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About the Lamé System in a Polygonal or a Polyhedral Domain and a Coupled Problem Between the Lamé System and the Plate Equation II: Exact Controllability

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7. - Introduction

This paper is the second part of a research, whose purpose is to study the regularity of the solutions of some problems related to the linear elasticity theory and the exact controllability of the associated dynamical problems. Part I [22] concerns the regularity, while Part II, the exact controllability. For convenience, we have numbered the paragraphs continuously: Paragraphs 1 to 6 form the first part, while Paraghraphs 7 to 12 forms this second part.

We use the notations and definitions of Part I without comment. But contrary to Part I, we consider here real Hilbert spaces; this means that all the functions we use are real-valued. Nevertheless, all the results given in Part I remain true in the real setting.

In order to avoid some repetitions and confusions, we divide this paper into two parts A and B, corresponding to the problem studied respectively in Paragraphs 2 to 4 and Paragraphs 5 to 6 of Part I.

In Part A, we study the linear elasticity system in a polygonal domain of the plane or a polyhedral domain of the space. Namely (see Paragraph 8 for more details), if Ω denotes this domain, we divide its boundary into two parts $\Gamma_{\mathrm{D}} \cup \Gamma_N$. We consider the following dynamical linear elasticity system (using the classical notations of linear elasticity theory):

$$u_i'' - \sum_{j=1}^{3} \mathrm{D}_j \sigma_{ij}(\vec{u}) = f_i \quad \text{in } \Omega \times (0,T), \ \ \forall i \in \{1,2,3\}, \tag{7.1}$$

$$\vec{u} = \vec{v} \quad \text{on } \Gamma_{\mathrm{D}} \times (0,T), \tag{7.2}$$

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$$\sum_{j=1}^{3} \sigma_{ij}(\vec{u})\nu_j = w_i \quad \text{on } \Gamma_N \times (0,T),\ \forall i \in \{1,2,3\}, \tag{7.3}$$

$$\vec{u}(0) = \vec{u}_0, \quad \vec{u}'(0) = \vec{u}_1. \tag{7.4}$$

The problem of exact controllability for (7.1)–(7.4) states as follows: given $T > 0$ large enough, for every initial data $\vec{u}_0$, $\vec{u}_1$ in suitable Hilbert spaces, is it possible to find controls $\vec{v}$ and $\vec{w}$ driving our system (7.1)–(7.4) to rest at time T, i.e., such that the solution $\vec{u}$ of (7.1)–(7.4) satisfies

$$\vec{u}(T) = \vec{u}'(T) = \vec{0}\,? \tag{7.5}$$

Using his Hilbert Uniqueness Method (HUM), J.-L. Lions in [15] answered to this question when Ω is smooth and when even Γ_N or Γ_D is empty. In this second case, he also assumed that Ω is star shaped with respect to a point x_0. Here, we remove all these assumptions and following Grisvard's technics of [10] (especially Paragraph 5), we adapt HUM to our setting. As in [10], we impose a regularity assumption, which means that the weak solution of the stationary Lamé system belongs to $(H^{3/2+\varepsilon}(\Omega))^3$, for some $\varepsilon > 0$. Let us notice that we gave in Part I geometrical hypotheses on Ω, which imply that this regularity assumption is fulfilled. Finally, the fact that Ω is not necessarily star shaped with respect to a point leads to some difficulties on Γ_N, the part of the boundary where we impose Neumann boundary conditions.

In Part B, we consider the exact controllability of a coupled problem between the linear elasticity system in the unit cube of $\mathbb{R}^3$ with a crack and the plate equation on this crack. Our motivation comes from a question raised in Paragraph 6 of [3] concerning the exact controllability of the problem they obtained in [3]. We answer partially to this question because we modify some boundary conditions of the problem obtained by [3] (we explain at the end of Part I the reasons of this modification). In view of the analogy between this problem and the problem studied in [21], we follow the method of [21] to adapt HUM to this new problem. As in [21], the main problem lies on the fact that the 3D-part of the weak solution of the stationary problem has never the regularity $H^{3/2+\varepsilon}$, for some $\varepsilon > 0$. Fortunately, since it has only edge singularities along the bottom of the crack, a good choice of the multiplier allows us to use HUM.

Since we use HUM, the general method of proof of the exact controllability of our dynamical problems is similar to those of [15], [10], [21]. This means that some results stated here seem to be identical with the previous ones; but of course they are different, especially their proofs. Moreover, some technical problems are solved in a different way (for instance, the identity with multiplier).

A. The elasticity system in dimension 2 or 3

8. - Estimate of the energy

In all this Part A, we shall use the notations of the beginning of Part I, i.e., from Paragraphs 1 to 4. So Ω denotes a bounded open connected subset of $\mathbb{R}^n$, $n \in \{2,3\}$. But here we suppose that Ω is only on one side of its boundary (so Ω has no slit!). For convenience, we also suppose that the set $\mathcal{D}$ is not empty.

We now introduce the operator A associated with the linear elasticity system

$$L\vec{u} = \vec{f} \quad \text{in } \Omega, \tag{8.1}$$

$$\gamma_k \vec{u} = \vec{0} \quad \text{on } \Gamma_k, \ \forall k \in \mathcal{D}, \tag{8.2}$$

$$T^{(k)}\vec{u} = \vec{0} \quad \text{on } \Gamma_k, \ \forall k \in \mathcal{N}. \tag{8.2}$$

We set $H = (L^2(\Omega))^n$ and we recall that

$$V = \{\vec{u} \in (H^1(\Omega))^n \text{ fulfilling (8.2)}\}.$$

Since Korn's inequality holds on Lipschitz domains (see [6]), we know that the bilinear form a_Ω is V-coercive. Therefore it induces an isomorphism $\mathcal{A}$ from V into V' defined by

$$(\mathcal{A}\vec{u})(\vec{v}) = a_\Omega(\vec{u},\vec{v}), \qquad \forall \vec{u},\ \vec{v} \in V.$$

Since V is dense and compactly imbedded into H and the form a_Ω is symmetric, it also induces a positive selfadjoint operator A from H into H, with a compact inverse, defined by

$$\begin{cases} \mathrm{D}_A = \{\vec{u} \in V : \mathcal{A}\vec{u} \in H\}, \\ \forall \vec{u} \in \mathrm{D}_A : A\vec{u} = \mathcal{A}\vec{u}. \end{cases}$$

Since H, V and the form a_Ω fulfil the hypotheses of Remark 4.4 of [21], Theorems 4.1 to 4.3 of [21] may be applied to A. In particular, we have existence and uniqueness of the following wave equation: Given $\vec{\varphi}_0 \in V$, $\vec{\varphi}_1 \in H$ and $f \in L^1(0,T;H)$, there exists a unique solution $\vec{\varphi}$ of

$$\begin{cases} \vec{\varphi} \in C([0,T],V) \cap C^1([0,T],H), \\ \vec{\varphi}''(t) + A\vec{\varphi}(t) = \vec{f}(t), \qquad t \in [0,T], \\ \vec{\varphi}(0) = \vec{\varphi}_0, \\ \vec{\varphi}'(0) = \vec{\varphi}_1. \end{cases} \tag{8.4}$$

As classically, the estimate of the energy of the solution $\vec{\varphi}$ of (8.4) with $\vec{f} = \vec{0}$ will be obtained by proving an appropriate identity with multiplier. This is our next goal.

THEOREM 8.1. *Let us assume that*

$$\mathrm{D}_A \subset (H^{3/2+\varepsilon}(\Omega))^n, \qquad \textit{for some } \varepsilon > 0. \tag{8.5}$$

Then $\vec{u} \in \mathrm{D}_A$ *fulfils* (8.1)–(8.3) *with* $\vec{f} = A\vec{u}$.

PROOF. To get (8.1), it suffices to notice that for all $\vec{\varphi} \in (\mathcal{D}(\Omega))^n$, we have

$$a_\Omega(\vec{u}, \vec{\varphi}) = -\langle L\vec{u}, \vec{\varphi} \rangle.$$

Now, applying Theorem 6.6, we see that

$$\sum_{k \in \mathcal{N}} \int_{\Gamma_k} (T^{(k)}\vec{u}, \gamma_k \vec{v})\, \mathrm{d}\sigma = 0, \qquad \forall \vec{v} \in V.$$

This implies that $\vec{u}$ fulfils (8.3). ∎

Let us recall that in Theorems 2.3 and 4.5, we give geometrical conditions on Ω to ensure that (8.5) holds. We looked for these conditions because, as we shall see later, (8.5) is sufficient to apply the Hilbert Uniqueness Method of J.-L. Lions [15] to this setting.

So in all this Part A, we shall assume, without repeating it, that the inclusion (8.5) holds.

LEMMA 8.2. *If* $\vec{u} \in \mathrm{D}_A$ *and* $m(x) = x - x_0$, *with some* $x_0 \in \mathbb{R}^n$, *then we have*

$$\begin{aligned} \int_\Omega (L\vec{u}, m \cdot \nabla_n \vec{u})\, \mathrm{d}x &= \left(\frac{n}{2} - 1\right) a_\Omega(\vec{u}, \vec{u}) \\ &+ \sum_{k \in \mathcal{D}} \int_{\Gamma_k} \frac{m \cdot \vec{\nu}^{(k)}}{2} \left\{ \mu \sum_{i=1}^n \left(\gamma_k \frac{\partial u_i}{\partial \nu} \right)^2 + (\lambda + \mu)(\operatorname{div} \vec{u})^2 \right\} \mathrm{d}\sigma \\ &- \sum_{k \in \mathcal{N}} \int_{\Gamma_k} \frac{m \cdot \vec{\nu}^{(k)}}{2} \left\{ \lambda (\operatorname{div} \vec{u})^2 + 2\mu \sum_{i,j=1}^n (\varepsilon_{ij}(\vec{u}))^2 \right\} \mathrm{d}\sigma, \end{aligned} \tag{8.6}$$

where we use the notations

$$m \cdot \nabla_n \vec{u} = (m \cdot \nabla_n u_i)_{i=1}^n = \left(\sum_{j=1}^n m_j \, \mathrm{D}_j u_i \right)_{i=1}^n, \qquad \operatorname{div} \vec{u} = \sum_{i=1}^n \mathrm{D}_i u_i,$$

and $\gamma_k \dfrac{\partial u_i}{\partial \nu}$ *denotes the outward normal derivative of* u_i *on the face* Γ_k.

PROOF. Let us assume for a moment that $\vec{u}$ belongs to $(H^2(\Omega))^n$; applying two times the Green identity, we obtain

$$\int_\Omega (L\vec{u}, m \cdot \nabla_n \vec{u})\,\mathrm{d}x = \left(\frac{n}{2} - 1\right) a_\Omega(\vec{u}, \vec{u}) \tag{8.7}$$

$$+\sum_{k\in\mathcal{F}} \int_{\Gamma_k} \left\{ (T^{(k)}\vec{u}, \gamma_k m \cdot \nabla_n \vec{u}) - \frac{m \cdot \vec{\nu}^{(k)}}{2} \left(\lambda(\operatorname{div}\vec{u})^2 + 2\mu \sum_{i,j=1}^{n} (\varepsilon_{ij}(\vec{u}))^2 \right) \right\} \mathrm{d}\sigma.$$

Since $H^2(\Omega)$ is dense in $H^{3/2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$, we see that $\vec{u} \in \mathrm{D}_A$ fulfils (8.7) because all the terms in (8.7) have a meaning. To prove that (8.7) is equivalent to (8.6), we have to transform appropriately the boundary terms of (8.7). For $k \in \mathcal{N}$, we directly obtain the result since $\vec{u}$ fulfils (8.3). On the other hand, the boundary condition (8.2) implies that

$$\gamma_k \mathrm{D}_j u_i = \nu_j^{(k)} \gamma_k \frac{\partial u_i}{\partial \nu} \qquad \text{on } \Gamma_k,\ \forall k \in \mathcal{D}. \tag{8.8}$$

Using this identity (8.8) in the definitions of $T^{(k)}\vec{u}$, $m \cdot \nabla_n \vec{u}$ and $\varepsilon_{ij}(\vec{u})$, we easily show that

$$(T^{(k)}\vec{u}, \gamma_k m \cdot \nabla_n \vec{u}) = m \cdot \vec{\nu}^{(k)} \left\{ \mu \sum_{i=1}^{n} \left(\gamma_k \frac{\partial u_i}{\partial \nu} \right)^2 + (\lambda + \mu)(\gamma_k \operatorname{div}\vec{u})^2 \right\},$$

$$\sum_{i,j=1}^{n} (\gamma_k \varepsilon_{ij}(\vec{u}))^2 = \frac{1}{2} \sum_{i=1}^{n} \left(\gamma_k \frac{\partial u_i}{\partial \nu} \right)^2 + \frac{1}{2} (\gamma_k \operatorname{div}\vec{u})^2.$$

Using these two identities in (8.7), we arrive to (8.6). ∎

Let us now fix $x_0 \in \mathbb{R}^n$. For $m(x) = x - x_0$, we set

$$\mathcal{D}^+ = \{k \in \mathcal{D} : m \cdot \vec{\nu}^{(k)} > 0 \text{ on } \Gamma_k\},$$

$$\mathcal{N}^+ \text{ (resp. } \mathcal{N}^-) = \{k \in \mathcal{N} : m \cdot \vec{\nu}^{(k)} > 0 \text{ (resp. } < 0) \text{ on } \Gamma_k\}.$$

We are now ready to establish the main result of this paragraph.

THEOREM 8.3. *Let* $\vec{\varphi} \in C([0,T], \mathrm{D}_A) \cap C^1([0,T], V) \cap C^2([0,T], H)$ *be a solution of* (8.4) *with* $\vec{f} = \vec{0}$. *Then there exists a minimal time* $T_0 > 0$ *and a constant* $C > 0$ *such that*

$$(T - T_0) E_0 \leq C |||\{\vec{\varphi}_0, \vec{\varphi}_1\}|||^2, \tag{8.9}$$

where E_0 denotes the energy of $\vec{\varphi}$ at time $t = 0$, namely

$$E_0 = \frac{1}{2}\{||\vec{\varphi}_1||_H^2 + a_\Omega(\vec{\varphi}_0, \vec{\varphi}_0)\}$$

and setting $\Sigma_k = \Gamma_k \times (0,T)$, for all $k \in \mathcal{F}$, we define

$$\begin{aligned}|||\{\vec{\varphi}_0, \vec{\varphi}_1\}|||^2 &= \sum_{k\in D^+}\int_{\Sigma_k} ||T^{(k)}\vec{\varphi}||^2 \, d\sigma \, dt + \sum_{k\in \mathcal{N}^+}\int_{\Sigma_k} ||D_t\vec{\varphi}||^2 \, d\sigma \, dt \\ &+ \sum_{k\in\mathcal{N}^-}\int_{\Sigma_k}\left\{\lambda(\operatorname{div}\vec{\varphi})^2 + 2\mu\sum_{i,j=1}^{n}(\varepsilon_{ij}(\vec{\varphi}))^2\right\} d\sigma \, dt.\end{aligned} \tag{8.10}$$

PROOF. By integration by parts with respect to the variable t in $(0,T)$ and using the Green identity in Ω, we show that

$$\begin{aligned}\int_Q (D_t^2\vec{\varphi}, m\cdot\nabla_n\vec{\varphi})\, dx\, dt &= \int_\Omega (D_t\vec{\varphi}, m\cdot\nabla_n\vec{\varphi})\, dx|_0^T \\ &+ \frac{n}{2}\int_Q ||D_t\vec{\varphi}||^2\, dx\, dt - \frac{1}{2}\sum_{k\in\mathcal{N}}\int_{\Sigma_k} m\cdot\vec{\nu}^{(k)}||D_t\vec{\varphi}||^2\, d\sigma\, dt,\end{aligned} \tag{8.11}$$

where we have set $Q = \Omega \times (0,T)$.

Integrating (8.6) over $(0,T)$ (applied to $\vec{\varphi}(t)$) and subtracting the obtained identity to (8.11), we get

$$\begin{aligned}0 &= \int_Q (\vec{\varphi}'' + A\vec{\varphi}, m\cdot\nabla_n\vec{\varphi})\, dx\, dt = \int_\Omega (D_t\vec{\varphi}, m\cdot\nabla_n\vec{\varphi})\, dx|_0^T \\ &+ \frac{n}{2}\int_Q ||D_t\vec{\varphi}||^2\, dx\, dt + \left(1-\frac{n}{2}\right)\int_0^T a_\Omega(\vec{\varphi}(t), \vec{\varphi}(t))\, dt \\ &- \sum_{k\in D}\int_{\Sigma_k}\frac{m\cdot\vec{\nu}^{(k)}}{2}\left\{\mu\sum_{i=1}^{n}\left(\gamma_k\frac{\partial\varphi_i}{\partial\nu}\right)^2 + (\lambda+\mu)(\operatorname{div}\vec{\varphi})^2\right\} d\sigma\, dt \\ &+ \sum_{k\in\mathcal{N}}\int_{\Sigma_k}\frac{m\cdot\vec{\nu}^{(k)}}{2}\left\{\lambda(\operatorname{div}\vec{\varphi})^2 + 2\mu\sum_{i,j=1}^{n}(\varepsilon_{ij}(\vec{\varphi}))^2 - ||D_t\vec{\varphi}||^2\right\} d\sigma\, dt.\end{aligned} \tag{8.12}$$

Using the identity (4.24) of [21], which holds in the abstract setting of Remark

4.4 of [21], we show that

$$(8.13)\qquad \begin{aligned} &\frac{n}{2}\int_Q ||\mathrm{D}_t\vec{\varphi}||^2\,\mathrm{d}x\,\mathrm{d}t + \left(1-\frac{n}{2}\right)\int_0^T a_\Omega(\vec{\varphi}(t),\vec{\varphi}(t))\,\mathrm{d}t \\ &\qquad = -\frac{n-1}{2}\int_\Omega (\mathrm{D}_t\vec{\varphi},\vec{\varphi}(t))\,\mathrm{d}x\Big|_0^T + TE_0. \end{aligned}$$

In order to conclude, we need the following inequality (8.14), which is a direct consequence of (8.8):

$$(8.14)\qquad \mu\left\{\mu\sum_{i=1}^n\left(\gamma_k\frac{\partial\varphi_i}{\partial\nu}\right)^2 + (\lambda+\mu)(\gamma_k\operatorname{div}\vec{\varphi})^2\right\} \le ||T^{(k)}\vec{\varphi}||^2 \text{ on } \Gamma_k,\ \forall k\in\mathcal{D}.$$

Inserting (8.13) into (8.12), using Schwarz's inequality, the coerciveness of the form a_Ω and (8.14), we obtain (8.9). ■

Let us now fix $T > T_0$ such that the inequality (8.9) holds. Then the application

$$\mathrm{D}_A\times V\to\mathbb{R}^+:\{\vec{\varphi}_0,\vec{\varphi}_1\}\to|||\{\vec{\varphi}_0,\vec{\varphi}_1\}|||$$

is a norm stronger than the norm induced by $V\times H$. As in [15], [10], [21], we define F as the closure of $\mathrm{D}_A\times V$ for this new norm (obviously, F depends on x_0 and T) and we have the algebraic and topological inclusions:

$$\mathrm{D}_A\times V\subset F\subset V\times H.$$

Arguing as in Theorem 5.6 of [10], Theorem 8.3 leads to the (by taking into account the hypothesis (8.5)).

PROPOSITION 8.4. Let $\{\vec{\varphi}_0,\vec{\varphi}_1\}\in F$ and $\vec{f}\in L^1(0,T;V)$, then the unique solution $\vec{\varphi}$ of (8.4) fulfils

$$(8.15)\qquad \gamma_k\,\mathrm{D}_t\vec{\varphi}\in(L^2(\Sigma_k))^n,\ \forall k\in\mathcal{N}^+;$$

$$(8.16)\qquad \lambda(\gamma_k\operatorname{div}\vec{\varphi})^2+2\mu\sum_{i,j=1}^n(\gamma_k\varepsilon_{ij}(\vec{\varphi}))^2\in L^1(\Sigma_k),\ \forall k\in\mathcal{N}^-;$$

$$(8.17)\qquad T^{(k)}\vec{\varphi}\in(L^2(\Sigma_k))^n,\ \forall k\in\mathcal{D}^+.$$

Moreover, there exists a constant $C>0$ (independent of $\{\vec{\varphi}_0,\vec{\varphi}_1\}$ and $\vec{f}$) such that

$$(8.18)\quad \left\{\sum_{k\in\mathcal{N}^+}\int_{\Sigma_k}||\gamma_k \mathrm{D}_t\vec{\varphi}||^2 \,\mathrm{d}\sigma\,\mathrm{d}t + \sum_{k\in\mathcal{N}^-}\int_{\Sigma_k}\left\{\lambda(\operatorname{div}\vec{\varphi})^2 + 2\mu\sum_{i,j=1}^{n}(\varepsilon_{ij}(\vec{\varphi}))^2\right\}\mathrm{d}\sigma\,\mathrm{d}t + \sum_{k\in\mathcal{D}^+}\int_{\Sigma_k}||T^{(k)}\vec{\varphi}||^2\,\mathrm{d}\sigma\,\mathrm{d}t\right\}^{1/2} \leq C\{|||\{\vec{\varphi}_0,\vec{\varphi}_1\}||| + ||\vec{f}||_{L^1(0,T;V)}\}.$$

9. - Weak solutions of the dynamical elasticity system

We transpose Proposition 8.4 to get the

THEOREM 9.1. *For all $\vec{u}_0 \in H$, $\vec{u}_1 \in V'$, $\vec{v}^{(k)} \in (L^2(\Sigma_k))^n$, $k \in \mathcal{D}^+ \cup \mathcal{N}^+$ and $v_{ij}^{(k)} \in L^2(\Sigma_k)$, $i,\ j \in \{1,\dots,n\}$, $k \in \mathcal{N}^-$, there exist unique $\vec{u} \in L^\infty(0,T;V')$, $\{\vec{\psi}_1,\vec{\psi}_0\} \in F'$, which are solutions of*

$$(9.1)\quad \begin{aligned}\int_0^T \langle\vec{u}(t),\vec{f}(t)\rangle_{V'-V}\,\mathrm{d}t + \langle\{\vec{\psi}_1,\vec{\psi}_0\},\{\vec{\varphi}_0,-\vec{\varphi}_1\}\rangle_{F'-F}\\ = \langle\vec{u}_1,\vec{\varphi}(0)\rangle_{V'-V} - \langle\vec{u}_0,\vec{\varphi}'(0)\rangle_{H'-H} - \sum_{k\in\mathcal{D}^+}\int_{\Sigma_k}(\vec{v}^{(k)},T^{(k)}\vec{\varphi})\,\mathrm{d}\sigma\,\mathrm{d}t\\ - \sum_{k\in\mathcal{N}^+}\int_{\Sigma_k}(\vec{v}^{(k)},\gamma_k \mathrm{D}_t\vec{\varphi})\,\mathrm{d}\sigma\,\mathrm{d}t\\ - \sum_{k\in\mathcal{N}^-}\int_{\Sigma_k}\sum_{i,j=1}^{n} v_{ij}^{(k)}\gamma_k\varepsilon_{ij}(\vec{\varphi})\,\mathrm{d}\sigma\,\mathrm{d}t,\end{aligned}$$

for all $\vec{f} \in L^1(0,T;V)$, $\{\vec{\varphi}_0,-\vec{\varphi}_1\} \in F$, where $\vec{\varphi}$ is the unique solution of

$$(9.2)\quad \begin{cases}\vec{\varphi} \in C([0,T],V)\cap C^1([0,T],H),\\ \vec{\varphi}''(t) + A\vec{\varphi}(t) = \vec{f}(t), \qquad t\in[0,T],\\ \vec{\varphi}(T) = \vec{\varphi}_0,\ \ \vec{\varphi}'(T) = \vec{\varphi}_1.\end{cases}$$

In order to give an interpretation to the equation (9.1), we show that on the faces where we impose homogeneous Neumann boundary conditions, the ourward normal derivative is a linear combination of the tangential derivatives.

LEMMA 9.2. *For all $k \in \mathcal{N}$, there exist coefficients $\mathrm{d}_{ilm}^k \in \mathbb{R}$, and $\mathrm{d}_{ijlm}^k \in \mathbb{R}$, $i,\ j,\ l \in \{1,\dots,n\}$ and $m \in \{1,\dots,n-1\}$ such that $\vec{u}$ fulfilling* (8.3) *satisfies*

$$\gamma_k \frac{\partial u_i}{\partial \nu} = \sum_{l=1}^{n} \sum_{m=1}^{n-1} \mathrm{d}_{ilm}^k \frac{\partial u_l}{\partial \tau_m^{(k)}} \qquad \textit{on } \Gamma_k, \tag{9.3}$$

$$\gamma_k \varepsilon_{ij}(\vec{u}) = \sum_{j,l=1}^{n} \sum_{m=1}^{n-1} \mathrm{d}_{ijlm}^k \frac{\partial u_l}{\partial \tau_m^{(k)}} \qquad \textit{on } \Gamma_k, \ \forall i, \ j = 1, \dots, n, \tag{9.4}$$

where $\{\vec{\nu}^{(k)}, \{\vec{\tau}_m^{(k)}\}_{m=1}^{n-1}\}$ *forms an orthonormal basis of* $\mathbb{R}^n$ *and* $\dfrac{\partial u_l}{\partial \tau_m^{(k)}}$ *denotes the tangential derivative of* $\gamma_k u_l$ *in the direction of* $\vec{\tau}_m^{(k)}$.

PROOF. By change of variables, there obviously exist coefficients $c_{im}^k \in \mathbb{R}$ such that

$$\gamma_k \mathrm{D}_i u_j = \nu_i^{(k)} \gamma_k \frac{\partial u_j}{\partial \nu} + \sum_{m=1}^{n-1} c_{im}^k \frac{\partial u_j}{\partial \tau_m^{(k)}} \qquad \text{on } \Gamma_k, \ \forall k \in \mathcal{F}. \tag{9.5}$$

Using the definition of $T^{(k)}\vec{u}$ and (9.5), we prove the existence of a vector $t^{(k)}(\vec{u})$, whose components are linear combinations of $\dfrac{\partial u_j}{\partial \tau_m^{(k)}}$, for every $j \in \{1, \dots, n\}$, $m \in \{1, \dots, n-1\}$, such that

$$T^{(k)}\vec{u} = (\lambda + \mu) \left(\vec{\nu}^{(k)}, \gamma_k \frac{\partial \vec{u}}{\partial \nu} \right) \vec{\nu}^{(k)} + \mu \gamma_k \frac{\partial \vec{u}}{\partial \nu} + t^{(k)}(\vec{u}) \qquad \text{on } \Gamma_k, \ \forall k \in \mathcal{F}, \tag{9.6}$$

where $\gamma_k \dfrac{\partial \vec{u}}{\partial \nu}$ is the vector $\left(\gamma_k \dfrac{\partial u_i}{\partial \nu} \right)_{i=1}^{n}$. Therefore, the boundary conditions (8.3) is equivalent to

$$(\lambda + \mu) \left(\vec{\nu}^{(k)}, \gamma_k \frac{\partial \vec{u}}{\partial \nu} \right) \vec{\nu}^{(k)} + \mu \gamma_k \frac{\partial \vec{u}}{\partial \nu} = -t^{(k)}(\vec{u}) \qquad \text{on } \Gamma_k, \ \forall k \in \mathcal{N}. \tag{9.7}$$

Taking the inner product of (9.7) with $\vec{\nu}^{(k)}$ and $\vec{\tau}_m^{(k)}$, $m \in \{1, \dots, n-1\}$, we see that (9.7) is equivalent to

$$\gamma_k \frac{\partial \vec{u}}{\partial \nu} = -(t^{(k)}(\vec{u}), \vec{\nu}^{(k)}) \vec{\nu}^{(k)} / (\lambda + 2\mu) - \sum_{m=1}^{n-1} (t^{(k)}(\vec{u}), \vec{\tau}_m^{(k)}) \vec{\tau}_m^{(k)} / \mu \quad \text{on } \Gamma_k, \ \forall k \in \mathcal{N}.$$

This proves (9.3). Moreover, (9.5) and (9.3) obviously imply (9.4). ∎

We can now say that a solution $\vec{u}$ of (9.1) fulfils formally (9.8) to (9.12) hereafter:

(9.8)
$$\vec{u}'' - L\vec{u} = \vec{0} \qquad \text{in } \Omega \times (0,T);$$

(9.9)
$$\vec{u}(0) = \vec{u}_0, \qquad \vec{u}'(0) = \vec{u}_1;$$

(9.10)
$$\gamma_k \vec{u} = \begin{cases} \vec{v}^{(k)} & \text{on } \Sigma_k,\ \forall k \in \mathcal{D}^+, \\ 0 & \text{on } \Sigma_k,\ \forall k \in \mathcal{D} \setminus \mathcal{D}^+; \end{cases}$$

(9.11)
$$T_l^{(k)} \vec{u} = \begin{cases} \mathrm{D}_t v_l^{(k)} & \text{on } \Sigma_k,\ \forall k \in \mathcal{N}^+, \\ \displaystyle\sum_{m=1}^{n-1} \sum_{i,j=1}^{n} \mathrm{d}_{ijlm}^k \frac{\partial v_{ij}^{(k)}}{\partial \tau_m^{(k)}} & \text{on } \Sigma_k,\ \forall k \in \mathcal{N}^-, \\ 0 & \text{on } \Sigma_k,\ \forall k \in \mathcal{N} \setminus (\mathcal{N}^+ \cup \mathcal{N}^-), \end{cases}$$

for all $l \in \{1, \dots, n\}$, where we set $T^{(k)}\vec{u} = (T_l^{(k)}\vec{u})_{l=1}^n$;

(9.12)
$$\vec{u}(T) = \vec{\psi}_0, \qquad \vec{u}'(T) = \vec{\psi}_1.$$

As classically, the solution $\vec{u}$ of (9.1) will be called a weak solution of the problem (9.8)–(9.11).

Analogously to [21], we now show that the solution $\vec{u}$ of Theorem 3.1 is more regular and fulfils (9.12). This is based on the following trace result.

THEOREM 9.3. *Let* $\vec{v}^{(k)} \in (\mathcal{D}(\Sigma_k))^n$, $k \in \mathcal{D}^+ \cup \mathcal{N}^+$ *and* $v_{ij}^{(k)} \in \mathcal{D}(\Sigma_k)$, $i, j \in \{1, \dots, n\}$, $k \in \mathcal{N}^-$. *Then there exists* $\vec{v} \in (\mathcal{D}(0, T, \mathcal{D}(\overline{\Omega})))^n$ *fulfilling* (9.10)–(9.11). *Moreover, for all* $t \in [0,T]$, $\vec{v}$ *is equal to zero in a neighbourhood of the vertices of* Ω *and, in dimension* 3, *also in a neighbourhood of the edges of* Ω.

PROOF. We actually solve the following more general trace result: Given $\vec{v}^{(k)}$, $\vec{w}^{(k)} \in (\mathcal{D}(\Sigma_k))^n$, for all $k \in \mathcal{F}$, find $\vec{v} \in (\mathcal{D}(0, T, \mathcal{D}(\overline{\Omega})))^n$ fulfilling

(9.13)
$$\begin{cases} \gamma_k \vec{v} = \vec{v}^{(k)} & \text{on } \Sigma_k, \\ T^{(k)} \vec{v} = \vec{w}^{(k)} & \text{on } \Sigma_k,\ \forall k \in \mathcal{F}, \end{cases}$$

having the same nullity property. But using the argumentations of Lemma 9.2, we notice that (9.13) is equivalent to

(9.14)
$$\begin{cases} \gamma_k \vec{v} = \vec{v}^{(k)} & \text{on } \Sigma_k, \\ \gamma_k \dfrac{\partial \vec{v}}{\partial \nu} = \vec{z}^{(k)} & \text{on } \Sigma_k,\ \forall k \in \mathcal{F}, \end{cases}$$

where $\vec{z}^{(k)} \in (\mathcal{D}(\Sigma_k))^n$, for all $k \in \mathcal{F}$ ($\vec{z}^{(k)}$ is a linear combination of $\vec{w}^{(k)}$ and of the tangential derivatives of $\vec{v}^{(k)}$).

Let us fix $k \in \mathcal{F}$. By an eventual rotation and a translation, we may suppose that Γ_k is included in the hypersurface $x_n = 0$ and that the intersection between Ω and the half-space $\{x_n > 0\}$ is nonempty. Moreover, the compactness of the support of $\vec{v}^{(k)}$ and $\vec{z}^{(k)}$ in Σ_k imply that there exists an open subset V_k of Γ_k satisfying $\overline{V}_k \subset \Gamma_k$ (where we consider Γ_k as an open set of $\mathbb{R}^{n-1}$ and we take the closure of V_k in $\mathbb{R}^{n-1}$) such that

$$\operatorname{supp} \vec{v}^{(k)}(t) \cup \operatorname{supp} \vec{z}^{(k)}(t) \subset V_k, \qquad \forall t \in [0, T].$$

Since Ω is nondegenerate, we can find $\eta_k > 0$ such that

$$V_k \times [0, \eta_k] \subset \overline{\Omega}, \text{ and}$$
$$(V_k \times [-\eta_k, \eta_k]) \cap \Gamma_j = \emptyset, \qquad \forall j \in \mathcal{F} \backslash \{k\}.$$

Let us take a cut-off function $\psi_k \in \mathcal{D}(\mathbb{R})$ satisfying

$$\begin{cases} \operatorname{supp} \psi_k \subset] - \eta_k/2, \ \eta_k/2[\text{ and} \\ \psi_k = 1 \qquad \text{on }] - \eta_k/4, \ \eta_k/4[. \end{cases}$$

We set

$$\vec{u}^{(k)}(x, t) = (\vec{v}^{(k)}(x_1, \dots, x_{n-1}, t) - \vec{z}^{(k)}(x_1, \dots, x_{n-1}, t) \cdot x_n)\psi_k(x_n).$$

Owing to the previous remarks, we easily check that $\vec{u}^{(k)}$ fulfils (9.14) on Σ_k, is equal to zero in a neighbourhood of the faces Σ_j, $j \in \mathcal{F} \backslash \{k\}$, and has the desired nullity property.

We conclude by taking

$$\vec{v} = \sum_{k \in \mathcal{F}} \vec{u}^{(k)}. \qquad \blacksquare$$

THEOREM 9.4. *Let* $\vec{u} \in L^\infty(0, T; V')$, $\{\psi_1, \psi_0\} \in F'$ *be the solutions of* (9.1) *with data* $\vec{u}_0 \in V$, $\vec{u}_1 \in H$, $\vec{v}^{(k)} \in (\mathcal{D}(\Sigma_k))^n$, $k \in \mathcal{D}^+ \cup \mathcal{N}^+$ *and* $v_{ij}^{(k)} \in \mathcal{D}(\Sigma_k)$, i, $j \in \{1, \dots, n\}$, $k \in \mathcal{N}^-$. *Then* $\vec{u} \in C([0, T], H^1(\Omega)) \cap C^1([0, T], H)$ *and fulfils* (9.8)–(9.10) *and* (9.12).

PROOF. We proceed as in Theorem 5.3 of [21]. Let $\vec{v} \in (\mathcal{D}(0, T, \mathcal{D}(\overline{\Omega}))^n$ be the function built in Theorem 9.3 satisfying (9.10)–(9.11). We set

$$\vec{f} = \vec{v}'' - L\vec{v}.$$

By Lemma I.3.4 of [15], there exists a unique solution $\vec{\psi} \in C([0, T], V) \cap C^1([0, T], H) \cap H^2(0, T; V')$ of

$$(9.15) \qquad \begin{cases} \langle \vec{\psi}''(t), \vec{w} \rangle + a_\Omega(\vec{\psi}(t), \vec{w}) = -\displaystyle\int_\Omega (\vec{f}(t), \vec{w}) \, dx, \quad \text{a.e. } t \in [0, T], \ \forall \vec{w} \in V, \\ \vec{\psi}(0) = \vec{u}_0, \qquad \vec{\psi}'(0) = \vec{u}_1. \end{cases}$$

We now show that $\vec{u} = \vec{v} + \vec{\psi}$ is the unique solution of (9.1) when

$$\psi_0 = \vec{u}(T)(= \vec{\psi}(T)), \qquad \vec{\psi}_1 = \vec{u}'(T)(= \vec{\psi}'(T)).$$

By Theorem 4.2 of [21], it suffices to check (9.1) for $\vec{\varphi} \in C([0,T], \mathrm{D}_A) \cap C^1([0,T], V) \cap C^2([0,T], H)$. By integration by parts over $]0,T[$, we get

$$\begin{aligned}\int_0^T \langle \vec{u}(t), \vec{\varphi}'' + A\vec{\varphi}\rangle \,\mathrm{d}t - \langle \vec{u}(T), \vec{\varphi}_1\rangle + \langle \vec{u}'(T), \vec{\varphi}_0\rangle &= \langle \vec{u}_1, \vec{\varphi}(0)\rangle - \langle \vec{u}_0, \vec{\varphi}'(0)\rangle \\ &+ \int_0^T \{\langle \vec{\psi}''(t), \vec{\varphi}(t)\rangle + a_\Omega(\vec{\psi}(t), \vec{\varphi}(t)) + \langle \vec{v}''(t), \vec{\varphi}(t)\rangle + (\vec{v}(t), A\vec{\varphi}(t))_H\} \,\mathrm{d}t.\end{aligned} \tag{9.16}$$

But the following Green identity has a meaning since $\vec{\varphi}(t) \in \mathrm{D}_A$ belongs to H^2 far from the vertices and, in dimension 3, also from the edges, while $\vec{v}(t)$ is precisely equal to zero near the vertices and the edges in dimension 3.

$$\begin{aligned}(\vec{v}(t), A\vec{\varphi}(t))_H = \int_\Omega (-L\vec{v}(t), \vec{\varphi}(t)) \,\mathrm{d}x + \sum_{k\in\mathcal{F}} \int_{\Gamma_k} \{(T^{(k)}\vec{v}, \gamma_k\vec{\varphi}) \\ - (\gamma_k\vec{v}, T^{(k)}\vec{\varphi})\} \,\mathrm{d}\sigma.\end{aligned} \tag{9.17}$$

Inserting this identity into the right-hand side of (9.16), using (9.15) and taking into account the boundary conditions fulfilled by $\vec{\varphi}$ and $\vec{v}$, the right-hand side of (9.16) is equal to

$$\begin{aligned}&\langle \vec{u}_1, \vec{\varphi}(0)\rangle - \langle \vec{u}_0, \vec{\varphi}'(0)\rangle + \sum_{k\in\mathcal{N}^+} \int_{\Sigma_k} (\mathrm{D}_t\vec{v}^{(k)}, \gamma_k\vec{\varphi}) \,\mathrm{d}\sigma \,\mathrm{d}t \\ &+ \sum_{k\in\mathcal{N}^-} \int_{\Sigma_k} \sum_{i,j,l=1}^{n} \sum_{m=1}^{n-1} \mathrm{d}^k_{ijlm} \frac{\partial v^k_{ij}}{\partial \tau^{(k)}_m} \gamma_k\varphi_l \,\mathrm{d}\sigma \,\mathrm{d}t \\ &- \sum_{k\in\mathcal{D}^+} \int_{\Sigma_k} (\vec{v}^{(k)}, T^{(k)}\vec{\varphi}) \,\mathrm{d}\sigma \,\mathrm{d}t.\end{aligned}$$

By integration by parts on Σ_k, for all $k \in \mathcal{N}^+ \cup \mathcal{N}^-$ (this is allowed since $\vec{v}^{(k)} \in (\mathcal{D}(\Sigma_k))^n$, $v^{(k)}_{ij} \in \mathcal{D}(\Sigma_k)$ and $\vec{\varphi} \in C([0,T], \mathrm{D}_A)$) and using the identity (9.4), we deduce that the previous expression is equal to the right-hand side of (9.1). So $\vec{u}$ fulfils (9.1).

Now, $\vec{u}$ fulfils (9.8) because (9.15) implies that

$$\vec{\psi}''(t) - L\vec{\psi}(t) = -\vec{f}(t) \qquad \text{in } \Omega \times (0,T),$$

in this distributional sense. This completes the proof of Theorem 9.4. ∎

Before going on, let us notice that the solution $\vec{u}$ of Theorem 9.4 does not fulfil (9.11) in a strong sense because $\vec{\psi}$ is not enough regular to fulfil (8.3).

Nevertheless, using the arguments of the end of Paragraph 5 of [21] replacing Proposition 4.10 of [21] by Proposition 8.4, we conclude the

THEOREM 9.5. *Under the assumption of Theorem* 9.1, *let* $\vec{u}$, $\{\vec{\psi}_1, \vec{\psi}_0\}$ *be the solutions of* (9.1), *then* $\vec{u} \in C([0,T], V') \cap C^1([0,T], \mathrm{D}'_A)$ *and satisfies* (9.12).

10. - Application of HUM

It is now easy to use HUM in order to conclude the exact controllability of the elasticity system (see [15], [10] and Paragraph 6 of [21]).

THEOREM 10.1. *For all* $\vec{u}_0 \in H$, $\vec{u}_1 \in V'$, *there exist* $\vec{v}^{(k)} \in (L^2(\Sigma_k))^n$, *for all* $k \in \mathcal{D}^+ \cup \mathcal{N}^+$, *and* $v_{ij}^{(k)} \in L^2(\Sigma_k)$, $i, j \in \{1, \dots, n\}$, $k \in \mathcal{N}^-$ *such that the weak solution* $\vec{u} \in C([0,T], V') \cap C^1([0,T], \mathrm{D}'_A)$ *of the elasticity system* (9.8)–(9.11) *(in the sense of* (9.1)*) satisfies*

$$\vec{u}(T) = \vec{u}'(T) = \vec{0}. \tag{10.1}$$

PROOF. By Proposition 8.4, for $\{\vec{\varphi}_0, \vec{\varphi}_1\} \in F$, there exists a unique solution $\vec{\varphi}$ of (8.4) with $\vec{f} = \vec{0}$, which fulfils (8.15) to (8.17).

We consider $\vec{\psi} \in L^\infty(0,T,V')$, $\{\vec{\chi}_1, -\vec{\chi}_0\} \in F'$, the solutions of

$$\begin{aligned}
&\int_0^T \langle \vec{\psi}(t), \vec{g}(t)\rangle \,\mathrm{d}t - \langle \{\vec{\chi}_1, -\vec{\chi}_0\}, \{\vec{\eta}_0, \vec{\eta}_1\}\rangle \\
&= -\sum_{k\in\mathcal{D}^+} \int_{\Sigma_k} (T^{(k)}\vec{\varphi}, T^{(k)}\vec{\eta})\,\mathrm{d}\sigma\,\mathrm{d}t - \sum_{k\in\mathcal{N}^+} \int_{\Sigma_k} (\gamma_k \mathrm{D}_t\vec{\varphi}, \gamma_k \mathrm{D}_t\vec{\eta})\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad - \sum_{k\in\mathcal{N}^-} \int_{\Sigma_k} \sum_{i,j=1}^n \gamma_k\sigma_{ij}(\vec{\varphi})\gamma_k\varepsilon_{ij}(\vec{\eta})\,\mathrm{d}\sigma\,\mathrm{d}t,
\end{aligned} \tag{10.2}$$

for all $\vec{g} \in L^1(0,T;V)$, $\{\vec{\eta}_0, \vec{\eta}_1\} \in F$, where $\vec{\eta}$ is the unique solution of

$$\begin{cases} \vec{\eta} \in C([0,T], V) \cap C^1([0,T], H), \\ \vec{\eta}''(t) + A\vec{\eta}(t) = \vec{g}(t), \qquad t \in [0,T], \\ \vec{\eta}(0) = \vec{\eta}_0, \qquad \vec{\eta}'(0) = \vec{\eta}_1. \end{cases} \tag{10.3}$$

This is possible owing to Theorem 9.1 by inverting the order of time and because (8.16) implies that $\gamma_k\sigma_{ij}(\vec{\varphi})$ belongs to $L^2(\Sigma_k)$, for all $k \in \mathcal{N}^-$.

On the other hand, Theorem 9.5 shows that

$$\vec{\psi}(0) = \vec{\chi}_0, \qquad \vec{\psi}'(0) = \vec{\chi}_1.$$

As classically, we now define the operator

$$\Lambda: F \to F': \{\vec{\varphi}_0, \vec{\varphi}_1\} \to \{\vec{\chi}_1, -\vec{\chi}_0\}.$$

The next lemma shows that Λ is an isomorphism and we conclude as in Paragraph 6 of [21].

LEMMA 10.2. $\Lambda = \Lambda^*$ *and for all* $\{\vec{\varphi}_0, \vec{\varphi}_1\} \in F$, *we have*

$$\langle \Lambda\{\vec{\varphi}_0, \vec{\varphi}_1\}, \{\vec{\varphi}_0, \vec{\varphi}_1\}\rangle = |||\{\vec{\varphi}_0, \vec{\varphi}_1\}|||^2. \tag{10.4}$$

PROOF. Applying the identity (10.2) with $\vec{g} = \vec{0}$ and using the definition of $\sigma_{ij}(\vec{\varphi})$, we see that

$$\begin{aligned}
&\langle \Lambda\{\vec{\varphi}_0, \vec{\varphi}_1\}, \{\vec{\eta}_0, \vec{\eta}_1\}\rangle \\
&= \sum_{k\in D^+} \int_{\Sigma_k} (T^{(k)}\vec{\varphi}, T^{(k)}\vec{\eta})\, d\sigma\, dt + \sum_{k\in \mathcal{N}^+} \int_{\Sigma_k} (\mathrm{D}_t\vec{\varphi}, \mathrm{D}_t\vec{\eta})\, d\sigma\, dt \\
&+ \sum_{k\in \mathcal{N}^-} \int_{\Sigma_k} \left\{ 2\mu \sum_{i,j=1}^{n} \varepsilon_{ij}(\vec{\varphi})\varepsilon_{ij}(\vec{\eta}) + \lambda \operatorname{div}\vec{\varphi} \operatorname{div}\vec{\eta} \right\} d\sigma\, dt.
\end{aligned} \tag{10.5}$$

This firstly proves that $\Lambda = \Lambda^*$. Secondly, taking $\{\vec{\eta}_0, \vec{\eta}_1\} = \{\varphi_0, \vec{\varphi}_1\}$ and hence $\vec{\eta} = \vec{\varphi}$, we deduce (10.4) in view of the definition (8.10) of the norm in F. ∎

B. The coupled problem

11. - An identity with multiplier

In all this Part B, we use the notations of Paragraphs 5 and 6 of Part I of this paper [22]. We want to establish the exact controllability of the dynamical problem associated with the boundary value problem (5.1)–(5.7). As we explain in the Introduction, we use the same method as in Paragraphs 4 to 6 of [21] or in Part A. Therefore, we only give the great lines and the differences with the previous results.

We gave in Paragraph 5 the variational formulation of problem (5.1)–(5.7). Since the next lemma proves that V is dense in H, we can associate with the form a (defined by (5.8)) a selfadjoint operator A from H to H with a compact inverse (see Paragraph 8). So Theorems 4.1 to 4.3 of [21] still hold for A. Moreover, Theorem 5.3 shows that for all $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$, we have

$$A\vec{U} = (-L\vec{u}, \rho\Delta^2\xi + \{\gamma_-\sigma_{22}(\vec{u}) - \gamma_+\sigma_{22}(\vec{u})\}\chi_\Gamma).$$

LEMMA 11.1. *V is dense in H.*

PROOF. Let $\vec{U} = (\vec{u}, \xi)$ in H and let us fix $\varepsilon > 0$. The density of $\mathcal{D}(\omega)$ in $L^2(\omega)$ gives the existence of $\eta \in \mathcal{D}(\omega)$ such that

$$||\xi - \eta||_{L^2(\omega)} \leq \varepsilon. \tag{11.1}$$

We now introduce cut-off functions $\varphi_\delta \in \mathcal{D}(\mathbb{R})$ depending on the real parameter $\delta \in]0, 1[$ (δ will be determined later) such that

$$0 \leq \varphi_\delta \leq 1, \ \ \varphi_\delta(0) = 1 \text{ and } \operatorname{supp} \varphi_\delta \subset \left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

We set

$$w_2(x_1, x_2, x_3) = \eta(x_1, x_3) \cdot \varphi_\delta(x_2).$$

It is immediate that

$$||w_2||_{L^2(\Omega)} \leq \sqrt{\delta}||\eta||_{L^2(\omega)}.$$

In view of (11.1), choosing δ such that

$$\sqrt{\delta} \leq \frac{\varepsilon}{\varepsilon + ||\xi||_{L^2(\omega)}},$$

we deduce that

$$||w_2||_{L^2(\Omega)} \leq \varepsilon. \tag{11.2}$$

Finally, owing to the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, there exists $\vec{\varphi} \in (\mathcal{D}(\Omega))^3$ such that

$$||\vec{\varphi} - \vec{u}||_{(L^2(\Omega))^3} \leq \varepsilon. \tag{11.3}$$

The conclusion follows by setting

$$\vec{v} = \vec{\varphi} + (0, w_2, 0).$$

Indeed, we easily check that $\vec{V} := (\vec{v}, \eta)$ belongs to V, while the inequalitites (11.1) to (11.3) imply that

$$||\vec{V} - \vec{U}||_H \leq \sqrt{5}\varepsilon. \qquad \blacksquare$$

The second step consists in establishing an identity with multiplier as in Lemma 8.2. Unfortunately, if $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$, then Theorem 6.2 shows that $\vec{u}$ admits the expansion (6.7) and therefore, $\vec{u}$ has never the regularity $H^s(\Omega)$, for some $s > \frac{3}{2}$. In spite of this lack of regularity, as in Theorem 4.5 of [21], by a good choice of the point x_0, we can prove this identity. Contrary to Theorem 4.5 of [21], where we cannot separate Ω and Γ to prove the identity with multiplier, here the best regularity of ξ allows us to separate Ω and ω. Let us start with the 3D-part, which is very technical. We postpone the proof of the 2D-part to Theorem 11.7.

THEOREM 11.2. *Let us assume that $m(x) = x - x_0$ with $x_0 = (0, 0, x_{03})$. Then for all $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$, we have:*

$$\begin{aligned}\int_\Omega (-L\vec{u}, m \cdot \nabla_3 \vec{u})\,\mathrm{d}x &= -\frac{1}{2}\, a_\Omega(\vec{u}, \vec{u}) \\ &\quad - \int_{\Gamma_1} \frac{m \cdot \vec{\nu}}{2} \left\{ \mu \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial \nu} \right)^2 + (\lambda + \mu)(\operatorname{div} \vec{u})^2 \right\} \mathrm{d}\sigma \\ &\quad - \int_{\Gamma_2} \frac{m \cdot \vec{\nu}}{2} \left\{ \lambda(\operatorname{div} \vec{u})^2 + 2\mu \sum_{i,j=1}^{3} (\varepsilon_{ij}(\vec{u}))^2 \right\} \mathrm{d}\sigma \\ &\quad - \int_\Gamma \{\gamma_- \sigma_{22}(\vec{u}) - \gamma_+ \sigma_{22}(\vec{u})\} \gamma_+ m \cdot \nabla_2 \xi \,\mathrm{d}x'.\end{aligned} \tag{11.4}$$

PROOF. For $\delta > 0$, let us denote (we recall that $z_i = (\theta_i - \pi/2, \varphi_i - \pi/2)$, for $i = 1$ or -1, see Paragraph 6)

$$C_\delta^1 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \le x_3 \le 1 \text{ and } |z_1| < \delta\},$$

$$C_\delta^{-1} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 1 \le x_3 \le 0 \text{ and } |z_{-1}| < \delta\},$$

$$D_\delta = C_\delta^1 \cup C_\delta^{-1}, \quad \Omega_\delta = \Omega \backslash \overline{\mathrm{D}}_\delta, \quad \Gamma_\delta = \Gamma \backslash \overline{\mathrm{D}}_\delta.$$

Owing to Theorem 6.2, we know that $\vec{u}$ admits the expansion (6.7); so it belongs to $(H^{3/2+\varepsilon}(\Omega_\delta))^3$, for some $\varepsilon > 0$ and for δ sufficiently small. Therefore $\vec{u}$ fulfils the identity (8.7) in Ω_δ; that is

$$\begin{aligned}\int_{\Omega_\delta} (-L\vec{u}, m \cdot \nabla_3 \vec{u})\,\mathrm{d}x &= -\frac{1}{2}\, a_{\Omega_\delta}(\vec{u}, \vec{u}) - \int_{\Gamma_1 \cup \partial \mathrm{D}_\delta \cup \Gamma_\delta} (T\vec{u}, m \cdot \nabla_3 \vec{u})\,\mathrm{d}\sigma \\ &\quad + \int_{\Gamma_1 \cup \partial \mathrm{D}_\delta \cup \Gamma_2} \frac{m \cdot \vec{\nu}}{2} \left\{ \lambda(\operatorname{div} \vec{u})^2 + 2\mu \sum_{i,j=1}^{3} (\varepsilon_{ij}(\vec{u}))^2 \right\} \mathrm{d}\sigma,\end{aligned} \tag{11.5}$$

by taking into account the boundary conditions fulfilled by $\vec{u}$ and the fact that

$$m \cdot \nu^{\pm} = 0 \qquad \text{on } \Gamma.$$

We now pass to the limit as δ goes to zero. Since $\vec{U} \in \mathrm{D}_A$, the left-hand side of (11.5) and the first term of the right-hand side of (11.5) tend respectively to the left-hand side of (11.4) and the first term of the right-hand side of (11.4). Moreover, as in Lemma 8.2, the boundary terms on Γ_1 in (11.5) are transformed into the boundary term on Γ_1 in (11.4). So it remains the boundary terms on Γ_δ and $\partial \mathrm{D}_\delta$. Let us denote them respectively by $I_{1\delta}$ and $I_{2\delta}$.

Using the definition of T, we easily check that

$$I_{1\delta} = \int\limits_{\Gamma_\delta} (\gamma_+\sigma_{22}(\vec{u}) - \gamma_-\sigma_{22}(\vec{u}))\gamma_+ m \cdot \nabla_2 \xi \, \mathrm{d}x'.$$

But Proposition 6.3 asserts that $\gamma_+\sigma_{22}(\vec{u}) - \gamma_-\sigma_{22}(\vec{u}) \in L^p(\Gamma)$, for all $p \in]1,2[$; while Theorem 6.6 and the Sobolev imbedding theorem imply that $\gamma_+ m \cdot \nabla_2 \xi \in L^q(\Gamma)$, for all $q > 1$. Therefore, Hölder's inequality and Lebesgue's bounded convergence theorem allow us to conclude that

$$I_{1\delta} \to \int\limits_{\Gamma} (\gamma_+\sigma_{22}(\vec{u}) - \gamma_-\sigma_{22}(\vec{u}))\gamma_+ m \cdot \nabla_2 \xi \, \mathrm{d}x', \qquad \text{as } \delta \to 0. \tag{11.6}$$

In view of the definition of $I_{2\delta}$, there exists a constant $M > 0$ such that

$$I_{2\delta} \leq M \Bigg\{ \Bigg(\int\limits_{\partial \mathrm{D}_\delta} \|\nabla_3 \vec{u}\|^2 \, \mathrm{d}\sigma \Bigg)^{1/2} \Bigg(\int\limits_{\partial \mathrm{D}_\delta} \|m \cdot \nabla_3 \vec{u}\|^2 \, \mathrm{d}\sigma \Bigg)^{1/2}$$
$$+ \int\limits_{\partial \mathrm{D}_\delta} |m \cdot \vec{\nu}| \, \|\nabla_3 \vec{u}\|^2 \, \mathrm{d}\sigma \Bigg\}.$$

But it is easily seen that

$$m \cdot \vec{\nu} \to 0, \qquad \text{as } \delta \to 0,$$

and using Theorem 11.3 hereafter, we conclude that

$$I_{2\delta} \to 0, \qquad \text{as } \delta \to 0. \tag{11.7}$$

So, (11.6) and (11.7) prove the identity (11.4). ∎

THEOREM 11.3. *Under the assumption of Theorem* 11.2, *if* $\vec{U} = (\vec{u}, \xi) \in D_A$ *there exist positive constants* K *and* δ_0 *such that for all* $\delta \in]0, \delta_0[$*:*

$$\int_{\partial D_\delta} ||\nabla_3 \vec{u}||^2 \, d\sigma \leq K, \tag{11.8}$$

$$\int_{\partial D_\delta} ||m \cdot \nabla_3 \vec{u}||^2 \, d\sigma \to 0, \qquad \textit{as } \delta \to 0. \tag{11.9}$$

PROOF. Let us denote by $\vec{u}_s$, the singular part of the decomposition (6.7) of $\vec{u}$. On one hand, Lemma 11.5 shows that $\vec{u}_s$ satisfies (11.8) and (11.9). On the other hand, Lemmas 6.3 and 11.4 imply that $\vec{u}_r$ fulfils

$$\int_{\partial D_\delta} ||\nabla_3 \vec{u}_r||^2 \, d\sigma \to 0, \qquad \text{as } \delta \to 0. \tag{11.10}$$

Since (11.10) is stronger than (11.8) and (11.9), we conclude by addition. ∎

LEMMA 11.4. *Let* $u \in H^1(\Omega)$, *then there exists a constant* $K > 0$ *such that for all* $\delta > 0$ *sufficiently small, we have*

$$\int_{\partial D_\delta} |u|^2 \, d\sigma \leq K \delta ln \delta ||u||^2_{H^1(\Omega)}. \tag{11.11}$$

In particular, u *fulfils*

$$\int_{\partial D_\delta} |u|^2 \, d\sigma \to 0, \qquad \textit{as } \delta \textit{ goes to } 0. \tag{11.12}$$

PROOF. It is similar to the proof of Lemma 4.7 of [21]. The difference is that we decompose ∂D_δ into two parts: $\partial D_\delta = \Gamma_\delta^1 \cup \Gamma_\delta^{-1}$, where we set

$$\Gamma_\delta^i = \partial D_\delta \cap \partial C_\delta^i, \qquad \text{for } i = 1 \text{ or } -1.$$

Using the coordinates $(r_i, r_{2i}, \theta_{2i})$ on C_δ^i and the arguments of Lemma 4.7 of [21] on C_δ^i where r is replaced by r_{2i}, we can prove (11.11). ∎

Obviously, we cannot apply Lemmas 6.3 and 11.4 to the singular part $\vec{u}_s$ of $\vec{u}$. Nevertheless, using its explicit expansion and the spherical coordinates, we prove directly that $\vec{u}_s$ fulfils (11.8) and (11.9).

LEMMA 11.5. *Under the assumption of Theorem* 11.2, *if* $\vec{U} = (\vec{u}, \xi) \in D_A$, *then the singular part* $\vec{u}_s$ *of* $\vec{u}$ *fulfils* (11.8) *and* (11.9).

PROOF. We only prove (11.8) and (11.9) for a part of the singular part $\vec{u}_s$, namely for

$$\vec{u}_{1s} = \eta_1 \mathcal{R}_1(c_1) A_1 \begin{pmatrix} \sigma_1(z_1) \\ 0 \\ 0 \end{pmatrix}. \tag{11.13}$$

The proof is similar for the other parts.

We remark that the matrix A_1 is the transition matrix from the gradient in cartesian coordinates to the gradient in spherical coordinates $(r_1, \theta_1, \varphi_1)$, this means that for a scalar function u, we have

$$\nabla_3 u = A_1 \nabla_s u, \tag{11.14}$$

where we set

$$\nabla_s u = \left(\frac{\partial u}{\partial r_1}, \frac{1}{r_1} \frac{\partial u}{\partial \theta_1}, \frac{1}{r_1 \sin \theta_1} \frac{\partial u}{\partial \varphi_1} \right)^T.$$

Since A_1 is uniformly bounded, in order to prove (11.8), it suffices to replace $\nabla_3 u$ by $\nabla_s u$. Let us firstly compute the gradient of $\mathcal{R}_1(c_1)$ (defined in Theorem 6.2). Using the notations introduced in Lemma 6.5 and the Leibniz rule, we see that

$$\partial \mathcal{R}_1(c_1)/\partial r_1 = \chi_1 r_1^{\varepsilon-1} \tag{11.15}$$

$$\{\varepsilon (c_1 *_t \phi)(ln r_1, z_1) + r_{21}^{-1} (c_1 *_t \psi)(ln r_1, z_1)\},$$

where for $r_{21} \leq r_0$, for some $r_0 > 0$, we have (see Remark 16.7 of [4]):

$$\psi(t, z_1) = r_{21}^{-1} \frac{\partial \hat{\varphi}}{\partial t} (t r_{21}^{-1}).$$

We shall now compute

$$I_\delta = \int_{\partial \mathrm{D}_\delta} \left\| \frac{\partial \vec{u}_{1s}}{\partial r_1} \right\|^2 \mathrm{d}\sigma.$$

As in Lemma 11.4, we split up this integral into two parts corresponding to the boundary of C_δ^1 and C_δ^{-1} i.e.

$$I_\delta = I_\delta^1 + I_\delta^{-1},$$

where we set

$$I_\delta^i = \int_{\Gamma_\delta^i} \left\| \frac{\partial \vec{u}_{1s}}{\partial r_1} \right\|^2 \mathrm{d}\sigma, \qquad i \in \{1, -1\}.$$

Using the coordinates $(r_1, r_{2i}, \theta_{2i})$ on C^i_δ, we see that Γ^i_δ is characterized by

$$\Gamma^i_\delta = \{(r_i, r_{2i}, \theta_{2i}) \text{ such that } 0 < r_i < r_\delta,\ r_{2i} = \delta \text{ and } \theta_{2i} \in [0, 2\pi[\}$$

for some $r_\delta > 1$ (notice that $r_\delta \to 1$, as $\delta \to 0$). Moreover, we easily check that the surface measure $d\sigma$ on Γ^i_δ is given by

$$d\sigma = \delta r_i \sqrt{\cos^2\theta_{2i} + \sin^2\theta_i \sin^2\theta_{2i}}\, dr_i\, d\theta_{2i}. \tag{11.16}$$

Recalling that

$$\sigma_i(z_i) = r_{2i}^{1/2} \sin\left(\frac{\theta_{2i}}{2}\right),$$

using (11.14), (11.15) and (11.16), we deduce that there exist $\delta_0 > 0$ and $K_1 > 0$ such that for all $\delta \in]0, \delta_0[$:

$$I^1_\delta \leq K_1 \int_0^2 r_1^{2(\varepsilon-1)} \{\varepsilon^2 |c_1 *_t \phi|^2 + \delta^{-2} |c_1 *_t \psi|^2\} \delta^2 r_1\, dr_1. \tag{11.17}$$

On one hand, inequality (6.16) shows that there exists $K_2 > 0$ such that for $\delta \in]0, \delta_0[$:

$$\int_0^2 r_1^{2\varepsilon-1} |(c_1 *_t \phi)(\ln r_1, z_1)|^2\, dr_1 \leq K_2. \tag{11.18}$$

On the other hand, Proposition 11.6 hereafter (applied with $\lambda = 0$) states that

$$\int_0^2 |(c_1 *_t \psi)(ln r_1, z_1)|^2 r_1^{-1}\, dr_1 \to 0, \qquad \text{as } |z_1| = r_{21} \to 0. \tag{11.19}$$

So we conclude that

$$I^1_\delta \to 0, \qquad \text{as } \delta \to 0. \tag{11.20}$$

Let us show the same result for I^{-1}_δ. Since the cut-off function η_1 in (11.13) is equal to zero in a neighbourhood of S_{-1} (see Theorem 6.2), we can say that there exists $K_3 > 0$ such that

$$I^{-1}_\delta \leq K_3 \int_{1/2}^{3/2} \int_0^{2\pi} \left\| \frac{\partial \vec{u}_{1s}}{\partial r_1} \right\|^2 \delta r_{-1}\, dr_{-1}\, d\theta_{2,-1}.$$

Using again (11.15), we obtain

$$I_\delta^{-1} \leq K_4 \int_{1/2}^{3/2} \int_0^{2\pi} r_1^{2\varepsilon-2} \{\varepsilon^2 |(c_1 *_t \phi)(ln r_1, z_1)|^2 + r_{21}^{-2} |(c_1 *_t \psi)(ln r_1, z_1)|^2\} \delta r_{21} r_{-1} \, dr_{-1} \, d\theta_{2,-1}. \tag{11.21}$$

But we remark that for δ small enough, there exist positive constants K_5 and K_6 such that for $x \in \Gamma_\delta^{-1}$ with $r_{-1}(x) \in \left[\frac{1}{2}, \frac{3}{2}\right]$, we have

$$K_5 \delta \leq r_{21}(x) \leq K_6 \delta. \tag{11.22}$$

Passing from the coordinates $(r_{-1}, r_{2,-1}, \theta_{2,-1})$ to the coordinates $(r_1, r_{21}, \theta_{21})$, we see that on Γ_δ^{-1}, we have

$$r_1 = \left(r_{-1}^2 + 4 - 4r_{-1} \sin\left(\frac{\pi}{2} - \delta \sin \theta_{2,-1}\right) \sin\left(\frac{\pi}{2} - \delta \cos \theta_{2,-1}\right)\right)^{\frac{1}{2}}. \tag{11.23}$$

But it is always possible to choose $\delta_{00} > 0$ such that for $\delta \in]0, \delta_{00}[$, the interval $r_{-1} \in]1/2, 3/2[$ is sent into an interval included into $]1/4, 2[$. So using this change of variable (11.23) into (11.21) and using (11.22), we get

$$I_\delta^{-1} \leq K_7 \int_0^{2\pi} \int_{1/4}^{2} \{\delta^2 |(c_1 *_t \phi)(ln r_1, z_1)|^2 + |(c_1 *_t \psi)(ln r_1, z_1)|^2\} \, dr_1 \, d\theta_{2,-1}.$$

Using again (11.18) and (11.19), by taking into account (11.22), we arrive to

$$I_\delta^{-1} \to 0, \qquad \text{as } \delta \to 0. \tag{11.24}$$

Joining together (11.20) and (11.24), we have proved that

$$\int_{\partial D_\delta} \left\| \frac{\partial \vec{u}_{1s}}{\partial r_1} \right\|^2 d\sigma \to 0, \qquad \text{as } \delta \to 0. \tag{11.25}$$

This is more than we need to prove (11.8) but we shall use it to show (11.9).

We argue as above for the derivatives with respect to θ_1 and φ_1, but here we only have boundedness. Indeed, for the derivative with respect to θ_1, for instance, it appears a term bounded by

$$\int_{\partial D_\delta} \eta_1^2 |\mathcal{R}_1(c_1)|^2 |\partial \sigma_1 / \partial \theta_1|^2 \, d\sigma.$$

Since $\partial\sigma_1/\partial\theta_1 = (-1/2)r_{21}^{-1/2}\cos(\theta_{21}/2)$, the previous expression is bounded by

$$\int_0^2 r_1^{2\varepsilon+1}|(c_1 *_t \phi)(lnr_1, z_1)|^2\,\mathrm{d}r_1.$$

Owing to inequality (6.16), it is only bounded.

We have just proved that $\vec{u}_{1s}$ fulfils (11.8). Now (11.14) shows that

$$\int_{\partial \mathrm{D}_\delta} ||m\cdot\nabla_3\vec{u}_{1s}||^2\,\mathrm{d}\sigma = \int_{\partial \mathrm{D}_\delta} ||m\cdot A_1\nabla_s\vec{u}_{1s}||^2\,\mathrm{d}\sigma.$$

For $x = (r_i, \delta, \theta_{2i})$ in Γ_δ^i, $i \in \{1, -1\}$, we easily check that

$$(m\cdot A_1)(x) \to -(r_i - i - x_{03}, 0, 0), \qquad \text{as } \delta \to 0.$$

Therefore (11.25), (11.8) for $\vec{u}_{1s}$ and Lebesgue's bounded convergence theorem imply (11.9). ∎

Let us now give the result we need in (11.19), concerning the regularization built with a function of mean zero.

PROPOSITION 11.6. *Let $\varphi \in \mathcal{D}(\mathbb{R})$ and set*

$$\lambda = \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x, \quad \textit{and } \mathcal{K} = \int_{\mathbb{R}} |\varphi(x)|\,\mathrm{d}x.$$

For all $\varepsilon > 0$, we set $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$. Then for all $u \in L^2(\mathbb{R})$, we have

$$\varphi_\varepsilon * u \to \lambda u \qquad \textit{in } L^2(\mathbb{R}),\ \textit{as } \varepsilon \to 0, \tag{11.26}$$

$$||\varphi_\varepsilon * u||_{L^2(\mathbb{R})} \le \mathcal{K}||u||_{L^2(\mathbb{R})}, \qquad \forall \varepsilon > 0. \tag{11.27}$$

PROOF. It is a direct consequence of the classical regularization theorem concerning nonnegative functions (see for instance Lemma 2.18 of [1]) by splitting up φ into its positive and negative parts. ∎

We now proceed to the proof of the identity with multiplier in the domain ω.

THEOREM 11.7. *Under the assumption of Theorem 11.2, all $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$ fulfils:*

$$\int_\omega \Delta^2\xi\gamma_+ m\cdot\nabla_2\xi\,\mathrm{d}x' = \int_\omega (\Delta\xi)^2\,\mathrm{d}x' - \frac{1}{2}\int_{\partial\omega}\gamma_+ m\cdot\vec{\nu}'(\Delta\xi)^2\,\mathrm{d}\sigma', \tag{11.28}$$

or equivalently

$$\rho \int_{\omega} \Delta^2 \xi \gamma_+ m \cdot \nabla_2 \xi \, \mathrm{d}x' = b_\omega(\xi, \xi) - \frac{\rho}{2} \int_{\partial\omega} \gamma_+ m \cdot \vec{\nu}' (\Delta \xi)^2 \, \mathrm{d}\sigma', \tag{11.29}$$

where $\vec{\nu}'$ denotes the outward normal unit vector on the boundary $\partial\omega$ of ω.

PROOF. Theorem 6.6 proves that $\xi \in H^{7/2+\varepsilon}(\omega)$, for some $\varepsilon > 0$. So Propositions 2 and 3 of [19] applied with $m(x) = x - x_0$ show that ξ fulfils (11.28), where the left-hand side is seen as a duality bracket. But we show in Theorem 6.6 that $\Delta^2 \xi \in L^p(\omega)$, for some $p \in]1, 2[$ and owing to the Sobolev imbedding theorem, $\nabla_2 \xi$ belongs to $(L^q(\omega))^2$, for all $q > 1$; therefore, we can replace this duality bracket by a classical integral.

Finally, (11.28) is equivalent to (11.29) because

$$\rho \int_{\omega} (\Delta \xi)^2 \, \mathrm{d}x' = b_\omega(\xi, \xi), \qquad \forall \xi \in H_0^2(\omega). \qquad \blacksquare$$

Adding the identities (11.4) and (11.29), we obtain the so-called identity with multiplier; which is analogous to the identity (4.5) obtained in [21].

COROLLARY 11.8. *Under the assumption of Theorem 11.2, $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$ satisfies*

$$\begin{aligned}
&\int_{\Omega} (-L\vec{u}, m \cdot \nabla_3 \vec{u}) \, \mathrm{d}x + \int_{\omega} \{\rho \Delta^2 \xi + \chi_\Gamma(\gamma_- \sigma_{22}(\vec{u}) - \gamma_+ \sigma_{22}(\vec{u}))\} \, \mathrm{d}x' \\
&= -\frac{1}{2} a_\Omega(\vec{u}, \vec{u}) + b_\omega(\xi, \xi) - \int_{\Gamma_1} \frac{m \cdot \vec{\nu}}{2} \left\{ \mu \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial \nu} \right)^2 + (\lambda + \mu)(\operatorname{div} \vec{u})^2 \right\} \mathrm{d}\sigma \\
&\quad + \int_{\Gamma_2} \frac{m \cdot \vec{\nu}}{2} \left\{ \lambda (\operatorname{div} \vec{u})^2 + 2\mu \sum_{i,j=1}^{3} (\varepsilon_{ij}(\vec{u}))^2 \right\} \mathrm{d}\sigma \\
&\quad - \frac{\rho}{2} \int_{\partial\omega} \gamma_+ m \cdot \vec{\nu}' (\Delta \xi)^2 \, \mathrm{d}\sigma'.
\end{aligned} \tag{11.30}$$

As in Paragraph 8, we are now able to give the estimate of the energy.

THEOREM 11.9. *Let $\vec{U} = (\vec{u}, \xi) \in C([0,T], \mathrm{D}_A) \cap C^1([0,T], V) \cap C^2([0,T], H)$ be a solution of*

$$\begin{cases} \vec{U}''(t) + A\vec{U}(t) = \vec{0}, & t \in [0, T], \\ \vec{U}(0) = \vec{U}_0, & \vec{U}'(0) = \vec{U}_1. \end{cases} \tag{11.31}$$

If $x_0 = (0,0,x_{03})$, *for some* $x_{03} \in \mathbb{R}$, *then there exists a minimal time* $T_0 > 0$ *and a constant* $C > 0$ *such that*

$$(T - T_0)E_0 \leq C|||\{\vec{U}_0, \vec{U}_1\}|||^2, \tag{11.32}$$

where, as classically, E_0 *denotes the energy of* $\vec{U}$ *at* $t = 0$, *and we define*

$$|||\{\vec{U}_0, \vec{U}_1\}|||^2 = \sum_{k\in \mathcal{D}^+} \int_{\Sigma_{1k}} ||T^{(1k)}\vec{u}||^2 \,\mathrm{d}\sigma\,\mathrm{d}t + \sum_{k\in \mathcal{N}^+} \int_{\Sigma_{2k}} ||\mathrm{D}_t\vec{u}||^2 \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+ \sum_{k\in \mathcal{N}^-} \int_{\Sigma_{2k}} \left\{ \lambda(\operatorname{div}\vec{u})^2 + 2\mu\sum_{i,j=1}^{3}(\varepsilon_{ij}(u))^2 \right\} \mathrm{d}\sigma\,\mathrm{d}t$$
$$+ \sum_{k\in \mathrm{d}^+} \int_{\sigma_k} (\gamma_k'\Delta\xi)^2 \,\mathrm{d}\sigma'\,\mathrm{d}t.$$

As previously, we set

$$\mathcal{D}^+ = \{k \in \{1,2\} : m\cdot \vec{\nu}^{(1k)} > 0 \text{ on } \Gamma_{1k}\},$$
$$\mathcal{N}^+ \text{ (resp. } \mathcal{N}^-) = \{k \in \{1,2,\ldots,5\} : m\cdot\vec{\nu}^{(2k)} > 0 \text{ (resp. } < 0) \text{ on } \Gamma_{2k}\},$$
$$\Sigma_{ik} = \Gamma_{ik} \times (0,T).$$

The boundary of ω *is split up as follows:*

$$\partial\omega = \bigcup_{k=1}^{4} \overline{\delta}_k,$$

where δ_k *is an open linear segment. We also denote*

$$\sigma_k = \delta_k \times (0,T) \quad \text{and} \quad \mathrm{d}^+ = \{k \in \{1,2,3,4\} : \gamma_+ m\cdot\vec{\nu}' > 0 \text{ on } \delta_k\}.$$

PROOF. The proof is almost identical with those of Proposition 4.8 of [21] or Theorem 8.3 above, i.e., by integrating (11.30) applied to $\vec{U}(t)$ over $(0,T)$ and adding it with an identity similar to (8.11). The only difference is the following one: we need to show that there exists a positive constant C_1 and a real number C_2 such that

$$\int_0^T \left\{ -\frac{1}{2}\, a_\Omega(\vec{u}(t),\vec{u}(t)) + b_\omega(\xi(t),\xi(t)) \right\} \mathrm{d}t + \int_{\Omega\times(0,T)} \frac{3}{2}\, ||\mathrm{D}_t\vec{u}||^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\omega\times(0,T)} |\mathrm{D}_t\xi|^2 \,\mathrm{d}x\,\mathrm{d}t \geq C_1 T E_0 + C_2 \int_0^T \{||\vec{U}'(t)||_H^2 - a(\vec{U}(t),\vec{U}(t))\}\,\mathrm{d}t.$$

Using the law of conservation of energy and the definition of a, this inequality holds if we have

$$\begin{cases} 1 \geq C_1 + C_2, \\ \dfrac{1}{2} \geq C_1 - C_2. \end{cases}$$

The best choice is $C_1 = 1/4$, $C_2 = 3/4$.

The remainder of the proof is left to the reader. ∎

Fixing once and for all $x_0 = (0, 0, x_{03})$ and $T_0 > 0$ such that (11.32) holds, we can define the space F as in Paragraph 8. And Proposition 8.4 is replaced by the

PROPOSITION 11.10. *Let $\{\vec{U}_0, \vec{U}_1\} \in F$ and $\vec{F} \in L^1(0,T;V)$, then the unique solution $\vec{U} = (\vec{u}, \xi) \in C([0,T], V) \cap C^1([0,T], H)$ of*

$$\begin{cases} \vec{U}''(t) + A\vec{U}(t) = \vec{F}(t), \qquad t \in [0,T], \\ \vec{U}(0) = \vec{U}_0, \qquad \vec{U}'(0) = \vec{U}_1 \end{cases} \tag{11.33}$$

fulfils

$$T^{(1k)}\vec{u} \in (L^2(\Sigma_{1k}))^3, \qquad \forall k \in \mathcal{D}^+, \tag{11.34}$$

$$\mathrm{D}_t\vec{u} \in (L^2(\Sigma_{2k}))^3, \qquad \forall k \in \mathcal{N}^+, \tag{11.35}$$

$$\operatorname{div}\vec{u}, \varepsilon_{ij}(\vec{u}) \in L^2(\Sigma_{2k}), \qquad \forall i,\ j \in \{1,2,3\},\ \forall k \in \mathcal{N}_{\bullet}^-, \tag{11.36}$$

$$\Delta\xi \in L^2(\sigma_k), \qquad \forall k \in \mathrm{d}^+. \tag{11.37}$$

Moreover, their norms depend continuously on the sum of the norm of $\{\vec{U}_0, \vec{U}_1\}$ in F with the norm of $\vec{F}$ in $L^1(0,T;V)$.

PROOF. There is no difference with the proof of Theorem 5.6 of [10] since we use the following lucky feature: if $\vec{U} = (\vec{u}, \xi) \in \mathrm{D}_A$, then $\vec{u}$ admits the decomposition (6.7), but the singular part has a support included into D_{δ_0}, for some $\delta_0 > 0$. Therefore, the trace of $\vec{u}$ (respectively of its derivatives) on $\Gamma_1 \cup \Gamma_2$ is equal to the trace of its regular part $\vec{u}_r$ (respectively of the derivatives of $\vec{u}_r$). ∎

12. - Exact controllability

At this step, there is no difficulty to follow Paragraphs 9 and 10 and to obtain the exact controllability of our coupled problem. The only point we have to explain is a trace result. To give it, we shall suppose that δ_2 and δ_4

correspond to the part of the boundary of ω such that $x_3 = -1$ and 1 respectively and we set

$$\delta_{k+} \text{ (resp. } \delta_{k-}) = \{(x_1, x_3) \in \delta_k : x_1 > 1 \text{ (resp. } x_1 < 1)\},$$

$$\sigma_{k\pm} = \delta_{k\pm} \times (0,T), \qquad \text{for } k = 2 \text{ or } 4.$$

THEOREM 12.1. *Let $\vec{v}^{(1k)} \in (\mathcal{D}(\Sigma_{1k}))^3$, $k \in \mathcal{D}^+$, $\vec{v}^{(2k)} \in (\mathcal{D}(\Sigma_{2k}))^3$, $k \in \mathcal{N}^+$, $v_{ij}^{(2k)} \in \mathcal{D}(\Sigma_{2k})$, i, $j \in \{1,2,3\}$, $k \in \mathcal{N}^-$ and $v^{(k)} \in \mathcal{D}(\sigma_k)$, $k \in \mathrm{d}^+$, fulfilling if $k = 2$ or 4, $v^{(k)} \in \mathcal{D}(\sigma_{k+}) \cap \mathcal{D}(\sigma_{k-})$. Then there exist $\vec{w} \in (\mathcal{D}(0,T,\mathcal{D}(\overline{\Omega})))^3$ and $\psi \in \mathcal{D}(0,T,D(\overline{\omega}))$ satisfying (12.1) to (12.6) below:*

$$\gamma_+ w_2 = \gamma_- w_2 = \psi \qquad \text{on } \Gamma, \tag{12.1}$$

$$\gamma_+ w_\alpha = \gamma_- w_\alpha = 0 \qquad \text{on } \Gamma,\ \alpha = 1,3, \tag{12.2}$$

$$\gamma_{1k}\vec{w} = \begin{cases} \vec{v}^{(1k)} & \text{on } \Sigma_{1k},\ \forall k \in \mathcal{D}^+, \\ 0 & \text{on } \Sigma_{1k},\ \text{else}, \end{cases} \tag{12.3}$$

$$T_l^{(2k)}(\vec{w}) = \begin{cases} \mathrm{D}_t v_l^{(2k)} & \text{on } \Sigma_{2k},\ \forall k \in \mathcal{N}^+, \\ \displaystyle\sum_{m=1}^{2} \sum_{i,j=1}^{3} \mathrm{d}_{ijlm}^{2k} \frac{\partial v_{ij}^{(2k)}}{\partial \tau_m^{(2k)}} & \text{on } \Sigma_{2k},\ \forall k \in \mathcal{N}^-, \\ 0 & \text{on } \Sigma_{2k},\ \textit{else}, \end{cases} \tag{12.4}$$

for all $l \in \{1,2,3\}$,

$$\gamma_k' \psi = 0 \qquad \text{on } \sigma_k,\ \forall k = 1,\dots,4, \tag{12.5}$$

$$\gamma_k' \frac{\partial \psi}{\partial \nu'} = \begin{cases} v^{(k)} & \text{on } \sigma_k,\ \forall k \in \mathrm{d}^+, \\ 0 & \text{on } \sigma_k,\ \textit{else}. \end{cases} \tag{12.6}$$

Moreover, $\vec{w}$ and ψ are equal to zero in a neighbourhood of the bottom of the crack of Ω.

PROOF. It is proved locally as Theorem 9.3. Let us only build two functions $\vec{w}$ and ψ fulfilling (12.1), (12.2), the exact boundary conditions on $\Gamma_{11} := \{x \in \Gamma_1 : x_3 = -1\}$ and σ_2, and homogeneous boundary conditions on the other faces. We claim (and let check to the reader) that these functions are

$$\vec{w}(x,t) = \{\vec{v}^{(11)}(x_1,x_2,t) - (0, v^{(2)}(t,x_1), 0) \cdot (x_3+1)\}\eta(x_3+1),$$

$$\psi(x_1,x_3,t) = -v^{(2)}(t,x_1)(x_3+1) \cdot \eta(x_3+1),$$

with the convention that $\vec{v}^{(11)} = 0$ if $1 \notin \mathcal{D}^+$ and $v^{(2)} = 0$ if $2 \notin \mathrm{d}^+$, and η is an appropriate cut-off function. $\vec{w}$ and ψ have the nullity property of the theorem since $\vec{v}^{(11)}$ and $v^{(2)}$ are equal to zero in a neighbourhood of the bottom of the crack. Moreover, the assumption $v^{(2)} \in \mathcal{D}(\sigma_{2+}) \cap \mathcal{D}(\sigma_{2-})$ implies that $\vec{u}$ is equal to zero in a neighbourhood of the face $\Sigma_{21} := \{x \in \Sigma_2 : x_1 = 1\}$.

Using the symmetry between σ_2 and σ_4 and the arguments of Theorem 9.3, we obtain the result. ∎

We can now give the weak formulation of the wave equation (11.33).

THEOREM 12.2. *For all $\vec{V}_0 \in H$, $\vec{V}_1 \in V'$, $\vec{v}^{(1k)} \in (L^2(\Sigma_{1k}))^3$, $k \in \mathcal{D}^+$; $\vec{v}^{(2k)} \in (L^2(\Sigma_{2k}))^3$, $k \in \mathcal{N}^+$; $v_{ij}^{(2k)} \in L^2(\Sigma_{2k})$, $i, j \in \{1,2,3\}$, $k \in \mathcal{N}^-$; and $v^{(k)} \in L^2(\sigma_k)$, $k \in \mathrm{d}^+$; there exist unique $\vec{V} \in L^\infty(0,T,V')$, $\{\vec{\psi}_1, \vec{\psi}_0\} \in F'$, which are solutions of*

$$(12.7)\qquad \begin{aligned} &\int_0^T \langle \vec{V}(t), \vec{F}(t)\rangle_{V'-V}\,\mathrm{d}t + \langle\{\vec{\psi}_1, \vec{\psi}_0\}, \{\vec{U}_0, -\vec{U}_1\}\rangle_{F'-F} \\ &= \langle \vec{V}_1, \vec{U}(0)\rangle - \langle \vec{V}_0, \vec{U}'(0)\rangle \\ &\quad - \sum_{k\in\mathcal{D}^+} \int_{\Sigma_{1k}} (\vec{v}^{(1k)}, T^{(1k)}\vec{u})\,\mathrm{d}\sigma\,\mathrm{d}t - \sum_{k\in\mathcal{N}^+} \int_{\Sigma_{2k}} (\vec{v}^{(2k)}, \gamma_{2k}\,\mathrm{D}_t\vec{u})\,\mathrm{d}\sigma\,\mathrm{d}t \\ &\quad - \sum_{k\in\mathcal{N}^-} \int_{\Sigma_{2k}} \sum_{i,j=1}^{3} v_{ij}^{(2k)}\gamma_{2k}\varepsilon_{ij}(\vec{u})\,\mathrm{d}\sigma\,\mathrm{d}t - \sum_{k\in\mathrm{d}^+} \int_{\sigma_k} v^{(k)}\gamma_k'\Delta\xi\,\mathrm{d}\sigma'\,\mathrm{d}t, \end{aligned}$$

for all $\vec{F} \in L^1(0,T;V)$, $\{\vec{U}_0, -\vec{U}_1\} \in F$, where $\vec{U} = (\vec{u}, \xi)$ is the unique solution of

$$(12.8)\qquad \begin{cases} \vec{U} \in C([0,T],V) \cap C^1([0,T],H), \\ \vec{U}'' + A\vec{U} = \vec{F}(t), \qquad t \in [0,T], \\ \vec{U}(T) = \vec{U}_0, \qquad \vec{U}'(T) = \vec{U}_1. \end{cases}$$

Moreover, $\vec{V} \in C([0,T],V') \cap C^1([0,T],\mathrm{D}_A')$ and satisfies

$$(12.9)\qquad \vec{V}(T) = \vec{\psi}_0, \qquad \vec{V}'(T) = \vec{\psi}_1.$$

PROOF. Direct consequence of Proposition 11.10 and Theorem 12.1, arguing as in Paragraph 5 of [21]. Let us remark that the nullity property in Theorem 12.1 for the function $\vec{w}$ is made in order to be able to apply the Green identity (9.17) to the pair $(\vec{w}(t), \vec{u}(t))$, when $\vec{U} = (\vec{u}, \xi) \in C([0,T],\mathrm{D}_A)$ (see Theorem 9.4). ∎

Formally, the solution $\vec{V}$ of (12.7) is equal to $(\vec{v}, \eta)$ and satisfies

$$(12.10)\qquad \vec{v}'' - L\vec{v} = \vec{0} \qquad \text{in } \Omega \times (0,T),$$

$$(12.11)\qquad \eta'' + \rho\Delta^2\eta + \chi_\Gamma(\gamma_-\sigma_{22}(\vec{v}) - \gamma_+\sigma_{22}(\vec{v})) = 0 \qquad \text{in } \omega \times (0,T),$$

the transmission and boundary conditions (12.1) to (12.6), the initial conditions

$$\vec{V}(0) = \vec{V}_0, \qquad \vec{V}'(0) = \vec{V}_1, \tag{12.12}$$

and the final condition (12.9).

Finally, applying the Hilbert Uniqueness Method of J.-L. Lions [15], we can conclude the exact controllability of our coupled problem:

THEOREM 12.3. *For all* $\vec{V}_0 \in H$, $\vec{V}_1 \in V'$, *there exist* $\vec{v}^{(1k)} \in (L^2(\Sigma_{1k}))^3$, $k \in \mathcal{D}^+$; $\vec{v}^{(2k)} \in L^2(\Sigma_{2k})$, $k \in \mathcal{N}^+$; $v_{ij}^{(2k)} \in L^2(\Sigma_{2k})$, $i, j \in \{1,2,3\}$, $k \in \mathcal{N}^-$ *and* $v^{(k)} \in L^2(\sigma_k)$, $k \in \mathrm{d}^+$, *such that the solution* $\vec{V}$ *of* (12.7) *fulfils*

$$\vec{V}(T) = \vec{V}'(T) = \vec{0}. \tag{12.13}$$

In Theorem 12.3, we establish the exact controllability for controls with support in Σ_{1k}, $k \in \mathcal{D}^+$, Σ_{2k}, $k \in \mathcal{N}^+ \cup \mathcal{N}^-$, and σ_k, $k \in \mathrm{d}^+$. From the mechanical point of view, it is interesting to notice that we get controls having their supports only concentrated on the external boundary of Ω, i.e., no control on the bottom of the crack Γ_0.

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