk B_1 of the complex z -plane such that:

- (i) $\Gamma(u) = (\partial\Omega(u)) \cap B_1$ contains the origin 0, where $\Omega(u) = \{z \in B_1 : u(z) > 0\}$;
- (ii) u is of class C^1 in B_1 ;
- (iii) $\Delta u(z) = 1$ in $\Omega(u)$ in the sense of distributions.

What can we say about regularity of the free boundary $\Gamma(u)$?

An accurate description of $\Gamma(u)$ was given by Caffarelli and Rivière in [1] and [2]. They showed that either:

- (1) 0 is a regular point; namely, for a small disk B_δ with radius $\delta > 0$ and center 0, $\Omega(u) \cap B_\delta$ is simply connected and $\Gamma(u) \cap B_\delta$ is a regular analytic simple arc passing through 0;

or

- (2) $B_\delta \setminus \Omega(u)$ is arranged along a straight line for small B_δ ; more precisely, there is an increasing function η defined on a half-open interval $[0, \delta)$ such that $\eta(0) = 0$ and

$$B_\delta \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B_\delta : |y| \leq \eta(|x|)\},$$

where α denotes a real number and $e^{i\alpha}E$ for a set E denotes $\{e^{i\alpha}z : z \in E\}$.

Furthermore they proved in [1] that in case (2) it follows that:

- (α) if 0 is not an isolated point of $\Gamma(u)$ and if the interior of $B_1 \setminus \Omega(u)$ is empty, then $\Gamma(u) \cap B_\delta$ is a real analytic simple arc, and
- (β) the boundary of each connected component of the interior of $B_\delta \setminus \Omega(u)$ is the union of a finite number of real analytic simple arcs.

Their results are fairly accurate, but there is still a possibility that an infinite number of connected components of the interior of $B_1 \setminus \Omega(u)$ exist and cluster around 0.

The purpose of this paper is to give a complete description of the free boundary $\Gamma(u)$ and to exclude such a possibility. Our main result is:

THEOREM 1.1. *Let u , $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$ be as above. Then the origin 0 is either a (1) regular, or a (2a) degenerate, or a (2b) double or a (2c) cusp point of Γ . Namely, there is a small disk $B = B_\delta$ such that one of the following occurs:*

- (1) $\Omega \cap B$ is simply connected and $\Gamma \cap B$ is a regular real analytic simple arc passing through 0;
- (2a) $\Gamma \cap B = \{0\}$ or $\Gamma \cap B$ is a regular real analytic simple arc passing through 0. $\Omega \cap B$ is equal to $B \setminus \Gamma$;
- (2b) $\Omega \cap B$ consists of two simply connected components Ω_1 and Ω_2 . $(\partial\Omega_1) \cap B$ and $(\partial\Omega_2) \cap B$ are distinct regular real analytic simple arcs passing through 0. They are tangent to each other at 0;
- (2c) $\Omega \cap B$ is simply connected and $\Gamma \cap B$ is a regular real analytic simple arc except for a cusp at 0. The cusp is pointing into $\Omega \cap B$. It is a very special one. There is a holomorphic function T defined on a closed disk $\overline{B}_\varepsilon$ such that:
 - i. $T(0) = 0$, $T'(0) = 0$ and $T''(0) \neq 0$;
 - ii. T is univalent on the closure $\overline{H}$ of a half disk $H = \{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$;
 - iii. T satisfies $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$ and $T(\overline{H}) \subset \Omega \cup \Gamma$, where $i(-\varepsilon, \varepsilon) = \{it : -\varepsilon < t < \varepsilon\}$.

Furthermore, all the second derivatives of u are continuous up to Γ , on Ω and u is real analytic up to Γ , on Ω except double and cusp points of Γ . If 0 is a double point, then there is a positive number γ such that

$$(1.1) \quad B \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B : |y| \leq \gamma x^2\},$$

where $e^{i\alpha}$ denotes the unit vector at 0 tangent to Γ . If 0 is a cusp point, then, for some positive number γ , it follows that

$$(1.2) \quad B \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B : x \leq 0 \text{ and } |y| \leq \gamma x^2\},$$

where $e^{i\alpha}$ denotes the unit vector at 0 tangent to Γ and pointing into Ω . Namely, we can take a quadratic function γt^2 as a function $\eta(t)$ in the argument given by Caffarelli and Rivière.

This regularity theorem holds also if we replace the constant function with value 1 in (iii) by a positive real analytic function φ defined in B_1 . This fact is quite interesting when we compare it with an example of the free boundary for the obstacle problem with C^∞ -obstacle due to Schaeffer [9]: if we replace the

constant function with value 1 in (iii) by some special positive C^∞ -function φ defined in B_1 , then there is a non-negative function u satisfying (i) to (iii) such that an infinite number of connected components of the interior of $B_1 \setminus \Omega(u)$ actually cluster around 0.

2. - Proof of Theorem 1.1

We shall first define classes of functions which appear in an obstacle problem. For the free boundary for the obstacle problem, we refer to Chapter V of Kinderlehrer -Stampacchia [4] and Chapter 6 of Rodrigues [6].

DEFINITION 2.1. We say that a non-negative function u in B_ρ is of class $P(1, B_\rho)$ if u satisfies (i) to (iii) of Section 1 with B_ρ replacing B_1 . We say that a real-valued function u in B_ρ is of class $R(1, B_\rho)$ if there exists an open subset $\Omega(u)$ of B_ρ such that u and $\Omega(u)$ satisfy (ii) and (iii) of Section 1 with B_ρ replacing B_1 and if

- (i') 0 is contained in $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$, and
- (iv) $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ on $\Gamma(u)$, where $z = x + iy$.

If u is of class $P(1, B_\rho)$, then $u(z) \geq 0$ in B_ρ and $u(z) = 0$ on $\Gamma(u)$, so u satisfies (iv). Thus, by taking $\Omega(u) = \{z \in B_\rho; u(z) > 0\}$, we see that $P(1, B_\rho) \subset R(1, B_\rho)$.

LEMMA 2.1. Let $u \in R(1, B_\rho)$. Then $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$, where $\bar{z}$ denotes the complex conjugate of z , is the Schwarz function of $\Omega(u) \cup \Gamma(u)$ in B_ρ ; namely, S is a function defined on $\Omega(u) \cup \Gamma(u)$ which is holomorphic in $\Omega(u)$, is continuous on $\Omega(u) \cup \Gamma(u)$ and satisfies $S(\zeta) = \bar{\zeta}$ on $\Gamma(u)$.

PROOF. By (ii), S is continuous on $\Omega(u) \cup \Gamma(u)$. Since by (iii)

$$\partial S(z) \setminus \partial \bar{z} = 1 - 4(\partial^2 u(z)) / (\partial z \partial \bar{z}) = 1 - \Delta u(z) = 0 \text{ in } \Omega(u),$$

S is holomorphic in $\Omega(u)$. By (iv), $\partial u / \partial z = 0$ on $\Gamma(u)$, and so $S(z) = \bar{z}$ on $\Gamma(u)$. Hence S is the Schwarz function of $\Omega(u) \cup \Gamma(u)$ in B_ρ . Q.E.D.

The next proposition is just an application of a regularity theorem proved in [8]; nevertheless, it is somewhat surprising when we look at the definition of $R(1, B_\rho)$. In particular, all the second derivatives of u are continuous up to $\Gamma(u)$, on $\Omega(u)$ for every $u \in R(1, B_\rho)$.

PROPOSITION 2.2. Let $u \in R(1, B_\rho)$, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$. Then the origin 0 is a regular, double, cusp point of Γ in the sense of Theorem 1.1 or a degenerate point in the sense that:

(2a') for a small disk $B = B_\delta$, $\Gamma \cap B = \{0\}$ or $\Gamma \cap B$ is an infinite set accumulating at 0 and is contained in a uniquely determined regular real analytic simple arc passing through 0. $\Gamma \cap B$ is a proper subset of the arc or the whole arc. $\Omega \cap B$ is equal to $B \setminus \Gamma$.

Furthermore, all the second derivatives of u are continuous up to Γ , on Ω and u is real analytic up to Γ , on Ω except double and cusp points of Γ .

PROOF. The first assertion follows from Lemma 2.1 and the Regularity Theorem for a boundary having a Schwarz function, see [8]. Let $U(z)$ and $V(z)$ be the real and imaginary parts of $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$, respectively. Then $U(z) = x - 2\partial u/\partial x$, $V(z) = -y + 2\partial u/\partial y$, $\partial U/\partial x = 1 - 2(\partial^2 u)/(\partial x^2) = -1 + 2(\partial^2 u)/(\partial y^2)$, $\partial V/\partial x = 2(\partial^2 u)/(\partial x \partial y)$ and $S' = (\partial U/\partial x) + i(\partial V/\partial x)$ in Ω . Since $\lim_{z \in \Omega, z \rightarrow \zeta} S'(z)$ exists for every ζ on Γ by Corollary 5.4 of [8], we see that all the second derivatives of u are continuous up to Γ , on Ω . The final assertion follows from the Regularity Theorem. Q.E.D.

Now we shall give a proof of our Theorem 1.1. Since $P(1, B_\rho) \subset R(1, B_\rho)$, from Proposition 2.2 it follows that the origin 0 falls in one of the situations (1), (2a'), (2b) or (2c). If the origin 0 is in (2a'), then $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ is holomorphic in B , and so $\frac{\partial u}{\partial z}(z) = 0$ on the arc determined by $\Gamma \cap B$ if 0 is not an isolated point of Γ . This means that $u(z) = 0$ on the arc. Hence $\Gamma \cap B$ is the whole arc determined by $\Gamma \cap B$ and (2a) holds. If (2b) holds, then (1.1) holds for some α and γ by Corollary 5.3 of [8]. To show that (1.2) holds if (2c) holds, we need the following lemma:

LEMMA 2.3. Let u and S be as in Lemma 2.1. Assume that 0 is a cusp point of Γ and let $z = T(\tau) = \kappa e^{i\alpha}(\tau^2 + a_3\tau^3 + a_4\tau^4 + \dots)$ be a one-to-one conformal mapping of $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$ into Ω such that $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$, where $\kappa > 0$. Then $\operatorname{Re} a_3 \leq 0$, and $\operatorname{Re} a_3 = 0$ if $u \in P(1, B_\rho)$.

PROOF. We can assume that $\kappa e^{i\alpha} = 1$. Set $\alpha_j = \operatorname{Re} a_j$ and $\beta_j = \operatorname{Im} a_j$. If $\tau = it$ and t is real, then $x = \operatorname{Re} T(\tau) = -t^2 + \beta_3 t^3 + \alpha_4 t^4 + \dots$ and $y = \operatorname{Im} T(\tau) = -\alpha_3 t^3 + \beta_4 t^4 + \dots$. Hence $x < 0$ for t with small $|t|$. If $\alpha_3 > 0$, then $y < 0$ for small $t > 0$ and $y > 0$ for $t < 0$ with small $|t|$. This contradicts the univalence of the mapping T . Therefore $\alpha_3 \leq 0$.

Since $S(z) = S(T(\tau)) = \overline{T(-\bar{\tau})} = \tau^2 - \bar{a}_3 \tau^3 + \bar{a}_4 \tau^4 + \dots$, we obtain

$$\begin{aligned} \frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z) &= 2 \frac{\partial u}{\partial z}(z) = \frac{1}{2}(\bar{z} - S(z)) \\ &= -i \operatorname{Im} \tau^2 + \bar{a}_3 \operatorname{Re} \tau^3 - i \bar{a}_4 \operatorname{Im} \tau^4 + \dots \\ &= (\alpha_3 r^3 \cos 3\theta + \dots) - i(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta + \dots), \end{aligned}$$

where $\tau = re^{i\theta}$. For small fixed $r > 0$, $\frac{\partial u}{\partial y}(z) = 0$ has a unique solution $\theta = \theta(r)$ in $(-\pi/2, \pi/2)$ and $\theta(r)$ is close to 0. Hence $u(T(re^{i\theta(r)})) - u(0) = \int_J \frac{\partial u}{\partial x}(z) dx$, where $J = \{T(se^{i\theta(s)}) : 0 \leq s \leq r\}$. If $\alpha_3 < 0$, then $\frac{\partial u}{\partial x}(z) < 0$ on J for small $r > 0$, and so $u(T(re^{i\theta(r)})) < u(0)$. Thus if $u \in P(1, B_\rho)$, namely, if $u(z) \geq 0$ in B_ρ , then $\alpha_3 = 0$. Q.E.D.

Now we shall show that (1.2) holds if (2c) holds. We use the notation of Lemma 2.3. From the lemma we see that $\alpha_3 = \operatorname{Re} a_3 = 0$. Hence, for $\tau = it$, $y = \beta_4 t^4 + \dots$. Since $x = -t^2 + \beta_3 t^3 + \dots$ for $\tau = it$, $x \leq 0$ and $|y| \leq 2|\beta_4|x^2$ for $z \in \Gamma \cap B_\delta$ with small $\delta > 0$. This completes the proof of Theorem 1.1.

3. - A function associated with a Schwarz function

In Section 2, to each $u \in R(1, B_1)$, we have assigned a Schwarz function S by $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$. In this section we shall discuss the converse. For a given Schwarz function S defined on $\Omega \cup \Gamma$, we shall construct a function $u \in R(1, B_\rho)$ such that $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ on $(\Omega \cup \Gamma) \cap B_\rho$.

In contrast with the arguments given in Chapter V of Kinderlehrer-Stampacchia [4], Lewy-Stampacchia [5], Schaeffer [9] and others, we do not assume that the free boundary Γ is a simple arc or a continuum.

Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1 , namely, let S be a function which is holomorphic in an open subset Ω of B_1 , is continuous on $\Omega \cup \Gamma$ and satisfies $S(\zeta) = \bar{\zeta}$ on Γ , where $\Gamma = (\partial\Omega) \cap B_1$ and $0 \in \Gamma$. Let D be a connected component of Ω . By the Regularity Theorem of [8], every point of $(\partial D) \cap B_1$ is an accessible boundary point of D . Let ζ be a fixed point of $(\partial D) \cap B_1$ and let $z \in D$. The integral $\int_J S(w)dw$ may depend on the choice of path J in

$D \cup \{\zeta\}$ joining ζ and z . We shall show that $\operatorname{Re} \int_J S(w)dw$ does not depend

on J . Let J' be another path in $D \cup \{\zeta\}$ joining ζ and z . To show that $\operatorname{Re} \int_{J-J'} S(w)dw = 0$, we may assume that $J - J'$ is a simple closed curve.

If $J - J'$ does not surround any part of $B_1 \setminus \Omega$, then, by the Cauchy theorem, $\int_{J-J'} S(w)dw = 0$. If $J - J'$ surrounds a part E of $B_1 \setminus \Omega$, then, by the Regularity Theorem, we may assume that ∂E consists of a finite number of

analytic simple closed curves having possibly double and cusp points. Hence

$$\int_{J-J'} S(w)dw = \int_{\partial E} S(w)dw = \int_{\partial E} \bar{w}dw = \int_E d\bar{w} \wedge dw = 2i \int_E dm,$$

and therefore the real part of this integral is 0. Thus $\operatorname{Re} \int_{\zeta}^z S(w)dw$ is single-valued and harmonic in D .

Next, for ρ with $0 < \rho < 1$, we shall define a function h harmonic in $\Omega \cap B_\rho$ and continuous on $\overline{\Omega \cap B_\rho}$. By the Regularity Theorem, we may assume that $\partial(\overline{\Omega \cap B_\rho})$ consists of a finite number of piecewise analytic simple closed curves. Take a connected component F_k of $\overline{\Omega \cap B_\rho}$ and let ζ_k be a fixed point on $(\partial F_k) \setminus (\partial B_\rho)$. Set

$$h_k(z) = 2 \operatorname{Re} \int_{\zeta_k}^z S(w)dw + |\zeta_k|^2$$

on F_k . Then h_k is harmonic in the interior of F_k , continuous on F_k and

$$\begin{aligned} h_k(z_2) - h_k(z_1) &= 2 \operatorname{Re} \int_{z_1}^{z_2} \bar{w}dw = \int_{z_1}^{z_2} (\bar{w}dw + w d\bar{w}) \\ (3.1) \quad &= \int_{z_1}^{z_2} d|w|^2 = (|z_2|^2 - |z_1|^2) \end{aligned}$$

if z_1 and z_2 belong to the same connected component of $(\partial F_k) \setminus (\partial B_\rho)$. The function h_k depends on the choice of $\zeta_k \in (\partial F_k) \setminus (\partial B_\rho)$ and is uniquely determined up to a real additive constant. Now we define a function h on $\overline{\Omega \cap B_\rho}$ by $h(z) = h_k(z)$ on F_k and set

$$u(z) = \frac{1}{4} (|z|^2 - h(z))$$

on $\overline{\Omega \cap B_\rho}$. Then $\Delta u(z) = 1$ in $\Omega \cap B_\rho$, u is continuous on $\overline{\Omega \cap B_\rho}$ and, by (3.1), u is constant on each connected component of $(\partial(\overline{\Omega \cap B_\rho})) \setminus \partial B_\rho$. We extend u onto B_ρ so that the extension, which we denote by u again, is continuous in B_ρ , is of class C^1 in the interior I of $B_\rho \setminus \Omega$ and satisfies

$$(3.2) \quad \lim_{z \in I, z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \in I, z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every $\zeta \in (\partial(\overline{\Omega \cap B_\rho})) \setminus \partial B_\rho$. This is possible, because u is constant on each connected component of $(\partial(\overline{\Omega \cap B_\rho})) \setminus \partial B_\rho$.

LEMMA 3.1. *The function u defined above satisfies $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ on $(\Omega \cup \Gamma) \cap B_\rho$ and is of class $R(1, B_\rho)$ if we set $\Omega(u) = \Omega \cap B_\rho$.*

PROOF. Since $0 \in \Gamma$, (i') of Definition 2.1 is satisfied. We have already seen that $\Delta u(z) = 1$ in $\Omega(u)$. To show (iv) of Definition 2.1, let $\Sigma_k(z) = \int_{\gamma_k}^z S(w)dw$ on a connected component F_k of $\overline{\Omega \cap B_\rho}$. Then

$$S(z) = \Sigma'_k(z) = \frac{\partial \Sigma_k(z)}{\partial z} + \frac{\partial \overline{\Sigma_k(z)}}{\partial z} = \frac{\partial 2 \operatorname{Re} \Sigma_k(z)}{\partial z}$$

in $\Omega \cap B_\rho$. Hence $\partial u(z)/\partial z = (1/4)(\bar{z} - S(z))$ in $\Omega \cap B_\rho$. Since $S(\zeta) = \bar{\zeta}$ on Γ ,

$$(3.3) \quad \lim_{z \in \Omega \cap B_\rho, z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \in \Omega \cap B_\rho, z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every $\zeta \in \Gamma \cap B_\rho = \Gamma(u)$. By (3.2) and (3.3),

$$(3.4) \quad \lim_{z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every $\zeta \in (\partial(\overline{\Omega \cap B_\rho})) \setminus \partial B_\rho$. If $\zeta \in \Gamma(u) \setminus \partial(\overline{\Omega \cap B_\rho})$, then ζ is a degenerate point of Γ . If ζ is an isolated degenerate point of Γ , then $S(\zeta) = \bar{\zeta}$ and (3.4) holds. If ζ is a non-isolated degenerate point of Γ , there exists a real analytic simple arc passing through ζ such that $S(z) = \bar{z}$ on the arc. Hence u is constant on the arc and (3.4) holds. Thus (iv) of Definition 2.1 holds and u is of class C^1 in B_ρ . Q.E.D.

To discuss the case $u(z) \geq 0$ in $\Omega \cap B_\rho$ for some $\rho > 0$, we shall have a more detailed discussion. We first note that, by the Regularity Theorem, $\overline{\Omega \cap B_\rho}$ is connected for sufficiently small $\rho > 0$. For such a small ρ , we see that $h(\zeta) = |\zeta|^2$ on $\Gamma \cap B_\rho$, and so $u(\zeta) = 0$ on $\Gamma \cap B_\rho$. We set $u_S(z) = u(z)$ on $\overline{\Omega \cap B_\rho}$ and $u_S(z) = 0$ on $B_\rho \setminus \overline{\Omega \cap B_\rho}$.

LEMMA 3.2. *Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1 . If 0 is a regular, non-isolated degenerate or double point of Γ , then $u_S \in P(1, B_\rho)$ for sufficiently small $\rho > 0$.*

PROOF. If 0 is a non-isolated degenerate point of Γ , then S is holomorphic in B_δ for small $\delta > 0$. Hence u_S is real analytic in B_δ , $\Delta u_S(z) = 1$ in B_δ and $u_S(z) = (\partial u_S / \partial x)(z) = (\partial u_S / \partial y)(z) = 0$ on the arc J determined by Γ . Let $(\partial u_S / \partial n)(z)$ be the derivative along the direction normal to J . Then $(\partial^2 u_S / \partial n^2)(z) = \Delta u_S(z) = 1$ on J . Since $u_S(z) = (\partial u_S / \partial n)(z) = 0$ on J , we see that $u_S(z) > 0$ on $B_\rho \setminus J$ and $u_S(z) = 0$ on $J \cap B_\rho$ for some small $\rho > 0$. Hence $u_S \in P(1, B_\rho)$ and $\Gamma(u_S) = J \cap B_\rho$.

The same argument works for the case that 0 is a regular or double point of Γ . Q.E.D.

REMARK. If 0 is a regular or double point of Γ , then $\Omega(u_S) \equiv \{z \in B_\rho : u_S(z) > 0\} = \Omega \cap B_\rho$ for some $\rho > 0$. If 0 is a non-isolated degenerate point of Γ , then $\Omega(u_S) = B_\rho \setminus J$ and $\Omega \cap B_\rho = B_\rho \setminus \Gamma$. Hence $\Omega(u_S) \subset \Omega \cap B_\rho$ and the equality does not hold in general.

Next we shall discuss the case that 0 is a cusp point of Γ . To do so, we shall define the index of a cusp point. Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1 and assume that 0 is a cusp point. Let $e^{i\alpha}$ be the unit vector at 0 tangent to Γ and pointing into Ω . Let C_ε be a half circle defined by $C_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon, |\arg z - \alpha| \leq \pi/2\}$. C_ε is oriented counterclockwise. Let $v(z) = S(z) - \bar{z}$. It is known that $\int_{C_\varepsilon} d \arg v(z) = \pm\pi + o(\varepsilon)$.

DEFINITION 3.1. We call the origin 0 a cusp with index $-1/2$ (resp. $+1/2$) if $\int_{C_\varepsilon} d \arg v(z) = \pi + o(\varepsilon)$ (resp. $\pi + o(\varepsilon)$).

Let J be the arc starting from 0 and defined by $J = \{z \in \Omega : \operatorname{Im} v(z) = 0\}$. Then J intersects ∂B_ε once for small $\varepsilon > 0$ and $\operatorname{Re} v(z) = v(z) \neq 0$ on $J \cap B_\varepsilon$. The index of the cusp point 0 is $-1/2$ if $\operatorname{Re} v(z) < 0$ on $J \cap B_\varepsilon$ and $+1/2$ if $\operatorname{Re} v(z) > 0$ on $J \cap B_\varepsilon$. For further details, see [7]. From arguments in Section 4 of [7] and Section 5 of Kinderlehrer-Nirenberg [3], we obtain:

LEMMA 3.3. Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1 . Let 0 be a cusp point of Γ and let u_S be the function defined before Lemma 3.2. Then $u_S \in P(1, B_\rho)$ for sufficiently small $\rho > 0$ if and only if the index of the cusp point is equal to $-1/2$.

PROOF. We can assume that $e^{i\alpha} = 1$. Let J be the arc starting from 0 defined by $J = \{z \in \Omega : \operatorname{Im} v(z) = 0\}$. Then $(\partial u_S / \partial y)(z) = 0$ on J , because $(\partial u_S / \partial z)(z) = (1/4)(\bar{z} - S(z)) = -(1/4)v(z)$ is real on J . Let $z(s)$ be a point on J , where s denotes the arc length of J from 0 to $z(s)$. Then, by the mean value theorem, $u_S(z(s)) = s(\partial u_S / \partial s)(z(\lambda s))$ for some λ with $0 < \lambda < 1$. Since $(\partial u_S / \partial s)(z(\lambda s)) = (\partial u_S / \partial x)(z(\lambda s))(\partial x / \partial s)(\lambda s)$, we obtain

$$\begin{aligned} (1/4)\operatorname{Re} v(z(\lambda s)) &= (1/2)(\partial u_S / \partial x)(z(\lambda s)) \\ &= (1/2)((\partial x / \partial s)(\lambda s))^{-1}(\partial u_S / \partial s)(z(\lambda s)) \\ &= (1/2)(s(\partial x / \partial s)(\lambda s))^{-1}u_S(z(s)). \end{aligned}$$

Since $(\partial x / \partial s)(\lambda s) > 0$ for small $s > 0$, $\operatorname{Re} v(z(\lambda s)) < 0$ if $u_S(z) > 0$ in $\Omega \cap B_\rho$. Hence the index is equal to $-1/2$ if $u_S \in P(1, B_\rho)$.

Conversely, if the index is equal to $-1/2$, then $\operatorname{Re} v(z(\lambda s)) < 0$, and so $u_S(z) > 0$ on J . Since

$$\lim_{z \in \Omega, z \rightarrow 0} S'(z) = e^{-2i\alpha} = 1$$

by Corollary 5.4 of [8], using the same argument as in the proof of Proposition 2.2 we can see that

$$\lim_{z \in \Omega, z \rightarrow 0} \frac{\partial^2 u_S}{\partial y^2}(z) = 1.$$

Hence the restriction of u_S to each vertical line $x = x_0$ is a convex function of y in a neighborhood of 0. It attains its minimum at $x_0 + iy_0 \in J$ if $x_0 > 0$ (because $(\partial u_S / \partial y)(z) = 0$ on J) and at $x_0 + iy_0 \in \Gamma$ if $x_0 \leq 0$, where $u_S(x_0 + iy_0) = (\partial u_S / \partial y)(x_0 + iy_0) = 0$. Thus $u_S(z) > 0$ on $\Omega \cap B_\rho$ for some $\rho > 0$. Q.E.D.

REMARK. Using the same argument as in the proof of Lemma 3.3, we can see that if 0 is an isolated point of Γ then $u_S \in P(1, B_\rho)$ for small $\rho > 0$ if and only if the index of $v(z) = S(z) - \bar{z}$ at 0 is equal to -1 . For the definition of the index of v at an isolated point of Γ , see [7].

4. - Cusp points

Let $u \in R(1, B_\rho)$ and assume that the origin 0 is a cusp point of $\Gamma(u)$. By Lemmas 2.1 and 3.1, we see that this cusp is the same as the cusp which appears on a boundary having a Schwarz function. It is precisely described by a conformal mapping as that given in (2c) of Theorem 1.1. In contrast with this fact, the cusp point for $u \in P(1, B_\rho)$ is a very special one. Here we shall have a more detailed discussion about the cusp point for $u \in P(1, B_\rho)$, which improves results due to Schaeffer [9] and Kinderlehrer-Nirenberg [3].

Let $u \in P(1, B_\rho)$, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$. Let 0 be a cusp point of Γ and let $e^{i\alpha} = -1$, where $e^{i\alpha}$ denotes the unit tangent vector to Γ at 0 pointing into Ω . For small $\delta > 0$, $\Gamma \cap B_\delta$ is a simple arc with cusp at 0. We divide $\Gamma \cap B_\delta$ into two regular real analytic simple arcs Γ_1 and Γ_2 : $\Gamma \cap B_\delta = \Gamma_1 \cup \Gamma_2$. We may assume for $j = 1, 2$ that Γ_j can be represented as the graph of $y = y_j(x)$ on $[0, \delta)$.

PROPOSITION 4.1. *For $j = 1, 2$ let*

$$y_j(x) = \gamma_j x^{n_j/2} + o(x^{n_j/2}),$$

where γ_j denotes a non-zero constant. Then $n_1 = n_2$ and if we write n for $n_1 = n_2$, then n is a natural number such that $n \geq 4$ and $n \not\equiv 3 \pmod{4}$; it follows that $\gamma_2 = (-1)^n \gamma_1$. Conversely, all integers n with $n \geq 4$ and $n \not\equiv 3 \pmod{4}$ actually occur in this situation for some $u \in P(1, B_\rho)$.

PROOF. To use the notation as in the proof of Lemma 2.3, we assume again that $e^{i\alpha} = 1$ and that $z = T(\tau) = \tau^2 + a_3 \tau^3 + a_4 \tau^4 + \dots$ is a one-to-one conformal mapping of $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$ into Ω such that $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$. For

$t \in (-\varepsilon, \varepsilon)$, $z = T(it) = -t^2 + \sum_{j=3}^{\infty} a_j i^j t^j$. Hence

$$x = -t^2 + \beta_3 t^3 + \alpha_4 t^4 - \beta_5 t^5 - \alpha_6 t^6 + \cdots$$

and

$$y = -\alpha_3 t^3 + \beta_4 t^4 + \alpha_5 t^5 - \beta_6 t^6 - \alpha_7 t^7 + \cdots,$$

where $a_j = \alpha_j + i\beta_j$. Thus we see that $n = n_1 = n_2$ is a natural number not less than 3 and $\gamma_2 = (-1)^n \gamma_1$.

Assume by contradiction that $n \equiv 3 \pmod{4}$; then $\alpha_j = 0$ for odd j with $3 \leq j < n$, $\beta_j = 0$ for even j with $4 \leq j < n$ and $\alpha_n \neq 0$. If $\alpha_n > 0$, then $y < 0$ for small $t > 0$ and $y > 0$ for $t < 0$ with small $|t|$. This contradicts the univalence of the mapping T in the half disk. Hence $\alpha_n < 0$. Since

$$S(z) - \bar{z} = v(z) = \operatorname{Re} v(z) = 2(-\alpha_n r^n \cos n\theta + \cdots)$$

on $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$ and $\theta(r)$ is close to 0 as in the proof of Lemma 2.3, we see that $v(z) = \operatorname{Re} v(z) > 0$ on J , and so $u(z) < 0$ on J . This contradicts the hypothesis $u \in P(1, B_\rho)$, and therefore we have proved that $n \not\equiv 3 \pmod{4}$ and $n \geq 4$.

Now we shall construct examples. Let $n \equiv 1 \pmod{4}$ and $n \geq 5$. Take a small $\alpha_n > 0$ and set $T(\tau) = \tau^2 + \alpha_n \tau^n$. Then $x = -t^2$, $y = \alpha_n t^n$ for $\tau = it$ and T is univalent in $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$ for some $\varepsilon > 0$, because $y > 0$ for $t > 0$ and $y < 0$ for $t < 0$. Set $S(z) = \overline{T(-\bar{\tau})}$ and $v(z) = S(z) - \bar{z}$. Then $v(z) = -2\alpha_n r^n \cos n\theta + i2r^2 \sin 2\theta$ and $v(z) = \operatorname{Re} v(z) = -2\alpha_n r^n < 0$ on $J = \{\zeta : 0 < \zeta = \operatorname{Re} \zeta < \delta\}$, where $\tau = re^{i\theta}$. Hence $u_S \in P(1, B_\rho)$ for some $\rho > 0$.

If $n \equiv 0 \pmod{4}$ and $n \geq 4$ we take small $\beta_3, \beta_n > 0$ and set $T(\tau) = \tau^2 + i\beta_3 \tau^3 + i\beta_n \tau^n$. It follows that $x = -t^2 + \beta_3 t^3$ and $y = \beta_n t^n$ for $\tau = it$. Let

$$\Gamma' = \{-t^2 + i\beta_n t^n : t \in (-\varepsilon, 0)\} = \{-t^2 + i\beta_n t^n : t \in (0, \varepsilon)\}.$$

Then Γ' is a regular real analytic simple arc contained in the second quadrant. We see that z lies on the right of Γ' for small $t > 0$ and on the left of Γ' for $t < 0$ with small $|t|$. Hence T is univalent on $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$ for some $\varepsilon > 0$. Defining S and v as above, we obtain $v(z) = 2\beta_n r^n \sin n\theta + i2(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta)$. Since $\beta_3 > 0$ and $\cos 3\theta(r) > 0$ for $\theta(r)$ close to 0, $\operatorname{Im} v(z) = 0$ only if $\theta(r) < 0$. Hence $\operatorname{Re} v(z) = 2\beta_n r^n \sin n\theta(r) < 0$ on $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$, and so $u_S \in P(1, B_\rho)$ for some $\rho > 0$.

If $n \equiv 2 \pmod{4}$ and $n \geq 6$, we take small $\beta_5, \beta_n > 0$ and set $T(\tau) = \tau^2 + i\beta_5 \tau^5 + i\beta_n \tau^n$. We notice that

$$\Gamma' = \{-t^2 - i\beta_n t^n : t \in (0, \varepsilon)\}$$

is contained in the third quadrant in this case. By repeating the same argument as above, we can see that $u_S \in P(1, B_\rho)$ for some $\rho > 0$. Q.E.D.

5. - Holomorphic functions having real analytic boundary values

Let $\varphi(x, y)$ be a complex-valued real analytic function of x and y in B_ρ , where $z = x + iy$. We note that

$$\varphi(x, y) = \varphi\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right).$$

To discuss the local property of the function $\varphi(x, y)$, we may assume that

$$f(z, w) = \varphi\left(\frac{z + w}{2}, \frac{z - w}{2i}\right)$$

is a holomorphic function of z and w in $B_\rho^2 = B_\rho \times B_\rho$. In what follows we write $f(z, \bar{z})$ for $\varphi(x, y)$.

Let $F(z, w)$ be a holomorphic function of two variables z and w in B_ρ^2 . Let Ω be an open subset of B_ρ such that $0 \in \partial\Omega$ and let $\Gamma = (\partial\Omega) \cap B_\rho$.

DEFINITION 5.1. Let S^F be a function defined on $\Omega \cup \Gamma$. We call S^F the *holomorphic function of $\Omega \cup \Gamma$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$* if:

- (i) S^F is holomorphic in Ω ;
- (ii) S^F is continuous on $\Omega \cup \Gamma$;
- (iii) $S^F(\zeta) = F(\zeta, \bar{\zeta})$ on Γ .

If 0 is a non-isolated point of Γ then S^F is uniquely determined (see remarks after Definition 3.1 of [8]).

If $F(z, \bar{z})$ is a holomorphic function of z in B_ρ , then $F(z, \bar{z})$ satisfies (i) to (iii) and it is a function S^F of $\Omega \cup \Gamma$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$. Namely, S^F exists for any open subset Ω of B_ρ if $F(z, \bar{z})$ is a holomorphic in B_ρ . The situation is quite different if $F(z, \bar{z})$ is not holomorphic in B_ρ .

We state the following lemma without proof.

LEMMA 5.1. Let f be a holomorphic function in B_ρ^2 . Then:

- (1) $f(z, \bar{z})$ is identically equal to zero in B_ρ if and only if $f(z, w)$ is identically equal to zero in B_ρ^2 ;
- (2) $f(z, \bar{z})$ is holomorphic in B_ρ if and only if $\partial f / \partial w$ is identically equal to zero as a function of two variables in B_ρ^2 .

In what follows, for the sake of simplicity, we consider the case $(\partial F / \partial w)(0, 0) \neq 0$.

PROPOSITION 5.2. *Let F be a holomorphic function in B_1^2 satisfying $(\partial F/\partial w)(0,0) \neq 0$. Let Ω be an open subset of B_1 such that $0 \in \partial\Omega$ and let $\Gamma = (\partial\Omega) \cap B_1$. Then there exists a holomorphic function S^F of $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$ for some $\rho > 0$ if and only if there exists a Schwarz function of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ for some $\delta > 0$.*

The proof of this Proposition follows immediately from the following two lemmas.

LEMMA 5.3. *Let F be a holomorphic function in B_1^2 . If there exists a Schwarz function S of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ , then $S^F(z) = F(z, S(z))$ is the holomorphic function of $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$ for some $\rho > 0$.*

PROOF. Take $\rho > 0$ so that $\rho < \delta$ and $|S(z)| < 1$ on $(\Omega \cup \Gamma) \cap B_\rho$. Then the function $F(z, S(z))$ is holomorphic in $\Omega \cap B_\rho$, is continuous on $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$ and satisfies $F(\zeta, S(\zeta)) = F(\zeta, \bar{\zeta})$ on $\Gamma \cap B_\rho$, because $S(\zeta) = \bar{\zeta}$ on $\Gamma \cap B_\rho$.
Q.E.D.

LEMMA 5.4. *Let S^F be the holomorphic function of $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$ and assume that $(\partial F/\partial w)(0,0) \neq 0$. Then there exists a Schwarz function of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ for some $\delta > 0$.*

PROOF. We introduce a new variable τ and consider a function $F(z, w) - \tau$ which is holomorphic in $B_\rho^2 \times B_\rho(\tau_0)$, where $\tau_0 = F(0,0)$ and $B_\rho(\tau_0)$ denotes the disk with radius ρ and center τ_0 . Since $(\partial/\partial w)(F(z, w) - \tau)(0,0,\tau_0) = (\partial F/\partial w)(0,0) \neq 0$ by the implicit function theorem there exists a unique holomorphic function $g(z, \tau)$ in $B_\varepsilon \times B_\varepsilon(\tau_0)$ for some $\varepsilon > 0$ such that $g(0, \tau_0) = 0$ and $F(z, g(z, \tau)) - \tau = 0$ in $B_\varepsilon \times B_\varepsilon(\tau_0)$. Take $\delta > 0$ so that $\delta < \varepsilon$ and $|S^F(z) - \tau_0| < \varepsilon$ on $(\Omega \cup \Gamma) \cap B_\delta$. Then $S(z) = g(z, S^F(z))$ is holomorphic in $\Omega \cap B_\delta$ and continuous on $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$. Since $F(\zeta, \bar{\zeta}) - S^F(\zeta) = 0$ on $\Gamma \cap B_\rho$, we obtain $\bar{\zeta} = g(\zeta, S^F(\zeta))$ on $\Gamma \cap B_\delta$, because $g(z, \tau)$ is uniquely determined by z and τ . Hence $S(\zeta) = g(\zeta, S^F(\zeta)) = \bar{\zeta}$ on $\Gamma \cap B_\delta$ and S is the Schwarz function of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ .
Q.E.D.

In the previous proof we have applied the implicit function theorem. The same idea can be found in Lewy-Stampacchia [5]. From Proposition 5.2 we see that if $(\partial F/\partial w)(0,0) \neq 0$ and if there exists a holomorphic function of $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$ for some $\rho > 0$, then the origin is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem.

Let $F(z, w) = F(0,0) + z^n F_1(z, w)$, where n denotes a non-negative integer and F_1 denotes a holomorphic function of z and w such that $F_1(0, w)$ is not identically equal to zero as a function of w . By applying the Fuchs theorem (see [8, Sec. 2]), we see that there exists a S^F of $\Omega \cup \Gamma$ in B_ρ if and only if there exists a S^{F_1} of $\Omega \cup \Gamma$ in B_ρ and $S^F(z) = F(0,0) + z^n S^{F_1}(z)$. Hence the same conclusion of Proposition 5.2 holds if $(\partial F_1/\partial w)(0,0) \neq 0$. If $(\partial F_1/\partial w)(0,0) = 0$,

then the function $g(z, \tau)$ for F_1 in the proof of Lemma 5.4 has a singularity at $(0, F_1(0, 0))$ and the conclusion of Lemma 5.4 would be complicated.

6. - A generalization of Theorem 1.1

We first state the following lemma without proof.

LEMMA 6.1. *Let f be a holomorphic function in B_ρ^2 . Then:*

- (1) *$f(z, \bar{z})$ is real-valued in B_ρ if and only if $\check{f} = f$ in B_ρ^2 , where:*

$$\check{f}(z, w) = \overline{f(\bar{w}, \bar{z})}.$$

In other words, $a_{jk} = \overline{a_{kj}}$ for every $j, k \geq 0$ if f has the Taylor expansion $\sum a_{jk} z^j w^k$ in B_ρ^2 ;

- (2) *$f(z, \bar{z})$ is real-valued in B_ρ if and only if $\Phi(z, \bar{z})$ is real-valued in B_ρ , where:*

$$\Phi(z, w) = \int_0^z \left\{ \int_0^w f(s, t) dt \right\} ds.$$

Next we define two classes of real-valued functions in B_ρ .

DEFINITION 6.1. Let f be a holomorphic function in B_ρ^2 such that $f(z, \bar{z})$ is real-valued and satisfies $f(z, \bar{z}) \neq 0$ in B_ρ . We say that a real-valued function u in B_ρ is of class $R(f, B_\rho)$ if there exists an open subset $\Omega(u)$ of B_ρ and if:

- (i) 0 is contained in $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$;
- (ii) u is of class C^1 in B_ρ ;
- (iii) $\Delta u(z) = f(z, \bar{z})$ in $\Omega(u)$ in the sense of distributions;
- (iv) $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ on $\Gamma(u)$.

DEFINITION 6.2. Let f be a holomorphic function in B_ρ^2 such that $f(z, \bar{z})$ is real-valued and satisfies $f(z, \bar{z}) > 0$ in B_ρ . Let u be a non-negative function in B_ρ and set $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$. We say that u is of class $P(f, B_\rho)$ if u and $\Omega(u)$ satisfy (i) to (iii) of Definition 6.1.

In Definition 6.1, the set $\Omega(u)$ may not be uniquely determined. For $u \in P(f, B_\rho)$, we take $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$; then u satisfies (iv), and so u is of class $R(f, B_\rho)$.

To prove a regularity theorem for functions of class $P(f, B_\rho)$, we start with two lemmas.

LEMMA 6.2. Let u be a function of class $R(f, B_\rho)$ and set

$$(6.1) \quad F(z, w) = \int_0^w f(z, t) dt.$$

Then

$$S^F(z) = F(z, \bar{z}) - 4 \frac{\partial u}{\partial \bar{z}}(z)$$

is the holomorphic function of $\Omega(u) \cup \Gamma(u)$ in B_ρ having the boundary values $F(\zeta, \bar{\zeta})$.

PROOF. By the chain rule, we obtain

$$\begin{aligned} \frac{\partial S^F}{\partial \bar{z}}(z) &= \frac{\partial F}{\partial z}(z, \bar{z}) \frac{\partial z}{\partial \bar{z}} + \frac{\partial F}{\partial w}(z, \bar{z}) \frac{\partial \bar{z}}{\partial \bar{z}} - 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}(z) \\ &= f(z, \bar{z}) - \Delta u(z) \end{aligned}$$

in $\Omega(u)$. Hence, by (iii) of Definition 6.1, S^F is holomorphic in $\Omega(u)$. It is continuous on $\Omega(u) \cup \Gamma(u)$, because u is of class C^1 in B_ρ . On $\Gamma(u)$, by (iv) of Definition 6.1, we obtain $S^F(\zeta) = F(\zeta, \bar{\zeta})$. This completes the proof. Q.E.D.

LEMMA 6.3. Let S^F be the holomorphic function of $\Omega \cup \Gamma$ in B_1 having the boundary values $F(\zeta, \bar{\zeta})$ and set $f = \partial F / \partial w$. If F satisfies (6.1) and if $f(z, \bar{z})$ is real-valued in B_1 and $f(0, 0) \neq 0$, then

$$u(z) = \begin{cases} \frac{1}{4} \left(\Phi(z, \bar{z}) - 2 \operatorname{Re} \int_0^z S^F(\tau) d\tau \right) & \text{in } \Omega \cap B_\rho \\ 0 & \text{on } B_\rho \setminus \Omega \end{cases}$$

is of class $R(f, B_\rho)$ for some $\rho > 0$, where

$$\Phi(z, w) = \int_0^z F(s, w) ds.$$

PROOF. Since $(\partial F)/(\partial w)(0, 0) = f(0, 0) \neq 0$, by Proposition 5.2 and the Regularity Theorem in [8], the origin 0 is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem. If 0 is not a degenerate point, then we can take a small $\rho > 0$ such that each connected component of $\Omega \cap B_\rho$ is simply connected. If 0 is a degenerate point, then we can take ρ such that S^F is holomorphic in B_ρ . In any case $\int_0^z S^F(\tau) d\tau$ is well-defined and

single-valued in $\Omega \cap B_\rho$. By (2) of Lemma 6.1, $\Phi(z, \bar{z})$ is real-valued in B_1 . Thus u is well-defined and real-valued in B_ρ . Set $\Omega(u) = \Omega \cap B_\rho$.

By definition, (i) of Definition 6.1 is satisfied. By the chain rule, we obtain

$$\frac{\partial \Phi(z, \bar{z})}{\partial z} = \frac{\partial \Phi}{\partial z}(z, \bar{z}) + \frac{\partial \Phi}{\partial w}(z, \bar{z}) \frac{\partial \bar{z}}{\partial z} = F(z, \bar{z}),$$

and so

$$4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z)$$

in $\Omega(u)$. Since u is real-valued, this equality implies (iv) of Definition 6.1. From this fact and condition $u(0) = 0$, we see that $u(z) = 0$ on $\Gamma(u) = \Gamma \cap B_\rho$ and hence (ii) of Definition 6.1 holds. Using the chain rule again, we obtain

$$\Delta u(z) = 4 \frac{\partial^2 u}{\partial \bar{z} \partial z}(z) = \frac{\partial F}{\partial w}(z, \bar{z}) = f(z, \bar{z})$$

in $\Omega(u)$. Hence u satisfies (iii) of Definition 6.1 and therefore it is of class $R(f, B_\rho)$. Q.E.D.

REMARK. We can construct a (complex-valued) u as above also in case $f(z, \bar{z})$ is complex-valued. In fact, $f(z, w)$ can be uniquely decomposed as $f_1(z, w) + i f_2(z, w)$, where f_j , $j = 1, 2$, are holomorphic functions in B_1^2 satisfying $\bar{f}_j = f_j$.

Lemmas 6.2 and 6.3 together with Proposition 5.2 imply that there exists a function $u \in R(f, B_\rho)$ with $\Omega(u) = \Omega \cap B_\rho$ for some $\rho > 0$ if and only if there exists a Schwarz function of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ for some $\delta > 0$, where Ω denotes an open subset of B_1 such that $0 \in \partial\Omega$ and $\Gamma = (\partial\Omega) \cap B_1$. As a consequence, Proposition 2.2 with $R(f, B_\rho)$ replacing $R(1, B_\rho)$ holds.

Finally we shall show the following theorem which is a generalization of our Theorem 1.1.

THEOREM 6.4. *Let $u \in P(f, B_\rho)$, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$. Then the same assertion as in Theorem 1.1 holds.*

PROOF. Since $P(f, B_\rho) \subset R(f, B_\rho)$, what we have to prove is that if 0 is a degenerate point in the sense of the Regularity Theorem, then (2a) of Theorem 1.1 holds and if 0 is a cusp point, then (1.2) holds.

From Lemmas 5.3 and 6.2 it follows that

$$(6.2) \quad 4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z) = F(z, \bar{z}) - F(z, S(z)),$$

in $\Omega \cap B_\delta$ for some $\delta > 0$, where F is the function defined by (6.1) and S denotes the Schwarz function of $\Omega \cup \Gamma$ in B_δ . Hence u is real analytic up to Γ on Ω except double and cusp points of Γ .

If 0 is a non-isolated degenerate point in the sense of the Regularity Theorem, then $u(\zeta) = 0$ on the arc determined by $\Gamma \cap B_\delta$, because $u(0) = 0$ and $\frac{\partial u}{\partial z}(\zeta) = 0$ on the arc. Hence $\Gamma \cap B_\delta = \{\zeta \in B_\delta : u(\zeta) = 0\}$ and $\Gamma \cap B_\delta$ is the whole arc.

Next assume that 0 is a cusp point and that $z = T(\tau) = \tau^2 + a_3\tau^3 + \dots$ is a one-to-one conformal mapping of $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$ into Ω such that $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$ for some δ and ε . Let $F(z, w) = \sum_{j,k \geq 0} (a_{jk}/(k+1)) z^j w^{k+1}$.

Since $S(z) = \overline{T(-\bar{\tau})} = \tau^2 - \bar{a}_3\tau^3 + \dots$, we obtain

$$\begin{aligned} F(z, \bar{z}) - F(z, S(z)) &= (a_{00}\bar{z} + O(|\tau|^4)) - (a_{00}S(z) + O(|\tau|^4)) \\ &= a_{00}(\bar{z} - S(z)) + O(|\tau|^4). \end{aligned}$$

From Definition 6.2, it follows that

$$a_{00} = \frac{\partial F}{\partial w}(0, 0) = f(0, 0) > 0.$$

By recalling (6.2) and applying the same argument as in the proof of Lemma 2.3, we see that $\operatorname{Re} a_3 = 0$ if $u(z) > 0$ in Ω , and therefore (1.2) holds. Q.E.D.

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