Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

V. Benci
D. Fortunato
F. Giannoni

On the existence of geodesics in static Lorentz manifolds with singular boundary

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 19, nº 2 (1992), p. 255-289
<http://www.numdam.org/item?id=ASNSP_1992_4_19_2_255_0>

On the Existence of Geodesics in Static Lorentz Manifolds with Singular Boundary[(1)]

V. BENCI - D. FORTUNATO - F. GIANNONI

1. - Introduction and statements of the results

In this paper we study some global geometric properties of certain Lorentz structures. More precisely we prove existence and multiplicity results about geodesics joining two given points in Lorentz manifolds having a singular boundary. We require that these geodesics do not touch the boundary.

Some particular solutions of the Einstein equations (for instance the Schwarzschild spacetime, see e.g. [9, page 149]), and of the Einstein-Maxwell equations (for instance the Reissner-Nordström spacetime, see e.g. [9, page 156]) are examples of those Lorentz structures which we consider.

Before stating the definitions of the geometrical structures, we need to recall some basic notions which can be found for example in [14]. A pseudo-Riemannian manifold is a smooth manifold $\mathcal{G}$ on which a nondegenerate $(0,2)$-tensor $g(z)[\cdot,\cdot]$ $(z \in \mathcal{G})$ is defined. This tensor is called metric tensor. If g is positive definite then $\mathcal{G}$ is a Riemannian manifold. A Lorentz manifold $\mathcal{L}$ is a pseudo-Riemannian manifold with the metric tensor g having index 1 (i.e. every matrix representation of g has exactly one negative eigenvalue). If a Lorentz manifold has dimension 4, it is called "spacetime". If no ambiguity can occur, we denote by $\langle\,,\,\rangle_R$ the metric on a Riemannian manifold and by $\langle\,,\,\rangle_L$ the metric on a Lorentz manifold.

We recall that a geodesic on a Lorentz manifold $\mathcal{L}$ is a curve

$$\gamma : [a,b] \to \mathcal{L} \text{ solving } D_s\dot{\gamma}(s) = 0 \text{ for all } s,$$

where $a, b \in \mathbb{R}$, $\dot{\gamma}(s)$ is the derivative of $\gamma(s)$, and $D_s\dot{\gamma}(s)$ is the covariant derivative of $\dot{\gamma}(s)$ with respect to the metric tensor g.

It is well known that a geodesic γ is a critical point of the "energy"

(1) Work supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40%-60%, 1989).

Pervenuto alla Redazione il 20 Marzo 1991 e in forma definitiva il 15 Ottobre 1991.

functional

$$(1.1) \qquad f(\gamma) = \int_a^b g(\gamma(s))[\dot\gamma(s), \dot\gamma(s)]\mathrm{d}s = \int_a^b \langle \dot\gamma(s), \dot\gamma(s)\rangle_L \,\mathrm{d}s.$$

If γ is a geodesic on $\mathcal{L}$ there exists a constant $E_\gamma \in \mathbb{R}$ such that

$$(1.2) \qquad E_\gamma = g(\gamma(s))[\dot\gamma(s), \dot\gamma(s)] \text{ for all } s.$$

A geodesic γ is called *space-like*, *null* or *time-like* if E_γ is respectively greater, equal or less then zero. A time-like geodesic is physically interpreted as the world line of a material particle under the action of a gravitational field, while a null geodesic is the world line of a light ray. Space-like geodesics have less physical relevance, however they are useful to the study of the geometrical properties of a Lorentz manifold.

Now we shall give some definitions.

DEFINITION 1.1. Let $(\mathcal{L}, \langle\ ,\ \rangle_L)$ be a Lorentz manifold. $\mathcal{L}$ is called *(standard) static* Lorentz manifold if:
there exist a Riemannian manifold $\mathcal{M}_0$ of class C^2 with metric $h(x)[\cdot,\cdot]$ of class C^2 and a scalar field $\beta \in C^2(\mathcal{M}_0,]0,+\infty[)$ such that $(\mathcal{L}, \langle\ ,\ \rangle_L)$ is isometric to $\mathcal{M}_0 \times \mathbb{R}$ equipped with the Lorentz metric $g(z)[\cdot,\cdot]$(2), defined by

$$(1.3) \qquad g(z)[\varsigma,\varsigma] = h(x)[\xi,\xi] - \beta(x)\tau^2,$$

where $z = (x,t) \in \mathcal{M}_0 \times \mathbb{R}$, $\varsigma = (\xi,\tau) \in T_z(\mathcal{M}_0 \times \mathbb{R}) = T_x(\mathcal{M}_0) \times \mathbb{R}$(3).

We shall identify $\mathcal{L}$ with $\mathcal{M}_0 \times \mathbb{R}$ and we shall write $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$. If $z \in \mathcal{L}$ we set $z = (x,t)$ with $x \in \mathcal{M}_0$ and $t \in \mathbb{R}$: x and t are called static coordinates of z. We refer to [12, page 328] for the physical interpretation of a static spacetime.

In a previous paper (see [5]), we have studied the existence and the multiplicity of geodesics in static Lorentz manifolds under the assumptions:

(i) *the Riemannian manifold* $(\mathcal{M}_0, h)$ *is complete,*

(ii) *there exist* $N, \nu > 0$ *such that* $N \geq \beta(x) \geq \nu$ *for all* $x \in \mathcal{M}_0$.

However in many physically relevant cases assumptions (i) and (or) (ii) are not satisfied.

Consider for example the solution of the Einstein equations corresponding to the exterior gravitational field produced by a static spherically symmetric

(2) This means that there is a diffeomorphism $\Psi: \mathcal{M}_0\times\mathbb{R} \to \mathcal{L}$ such that $g(z)[\varsigma,\varsigma] = \langle \mathrm{d}\Psi(z)\varsigma, \mathrm{d}\Psi(z)\varsigma\rangle_L$, $\mathrm{d}\Psi$ denoting the differential map.

(3) Here $T_z(\mathcal{M}_0\times\mathbb{R})$ denotes the tangent space to $\mathcal{M}_0\times\mathbb{R}$ at z, $T_x(\mathcal{M}_0)$ denotes the tangent space to $\mathcal{M}_0$ at x and $\mathbb{R}$ is identified with its tangent space.

massive body. This solution, called Schwarzschild metric, can be written (using polar coordinates) in the form:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 - c^2 \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2, \tag{1.4}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \cdot \mathrm{d}\varphi^2$ is the standard metric of the unit 2-sphere in the Euclidean 3-space, $m = GM/c^2$, G is the universal gravitation constant, M is the mass of the body and c is the speed of the light.

The Schwarzschild spacetime is the Lorentz manifold

$$\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}, \quad \mathcal{M}_0 = \{(r, \vartheta, \varphi) : r > 2m\}$$

equipped with the metric (1.4). Notice that the Riemannian metric

$$\mathrm{d}x^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

has no meaning on the region

$$\partial \mathcal{M}_0 = \{(r, \vartheta, \varphi) : r = 2m\}.$$

Moreover it is easy to see that the radial geodesics $(r(s), \vartheta_0, \varphi_0)$ $(\vartheta_0, \varphi_0 \in \mathbb{R})$ on $\mathcal{M}_0$ with respect to the Riemannian metric $\mathrm{d}x^2$ "reach" the region $\{(r, \vartheta, \varphi) : r = 2m\}$ within a finite value of the parameter s. Therefore $\mathcal{M}_0$ with the metric $\mathrm{d}x^2$ is not complete. Moreover

$$\beta(r) = c^2 \left(1 - \frac{2m}{r}\right) \to 0 \text{ as } r \to 2m.$$

Then both conditions (i) and (ii) are not satisfied by the Schwarzschild spacetime.

The metric (1.4) is singular on $\partial \mathcal{M}_0 \times \mathbb{R}$ in the sense that it cannot be smoothly extended on $\partial \mathcal{M}_0 \times \mathbb{R}$. However the singularity is not intrinsic, but it is a consequence of the choice of the static coordinates.

In fact if we denote by $(\mathcal{K}, g)$ the Kruskal spacetime (which is the maximal analytical extension of the Schwarzschild spacetime, cf. [11] or [9, pp. 153-155]), there is an injective isometry

$$\Psi : \mathcal{M}_0 \times \mathbb{R} \to \mathcal{K},$$

and g is not singular on $\partial(\Psi(\mathcal{M}_0 \times \mathbb{R}))$. However $\partial(\Psi(\mathcal{M}_0 \times \mathbb{R}))$ is not a smooth 3-manifold. In this way the singularity has been "transferred" from the metric to the geometry of the boundary, and this justifies the title of this paper.

We recall that (1.4) solves the Einstein equations for $r > r_M$, r_M being the radius of the body responsible for the gravitational curvature. Then, it is physically meaningful to equip all $\mathcal{M}_0 \times \mathbb{R}$ with the metric (1.4) only if

$r_M \leq 2m$. In this case the matter of the body is "contained" within the event horizon $\{r = 2m\}$ and we are in the presence of a universe with a black hole.

The name is justified by the fact that a light ray cannot leave the region $\{r \leq 2m\}$. If an astronaut "falls" in the black hole, he spends a finite "proper" time, but an observer far from the black hole does not see the astronaut to fall in it in a finite time. More precisely any time-like geodesic in the Schwarzschild spacetime can reach the region $\{r = 2m\}$ only if the time coordinate t goes to $\pm\infty$ (see the appendix).

Having in mind, as model, the Schwarzschild spacetime we are led to introduce the following definitions:

DEFINITION 1.2. Let U be an open connected subset of a manifold $\mathcal{M}$ and let ∂U be its topological boundary. U is said to be a *static universe* if

(i) $U = \mathcal{M}_0 \times \mathbb{R}$ is a static Lorentz manifold (see Definition 1.1);

(ii) $\sup_{\mathcal{M}_0} \beta < +\infty$, where β is the function in (1.3);

(iii) $\lim_{k\to+\infty} \beta(x_k) = 0$, for any $(x_k, t_k) \underset{k}{\to} z \in \partial U$;

(iv) for every $\delta > 0$ the set $\{x \in \mathcal{M}_0 : \beta(x) \geq \delta\}$ is complete (with respect to the Riemannian structure of $\mathcal{M}_0$);

(v) for every time-like geodesic $\gamma(s) = (x(s), t(s))$ in U such that $\liminf_{s\to s_0^-} \beta(x(s)) = 0$, we have $\limsup_{s\to s_0^-} |t(s)| = +\infty$.

REMARK 1.3. Condition (v) says that if a material particle reaches the topological boundary of U, an observer far from the boundary (Schwarzschild observer) does not see this event in a finite time, since his proper time is a reparametrization of the universal time t. This condition justifies the name of the structure introduced in Definition 1.2.

Notice that in general (v) does not follow from (iii). In fact consider the Lorentz manifold

$$\{x \in \mathbb{R} : x > 1\} \times \mathbb{R} \text{ with metric } \mathrm{d}s^2 = \mathrm{d}x^2 \beta(x) \mathrm{d}t^2,$$

$$\text{where } \beta \text{ is bounded and } \beta(x) = x - 1 \text{ if } x \leq 2.$$

A straightforward calculation shows that it does not satisfy (v).

In the Appendix we verify that the Schwarzschild spacetime satisfy (v) of Definition 1.2. Then, clearly, it is a static universe.

Another example of static universe is given by the Reissner-Nordström spacetime

$$\left\{r > m + \sqrt{m^2 - e^2}\right\} \times \mathbb{R}$$

when $m^2 > e^2$. Here m represents the gravitational mass and e the electric charge of the body responsible of the gravitational curvature. This can be seen by the same computations used for the Schwarzschild spacetime (see the Appendix).

Whenever U is a static universe we have the following results about the existence of time-like geodesics joining two given events.

THEOREM 1.4. *Let* $U = \mathcal{M}_0 \times \mathbb{R}$ *be a static universe (see Definition* 1.2*). Let* $z_0 = (x_0, t_0)$ *and* $z_1 = (x_1, t_1)$ *be events in* U*. There exists a time-like geodesic* γ *in* U *such that* $\gamma(0) = z_0$ *and* $\gamma(1) = z_1$ *if and only if*

$$(1.5)\qquad \begin{gathered} \exists\ x \in C^1([0,1], \mathcal{M}_0) : x(0) = x_0,\ \ x(1) = x_1 \ \text{and} \\ \left[\int_0^1 \frac{1}{\beta(x(s))}\,\mathrm{d}s\right] \cdot \left[\int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle_R \,\mathrm{d}s\right] < (t_1 - t_0)^2. \end{gathered}$$

REMARK 1.5. When we fix x_0 and x_1, the condition (1.5) is certainly satisfied if $|t_1 - t_0|$ is large enough, while it does not hold whenever $|t_1 - t_0|$ is small.

REMARK. Condition (ii) of Definition 1.2 is essential to obtain our existence results. In fact the Anti-de Sitter space (see e.g. [9,16]) furnishes counterexamples to the existence of geodesics between two given events. However if $\beta(x)$ goes to $+\infty$ with a mild rate as x goes to ∞, Theorems 1.4 and 1.6 still hold.

Now let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold and (x_0, t_0), (x_1, t_1) two events in $\mathcal{L}$. If $(x(s), t(s))$ is a geodesic joining (x_0, t_0) and (x_1, t_1), since the metric tensor is independent of t, $(x(s), t(s) + \tau)$ is a geodesic joining $(x_0, t_0 + \tau)$ and $(x_1, t_1 + \tau)$. Then the number of geodesics in $\mathcal{L}$ joining two events (x_0, t_0) and (x_1, t_1) depends only on x_0, x_1 and $|t_1 - t_0|$.

We denote by $N(x_0, x_1, |t_1 - t_0|)$ the number of time-like geodesics in U joining (x_0, t_0) and (x_1, t_1). If U has a non-trivial topology we get the following multiplicity result of geodesics joining z_0 and z_1.

THEOREM 1.6. *Let* $U = \mathcal{M}_0 \times \mathbb{R}$ *be a static universe and* $(\mathcal{M}_0, h)$ *a* C^3*-Riemannian manifold which is not contractible in itself. Moreover assume that* (1.5) *holds.*

Then

$$(1.6)\qquad \lim_{|t_1 - t_0| \to +\infty} N(x_0, x_1, |t_1 - t_0|) = +\infty.$$

About other existence results for time-like geodesics joining two given points in Lorentz manifolds we refer to [1,18,19], where the Lorentz manifolds are assumed to be globally hyperbolic.

In this paper we deal also with the problem of the geodesical connectivity for a Lorentz manifold. We recall that

A Lorentz manifold $\mathcal{L}$ *is called geodesically connected if for every* $z_0, z_1 \in \mathcal{L}$ *there exists a geodesic* $\gamma : [0,1] \to \mathcal{L}$ *such that* $\gamma(0) = z_0$ *and* $\gamma(1) = z_1$.

Clearly, for studying the geodesical connectivity it is necessary to consider also space-like geodesics which are more difficult to deal with. The geodesical connectivity has not been treated in the works [1,18,19], which deal only with time-like and null geodesics. This problem has been faced in [3,4] for stationary complete Lorentz manifolds(4) without boundary. Here we consider the case of static Lorentz manifolds with singular boundary, in order to cover the case of the Schwarzschild spacetime.

For the study of the geodesical connectivity the condition of being a static universe (see Definition 1.2) is not appropriate. Indeed consider the Lorentz manifold

$$\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\}\times\mathbb{R} \tag{1.7}$$

$$\text{with metric } ds^2 = dx^2+dy^2-\beta(x,y)dt^2,$$

where β is bounded and $\beta(x,y)=\left(\sqrt{x^2+y^2}-1\right)^2$ if $\sqrt{x^2+y^2}\le 2$.

Simple calculations show that (1.7) is a static universe while it is not geodesically connected (the events of the type (x_1,x_2,t_0) and $(-x_1,-x_2,t_0)$ cannot be joined by geodesics lying in the Lorentz manifold (1.7)). For this reason we introduce the following geometrical condition:

DEFINITION 1.7. Let $\mathcal{L}$ be an open connected subset of a manifold $\mathcal{M}$ and $\partial\mathcal{L}$ its topological boundary. $\mathcal{L}$ is said to be a *static Lorentz manifold with convex boundary* if

(i) $\mathcal{L}=\mathcal{M}_0\times\mathbb{R}$ is a static Lorentz manifold (see Definition 1.1));

(ii) $\sup\limits_{\mathcal{M}_0}\beta<+\infty$, where β is the function in (1.3);

(iii) there exists $\phi\in C^2(\mathcal{L},\mathbb{R}^+\setminus\{0\})$ such that

$$\lim_{(x,t)\to z\in\partial\mathcal{L}}\phi(x,t)=0 \text{ and}$$

$$\phi(x,t)=\phi(x,0)\equiv\phi(x)\quad\forall(x,t)\in\mathcal{L};$$

(iv) for every $\eta>0$ the set $\{x\in\mathcal{M}_0:\phi(x)\ge\eta\}$ is complete (with respect to the Riemannian structure of $\mathcal{M}_0$);

(v) there exist $N,M,\nu,\delta\in\mathbb{R}^+\setminus\{0\}$ such that the function ϕ of (iii) satisfies:

(4) I.e. with the metric tensor not depending by the time variable.

(1.8) $$z \in \mathcal{L},\ \phi(z) < \delta \Rightarrow N \geq \langle \nabla_L \phi(z), \nabla_L \phi(z) \rangle_L \geq \nu^{(5)},$$

(1.9) $$z \in \mathcal{L},\ \phi(z) < \delta \Rightarrow H_L^\phi(z)[v,v] \leq M \cdot |\langle v, v \rangle_L| \cdot \phi(z) \quad \forall v \in T_z(\mathcal{L})^{(6)}.$$

In the appendix we prove that the Schwarzschild spacetime is a static Lorentz manifold with convex boundary using the function ϕ given by

$$\phi(r, \vartheta, \varphi, t) = \sqrt{1 - \frac{2m}{r}}.$$

Also the Reissner-Nordström spacetime $\left\{r > m + \sqrt{m^2 - e^2}\right\} \times \mathbb{R}$ is a static Lorentz manifold with convex boundary provided that $m^2 > \frac{9}{5} \cdot e^2$, as we have proved in the appendix using the function

$$\phi(r, \vartheta, \varphi, t) = \sqrt{1 - \frac{2m}{r} + \frac{e^2}{r^2}}.$$

Definition 1.7 allows us to obtain the following result:

THEOREM 1.8. *A static Lorentz manifold with convex boundary is geodesically connected.*

REMARK. Notice that ϕ becomes zero on $\partial\mathcal{L}$, so (1.9) implies

(1.10) $$\limsup_{z \to z_0 \in \partial \mathcal{M}} H_L^\phi(z)[v,v] \leq 0 \quad \text{for all } v \text{ such that } |\langle v, v \rangle_L| \leq 1.$$

However, in order to get the geodesic connectivity of $\mathcal{L}$, it seems we need a control of the rate for which the limit in (1.10) is achieved. The assumption (1.9) provides this control.

Whenever the topology of $\mathcal{L}$ is not trivial we get the following multiplicity result about space-like geodesics. This result has been proved in [4] in the case of stationary Lorentz manifolds $\mathcal{M}_0 \times \mathbb{R}$ with $\mathcal{M}_0$ compact.

THEOREM 1.9. *Let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold with convex boundary, and $(\mathcal{M}_0, h)$ a C^3 Riemannian manifold which is not contractible.*

(5) $\nabla_L \phi(z)$ denotes the gradient of the function ϕ with respect to the Lorentz structure, i.e. it is the unique vector field $F(z)$ on $\mathcal{L}$ such that $\langle F(z), v \rangle_L = d\phi(z)v\ \forall v \in T_z(\mathcal{M})$.

(6) $H_L^\phi(z)[v,v]$ denotes the Hessian of the function ϕ at z in the direction v, i.e. $\frac{d^2}{ds^2}(\phi(\gamma(s)))_{|s=0}$ where γ is a geodesic in $\mathcal{L}$ such that $\gamma(0)=z$ and $\dot\gamma(0)=v$.

Then, for every $z_0, z_1 \in \mathcal{L}$, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of space-like geodesics in $\mathcal{L}$ joining z_0 and z_1 such that

$$\lim_{n\to+\infty} E_{\gamma_n} = +\infty.$$

REMARK 1.10. Theorems 1.4 and 1.6 hold even for a static Lorentz manifold with convex boundary, while Theorems 1.8 and 1.9 in general do not hold for a static universe, as we can see using the Lorentz manifold (1.7).

The results proved in this paper have been announced in [6].

2. - Technical preliminaries

Let $\mathcal{L}$ be a static Lorentz manifold. Then (see Definition 1.1) $\mathcal{L}$ is isometric to $\mathcal{M}_0 \times \mathbb{R}$ equipped with the warped product

$$g(z)[\varsigma,\varsigma] = h(x)[\xi,\xi] - \beta(x)\tau^2, \tag{2.1}$$

where $z = (x,t)$, $\varsigma = (\xi,\tau) \in (T_x\mathcal{M}_0) \times \mathbb{R}$.

In the following we set for simplicity

$$\langle \xi,\xi\rangle_R = h(x)[\xi,\xi]$$

and

$$\langle \varsigma,\varsigma\rangle_L = g(z)[\varsigma,\varsigma].$$

Let $z_0 = (x_0,t_0)$, $z_1 = (x_1,t_1)$ be two events in $\mathcal{M}_0 \times \mathbb{R}$. We put

$$W^{1,2}([0,1],\mathcal{M}_0)$$
$$= \left\{ x : [0,1] \to \mathcal{M}_0,\ x \text{ absolutely continuous},\ \int_0^1 \langle \dot{x},\dot{x}\rangle_R < +\infty \right\},$$

and

$$\Omega^1 \equiv \Omega^1(\mathcal{M}_0,x_0,x_1) = \left\{ x \in W^{1,2}([0,1],\mathcal{M}_0) : x(0) = x_0,\ x(1) = x_1 \right\}. \tag{2.2}$$

Ω^1 is a Hilbert manifold (see e.g. [10,17]) and its tangent space at $x \in \Omega^1$ is given by

$$T_x(\Omega^1)$$
$$= \left\{ \xi \in W^{1,2}([0,1],T\mathcal{M}_0) \middle| \xi(0) = \xi(1) = 0,\ \xi(s) \in T_{x(s)}\mathcal{M}_0 \text{ for all } s \in [0,1] \right\}.$$

Here $T\mathcal{M}_0$ is the tangent bundle of $\mathcal{M}_0$ while $W^{1,2}([0,1], T\mathcal{M}_0)$ is the set of the absolutely continuous curves $\xi : [0,1] \to T\mathcal{M}_0$, such that

$$\langle \xi, \xi \rangle_1 = \int_0^1 \langle \mathrm{D}_s\xi(s), \mathrm{D}_s\xi(s)\rangle_R < +\infty,$$

where D_s is the covariant derivative with respect to the Riemannian structure. Notice that $\langle \cdot, \cdot \rangle_1$ is the Riemannian structure of Ω^1 inherited by that of $\mathcal{M}_0$.

We denote by $C([0,1], \mathcal{M}_0)$ the space of the continuous curves $x : [0,1] \to \mathcal{M}_0$ endowed with the metric

$$\mathrm{d}_\infty(x, x') = \sup_{s \in [0,1]} \mathrm{d}\left(x(s), x'(s)\right), \tag{2.3}$$

where d is the distance derived from the Riemannian metric on $\mathcal{M}_0$. Consider now the Riemannian manifold

$$Z = \Omega^1 \times \left\{ t \in W^{1,2}([0,1], \mathbb{R}) \middle| t(0) = t_0,\ \ t(1) = t_1 \right\}. \tag{2.4}$$

It is easy to see that the "energy" functional

$$f(z) = \int_0^1 \langle \dot{z}(s), \dot{z}(s)\rangle_L \, \mathrm{d}s \tag{2.5}$$

is C^1 on Z. The geodesics on $\mathcal{L}$ joining z_0, z_1 are the critical points of f on Z, namely $\gamma \in Z$ is a geodesic if and only if, for all

$$\begin{gathered} \varsigma = (\xi, \tau) \in T_x\Omega^1 \times W_0^{1,2}([0,1], \mathbb{R}), \\ \langle f'(\gamma), \varsigma \rangle = \int_0^1 \langle \dot{\gamma}(s), \mathrm{D}_s\varsigma(s)\rangle_L \, \mathrm{d}s = 0 \end{gathered} \tag{2.6}$$

where $\mathrm{D}_s\varsigma(s)$ denotes the covariant derivative of ς in the direction $\dot{\gamma}(s)$ with respect to the metric (2.1) and

$$W_0^{1,2}([0,1], \mathbb{R}) = \left\{ \tau \in W^{1,2}([0,1], \mathbb{R}) \middle| \tau(0) = \tau(1) = 0 \right\}.$$

We are interested in studying situations in which $(\mathcal{M}_0, h)$ is not complete (see Section 1). In these cases also $C([0,1], \mathcal{M}_0)$ and Ω^1 are not complete. To overcome this lack of completeness we introduce a suitable penalization term in (2.5). More precisely we shall set

$$f_\varepsilon(z) = f(z) + \int_0^1 V_\varepsilon(z)\mathrm{d}s, \quad z = (x, t) \in Z, \tag{2.7}$$

where

$$\text{(2.8)} \qquad \begin{cases} \text{for all } \varepsilon \geq 0, \ V_\varepsilon(z) = V_\varepsilon(x,0) \equiv V_\varepsilon(x) \text{ is a non-negative } C^2 \\ \text{scalar field on } \mathcal{M} \text{ depending only on } x \in \mathcal{M}_0 \text{ and with } V_0 \equiv 0. \end{cases}$$

We shall specify the penalization function V_ε in Sections 3 and 4 where we shall prove Theorems 1.4, 1.6, 1.8 and 1.9.

Since the metric g is indefinite, the functional (2.7) is unbounded both from below and from above. Nevertheless the study of the critical points of f_ε ($\varepsilon \geq 0$) can be reduced to the study of the critical points of a suitable functional which is bounded from below when β is bounded from above.

In fact let $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1)$ be two points in $\mathcal{L}$ and consider the functional

$$\text{(2.9)} \qquad J_\varepsilon(x) = J(x) + \int_0^1 V_\varepsilon(x) \mathrm{d}s, \quad x \in \Omega^1,$$

$J(x)$ being defined by

$$J(x) = \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_R - \Delta^2 \left[\int_0^1 \frac{\mathrm{d}s}{\beta(x(s))} \right]^{-1}$$

where $\Delta = t_1 - t_0$. Observe that (2.9) is bounded from below if β is bounded from above.

The following theorem holds:

THEOREM 2.1. *Let $z(s) = (x(s), t(s)) \in Z$ (see (2.4)). Then the following statements are equivalent:*

(i) *z is a critical point of f_ε on Z;*

(ii) *x is a critical point of J_ε on Ω^1, i.e.*

$$\text{(2.10)} \qquad \begin{aligned} \langle J'(x), \xi \rangle = \int_0^1 2 \langle \dot{x}, \mathrm{D}_s \xi \rangle_R \, \mathrm{d}s - \Delta^2 \left[\int_0^1 \frac{\mathrm{d}s}{\beta(x)} \right]^{-2} \cdot \int_0^1 \frac{\langle \beta'(x), \xi \rangle_R}{\beta^2(x)} \mathrm{d}s \\ + \int_0^1 \langle V_\varepsilon'(x), \xi \rangle_R \, \mathrm{d}s = 0^{(7)} \quad \textit{for all } \xi \in T_x\Omega^1 \end{aligned}$$

(7) Here β' and V_ε' denote the Riemann gradient of β and V_ε respectively.

and $t = t(s)$ *solves the Cauchy problem*

$$(2.11) \qquad \begin{cases} \dot{t} = \Delta \left[\int_0^1 \frac{1}{\beta(x)} \, \mathrm{d}s \right]^{-1} \frac{1}{\beta(x(s))} \\ t(0) = t_0. \end{cases}$$

Moreover if (i) (*or* (ii)) *is satisfied, we have*

$$(2.12) \qquad f_\varepsilon(z) = J_\varepsilon(x).$$

In particular z *is a critical point of* f *iff* x *is a critical point of* J.

When $\varepsilon = 0$, Theorem 2.1 has been proved in [5]. Nevertheless, for the convenience of the reader, we shall give here a proof of Theorem 2.1.

PROOF OF THEOREM 2.1.

(i) $\Rightarrow$ (ii). Let $z(s) = (x(s), t(s))$ be a critical point of f_ε on Z. Then

$$f'_\varepsilon(z)\left[\begin{pmatrix}\xi\\ \tau\end{pmatrix}\right] =$$

$$(2.13) \qquad \int_0^1 \left(\langle \dot{x}, \mathrm{D}_s\xi \rangle_R \dot{t}^2 \langle \beta'(x), \xi \rangle_R - 2\beta(x)\dot{t}\dot{\tau} + \langle V'_\varepsilon(x), \xi \rangle_R \right) \mathrm{d}s = 0$$

$$\text{for all } \begin{pmatrix}\xi\\ \tau\end{pmatrix} \in T_z Z.$$

Taking $\xi = 0$ in (2.13) we get

$$(2.14) \qquad \int_0^1 \beta(x)\dot{t}\dot{\tau} \; \mathrm{d}s = 0 \qquad \text{for all } \tau \in H_0^1([0,1], \mathbb{R}),$$

then there exists a constant $K \in \mathbb{R}$ such that

$$(2.15) \qquad \dot{t}(s) = \frac{K}{\beta(x(s))} \qquad \text{for all } s \in [0,1].$$

Integrating in $[0,1]$ we get

$$(2.16) \qquad K = \Delta \left[\int_0^1 \frac{1}{\beta(x)} \, \mathrm{d}s \right]^{-1}, \quad \Delta = t_1 - t_0.$$

By (2.16) and (2.15) we deduce that $t = t(s)$ solves (2.11). Now if we substitute (2.11) in (2.13) and choose $\tau = 0$, we see that (2.10) is satisfied.

(ii) $\Rightarrow$ (i). Suppose that $x \in \Omega^1$ solves (2.10) and t solves (2.11). Obviously t solves (2.14). Now if in (2.10) we add (2.14) and substitute $\Delta^2 \left[\int_0^1 \frac{\mathrm{d}s}{\beta(x)}\right]^{-2}$ by (2.11), we see that $z = (x, t)$ satisfies (2.13), namely it is a critical point of f_ε on Z.

Finally (2.12) is immediately checked. ■

The following Lemma will be useful

LEMMA 2.2. *Let $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Z$ be a critical point of f_ε. Then there exists $K_\varepsilon \in \mathbb{R}$ s.t.*

$$K_\varepsilon = \langle \dot{z}(s), \dot{z}(s) \rangle_L - V_\varepsilon(z_\varepsilon(s)) \quad \textit{for all } s \in [0,1]. \tag{2.17}$$

Moreover

$$K_\varepsilon = J_\varepsilon(x_\varepsilon(s)) - 2 \int_0^1 V_\varepsilon(x_\varepsilon(s))\mathrm{d}s. \tag{2.18}$$

PROOF. Since $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Z$ is a critical point of f_ε we have

$$\mathrm{D}_s \dot{z}_\varepsilon(s) - \frac{1}{2} \nabla_L V_\varepsilon(z_\varepsilon(s)) = 0 \quad \text{for all } s \in [0,1], \tag{2.19}$$

where D_s and ∇_L denote respectively the covariant derivative and the gradient with respect to the Lorentz metric $\langle \cdot, \cdot \rangle_L$ defined in (2.1). From (2.19) we deduce that, for all s,

$$\langle \mathrm{D}_s \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s) \rangle_L - \frac{1}{2} \langle \nabla_L V_\varepsilon(z_\varepsilon(s)), \dot{z}_\varepsilon(s) \rangle_L = 0,$$

then, for all s,

$$\frac{\mathrm{d}}{\mathrm{d}s} \left[\langle \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s) \rangle_L - V_\varepsilon(z_\varepsilon(s)) \right] = 0,$$

from which we deduce (2.17). Now integrating (2.17) from 0 to 1 we have

$$K_\varepsilon = f(z_\varepsilon) - \int_0^1 V_\varepsilon(z_\varepsilon(s))\mathrm{d}s = f_\varepsilon(z_\varepsilon) - 2 \int_0^1 V_\varepsilon(z_\varepsilon(s))\mathrm{d}s. \tag{2.20}$$

Then by using (2.12) of Theorem 2.1 we get (2.18). ■

We conclude this section with a lemma which allows us to overcome the difficulty due to the lack of completeness of $\mathcal{M}_0$.

LEMMA 2.3. *Let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold and*

$$\phi \in C^0\ (\mathcal{L} \cup \partial\mathcal{L}, \mathbb{R}^+) \cap C^1(\mathcal{L}, \mathbb{R}^+)$$

such that

$$\phi(z) = 0 \quad \textit{iff } z \in \partial\mathcal{L}; \tag{2.21}$$

$$\phi(x,t) = \phi(x,0) \equiv \phi(x) \quad \forall (x,t) \in \mathcal{L}; \tag{2.22}$$

there exist $N, \delta \in \mathbb{R}^+ \setminus \{0\}$ such that

$$z \in \mathcal{L},\ \phi(z) < \delta \Rightarrow N \geq \langle \nabla_L \phi(z), \nabla_L \phi(z) \rangle_L \left(= \langle \nabla_R \phi(x), \nabla_R \phi(x) \rangle_R\right)^{(8)}. \tag{2.23}$$

Now let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\Omega^1(\mathcal{M}_0, x_0, x_1)$ such that

$$\int_0^1 \langle \dot{x}_n(s), \dot{x}_n(s) \rangle_R \, \mathrm{d}s \quad \textit{is bounded} \tag{2.24}$$

and there exists $s_n \in]0,1[$ such that

$$\lim_{n\to+\infty} \phi(x_n(s_n)) = 0.$$

Then

$$\lim_{n\to+\infty} \int_0^1 \frac{1}{\phi^2(x_n(s))} \, \mathrm{d}s = +\infty.$$

The proof of Lemma 2.3 is essentially contained in [2]. Nevertheless for the convenience of the reader we shall give here the proof.

PROOF OF LEMMA 2.3. From (2.24) we deduce that

$$\{x_n(s) : n \in \mathbb{N}, s \in [0,1]\}$$

is a bounded subset of $\mathcal{M}_0$. Then, by using (2.23), we deduce that there exists a real constant c_1, independent of n, such that

$$\|\nabla_R \phi(x_n(s))\|^2 = \langle \nabla_R \phi(x_n(s)), \nabla_R \phi(x_n(s)) \rangle_R \leq c_1 \tag{2.25}$$

(8) Here $\nabla_L \phi(z)$ denotes the gradient of the function ϕ with respect to the Lorentz structure $g(z)[\cdot,\cdot]$, while $\nabla_R \phi(x)$ denotes the gradient of the function ϕ with respect to the Riemann structure $h(x)[\cdot,\cdot]$.

for all $n \in \mathbb{N}$, for all $s \in [0,1]$. (Here $\|\cdot\|$ denotes the norm induced by the Riemannian structure). From (2.25), for $s > s_n$, we have

$$\phi(x_n(s)) - \phi(x_n(s_n)) = \int_{s_n}^{s} \langle \nabla_R \phi(x_n(\tau)), \dot{x}_n(\tau)\rangle_R \, \mathrm{d}\tau$$

$$(2.26) \qquad \leq \int_{s_n}^{s} \|\nabla_R \phi(x_n(\tau))\| \cdot \|\dot{x}_n(\tau)\| \, \mathrm{d}\tau \leq \sqrt{c_1} \cdot \sqrt{s - s_n} \cdot \left(\int_0^1 \|\dot{x}_n(s)\|^2 \, \mathrm{d}\tau \right)^{1/2}$$

$$\leq (\text{by } 2.24) \leq c_2 \cdot \sqrt{s - s_n},$$

where c_2 is a constant independent of n. Since $\phi(x_n(1)) = \phi(x_1) > 0$ for all $n \in \mathbb{N}$, there exists $\mu > 0$ such that

$$(2.27) \qquad \begin{gathered} \forall\, n \in \mathbb{N}, \ \exists\, \tau_n > s_n : \\ \phi(x_n(\tau_n)) - \phi(x_n(s_n)) \geq \mu > 0 \end{gathered}$$

independently of n. Then, from (2.26) (with $s = \tau_n$) and (2.27), we get

$$(2.28) \qquad \tau_n - s_n \geq \left(\frac{\mu}{c_2} \right)^2 > 0.$$

Moreover using again (2.26) we get

$$(2.29) \qquad \int_{s_n}^{\tau_n} \frac{\mathrm{d}\tau}{(\phi(x_n(s_n)) + c_2\sqrt{\tau - s_n})^2} \leq \int_0^1 \frac{1}{\phi^2(x_n(\tau))} \, \mathrm{d}\tau.$$

Since $\lim_{n \to +\infty} \phi(x_n(s_n)) = 0$, by (2.28) the left-hand side in (2.29) diverges, so by (2.29) we get the conclusion. ∎

3. - Proof of Theorems 1.4, 1.6

In order to prove Theorems 1.4 and 1.6 we shall study the penalized functional J_ε defined by (2.9), i.e.

$$(3.1) \qquad J_\varepsilon(x) = \int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle_R \, \mathrm{d}s - \frac{\Delta^2}{\displaystyle\int_0^1 \frac{\mathrm{d}s}{\beta(x(s))}} + \int_0^1 V_\varepsilon(x(s)) \mathrm{d}s, \quad x \in \Omega^1$$

where

$$V_\varepsilon(x) = \Psi_\varepsilon\left(\frac{1}{\phi^2(x)}\right);$$

$\phi(x) = \phi(x, 0)$ with ϕ satisfying (iii) of Definition 1.7[9];

and,

$$\text{for all } \varepsilon > 0,\ \Psi_\varepsilon \in C^2(\mathbb{R}, \mathbb{R}) \text{ satisfies}$$

$$\text{(3.2)} \qquad \begin{cases} \Psi_\varepsilon(\tau) = 0 & \text{for } \tau \le 1/\varepsilon \\ \Psi'_\varepsilon(\tau) > 0 & \text{for } \tau > 1/\varepsilon \\ \Psi'_\varepsilon(\tau) = 1 & \text{for } \tau \ge 1 + 1/\varepsilon \\ \Psi_\varepsilon(\tau) \le \Psi_{\varepsilon'}(\tau) & \text{for all } \tau \text{ and } \varepsilon \le \varepsilon'. \end{cases}$$

REMARK 3.1. By standard methods we can modify the function ϕ in order to get another function of class C^2 (which we continue to call ϕ) satisfying (2.23) and

$$\phi(x) \to 0 \text{ if and only if } \beta(x) \to 0.$$

In all this section we shall assume ϕ to satisfy these properties.

Now consider

$$\text{(3.3)} \qquad A_\mu = \left\{ x \in \mathcal{M}_0 \middle| \phi(x) \ge \mu \right\}.$$

By (iv) of Definition 1.2 and Remark 3.1 it follows that

$$A_\mu \text{ is complete.}$$

Now we shall prove a lemma which will play a fundamental role in the proof of Theorems 1.4 and 1.6.

LEMMA 3.2. *Let $\mathcal{U} = \mathcal{M}_0 \times \mathbb{R}$ be a smooth static universe and let J_ε be as in* (3.1). *For every $\varepsilon \in]0, 1]$, let x_ε be a critical point of J_ε on $\Omega^1 = \Omega^1(x_0, x_1, \mathcal{M}_0)$. Assume that*

$$\text{(3.4)} \qquad \forall\ \varepsilon \in]0, 1], \quad J_\varepsilon(x_\varepsilon) \le -\mu < 0.$$

Then, if ε is small enough, x_ε is a critical point of J on Ω^1 and $J(x_\varepsilon) \le -\mu$.

[9] Notice that we can choose $\phi: \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ such that $\phi(x,t) = \beta(x)$ for all $(x,t) \in \mathcal{M}_0 \times \mathbb{R}$, because of (iii) of Definition 1.2.

PROOF. The penalization term V_ε is zero when $\phi^2(x) \geq \varepsilon$ (see 3.2): then, in order to get the conclusion, it will be sufficient to prove that

$$\text{there exists } \delta > 0 \text{ such that, for } \varepsilon \text{ small enough,} \quad \phi(x_\varepsilon(s)) \geq \delta \text{ for all } s \in [0,1]. \tag{3.5}$$

Arguing by contradiction, assume that there exists a sequence $\varepsilon_n \underset{n}{\rightarrow} 0$ with the corresponding critical points x_{ε_n} such that

$$\phi\left(x_{\varepsilon_n}(s_n)\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{3.6}$$

where $s_n \in [0,1]$ for all n. Let $\eta > 0$ such that $\phi(x_1)$, $\phi(x_2) > \eta$. Then

$$\phi\left(x_{\varepsilon_n}(s_n)\right) < \eta \text{ for } n \text{ large enough.} \tag{3.7}$$

Let $s_n(\eta)$ be the "first" instant such that $\phi\left(x_{\varepsilon_n}(s_n(\eta))\right) = \eta$. Up to consider a subsequence we have

$$s_n(\eta) \longrightarrow s_\eta \quad \text{for } n \rightarrow \infty.$$

Since β is bounded from above, (3.4) implies that $\int_0^1 \langle \dot{x}_{\varepsilon_n}(s), \dot{x}_{\varepsilon_n}(s)\rangle_R \, \mathrm{d}s$ is bounded (independently of n). Then for n large enough we have that

$$\phi\left(x_{\varepsilon_n}(s)\right) \geq \eta/2 \quad \text{for all } s \in [0, s_\eta]. \tag{3.8}$$

Moreover by virtue of the boundary conditions, we have that x_{ε_n} is bounded in $W^{1,2}(0,1,\mathcal{M}_0)$ (and therefore also in $C(0,1,\mathcal{M}_0)$) so by virtue of Remark 3.1 there exists $\overline{\eta} > 0$ such that

$$\beta\left(x_{\varepsilon_n}(s)\right) \geq \overline{\eta} \quad \text{for all } s \in [0, s_\mu]. \tag{3.9}$$

Since x_{ε_n} is bounded in $W^{1,2}$ and $A_{\eta/2}$ is complete we have

$$x_{\varepsilon_n} \rightarrow x_\eta \quad \text{weakly in } W^{1,2}\left([0,s_\eta], A_{\eta/2}\right) \tag{3.10}$$

and

$$x_{\varepsilon_n} \rightarrow x_\eta \quad \text{in } C\left([0,s_\eta], A_{\eta/2}\right), \tag{3.11}$$

where $C\left([0,s_\eta], A_{\eta/2}\right)$ is the (complete) space of the continuous curves defined in $[0,s_\eta]$ and taking values in $A_{\eta/2}$, equipped with the distance (2.3). Now consider $t_{\varepsilon_n} = t_{\varepsilon_n}(s)$, $s \in [0,1]$, defined by

$$\begin{cases} \dot{t}_{\varepsilon_n} = \Delta \left[\displaystyle\int_0^1 \frac{\mathrm{d}s}{\beta(x_{\varepsilon_n})}\right]^{-1} \dfrac{1}{\beta(x_{\varepsilon_n})} \\ t_{\varepsilon_n} = t_0. \end{cases} \tag{3.12}$$

Since β is bounded from above, $\left[\int_0^1 \frac{\mathrm{d}s}{\beta(x_{\varepsilon_n})}\right]^{-1}$ is bounded, so by (3.9) and (3.12) we deduce that $\{t_n\}$ is bounded in $W^{1,2}\left([0,s_\eta],\mathbb{R}\right)$ and therefore, passing to a subsequence, we have

$$t_{\varepsilon_n} \to t_\eta \quad \text{weakly in } W^{1,2}\left([0,s_\eta],\mathbb{R}\right). \tag{3.13}$$

Since x_{ε_n} is a critical point of J_{ε_n} we have

$$\mathrm{D}_s\dot{x}_{\varepsilon_n} = -\frac{\Delta^2}{\left(\int_0^1 \frac{1}{\beta(x_{\varepsilon_n})}\right)^2} \cdot \frac{\beta'(x_{\varepsilon_n})}{\beta^2(x_{\varepsilon_n})} + V'_{\varepsilon_n}(x_{\varepsilon_n}). \tag{3.14}$$

Moreover $\left[\int_0^1 \frac{\mathrm{d}s}{\beta(x_{\varepsilon_n})}\right]^{-1}$ is bounded; hence by (3.9) and (3.11), eventually passing to a subsequence, the right-hand side in (3.14) converges uniformly in $[0,s_\eta]$. Then

$$\mathrm{D}_s\dot{x}_{\varepsilon_n} \to \mathrm{D}_s\dot{x}_\eta \quad \text{in } C\left([0,s_\eta],A_{\eta/2}\right). \tag{3.15}$$

Now, from (3.12) we have

$$|\dot{t}_{\varepsilon_n}| \le |\Delta| \left[\int_0^{s_n} \frac{\mathrm{d}s}{\beta(x_{\varepsilon_n}(s))}\right]^{-1} \frac{1}{\beta(x_{\varepsilon_n})}, \tag{3.16}$$

so by (3.9) and (3.16) we deduce that

$$\{\dot{t}_{\varepsilon_n}\} \text{ is bounded in } L^\infty\left([0,s_\eta],\mathbb{R}\right). \tag{3.17}$$

Moreover from (3.12) we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\beta(x_{\varepsilon_n}(s))\cdot\dot{t}_{\varepsilon_n}(s)\right) = 0 \quad \text{for all } s\in[0,1],$$

then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(t_{\varepsilon_n}(s)\right) = -\frac{\langle\beta'(x_{\varepsilon_n}(s)),\dot{x}_{\varepsilon_n}(s)\rangle_R}{\beta(x_{\varepsilon_n}(s))}\cdot\dot{t}_{\varepsilon_n}(s) \quad \text{for all } s\in[0,s_\eta]. \tag{3.18}$$

Since $t_{\varepsilon_n}(0) = t_0$ for all n, from (3.17), (3.18), (3.9) and (3.15) we easily have

$$t_{\varepsilon_n} \to t_\eta \quad \text{in } C^2\left([0,s_\eta],\mathbb{R}\right). \tag{3.19}$$

Moreover from (3.12) we get that t_{ε_n} is monotone. Then, if for instance $t_0 \leq t_1$,

(3.20) $$t_0 = t_{\varepsilon_n}(0) \leq t_{\varepsilon_n}(s) \leq t_{\varepsilon_n}(1) = t_1 \text{ for all } n \in \mathbb{N} \text{ and } s \in [0,1].$$

Now, by Theorem 2.1, $\gamma_{\varepsilon_n} = \left(x_{\varepsilon_n}, t_{\varepsilon_n}\right)$ solves the equation

$$\mathrm{D}_s \dot{\gamma}_{\varepsilon_n}(s) = \nabla_L V_{\varepsilon_n}\left(\gamma_{\varepsilon_n}(s)\right) \quad \text{for all } s \in [0, s_\eta].$$

Then, by (3.8), we obtain, for n sufficiently large,

(3.21) $$\mathrm{D}_s \dot{\gamma}_{\varepsilon_n}(s) = 0 \quad \text{for all } s \in [0, s_\eta],$$

so by (3.15) and (3.19) we get

(3.22) $$\mathrm{D}_s \dot{\gamma}_\eta(s) = 0 \quad \text{for all } s \in [0, s_\eta],$$

where $\gamma_\eta = \left(x_\eta, t_\eta\right)$. Then γ_η is a geodesic in the interval $[0, s_\eta]$. In order to prove that γ_η is time-like, observe that, by virtue of (3.4), (2.17) and (2.18), we have, for all $s \in [0,1]$ and for all n,

(3.23) $$\begin{aligned} -\mu \geq J_{\varepsilon_n}(x_{\varepsilon_n}) &= \langle \dot{\gamma}_{\varepsilon_n}(s), \dot{\gamma}_{\varepsilon_n}(s)\rangle_L - V_{\varepsilon_n}(\gamma_{\varepsilon_n}(s)) + 2\int_0^1 V_{\varepsilon_n}(\gamma_{\varepsilon_n}) \\ &\geq \langle \dot{\gamma}_{\varepsilon_n}(s), \dot{\gamma}_{\varepsilon_n}(s)\rangle_L - V_{\varepsilon_n}(\gamma_{\varepsilon_n}(s)). \end{aligned}$$

Taking in (3.23) the limit in the interval $[0, s_\eta]$, we get

$$-\mu \geq \langle \dot{\gamma}_\eta(s), \dot{\gamma}_\eta(s)\rangle_L \quad \text{for all } s \in [0, s_\eta],$$

therefore γ_η is time-like.

Summarizing, in dependence of $\eta > 0$, we have constructed a subsequence (of $\gamma_{\varepsilon_n} = (x_{\varepsilon_n}, t_{\varepsilon_n})$)

(3.24) $$\gamma_n^\eta = \left(x_n^\eta, t_n^\eta\right)$$

which converges, in a suitable interval $[0, s_\eta] \subset [0,1]$, to a time-like geodesic

$$\gamma_\eta = (x_\eta, t_\eta),$$

such that

$$\gamma_\eta(0) = \gamma_0, \quad \phi(\gamma_\eta(s_\eta)) = \eta$$

and

$$t_0 \leq t_\eta(s) \leq t_1 \quad \text{for all } s \in [0, s_\eta].$$

Repeating the above procedure in correspondence of $\eta/2$, we can select from (3.24) a subsequence

$$\gamma_n^{\eta/2} = \left(x_n^{\eta/2}, t_n^{\eta/2}\right)$$

which approaches (in a suitable interval $[0, s_{\eta/2}]$ with $s_{\eta/2} > s_\eta$) a time-like geodesic

$$\gamma_{\eta/2} = \left(x_{\eta/2}, t_{\eta/2}\right)$$

such that

$$\gamma_{\eta/2} \text{ extends } \gamma_\eta,$$

$$\gamma_{\eta/2}(0) = \gamma_0, \ \ \phi(\gamma_{\eta/2}(s_{\eta/2})) = \eta/2,$$

and

$$t_0 \leq t_{\eta/2}(s) \leq t_1 \qquad \text{for all } s \in [0, s_{\eta/2}].$$

Following this procedure we can find a geodesic for any η/k ($k \in \mathbb{N}$). Taking the limit when k goes to $+\infty$, we obtain a time-like geodesic

$$\gamma = (x, t) : [0, s_0[\to \mathcal{U}$$

such that

$$s_0 = \sup\left\{s_{\eta/k} : k \in \mathbb{N}\right\} < 1,$$

$$\gamma_{|[0,s_{\eta/k}]} = \gamma_{\eta/k} \qquad \text{for all } k \in \mathbb{N},$$

$$\gamma(0) = \gamma_0, \ \ \phi(\gamma(s_{\eta/k})) = \eta/k,$$

and

(3.25)
$$t_1 \leq t(s) \leq t_2 \qquad \text{for all } s \in [0, s_0[.$$

Now,

$$\liminf_{s \to s_0^-} \phi(x(s)) = \lim_{k \to +\infty} \phi\left(\gamma(s_{\eta/k})\right) = \lim_{k \to +\infty} \phi\left(\gamma_{\eta/k}(s_{\eta/k})\right) = 0$$

because of (3.6). Then, by virtue of Remark 3.1,

(3.26)
$$\liminf_{s \to s_0^-} \beta(x(s)) = 0.$$

Since γ is time-like, (3.25) and (3.26) contradict property (v) of Definition 1.2. Then (3.5) (and therefore Lemma 3.2) is proved. ∎

LEMMA 3.3. *Let $\varepsilon > 0$. Then for any $a \in \mathbb{R}$ the sublevels*

$$J_\varepsilon^a = \left\{x \in \Omega^1(\mathcal{M}_0, x_0, x_1) \middle| J_\varepsilon(x) \leq a\right\}$$

are complete metric spaces. Moreover if $(\mathcal{M}_0, h)$ *is of class* C^3, J_ε *satisfies the Palais-Smale condition, i.e. any sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \Omega^1$ *such that*

$$J_\varepsilon(x_n) \text{ is bounded} \tag{3.27}$$

and

$$J'_\varepsilon(x_n) \to 0 \tag{3.28}$$

contains a subsequence convergent (in $W^{1,2}$*) to* $x_\varepsilon \in \Omega^1(\mathcal{M}_0, x_0, x_1)$.

PROOF. Clearly the sets

$$\left\{ \int_0^1 \langle \dot{x}, \dot{x} \rangle_R \, \mathrm{d}s : x \in J_\varepsilon^a \right\}, \text{ and } \left\{ \int_0^1 \frac{\mathrm{d}s}{\beta(x)} : x \in J_\varepsilon^a \right\}$$

are bounded. Then by Lemma 2.3 we deduce that there exists $\mu > 0$, such that

$$J_\varepsilon^a \subset \Omega^1 \left(A_\mu, x_0, x_1 \right) \tag{3.29}$$

where

$$A_\mu = \left\{ x \in \mathcal{M}_0 \middle| \phi(x) \geq \mu \right\}.$$

Now A_μ, with the metric $\langle \cdot, \cdot \rangle_R$, is complete. Then also the closed subset J_ε^a of $\Omega^1(A_\mu, x_0, x_1)$ (see 3.29) is complete. Now assume $(\mathcal{M}_0, h)$ to be of class C^3 in order to use the Nash embedding theorem (see [13]). Let $\{x_n\} \subset \Omega^1$ satisfy (3.27) and (3.28). Clearly $\{x_n\} \subset J_\varepsilon^a$ for some $a \in \mathbb{R}$, then from (3.29)

$$\{x_n\} \subset \Omega^1 \left(A_\mu, x_0, x_1 \right) \text{ for some } \mu > 0.$$

By Nash embedding theorem (see [13]), A_μ can be isometrically embedded into $\mathbb{R}^N$ (with N sufficiently large) equipped with the Euclidean metric. Then, using Lemma (2.1) in [4] and arguing as in the proof of Theorem 1.1 in [5], we can deduce that $\{x_n\}$ contains a subsequence convergent to

$$x_\varepsilon \in \Omega^1 \left(A_\mu, x_0, x_1 \right) \subset \Omega^1 \left(\mathcal{M}_0, x_0, x_1 \right). \qquad \blacksquare$$

We are now ready to prove Theorems 1.4 and 1.6.

PROOF OF THEOREM 1.4. By virtue of Theorem 2.1 we see immediately that condition (1.5) is necessary to guarantee the existence of a time-like geodesic joining z_0 and z_1.

In order to prove the sufficiency, observe that by the assumption (1.5) there exists

$$\overline{x} \in \Omega^1(\mathcal{M}_0, x_0, x_1) \equiv \Omega^1$$

such that

$$J(\overline{x}) < 0. \tag{3.30}$$

Since $\phi(\overline{x}(s)) > 0$ for all $s \in [0,1]$, by the definition of V_ε, it is easy to see that

$$J_\varepsilon(\overline{x}) = J(\overline{x}) < 0 \quad \text{for } \varepsilon \text{ small enough.} \tag{3.31}$$

Clearly a minimizing sequence $\{x_n\}$ for J_ε is contained in some sublevel J_ε^a, which, by Lemma 3.3, is complete. Then, since J_ε is weakly lower semicontinuous, the infimum of J_ε on Ω^1 is attained at some $x_\varepsilon \in J_\varepsilon^a \subset \Omega^1(\mathcal{M}_0, x_0, x_1)$. Moreover by (3.31) we have

$$J_\varepsilon(x_\varepsilon) \le J_\varepsilon(\overline{x}) = J(\overline{x}) < 0 \quad \text{for } \varepsilon \text{ small.}$$

Then by Lemma 3.2 we deduce that x_ε, for ε small enough, is also a critical point for J on Ω^1 and $J(x_\varepsilon) < 0$. Then, using Theorem 2.1, Theorem 1.4 is proved. ∎

PROOF OF THEOREM 1.6. Assume that $\mathcal{M}_0$ is not contractible in itself. Fadell and Husseini have recently proved that there exists a sequence $\{K_m\}_{m\in\mathbb{N}}$ of compact subsets of Ω^1 such that

$$\lim_{m\to+\infty} \operatorname{cat}_{\Omega^1}(K_m) = +\infty, \tag{3.32}$$

(see [7, Corollary 3.3] and [8, Remark 2.23]). Here $\operatorname{cat}_{\Omega^1}(K_m)$ denote the Ljusternik-Schnirelman category of K_m in Ω^1 (see e.g. [17]), i.e. the minimal number of closed, contractible subsets of Ω^1 covering K_m. Now let $m \in \mathbb{N}$. By (3.32) there exists a compact subset $K \subset \Omega^1$ with

$$\operatorname{cat}_{\Omega^1}(K) \ge m.$$

Clearly there exists $\overline{\Delta}$ such that

$$\forall\, \Delta > \overline{\Delta} \text{ and } \forall\, \varepsilon \in]0,1], \quad \sup J_\varepsilon(K) \le \sup J_1(K) \le -1.$$

Now for every $\varepsilon \in]0,1]$ we set

$$c_{j\varepsilon} = \inf_{\operatorname{cat} A \ge j} \sup J_\varepsilon(A) \quad j = 1, \ldots, m.$$

Obviously

$$\forall\, \varepsilon \in]0,1],\ \forall\, j \in \{1,\ldots,m\}, \quad c_{j\varepsilon} \ge \inf_{\mathcal{M}_0} J_\varepsilon \ge \inf_{\mathcal{M}_0} J > -\infty$$

and

$$c_{j\varepsilon} \le -1. \tag{3.33}$$

By Lemma 3.3 and well known methods in critical point theory (see e.g. [15, 17]) we deduce that every $c_{j\varepsilon}$ is a critical value for the functional J_ε. Moreover if $c_{i\varepsilon} = c_{j\varepsilon}$ for some $i \neq j$, there are infinitely many critical points of J_ε at the level $c_{j\varepsilon}$. Therefore
for every $\varepsilon \in]0,1]$, there exist $x_{1\varepsilon}, \dots, x_{m\varepsilon}$ distinct critical points of J_ε, with critical values ≤ -1.

Using Lemma 3.2 we deduce that, if ε is small enough, $x_{1\varepsilon}, \dots, x_{m\varepsilon}$ are also critical points of J with critical values ≤ -1. Then, since m is arbitrary, (1.6) is obtained by virtue of Theorem 2.1. ∎

4. - Proof of Theorems 1.8, 1.9

In order to prove Theorems 1.8 and 1.9 we shall study the penalized functional J_ε defined by (2.9) i.e.

$$J_\varepsilon(x) = \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_R \, ds - \frac{\Delta^2}{\displaystyle\int_0^1 \frac{ds}{\beta(x(s))}} + \int_0^1 V_\varepsilon(x(s)) ds, \tag{4.1}$$

where $x \in \Omega^1 \equiv \Omega^1(\mathcal{M}_0, x_0, x_1)$,

$$V_\varepsilon(x) = \frac{\varepsilon}{\phi^2(x)}, \quad \varepsilon \in]0,1],$$

and ϕ satisfies (iii) and (v) of Definition 1.7.

Now we shall prove some preliminary lemmas.

LEMMA 4.1. *Let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold with convex boundary $\partial\mathcal{L}$ (see Definition 1.7). For any $\varepsilon \in]0,1]$ let x_ε be a critical point of J_ε on $\Omega^1(\mathcal{M}_0, x_0, x_1)$ and assume that there exists $c_1 > 0$ such that*

$$\text{for all } \varepsilon \in]0,1], \quad J_\varepsilon(x_\varepsilon) \leq c_1. \tag{4.2}$$

Then

$$\phi(x_\varepsilon(s)) \geq c_0 > 0, \quad \text{for all } s \in [0,1] \tag{4.3}$$

where c_0 is independent of ε.

PROOF. Arguing by contradiction, assume that there exists a sequence $\{x_{\varepsilon_n}\}_{n \in \mathbb{N}}$ of critical points of $J_{\varepsilon_n} (\varepsilon_n \to 0)$ such that

$$u_n(s_n) = \phi(x_{\varepsilon_n}(s_n)) \to 0 \quad \text{for } n \to +\infty \tag{4.4}$$

where s_n is a minimum point for the map

$$u_n(s) \equiv \phi(x_{\varepsilon_n}(s)) \quad s \in [0,1].$$

Now we set

$$\gamma_n(s) = \big(x_{\varepsilon_n}(s), t_{\varepsilon_n}(s)\big)$$

where $t_{\varepsilon_n} = t_{\varepsilon_n}(s)$ is defined by

$$\begin{cases} t_{\varepsilon_n}(s) = \Delta \left[\int_0^1 \frac{\mathrm{d}s}{\beta(x_{\varepsilon_n}(s))} \right]^{-1} \frac{1}{\beta(x_{\varepsilon_n}(s))} \\ t_{\varepsilon_n}(0) = t_0. \end{cases}$$

By Theorem 2.1, γ_n is a critical point of f_{ε_n}, then

$$\begin{aligned} &\mathrm{D}_s\dot\gamma_n(s) = \frac{1}{2}\nabla_L V_n(\gamma_n(s)), \quad \text{for all } s \in [0,1] \\ &\text{where } V_n(\gamma_n(s)) = V_{\varepsilon_n}(x_{\varepsilon_n}(s)). \end{aligned} \tag{4.5}$$

Then for all $s \in [0,1]$ we have

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}s^2}(u_n(s)) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\langle \nabla_L \phi(\gamma_n(s)), \dot\gamma(s)\rangle_L\right] \\ &= H_L^\phi(\gamma_n(s))[\dot\gamma_n(s), \dot\gamma_n(s)] + \langle \nabla_L\phi(\gamma_n(s)), \mathrm{D}_s\dot\gamma_n(s)\rangle_L \\ &= H_L^\phi(\gamma_n(s))[\dot\gamma_n(s), \dot\gamma_n(s)] + \frac{1}{2}\langle \nabla_L\phi(\gamma_n(s)), \nabla_L V_n(\gamma_n(s))\rangle_L. \end{aligned} \tag{4.6}$$

Eventually passing to a subsequence let

$$s_0 = \lim_{n\to+\infty} s_n.$$

Since $\phi(\gamma_n(0)) = \phi(x_0) > 0$, $\phi(\gamma_n(1)) = \phi(x_1) > 0$, and $\int_0^1 \langle \dot x_{\varepsilon_n}, \dot x_{\varepsilon_n}\rangle_R \,\mathrm{d}s$ is bounded independently of n, we have

$$s_0 \neq 0, \;\; s_0 \neq 1.$$

Moreover, by (4.4) and the boundness of $\int_0^1 \langle \dot x_{\varepsilon_n}, \dot x_{\varepsilon_n}\rangle_R \,\mathrm{d}s$, there is $\mu > 0$ such that

$$u_n(s) = \phi(\gamma_n(s)) < \delta \text{ for } s \in [s_0 - \mu, s_0 + \mu] \text{ and } n \text{ large enough,}$$

where $\delta > 0$ is introduced in (1.8) and (1.9). Then from (4.6), (1.8) and (1.9) we deduce that, for n large enough and for all $s \in [s_0 - \mu, s_0 + \mu]$,

$$\begin{aligned} \frac{d^2 u_n}{ds^2}(s) &\leq M\phi(\gamma_n(s))|\langle\dot\gamma_n(s), \dot\gamma_n(s)\rangle_L| \\ &\qquad - \varepsilon_n \langle \nabla_L \phi(\gamma_n(s)), \nabla_L \phi(\gamma_n(s))\rangle_L \cdot \frac{1}{\phi^3(\gamma_n(s))} \\ &\leq M u_n(s)|\langle\dot\gamma_n(s), \dot\gamma(s)\rangle_L| - \nu\varepsilon_n \cdot \frac{1}{u_n^3(s)}. \end{aligned} \tag{4.7}$$

Now by (2.17) and (2.18) in Lemma 2.2 we have, for all $s \in [0, 1]$,

$$|\langle\dot\gamma_n(s), \dot\gamma_n(s)\rangle_L| = |J_{\varepsilon_n}(x_{\varepsilon_n}) - 2\int_0^1 V_{\varepsilon_n}(x_{\varepsilon_n})ds + V_{\varepsilon_n}(x_{\varepsilon_n}))|. \tag{4.8}$$

Moreover

$$\begin{aligned} &\int_0^1 \langle \dot x_{\varepsilon_n}(s), \dot x_{\varepsilon_n}(s)\rangle_R \, ds - \Delta^2 \sup_{\mathcal{M}_0} \beta + \int_0^1 V_{\varepsilon_n}(x_{\varepsilon_n}(s))ds \\ &\leq \int_0^1 \langle \dot x_{\varepsilon_n}(s), \dot x_{\varepsilon_n}(s)\rangle_R \, ds - \frac{\Delta^2}{\displaystyle\int_0^1 \frac{ds}{\beta(x_{\varepsilon_n}(s))}} + \int_0^1 V_{\varepsilon_n}(x_{\varepsilon_n}(s))ds \\ &= (\text{by } (4.1)) = J_{\varepsilon_n}(x_{\varepsilon_n}) \leq (\text{by } (4.2)) \leq c_1. \end{aligned} \tag{4.9}$$

From (4.9) we deduce that

$$\int_0^1 V_{\varepsilon_n}(x_{\varepsilon_n}(s))ds \leq c_1 + \Delta^2 \sup_{\mathcal{M}_0} \beta \equiv c_2,$$

so by (4.8) and (4.2) we have, for all $s \in [0, 1]$,

$$|\langle\dot\gamma_n(s), \dot\gamma_n(s)\rangle_L| \leq c_1 + 2c_2 + \frac{\varepsilon_n}{u_n^2(s)}. \tag{4.10}$$

Then, inserting (4.10) in (4.7), we get

$$\begin{aligned} &\frac{d^2 u_n}{ds^2}(s) \leq c_3 u_n(s) + c_4 \frac{\varepsilon_n}{u_n^2(s)} - \nu\varepsilon_n \cdot \frac{1}{u_n^3(s)} \\ &\qquad \text{for all } s \in [s_0 - \mu, s_0 + \mu], \end{aligned} \tag{4.11}$$

where c_3, c_4 are real constants independent of n. Since $u_n(s) < \delta$, if δ is small enough, we obtain from (4.11),

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}s^2}(s) \le c_3 u_n(s) \quad \text{for all } s \in [s_0 - \mu, s_0 + \mu]. \tag{4.12}$$

Then, since $u_n'(s_n) = 0$, we get, by Gronwall Lemma, that $u_n(s)$ converges uniformly to zero and this contradicts the boundary conditions

$$u_n(0) = \phi(x_0) > 0, \;\; u_n(1) = \phi(x_1) > 0.$$ ∎

By the same proof of Lemma 3.3 we have the following

LEMMA 4.2. *Let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold and denote by $\partial\mathcal{L}$ the (topological) boundary of $\mathcal{L}$. Assume that there exists a map $\phi \in C^0(\mathcal{L} \cup \partial\mathcal{L}, \mathbb{R}^+) \cap C^1(\mathcal{L}, \mathbb{R}^+)$ satisfying* (2.21), (2.22) *and* (2.23). *Assume moreover that* (ii) *and* (iv) *of Definition* 1.7 *hold. Then for any $a \in \mathbb{R}$ the sublevels*

$$J_\varepsilon^a = \left\{ x \in \Omega^1(\mathcal{M}_0, x_0, x_1) \middle| J_\varepsilon^a(x) \le a \right\}$$

are complete metric spaces. Moreover, if $(\mathcal{M}_0, h)$ is of class C^3, J_ε satisfies the Palais-Smale condition on $\Omega^1(\mathcal{M}_0, x_0, x_1)$.

LEMMA 4.3. *Assume* (ii), (iii), (iv) *of Definition* 1.7. *Assume moreover that* (1.8) *holds. Then, for any $c \in \mathbb{R}$,*

$$\mathrm{cat}_{\Omega^1}(J^c) < +\infty. \tag{4.13}$$

PROOF. For $\mu > 0$, we set

$$V_\mu = \left\{ x \in \mathcal{M}_0 \middle| \phi(x) < \mu \right\}, \quad A_\mu = \mathcal{M}_0 \backslash V_\mu. \tag{4.14}$$

Now, if $\mu > 0$ is small enough, there is a diffeomorphism

$$\Psi : \mathcal{M}_0 \to A_\mu \tag{4.15}$$

which can be constructed by means of the solutions of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}s} = \chi(\eta(s)) \dfrac{\phi'(\eta(s))}{1 + \langle \phi'(\eta(s)), \phi'(\eta(s)) \rangle_R^{1/2}} \\ \eta(0) = x \in \mathcal{M}_0 \end{cases} \tag{4.16}$$

where χ is a real C^2 map on $\mathcal{M}_0$ such that

$$\chi(x) = \begin{cases} 1 & \text{for } x \in V_{\delta_2} \\ & \qquad\qquad \chi(x) \in]0,1] \text{ if } x \notin A_{\delta_1} \\ 0 & \text{for } x \in A_{\delta_1}, \end{cases}$$

with $0 < \delta_2 < \delta_1 < \delta$, δ being introduced in (1.8). Denote by $\eta = \eta(s, x)$, $s \in \mathbb{R}$ the solution of (4.16). Then, by (1.8) and standard arguments (see e.g. [15, 17]), it can be shown that there exist $\overline{s}$ and $\mu > 0$ such that

$$\forall\ x \in \mathcal{M}_0 \quad \Psi(x) \equiv \eta(\overline{s}, x) \in A_\mu. \tag{4.17}$$

Obviously we can choose $\mu > 0$ so small that

$$x_0, x_1 \in A_\mu. \tag{4.18}$$

Let us set

$$\forall\ c \in \mathbb{R} \quad J^{c,\mu} = \left\{ x \in J^c \middle| \forall\ s \in [0,1]\ x(s) \in A_\mu \right\}. \tag{4.19}$$

It is not difficult to see that

$$\forall\ c \in \mathbb{R}\ \exists\ \alpha = \alpha(c) > 0 \text{ s.t. } \forall\ x \in J^c, \quad \Psi(x) \in J^{c+\alpha,\mu}, \tag{4.20}$$

because Ψ is Lipschitz continuous and β is bounded. Now consider the penalized functional

$$J_*(x) = J(x) + \int_0^1 \frac{\vartheta(x)}{\phi^2(x)}\, \mathrm{d}s, \quad x \in \Omega^1(\mathcal{M}_0, x_0, x_1)$$

where ϑ is a C^2 positive scalar field on $\mathcal{M}_0$ such that

$$\vartheta(x) = \begin{cases} 0 & \text{for } x \in A_{\mu/2} \\ 1 & \text{for } x \in V_{\mu/3} \end{cases} \qquad \vartheta(x) \in [0,1] \text{ for all } x \in \mathcal{M}_0.$$

Clearly for all $x \in \Omega^1(A_\mu, x_0, x_1)$ we have

$$J_*(x) = J(x),$$

then

$$J^{c+\alpha,\mu} \subset J_*^{c+\alpha} \equiv \left\{ x \in \Omega^1(\mathcal{M}_0, x_0, x_1) \middle| J_*(x) \le c + \alpha \right\}. \tag{4.21}$$

Therefore, from (4.20) and (4.21) we deduce that

$$\Psi(J^c) \subset J_*^{c+\alpha}. \tag{4.22}$$

Now, as in Lemma 4.2, the penalized functional J_* satisfies the Palais-Smale condition on $\Omega^1(\mathcal{M}_0, x_0, x_1)$ and its sublevels are complete. Then, arguing as in the proof of Theorem 1.1 in [5], we get

$$\operatorname{cat}_{\Omega^1}(J_*^{c+\alpha}) < +\infty. \tag{4.23}$$

At this point (4.22) and (4.23) and well known properties of the Ljusternik-Schnirelman category imply that

$$\mathrm{cat}_{\Omega^1}(J^c) \leq \mathrm{cat}_{\Omega^1}(J_*^{c+\alpha})$$

so

$$\mathrm{cat}_{\Omega^1}(J^c) < +\infty.$$

■

Now we are ready to prove Theorems 1.8 and 1.9.

PROOF OF THEOREM 1.8. Since the sublevels of J_ε are complete (see Lemma 4.2) it is not difficult to show, as in the proof of Theorem 1.4 that, for all $\varepsilon \in]0,1]$, J_ε has a minimum point x_ε on $\Omega^1(\mathcal{M}_0, x_0, x_1)$. Clearly there exists c_1 (independent of ε) such that

$$J_\varepsilon(x_\varepsilon) \leq c_1 \quad \text{for all } \varepsilon \in]0,1]. \tag{4.24}$$

Then, by Lemma 4.1, we obtain that there exists $c_0 > 0$ such that

$$\forall\ \varepsilon \in]0,1],\ \forall\ s \in [0,1], \quad \phi(x_\varepsilon(s)) \geq c_0 > 0. \tag{4.25}$$

Moreover, using again (4.24), we deduce that x_ε is bounded in the $W^{1,2}$ norm independently of ε. Since, by Lemma 4.2, J^{c_1} is complete, we obtain a sequence $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}}$ $(\varepsilon_n \to 0)$ such that

$$x_{\varepsilon_n} \to \overline{x} \in \Omega^1(\mathcal{M}_0, x_0, x_1) \text{ weakly in } W^{1,2}. \tag{4.26}$$

Now, by (4.25) we can take the weak limit in the equation

$$J'_{\varepsilon_n}(x_{\varepsilon_n}) = 0$$

and we get, by (4.26), that $\overline{x}$ is a critical point of J on $\Omega^1(\mathcal{M}_0, x_0, x_1)$. Then Theorem 1.8 is proved using Theorem 2.1 (with $\varepsilon = 0$). ■

PROOF OF THEOREM 1.9. Let $\alpha \in \mathbb{R}$ and set

$$J_\alpha = \left\{x \in \Omega^1 \,\middle|\, J(x) \geq \alpha\right\}, \quad J_{\varepsilon,\alpha} = \left\{x \in \Omega^1 \,\middle|\, J_\varepsilon(x) \geq \alpha\right\}.$$

By Lemma 4.3 there exists $k = k(\alpha) \in \mathbb{N}$ such that

$$B \cap J_\alpha \neq \emptyset \quad \text{if } B \subset \Omega^1 \text{ and } \mathrm{cat}_{\Omega^1}(B) \geq k. \tag{4.27}$$

Then, since $J_\alpha \subset J_{\varepsilon,\alpha}$, we have

$$\begin{gathered} B \cap J_{\varepsilon,\alpha} \neq \emptyset \\ \text{for all } \varepsilon > 0, \text{ for all } B \subset \Omega^1 \text{ with } \mathrm{cat}_{\Omega^1}(B) \geq k, \end{gathered} \tag{4.28}$$

from which we deduce that

$$c_{k,\varepsilon} \equiv \inf \{\sup J_\varepsilon(A) : \mathrm{cat}_{\Omega^1}(A) \geq k\} \geq \alpha. \tag{4.29}$$

Since $\mathcal{M}_0$ is not contractible in itself, (3.32) holds, hence there exists a compact subset K of Ω^1 such that $\mathrm{cat}_{\Omega^1}(K) \geq k$. Therefore, for all $\varepsilon \in]0,1]$, we have

$$c_{k,\varepsilon} \leq \sup J_\varepsilon(K) \leq \sup J_1(K) \equiv c_1 < +\infty.$$

Therefore by Lemma 4.2 and well known arguments in critical point theory (see e.g. [15, 17]), we deduce that every $c_{k,\varepsilon}$ in (4.29) is a critical value of J_ε, so for all $\varepsilon \in]0,1]$, there exists

$$\begin{aligned} &x_\varepsilon \in \Omega^1 \text{ critical point of } J_\varepsilon \text{ s.t.} \\ &c_1 \geq J_\varepsilon(x_\varepsilon) = c_{k,\varepsilon} \geq \alpha. \end{aligned} \tag{4.30}$$

Now, by Lemma 4.1, we have that

$$\phi(x_\varepsilon(s)) \geq c_0 > 0 \qquad \text{for all } \varepsilon \in]0,1] \text{ and } s \in [0,1], \tag{4.31}$$

so, following the same arguments used in proving Theorem 1.8, we get the existence of a sequence $\varepsilon_n \to 0$ with the corresponding critical point x_{ε_n} such that

$$x_{\varepsilon_n} \to \overline{x} \in \Omega^1(\mathcal{M}_0, x_0, x_1) \text{ weakly in } W^{1,2} \tag{4.32}$$

and $\overline{x}$ is a critical point of J on Ω^1.

Now we want to show that

$$J(\overline{x}) \geq \alpha. \tag{4.33}$$

Since α is arbitrary, from (4.33) we easily get the conclusion using Theorem 2.1. Clearly, (4.33) is a consequence of (4.30) if we show that

$$x_{\varepsilon_n} \to \overline{x} \quad \text{in } C^1. \tag{4.34}$$

Since x_{ε_n} is a critical point of J_{ε_n} we have

$$2D_s\dot{x}_{\varepsilon_n} = \Delta^2 \left[\int_0^1 \frac{ds}{\beta(x_{\varepsilon_n}(s))} \right]^{-2} \frac{\beta'(x_{\varepsilon_n})}{\beta^2(x_{\varepsilon_n})} - 2\varepsilon_n \frac{\phi'(x_{\varepsilon_n})}{\phi^3(x_{\varepsilon_n})} \equiv F(x_n), \tag{4.35}$$

where the derivatives are taken in the distributional sense. By (4.31), (4.32) and the completeness of A_{c_0}, we obtain that

$$F(x_{\varepsilon_n}) \to \Delta^2 \left[\int_0^1 \frac{\mathrm{d}s}{\beta(\overline{x})}\right]^{-2} \frac{\beta'(\overline{x})}{\beta^2(\overline{x})} \quad \text{uniformly.} \tag{4.36}$$

Then from (4.35) we have that

$$\{\mathrm{D}_s \dot{x}_{\varepsilon_n}\}_{n\in\mathbb{N}} \text{ converges uniformly}$$

from which we deduce (4.34). ∎

REMARK 4.4. Clearly if two events (x_0, t_0), (x_1, t_1) are simultaneous (i.e. $t_1 = t_2$) we have

$$J(x) = \int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle_R \, \mathrm{d}s, \qquad x \in \Omega^1 \equiv \Omega^1(\mathcal{M}_0, x_0, x_1),$$

then the critical points of J are the geodesics on $\mathcal{M}_0$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle_R$.

Appendix

In this appendix we will verify that the Schwarzschild spacetime

$$\mathcal{L} = \{(r, \vartheta, \varphi) : r > 2m\} \times \mathbb{R} \text{ with metric (1.4)}$$

is a static universe and a static Lorentz manifold with convex boundary.

The same computations will show that, when $m^2 > e^2$, the Reissner-Nordström spacetime $\{r > m + \sqrt{m^2 - e^2}\} \times \mathbb{R}$ is a static universe and that it is a static Lorentz manifold with convex boundary provided that $m^2 > \frac{9}{5} \cdot e^2$.

PROPOSITION A.1. *The Schwarzschild spacetime is a static universe.*

PROOF. Clearly, in order to prove Proposition A.1, it suffices to prove that the Schwarzschild spacetime verifies (v) of Definition 1.2. To this aim let $\gamma(s) = (r(s), \varphi(s), \vartheta(s), t(s))$ a *time-like* geodesic with respect to the Lorentz metric (1.4): γ is a critical point for the functional

$$f(\gamma) = \int_a^b \left[\frac{1}{\beta(r)} \dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2 \vartheta \cdot \dot{\varphi}^2) - c^2 \beta(r) \dot{t}^2\right] \mathrm{d}s, \tag{A.1}$$

where $\beta(r) = 1 - \dfrac{2m}{r}$, on the space of the smooth curves $\gamma(s)$ on $\mathcal{L}$. Then a geodesic $\gamma(s) = (r(s), \varphi(s), \vartheta(s), t(s))$ solves the following system of differential equations:

$$\text{(A.2)} \quad \begin{cases} -\dfrac{1}{\beta^2(r)}\beta'(r)\dot{r}^2 - \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{2}{\beta(r)}\dot{r}\right) + 2r(\dot{\vartheta}^2 + \sin^2\vartheta \cdot \dot{\varphi}^2) - c^2\beta'(r)\dot{t}^2 = 0 \\ -\dfrac{\mathrm{d}}{\mathrm{d}s}\left(2r^2\dot{\vartheta}\right) + 2\sin\vartheta\cos\vartheta \cdot \dot{\varphi}^2 = 0 \\ r^2\sin^2\vartheta \cdot \dot{\varphi} = L \\ \beta(r)\dot{t} = K \end{cases}$$

where L and K are real constants.

Let $\gamma(s) = (r(s), \varphi(s), \vartheta(s), t(s))$ be a solution of (A.2). Obviously, up to a rotation, we can assume $\vartheta(0) = \pi/2$ and $\dot{\vartheta}(0) = 0$. Then by the second equation of (A.2) and the uniqueness of the Cauchy problem we have that

$$\gamma(s) = (r(s), \varphi(s), \pi/2, t(s)) \qquad \text{for all } s,$$

and $(r(s), \varphi(s), t(s))$ solves

$$\begin{cases} -\dfrac{1}{\beta^2(r)}\beta'(r)\dot{r}^2 - \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{2}{\beta(r)}\dot{r}\right) + 2r\dot{\varphi}^2 - c^2\beta'(r)\dot{t}^2 = 0 \\ r^2\dot{\varphi} = L \\ \beta(r)\dot{t} = K \end{cases}$$

and therefore

$$\text{(A.3)} \quad \begin{cases} \dfrac{1}{\beta^2(r)}\beta'(r)\dot{r}^2 - \dfrac{2}{\beta(r)}\dfrac{\mathrm{d}^2 r}{\mathrm{d}s^2} + 2r\dot{\varphi}^2 - c^2\beta'(r)\dot{t}^2 = 0 \\ r^2\dot{\varphi} = L \\ \beta(r)\dot{t} = K. \end{cases}$$

Since $\gamma(s) = (r(s), \varphi(s), \pi/2, t(s))$ is a time-like geodesic we have

$$\text{(A.4)} \qquad \frac{1}{\beta(r)}\dot{r}^2 + r^2\dot{\varphi}^2 - c^2\beta(r)\dot{t}^2 = E < 0$$

so the constant K in the third equation of (A.3) is different from zero. Obviously we can assume

$$\text{(A.5)} \qquad K > 0.$$

Now assume that the time-like geodesic $\gamma : [a, s_0[\to \mathcal{L}$ satisfies

$$\liminf_{s \to s_0^-} \beta(r(s)) = 0.$$

Now, replacing in (A.4) $\beta(r)\dot{t}$ with K and $r^2\dot{\varphi}$ with L, (see (A.3)) we see that there exists $\beta_0 > 0$ such that

$$\dot{r}(s) \neq 0 \text{ for all } s \text{ such that } \beta(r(s)) \leq \beta_0.$$

Moreover $\beta'(r) \neq 0$ for all $r \neq 0$, hence the function $s \mapsto \beta(r(s))$ is monotone in a left neighbourhood of s_0 and therefore

$$\lim_{s \to s_0^-} \beta(r(s)) = 0,$$

i.e.

$$\text{(A.6)} \qquad \lim_{s \to s_0^-} r(s) = 2m.$$

Now, by (A.4) and (A.6) we get $\lim_{s \to s_0^-} \dot{r}(s) = \pm Kc \neq 0$. So, since $r(s) > 2m$ if $s < s_0$, there exist a left neighbourhood N^- of s_0 and two positive real constants δ_1 and δ_2 such that

$$\text{(A.7)} \qquad -\delta_1 \leq \dot{r}(s) \leq -\delta_2 < 0 \text{ for all } s \in N^-.$$

To conclude the proof of Proposition A.1 we shall prove that

$$\text{(A.8)} \qquad \lim_{s \to s_0^-} t(s) = +\infty.^{(10)}$$

To this aim, notice that, by the third equation of (A.3), we have

$$t(s) = t(a) + K \int_a^s \frac{1}{\beta(r(\tau))} \, \mathrm{d}\tau,$$

and putting $r(\tau) - 2m = \sigma$ and $r_0 = r(a) - 2m > 0$, we have

$$t(s) = t(a) + K \int_{r(s)-2m}^{r_0} \left(1 + \frac{2m}{\sigma}\right) \left(\frac{-1}{\dot{r}(\tau)}\right) \mathrm{d}\sigma.$$

Then by (A.5), (A.6) and (A.7), we get (A.8), because $r(s_0) - 2m > 0$ and $r(s) - 2m \to 0$ as $s \to s_0^-$. ∎

(10) We recall that we have assumed $K>0$.

REMARK. Consider the Reissner-Nordström spacetime and, when $m^2 > e^2$, put

$$r_+ = m + \sqrt{m^2 - e^2}, \text{ and } r_- = m - \sqrt{m^2 - e^2}.$$

By the same computations of the Proposition A.1, we see that

$$t(s) = t(a) + K \int_{r(s)-r_+}^{r(a)-r_+} \left(\frac{1}{\sigma}\right) \cdot \frac{(\sigma + r_+)^2}{(\sigma + r_+ - r_-)} \cdot \left(\frac{-1}{\dot{r}(\tau)}\right) \mathrm{d}\sigma,$$

so $t(s)$ diverges when a time-like geodesic approaches the topological boundary (where $r = r_+$). Therefore the Reissner-Nordström spacetime $\{r > r_+\} \times \mathbb{R}$ is a static universe.

PROPOSITION A.2. *The Schwarzschild spacetime is a static Lorentz manifold with convex boundary according to Definition* 1.7.

PROOF. Consider the function ϕ given by

$$\phi(r, \vartheta, \varphi, t) = \sqrt{\beta(r)}. \tag{A.9}$$

A simple calculation shows that ϕ satisfies (1.8). Then, clearly, to prove Proposition A.2 it suffices to see that ϕ satisfies (1.9). To this aim let $\gamma(s) = (r(s), \varphi(s), \pi/2, t(s))$ be a geodesic with respect to the Lorentz metric (1.4), i.e. a solution of (A.3). We have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} (\phi(\gamma(s))) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left(\sqrt{\beta(r(s))}\right)$$

where r solves the first equation of (A.3). Then

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}s^2} (\phi(\gamma(s))) &= \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{2} \beta^{-1/2}(r(s))\beta'(r(s))r'(s)\right) \\ &= -\frac{1}{4} \beta^{-3/2}(\beta')^2\dot{r}^2 + \frac{1}{2} \beta^{-1/2}\beta''\dot{r}^2 + \frac{1}{2} \beta^{-1/2}\beta' \cdot \frac{\mathrm{d}^2 r}{\mathrm{d}s^2} \\ &= \frac{1}{2} \beta^{-1/2}\beta''\dot{r}^2 + \frac{1}{4} \beta^{1/2}\beta' \left(2r\dot{\varphi}^2 - c^2\beta'(r)\dot{t}^2\right). \end{aligned}$$

Moreover there exists $E \in \mathbb{R}$ such that

$$\frac{1}{\beta(r)} \dot{r}^2 + r^2\dot{\varphi}^2 - c^2\beta(r)\dot{t}^2 = E, \tag{A.10}$$

so we have

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,(\phi(\gamma(s))) \\
&= \frac{1}{2}\,\beta^{-1/2}\left(\beta'' - \frac{\beta'}{r}\right)\dot r^2 + \frac{c^2}{4}\,\beta^{1/2}\beta'\left(\frac{2\beta}{r} - \beta'\right)\dot t^2 + \frac{1}{2}\,\beta^{1/2}\frac{\beta'}{r}\,c^2E \\
&\le \frac{c^2}{4}\,\beta^{1/2}\beta'\left(\frac{2\beta}{r} - \beta'\right)\dot t^2 + \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2E
\end{aligned}$$

because $\beta'' < 0$ and $\beta' > 0$.
Moreover when $r \le 3m$, $\dfrac{2\beta}{r} - \beta' \le 0$. Therefore if $\phi(\gamma) \le \sqrt{1/3}$, we have

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,(\phi(\gamma(s))) &\le \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2E \le \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2|E| \\
&\le \text{(by (A.10))}\ \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2\left|\frac{1}{\beta(r)}\,\dot r^2 + r^2\dot\varphi^2 - c^2\beta(r)\dot t^2\right|.
\end{aligned}$$

Now $\dfrac{1}{\beta(r)}\,\dot r^2 + r^2\dot\varphi^2 - c^2\beta(r)\dot t^2 = \langle\dot\gamma,\dot\gamma\rangle_L$ with respect to the Lorentz structure (1.4) because $\dot\vartheta \equiv 0$. Therefore we have, when $\phi(\gamma) \le \sqrt{1/3}$,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,(\phi(\gamma(s))) &\le \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2|\langle\dot\gamma(s),\dot\gamma(s)\rangle_L| \\
&= \frac{1\beta'}{2r}\,c^2|\langle\dot\gamma(s),\dot\gamma(s)\rangle_L|\cdot\phi(\gamma(s)).
\end{aligned}$$

Finally, since $r \ge 2m$ implies that $\dfrac{\beta'}{r} = \dfrac{2m}{r^3}$, we get (1.9) and the proof of Proposition A.2. ∎

REMARK. Performing similar computations for the Reissner-Nordström spacetime $\{r > r_+\} \times \mathbb{R}$, with

$$\phi = \sqrt{1 - \frac{2m}{r} + \frac{e^2}{r^2}},$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,(\phi(\gamma(s))) \\
&= \frac{1}{2}\,\beta^{-1/2}\left(\beta'' - \frac{\beta'}{r}\right)\dot r^2 + \frac{c^2}{4}\,\beta^{1/2}\beta'\left(\frac{2\beta}{r} - \beta'\right)\dot t^2 + \frac{1}{2}\,\beta^{1/2}\,\frac{\beta'}{r}\,c^2E,
\end{aligned}$$

where $\beta = 1 - \dfrac{2m}{r} + \dfrac{e^2}{r^2}$ and $E = \dfrac{1}{\beta(r)}\,\dot r^2 + r^2\dot\varphi^2 - c^2\beta(r)\dot t^2$. From this formula we deduce that the Reissner-Nordström spacetime $\{r > r_+\} \times \mathbb{R}$ is a static Lorentz manifold with convex boundary provided that $m^2 \geq \frac{9}{5} \cdot e^2$.

REFERENCES

[1] A. AVEZ, *Essais de géométrie Riemanniene hyperbolique: Applications to the relativité générale*, Ann. Inst. Fourier **132**, (1963), 105-190.

[2] V. BENCI, *Normal modes of a Lagrangian system constrained in a potential well*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **1**, (1984), 379-400.

[3] V. BENCI - D. FORTUNATO, *Existence of geodesics for the Lorentz metric of a stationary gravitational field*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **7**, (1990), 27-35.

[4] V. BENCI - D. FORTUNATO, *On the existence of infinitely many geodesics on space-time manifolds*, to appear in Adv. Math.

[5] V. BENCI - D. FORTUNATO - F. GIANNONI, *On the existence of multiple geodesics in static space-times*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **8**, (1991), 79-102.

[6] V. BENCI - D. FORTUNATO - F. GIANNONI, *Some results on the existence of geodesics in Lorentz manifolds with nonsmooth boundary*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **2**, (1991), 17-23.

[7] E. FADELL - A. HUSSEINI, *Category of loop spaces of open subsets in Euclidean space*, preprint.

[8] E. FADELL - A. HUSSEINI, *Infinite cup length in free loop spaces with application to a problem of the N-body type*, preprint.

[9] S.W. HAWKING - G.F. ELLIS, *The large scale structure of space-time*, Cambridge Univ. Press, London/New-York, 1973.

[10] W. KLINGENBERG, *Lectures on closed geodesics*, Springer-Verlag, Berlin/Heidelberg/New-York, 1978.

[11] M.D. KRUSKAL, *Maximal extension of Schwarzschild metric*, Phys. Rev., **119**, (1960), 1743-1745.

[12] L. LANDAU - E. LIFCHITZ, *Theorie des champs*, Mir, Moscou, 1970.

[13] J. NASH, *The embedding problem for Riemannian manifolds*, Ann. of Math. **63**, (1956), 20-63.

[14] B. O'NEILL, *Semi-Riemannian geometry with applications to relativity*, Academic Press Inc., New York/London, 1983.

[15] R.S. PALAIS, *Critical point theory and the minimax principle*, Global Anal., Proc. Sym. "Pure Math." **15**, Amer. Math. Soc. (1970), 185-202.

[16] R. PENROSE, *Techniques of differential topology in relativity*, Conf. Board Math. Sci. **7**, S.I.A.M. Philadelphia, 1972.

[17] J.T. SCHWARTZ, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

[18] H.J. SEIFERT, *Global connectivity by time-like geodesics*, Z. Naturforsch. **22a**, (1967), 1356-60.

[19] K. UHLENBECK, *A Morse theory for geodesics on a Lorentz manifold*, Topology **14**, (1975), 69-90.

Istituto di Matematica Applicata
Facoltà di Ingegneria
Università di Pisa
PISA

Dipartimento di Matematica
Università di Bari
BARI

Istituto di Matematica Applicata
Facoltà di Ingegneria
Università di Pisa
PISA