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# On a New Class of Generalized Solutions for the Stokes Equations in Exterior Domains

HIDEO KOZONO - HERMANN SOHR

## Introduction

Let  $n \geq 2$  and let  $\Omega$  be an exterior domain in  $\mathbb{R}^n$ , i.e., a domain having a compact complement  $\mathbb{R}^n/\Omega$ , and assume that the boundary  $\partial\Omega$  is of class  $C^{2+\mu}$  with  $0 < \mu < 1$ . Consider the following boundary value problem for the Stokes equations in  $\Omega$ :

$$(S) \quad -\Delta u + \nabla p = f, \quad \operatorname{div} u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where  $u = (u^1(x), \dots, u^n(x))$  and  $p = p(x)$  denote the unknown velocity and pressure, respectively;  $f = (f^1(x), \dots, f^n(x))$  and  $g = g(x)$  denote the given external force and the scalar divergence, respectively.

The purpose of the present paper is to extend the well-known concept of generalized solutions  $u$  of (S) having a finite Dirichlet integral

$$(D) \quad \int_{\Omega} |\nabla u(x)|^2 dx < \infty$$

(see, e.g., Chang-Finn [11], Finn [14], Fujita [16], Heywood [21]). We consider here a much larger class of the generalized solutions  $u$  of (S) satisfying

$$(CL_q) \quad \int_{\Omega} |\nabla u(x) - A|^q dx < \infty \text{ with some matrix } A,$$

where  $1 < q < \infty$ . In particular, setting  $A \equiv 0$ , we treat the class

$$(D_q) \quad \int_{\Omega} |\nabla u(x)|^q dx < \infty$$

which generalizes the Dirichlet integral to  $L^q$ -spaces.

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In the class  $(CL_q)$ , we can investigate the motion of the fluid past an obstacle rotating around its axis. Such a fluid motion is governed by  $(S)$  with the boundary condition at infinity

$$(B.C.)_\infty \quad u(x) \rightarrow Ax + a \quad \text{as } x \rightarrow \infty,$$

where  $A$  denotes a skew-symmetric matrix and the vector  $a$  is a constant. Another physical phenomenon described in the class  $(CL_q)$  is the flow due to an obstacle embedded in a pure straining tensor: far from the obstacle the fluid is in a pure stretching specified by the rate-of-strain tensor  $A$ , with  $\text{Tr } A = 0$ . Then the velocity  $u$  can be written as

$$u(x) = Ax + u_0(x),$$

where  $u_0$  represents the changes due to the presence of the obstacle, with  $u_0(x)$  small for large  $|x|$ . Such a solution describes a *suspension*, i.e., the motion of a small particle in the fluid, by which one can calculate an effective viscosity being different from that of the original fluid and determine the radius of particles. Einstein [12] calculated their quantities when the obstacle is a sphere (see Batchelor [3] and Landau-Lifschitz [23]).

In 1850, G.G. Stokes showed that, in general, in two-dimensional exterior domains, there is no solution  $u$  of  $(S)$  tending to a prescribed *non-zero constant vector at infinity*. We shall first generalize the “Stokes paradox” to higher dimensions and determine the exact class of solutions in which the paradox holds. Indeed, we shall treat the simpler class  $(D_q)$  and show that  $u \equiv 0$  is the only solution of  $(S)$  with  $f \equiv g \equiv 0$ , if  $1 < q \leq n/(n-1)$ . In the two-dimensional case, Finn [14] and Heywood [20] obtained similar results. Secondly, we shall give a concrete characterization of the *null-space* for the solutions of  $(S)$  in the class  $(CL_q)$ . Here we shall see that a non-trivial null-space appears when  $q$  varies and that the case  $q = n/(n-1)$  is critical. Finally, based on these results of the null-space, we shall give a theorem on the existence and uniqueness for the solutions of  $(S)$  in the class  $(CL_q)$ . This theorem holds if one can solve  $(S)$  with the boundary condition  $(B.C.)_\infty$  at infinity.

Our basic tool consists of the two fundamental facts, a *regularity theory* (Theorem 3.1) and an *a priori estimate* (Theorem 3.3) in  $L^q$ -spaces for the gradient of solutions of  $(S)$ . The regularity theorem is useful to show the Stokes paradox in higher dimensions and enables us to see why the critical value  $q = n/(n-1)$  appears in solvability of  $(S)$ . The a priori estimate plays a basic role in characterization of the null-space and range of solutions. Such an estimate has been got by several authors for  $n \geq 3$  (Kozono-Sohr [22], Borchers-Miyakawa [7]). Recently, Galdi-Simader [18] obtained a similar result to ours by using the hydrodynamic potentials. Our method is however different from Galdi-Simader’s [18]: we are based on the cut-off procedure. Making use of a simple embedding argument about a certain functional space, we shall show the same a priori estimate holding for all  $n \geq 2$ .

Concerning characterization of the null-space in the class  $(CL_q)$ , Maslennikova-Timoshin [25] solved  $(S)$  explicitly in an exterior domain of the unit sphere in  $\mathbb{R}^3$  and announced a similar result to ours. They have used the special functions (the Legendre functions) for representation of the solution. We shall give a more systematic treatment for generalized solutions of  $(S)$ . Our approach is so different from [25] that we can apply it to all dimensions  $n \geq 2$ . For another investigation such as strong solutions, see, *e.g.*, Sohr-Varnhorn [30].

## 1. - Main Results

**1.1.** Before stating our results we introduce some notations. For  $1 < q < \infty$  ( $q' = q/(q-1)$ ),  $\|\cdot\|_q$  and  $(\cdot, \cdot)$  denote the usual norm of  $L^q(\Omega)$  and the inner product between  $L^q(\Omega)$  and  $L^{q'}(\Omega)$ , respectively. In general we shall denote by  $(f, \phi)$  the value of the distribution  $f$  at  $\phi \in C_0^\infty(\Omega)$ .  $\hat{H}_0^{1,q}(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with respect to the norm  $\|\nabla u\|_q$ . Since  $\Omega$  is an exterior domain,  $\hat{H}_0^{1,q}(\Omega)$  is larger than  $H_0^{1,q}(\Omega)$ . Having introduced  $\hat{H}_0^{1,q}(\Omega)$ , it is also useful to define  $\hat{H}^{-1,q}(\Omega) := \hat{H}_0^{1,q}(\Omega)^*$  ( $X^*$ ; dual space of  $X$ ), and  $\|\cdot\|_{-1,q}$  denotes the norm of  $\hat{H}^{-1,q}(\Omega)$  defined by  $\|f\|_{-1,q} := \sup \{ |(f, \phi)| / \|\nabla \phi\|_{q'}; \phi \in C_0^\infty(\Omega), \phi \neq 0 \}$ . We shall denote by  $C_0^\infty(\Omega)^n$ ,  $L^q(\Omega)^n, \dots$ , and  $C_0^\infty(\Omega)^{n^2}$ ,  $L^q(\Omega)^{n^2}, \dots$  the corresponding spaces for the vector-valued and the matrix-valued functions, respectively. In such spaces, we shall also use the same notations  $\|\cdot\|_q$  and  $(\cdot, \cdot)$ .

Let  $f \in \hat{H}^{-1,q}(\Omega)^n$  and  $g \in L_{\text{loc}}^q(\bar{\Omega})$ , where  $g \in L_{\text{loc}}^q(\bar{\Omega})$  means that  $\int_{\Omega \cap B} |g(x)|^q dx < \infty$  for all open balls  $B$  in  $\mathbb{R}^n$  with  $\Omega \cap B \neq \emptyset$ . A pair  $\{u, p\} \in H_{\text{loc}}^{1,q}(\bar{\Omega})^n \times L^q(\Omega)$  with  $u|_{\partial\Omega} = 0$  (in the trace sense) is called a *generalized solution* of  $(S)$  if

$$(\nabla u, \nabla \Phi) - (p, \text{div } \Phi) = (f, \Phi), \quad -(u, \nabla \phi) = (g, \phi)$$

for all  $\Phi \in C_0^\infty(\Omega)^n$  and all  $\phi \in C_0^\infty(\Omega)$ , respectively.

**1.2.** Our result on the generalized Stokes paradox now reads:

**THEOREM A.** (Stokes paradox). *Let  $n \geq 2$  and  $1 < q \leq n'$  ( $n' \equiv n/(n-1)$ ). Suppose that  $\{u, p\} \in H_{\text{loc}}^{1,q}(\bar{\Omega})^n \times L^q(\Omega)$  is a generalized solution of  $(S)$  with  $f \equiv 0$ ,  $g \equiv 0$  satisfying  $\nabla u \in L^q(\Omega)^{n^2}$ . Then it follows that  $u \equiv 0$ ,  $p \equiv 0$  in  $\Omega$ .*

By Bogovskii's result [6], the pressure  $p$  is determined by  $u$ , and hence we can restate the above theorem without  $p$ .

**THEOREM A'.** *Let  $n \geq 2$  and  $1 < q \leq n'$ . Suppose that  $u \in H_{\text{loc}}^{1,q}(\bar{\Omega})^n$  satisfies  $\text{div } u = 0$  in  $\Omega$ ,  $u|_{\partial\Omega} = 0$ , and  $(\nabla u, \nabla \Phi) = 0$  for all  $\Phi \in C_0^\infty(\Omega)^n$  with  $\text{div } \Phi = 0$ . If, in addition,  $\nabla u \in L^q(\Omega)^{n^2}$ , then we have  $u \equiv 0$  in  $\Omega$ .*

REMARKS. 1. In the above theorem, we do not assume any integrability condition on  $u$  itself. It follows that there is no solution  $u$  of  $(S)$  with  $f \equiv 0$ ,  $g \equiv 0$  in the class  $(D_q)$  for  $1 < q \leq n'$  such that  $u(x) \rightarrow a$  as  $x \rightarrow \infty$ , where  $a$  is a *non-zero* constant vector in  $\mathbb{R}^n$ .

2. Heywood [20] showed the same result in the special case  $n = q = 2$ . Chang-Finn [11] gave a similar result for  $n = 2$  in the class  $u(x) = o(\log |x|)$  as  $x \rightarrow \infty$ .

**1.3.** We next proceed to the characterization of the null-space for  $(S)$  in the class  $(CL_q)$ .

Let us denote by  $\mathbb{N}_q$  the set of all generalized solutions  $\{u, p\} \in H_{\text{loc}}^{1,q}(\overline{\Omega})^n \times L^q(\Omega)$  of  $(S)$  with  $f \equiv 0$ ,  $g \equiv 0$  satisfying  $\nabla u - A \in L^q(\Omega)^{n^2}$  for some matrix  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$ .  $\mathbb{N}_q^0$  is the subspace of  $\mathbb{N}_q$  defined by  $\mathbb{N}_q^0 \equiv \left\{ \{u, p\} \in \mathbb{N}_q; \nabla u \in L^q(\Omega)^{n^2} \right\}$ .

Our second result now reads:

**THEOREM B.** (Characterization of the null-space). (i) *Let  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Then  $\dim \mathbb{N}_q = n^2 - 1$  and  $\dim \mathbb{N}_q^0 = 0$ . For every  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$  and  $a \in \mathbb{R}^n$  satisfying the condition*

$$(1.1) \quad \int_{\partial\Omega} \left\{ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right\} dS = 0$$

*for all  $\{v, \chi\} \in \mathbb{N}_q^0$ , there exists a unique  $\{u, p\} \in \mathbb{N}_q$  such that*

$$(1.2) \quad \nabla u - A \in L^q(\Omega)^{n^2},$$

$$(1.3) \quad u \in C^0(\overline{\Omega})^n, \quad \lim_{x \rightarrow \infty} |u(x) - (Ax + a)| = 0,$$

*where  $\nu$  denotes the unit outer normal to  $\partial\Omega$  and  $dS$  is the surface element of  $\partial\Omega$ . Conversely, for every  $\{u, p\} \in \mathbb{N}_q$ , there are unique  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$  and  $a \in \mathbb{R}^n$  such that (1.2) and (1.3) hold.*

(ii) *Let  $n' < q < \infty$ ,  $n \geq 2$ . Then  $\dim \mathbb{N}_q = n^2 + n - 1$  and  $\dim \mathbb{N}_q^0 = n$ . For every  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$  and  $a \in \mathbb{R}^n$ , there exists a unique  $\{u, p\} \in \mathbb{N}_q$  such that (1.2) and (1.3) hold if  $n \geq 3$ , and such that (1.2) and*

$$(1.3') \quad \int_{\Omega} |\nabla[u(x) - Ax - E(x)a]|^2 dx < \infty$$

*hold if  $n = 2$ , where  $E = (E_{ij}(x))_{i,j=1,2}$  denotes the fundamental tensor of the Stokes equations in  $\mathbb{R}^2$ :  $E_{ij}(x) = (4\pi)^{-1} \left[ \log \left( \frac{\delta_{ij}}{|x|} \right) + \frac{x_i x_j}{|x|^2} \right]$ . Conversely, for every  $\{u, p\} \in \mathbb{N}_q$ , there are unique  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$  and  $a \in \mathbb{R}^n$  such that (1.2)-(1.3) hold if  $n \geq 3$ , and such that (1.2)-(1.3') hold if  $n = 2$ .*

(iii) Let  $n = q = 2$ . Then  $\dim \mathbb{N}_2 = n^2 - 1 = 3$  and  $\dim \mathbb{N}_2^0 = 0$ . For every  $A \in \mathbb{R}^{2^2}$  with  $\text{Tr } A = 0$ , there is a unique  $\{u, p\} \in \mathbb{N}_2$  such that (1.2) holds with  $q = 2$ . Conversely, for every  $\{u, p\} \in \mathbb{N}_2$ , there is a unique  $A \in \mathbb{R}^{2^2}$  with  $\text{Tr } A = 0$  such that (1.2) holds with  $q = 2$ .

REMARK. For  $\{v, \chi\} \in \mathbb{N}_q^0$ , we have  $\nabla v - \chi I \in L^{q'}(\Omega)^{n^2}$  and  $\text{div}(\nabla v - \chi I) = 0$  in the sense of distributions in  $\Omega$ , where  $I$  is the identity matrix in  $\mathbb{R}^{n^2}$ . Using the trace theorem as in Miyakawa [26, Proposition 1.2] and Simader-Sohr [29], we see that  $\frac{\partial v}{\partial \nu} - \chi I \cdot \nu \in \left(H^{1/q', q}(\partial\Omega)^n\right)^*$  and hence (1.1) should be understood in such a generalized sense as the duality between  $\frac{\partial v}{\partial \nu} - \chi I \cdot \nu \in \left(H^{1/q', q}(\partial\Omega)^n\right)^*$  and  $Ax + a \in H^{1/q', q}(\partial\Omega)^n$ . However, from the regularity theorem in bounded domains (as Cattabriga [10] shows), we get  $v \in H_{\text{loc}}^{2, q'}(\overline{\Omega})^n$ ,  $\chi \in H_{\text{loc}}^{2, q'}(\overline{\Omega})$ ; therefore (1.1) may be also regarded in the usual sense.

**1.4.** We are next concerned with the necessary and sufficient condition for the solvability of (S) in the class  $(CL_q)$ .

THEOREM C. (Inhomogeneous case). (i) Let  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Then for every  $f \in \hat{H}^{-1, q}(\Omega)^n$ ,  $g \in L_{\text{loc}}^q(\overline{\Omega})$ ,  $A \in \mathbb{R}^{n^2}$  with  $g - \text{Tr } A \in L^q(\Omega)$  and  $a \in \mathbb{R}^n$ , there exists a generalized solution  $\{u, p\} \in H_{\text{loc}}^{1, q}(\overline{\Omega})^n \times L^q(\Omega)$  of (S) satisfying (1.2) and

$$(1.4) \quad \int_{\Omega} |u(x) - (Ax + a)|^{nq/(n-q)} dx < \infty,$$

if and only if the compatibility condition

$$(1.5) \quad (f, v) - (g - \text{Tr } A, \chi) + \int_{\partial\Omega} \left\{ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right\} dS = 0$$

holds for all  $\{v, \chi\} \in \mathbb{N}_q^0$ . Such  $\{u, p\}$  is unique and subject to the inequality

$$(1.6) \quad \|\nabla u - A\|_q + \|p\|_q \leq C (\|f\|_{-1, q} + \|g - \text{Tr } A\|_q + |A| + |a|)$$

with  $C = C(\Omega, n, q) > 0$  independent of  $u$  and  $p$ , where  $|A|$  and  $|a|$  denote the standard Euclidian norms in  $\mathbb{R}^{n^2}$  and  $\mathbb{R}^n$ , respectively.

(ii) Let  $n' < q < n$ ,  $n \geq 3$ . Then for every  $f \in \hat{H}^{-1, q}(\Omega)^n$ ,  $g \in L_{\text{loc}}^q(\overline{\Omega})$ ,  $A \in \mathbb{R}^{n^2}$  with  $g - \text{Tr } A \in L^q(\Omega)$  and  $a \in \mathbb{R}^n$ , there exists a unique generalized solution  $\{u, p\} \in H_{\text{loc}}^{1, q}(\overline{\Omega})^n \times L^q(\Omega)$  of (S) such that (1.2) and (1.4) hold. Such  $\{u, p\}$  is subject to the inequality (1.6). If in addition  $f \in \hat{H}^{-1, r}(\Omega)^n$ ,  $g - \text{Tr } A \in L^r(\Omega)$  for some  $r > n$ , we have also  $\nabla u - A \in L^r(\Omega)^{n^2}$ ,  $p \in L^r(\Omega)$  and (1.3).

(iii) Let  $n \leq q < \infty$  for  $n \geq 3$  and  $2 < q < \infty$  for  $n = 2$ . Then for every  $f \in \hat{H}^{-1,q}(\Omega)^n$ ,  $g \in L^q_{\text{loc}}(\bar{\Omega})$  and  $A \in \mathbb{R}^{n^2}$  with  $g - \text{Tr } A \in L^q(\Omega)$ , there exists at least one generalized solution  $\{u, p\} \in H^{1,q}_{\text{loc}}(\bar{\Omega})^n \times L^q(\Omega)$  of (S) satisfying (1.2). Such  $\{u, p\}$  is unique modulo  $\mathbb{N}_q^0$  and subject to the inequality

$$(1.7) \quad \inf \{ \|\nabla u - A - \nabla v\|_q + \|p - \chi\|_q; \{v, \chi\} \in \mathbb{N}_q^0 \} \\ \leq C (\|f\|_{-1,q} + \|g - \text{Tr } A\|_q + |A|),$$

where  $C = C(\Omega, n, q) > 0$ .

(iv) Let  $n = q = 2$ . Then for every  $f \in \hat{H}^{-1,2}(\Omega)^2$ ,  $g \in L^2_{\text{loc}}(\bar{\Omega})$  and  $A \in \mathbb{R}^{2^2}$  with  $g - \text{Tr } A \in L^2(\Omega)$ , there exists a unique generalized solution  $\{u, p\} \in H^{1,2}_{\text{loc}}(\bar{\Omega})^2 \times L^2(\Omega)$  of (S) satisfying (1.2) with  $q = 2$ . Such  $\{u, p\}$  is subject to the inequality

$$(1.8) \quad \|\nabla u - A\|_2 + \|p\|_2 \leq C (\|f\|_{-1,2} + \|g - \text{Tr } A\|_2 + |A|),$$

where  $C = C(\Omega) > 0$ .

REMARKS. 1. In case (i), the compatibility condition (1.5) is *necessary and sufficient* for the solvability of (S).

2. In case (ii), the additional condition  $f \in \hat{H}^{-1,r}(\Omega)^n$ ,  $g - \text{Tr } A \in L^r(\Omega)$  ( $r > n$ ) enables us to get the smoothness of  $u$  and its asymptotic behaviour (1.3). In case (iii) we cannot prescribe  $a \in \mathbb{R}^n$  so that the uniqueness follows. However, if we assume in addition that  $f \in \hat{H}^{-1,\gamma}(\Omega)^n$ ,  $g - \text{Tr } A \in L^\gamma(\Omega)$  for some  $n' < \gamma < n$ , then we can prescribe  $a \in \mathbb{R}^n$  so as to get the unique solvability under the condition (1.3).

## 2. - Preliminaries

### 2.1. Homogeneous Sobolev space $\hat{H}_0^{1,q}(\Omega)$ .

In this subsection we shall give a concrete characterization of  $\hat{H}_0^{1,q}(\Omega)$  and some elementary lemmas for the proof of the main results.

Let  $D$  be a domain in  $\mathbb{R}^n$  ( $n \geq 2$ ). We denote by  $\|\cdot\|_{q,D}$  and  $(\cdot, \cdot)_D$  the norm of  $L^q(D)$  and the inner product between  $L^q(D)$  and  $L^{q'}(D)$ , respectively.  $\hat{H}_0^{1,q}(D)$  is the completion of  $C_0^\infty(D)$  with respect to the norm  $\|\nabla u\|_{q,D}$ . If  $D$  is bounded, the Poincaré inequality states that  $\hat{H}_0^{1,q}(D) = H_0^{1,q}(D)$ , but, in general,  $\hat{H}_0^{1,q}(D)$  is larger than  $H_0^{1,q}(D)$ .  $\hat{H}^{-1,q}(D)$  is the dual space of  $\hat{H}_0^{1,q'}(D)$  ( $1/q + 1/q' = 1$ ) whose norm is denoted by  $\|\cdot\|_{-1,q,D}$ . In case  $D = \Omega$ , we shall call these norms  $\|\cdot\|_q$ ,  $(\cdot, \cdot)$ , and  $\|\cdot\|_{-1,q}$ . In what follows  $C$  denotes a constant which may change from line to line. In particular,  $C = C(*, \dots, *)$  denotes a constant depending only on the quantities appearing in the parentheses.

The following inequality is simple but very useful for the forthcoming arguments (see also Simader-Sohr [29]).

*Variational inequality in  $L^q$ .* Let  $n \geq 2$  and  $1 < q < \infty$ . Suppose that  $u \in L^q_{\text{loc}}(\mathbb{R}^n)$  with  $\nabla u \in L^q(\mathbb{R}^n)^n$ . Then we have

$$(2.1) \quad \begin{aligned} & \|\nabla u\|_{q, \mathbb{R}^n} \\ & \leq C \sup \left\{ \frac{|(\nabla u, \nabla \phi)_{\mathbb{R}^n}|}{\|\nabla \phi\|_{q', \mathbb{R}^n}}; \phi \in C_0^\infty(\mathbb{R}^n), \phi \neq 0 \right\} \end{aligned}$$

with  $C = C(n, q)$  independent of  $u$ .

Indeed, the Calderon-Zygmund inequality gives

$$\|\nabla \nabla \psi\|_{q', \mathbb{R}^n} \leq C \|\Delta \psi\|_{q', \mathbb{R}^n} \quad (\psi \in C_0^\infty(\mathbb{R}^n)),$$

Then, since the space  $H \equiv \{\Delta \psi; \psi \in C_0^\infty(\mathbb{R}^n)\}$  is dense in  $L^{q'}(\mathbb{R}^n)$ , we have for each  $i = 1, \dots, n$

$$\begin{aligned} & \sup \left\{ \frac{|(\nabla u, \nabla \phi)_{\mathbb{R}^n}|}{\|\nabla \phi\|_{q', \mathbb{R}^n}}; \phi \in C_0^\infty(\mathbb{R}^n), \phi \neq 0 \right\} \\ & \geq \sup \left\{ \frac{|(\nabla u, \nabla(\partial_i \psi))_{\mathbb{R}^n}|}{\|\nabla(\partial_i \psi)\|_{q', \mathbb{R}^n}}; \psi \in C_0^\infty(\mathbb{R}^n), \psi \neq 0 \right\} \\ & \geq C \sup \left\{ \frac{|(\partial_i u, \Delta \psi)_{\mathbb{R}^n}|}{\|\Delta \psi\|_{q', \mathbb{R}^n}}; \psi \in C_0^\infty(\mathbb{R}^n), \psi \neq 0 \right\} \\ & = C \sup \left\{ \frac{|(\partial_i u, g)_{\mathbb{R}^n}|}{\|g\|_{q', \mathbb{R}^n}}; g \in L^{q'}(\mathbb{R}^n), g \neq 0 \right\} \\ & = C \|\partial_i u\|_{q, \mathbb{R}^n} \end{aligned}$$

with  $C = C(n, q)$ , and (2.1) follows.

Based on the above variational inequality, we get the following approximation lemma.

**LEMMA 2.1.** *Let  $n \geq 2$  and  $1 < q < \infty$ . Then for every  $u \in L^q_{\text{loc}}(\overline{\Omega})$  with  $\nabla u \in L^q(\Omega)^n$ , there is a sequence  $\{u_j\}_{j=1}^\infty$  in  $C_0^\infty(\overline{\Omega})$  such that  $\nabla u_j \rightarrow \nabla u$  in  $L^q(\Omega)^n$ , where  $C_0^\infty(\overline{\Omega})$  is the set of all  $C^\infty$ -functions  $\phi$  with compact support in  $\overline{\Omega}$  ( $\phi$  may not vanish on  $\partial\Omega$ ). The same assertion is true with  $\Omega$  replaced by  $\mathbb{R}^n$ .*

**PROOF.** By the extension theorem (Adams [1]), for each  $u \in L^q_{\text{loc}}(\overline{\Omega})$  with  $\nabla u \in L^q(\Omega)^n$ , there is a function  $\tilde{u} \in L^q_{\text{loc}}(\mathbb{R}^n)$  with  $\nabla \tilde{u} \in L^q(\mathbb{R}^n)^n$  such that  $\tilde{u} = u$  in  $\Omega$ , so we may only prove the assertion on  $\mathbb{R}^n$ . Let  $L^{1,q} = \{u \in L^q_{\text{loc}}(\mathbb{R}^n); \nabla u \in L^q(\mathbb{R}^n)^n\}$ . We denote by  $[u]$  the set of all  $v \in L^{1,q}$  such that  $u - v$  is a constant function on  $\mathbb{R}^n$ , and set  $L^{1,q}/\mathbb{R} \equiv \{[u]; u \in L^{1,q}\}$  and  $G_q \equiv \{\nabla u \in L^q(\mathbb{R}^n)^n; [u] \in L^{1,q}/\mathbb{R}\}$ . We may regard  $G_q$  as a closed



subspace of  $L^q(\mathbb{R}^n)^n$ ; equipped with the norm  $\| [u] \|_{L^{1,q}/\mathbb{R}} := \| \nabla u \|_{q,\mathbb{R}^n}$ ,  $L^{1,q}/\mathbb{R}$  is a Banach space isometric to  $G_q$ . Hence it suffices to prove that the space  $W \equiv \{ \nabla \phi; \phi \in C_0^\infty(\mathbb{R}^n) \}$  is dense in  $G_q$ . To this end, let us consider a map  $A_q : \nabla u \in G_q \rightarrow A_q(\nabla u) \in G_q^*$  defined by  $\langle A_q(\nabla u), \nabla v \rangle = (\nabla u, \nabla v)_{\mathbb{R}^n}$  for  $\nabla v \in G_q$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $G_q^*$  and  $G_q$ . Then by (2.1) we see that  $A_q$  is injective and that its range is closed in  $G_q^*$ . Since  $A_q^*$  coincides with  $A_q$  ( $T^*$ ; adjoint operator of  $T$ ), it follows from the closed range theorem that  $A_q$  is also surjective and hence bijective. Now, suppose that  $F \in G_q^*$  satisfies  $\langle F, \nabla \phi \rangle = 0$  for all  $\phi \in C_0^\infty(\mathbb{R}^n)$ . Since  $A_q$  is also bijective, there is a unique  $\nabla u \in G_q$  such that  $\langle F, \nabla u \rangle = \langle A_q(\nabla u), \nabla v \rangle = (\nabla u, \nabla v)_{\mathbb{R}^n}$  holds for all  $\nabla v \in G_q$ . Then by the assumption and (2.1) we get  $\nabla u = 0$  and hence  $F = 0$ , which implies that  $W$  is dense in  $G_q$ .  $\square$

REMARK. Simader [28] gave another proof of this lemma by using the Poincaré inequality on annular domains and a scaling argument.

The following concrete characterization of the space  $\hat{H}_0^{1,q}(\Omega)$  is essentially due to Galdi-Simader [18, Theorem 1.1]. Based on Lemma 2.1, we give here another proof.

LEMMA 2.2 (Galdi-Simader). (i) For  $1 < q < n$ , we have

$$\hat{H}_0^{1,q}(\Omega) = \left\{ u \in L^{nq/(n-q)}(\Omega); \nabla u \in L^q(\Omega)^n, u|_{\partial\Omega} = 0 \right\}.$$

(ii) For  $n \leq q < \infty$ , we have

$$\hat{H}_0^{1,q}(\Omega) = \left\{ u \in L_{\text{loc}}^q(\overline{\Omega}); \nabla u \in L^q(\Omega)^n, u|_{\partial\Omega} = 0 \right\}.$$

If  $n < q$ , the function  $u \in \hat{H}_0^{1,q}(\Omega)$  is continuous on  $\overline{\Omega}$  (after redefinition on a set of measure zero of  $\Omega$ ) and satisfies

$$u(x) = O\left(|x|^{1-n/q}\right) \text{ as } x \rightarrow \infty.$$

PROOF. Let  $H_q$  be the space defined by the right-hand side of (i) and (ii). By the Sobolev inequality, it is easy to see that  $\hat{H}_0^{1,q}(\Omega) \subset H_q$  and so we may only prove the converse inclusion. To this end, we introduce an extension operator  $\Gamma$ . Take  $R > 0$  so that  $\partial\Omega \subset B_R \equiv \{x \in \mathbb{R}^n; |x| < R\}$  and consider a continuous extension operator  $\Gamma : H^{1-1/q,q}(\partial\Omega) \rightarrow H^{1,q}(\Omega)$  satisfying  $\text{supp } \Gamma\phi \subset B_R$  for all  $\phi \in H^{1-1/q,q}(\partial\Omega)$ .

(i) Case  $1 < q < n$ . Let  $u \in H_q$ . Then by Lemma 2.1, there is a sequence  $\{u_j\}_{j=1}^\infty$  in  $C_0^\infty(\overline{\Omega})$  such that  $\nabla u_j \rightarrow \nabla u$  in  $L^q(\Omega)^n$ . Since  $u \in L^{nq/(n-q)}(\Omega)$ , it follows from the Sobolev inequality that  $u_j \rightarrow u$  in  $L^{nq/(n-q)}(\Omega)$ . Then by the trace theorem, we get  $u_j|_{\partial\Omega} \rightarrow 0$  in  $H^{1-1/q,q}(\partial\Omega)$ . Setting  $w_j = u_j - \Gamma(u_j|_{\partial\Omega})$ , we get  $w_j \in \hat{H}_0^{1,q}(\Omega)$  and it follows from the continuity of  $\Gamma$  that

$$\| \nabla w_j - \nabla u \|_q \leq \| \nabla u_j - \nabla u \|_q + C \| u_j|_{\partial\Omega} \|_{H^{1-1/q,q}(\partial\Omega)}$$

with  $C$  independent of  $j$ . Hence  $\nabla w_j \rightarrow \nabla u$  in  $L^q(\Omega)^n$ , and since  $C_0^\infty(\Omega)$  is dense in  $\hat{H}_0^{1,q}(\Omega)$ , we obtain  $u \in \hat{H}_0^{1,q}(\Omega)$ .

(ii) Case  $n \leq q < \infty$ . Let  $u \in H_q$ . Then it follows from Lemma 2.1 and a standard argument that there are sequences  $\{u_j\}_{j=1}^\infty$  in  $C_0^\infty(\bar{\Omega})$  and  $\{c_j\}_{j=1}^\infty$  in  $\mathbb{R}$  such that  $\nabla u_j \rightarrow \nabla u$  in  $L^q(\Omega)^n$  and  $u_j + c_j \rightarrow u$  in  $L_{\text{loc}}^q(\bar{\Omega})$ . We shall next approximate the sequence  $\{c_j\}_{j=1}^\infty$  in terms of a sequence of functions in  $C_0^\infty(\Omega)$  with respect to the norm  $\|\nabla \cdot\|_q$ . Take  $\zeta \in C_0^\infty(\mathbb{R}^n)$  satisfying  $0 \leq \zeta \leq 1$ ,  $\zeta(x) = 1$  for  $|x| \leq 1$  and  $\zeta(x) = 0$  for  $|x| \geq 2$  and set  $\zeta_k(x) = \zeta(x/k)$  ( $k = 1, 2, \dots$ ). The sequence  $\{\zeta_k\}_{k=1}^\infty$  will be called a sequence of  $n$ -dimensional cut-off functions. Then we have  $\zeta_k(x) = 1$  for  $|x| \leq k$  and  $\|\nabla \zeta_k\|_{q, \mathbb{R}^n} \leq Ck^{-1+n/q}$  ( $k = 1, 2, \dots$ ) with  $C$  independent of  $k$ . Since  $n \leq q$ , by Mazur's theorem ([33, p. 120 Theorem 2]), we can choose a sequence  $\{\bar{\zeta}_k\}_{k=1}^\infty$  of convex combinations of  $\zeta_k$ 's so that

$$\nabla \bar{\zeta}_k \rightarrow 0 \text{ in } L^q(\mathbb{R}^n)^n, \quad \bar{\zeta}_k \rightarrow 1 \text{ locally uniformly in } \mathbb{R}^n.$$

Hence there is a subsequence  $\{\bar{\zeta}_{k(j)}\}_{j=1}^\infty$  of  $\{\bar{\zeta}_k\}_{k=1}^\infty$  such that  $c_j \|\nabla \bar{\zeta}_{k(j)}\|_{q, \mathbb{R}^n} \rightarrow 0$  as  $j \rightarrow \infty$ . Defining  $\bar{u}_j = u_j + c_j \bar{\zeta}_{k(j)}$  ( $j = 1, 2, \dots$ ), we have  $\bar{u}_j \in C_0^\infty(\bar{\Omega})$  and  $\nabla \bar{u}_j \rightarrow \nabla u$  in  $L^q(\Omega)^n$ ,  $\bar{u}_j \rightarrow u$  in  $L_{\text{loc}}^q(\bar{\Omega})$ . Now, making use of a sequence  $w_j = \bar{u}_j - \Gamma(\bar{u}_j|_{\partial\Omega})$  ( $j = 1, 2, \dots$ ) as in the case of (i), we can prove similarly as above that  $u \in \hat{H}_0^{1,q}(\Omega)$ .

Finally, the asymptotic behaviour  $u(x) = O(|x|^{1-n/q})$ ,  $x \rightarrow \infty$  for  $u \in \hat{H}_0^{1,q}(\Omega)$  with  $q > n$  follows from Friedman [15, p. 23 Theorem 9.2].  $\square$

We shall next consider the complex interpolation space  $[X, Y]_\theta$  ( $0 \leq \theta \leq 1$ ). For all  $1 < q, r < \infty$ , the norms  $\|\nabla u\|_q$  and  $\|\nabla u\|_r$  are consistent on  $C_0^\infty(\Omega)$ , so the pair  $\{\hat{H}_0^{1,q}(\Omega), \hat{H}_0^{1,r}(\Omega)\}$  is interpolation couple. See Reed-Simon [27, p. 35]. Using the Riesz-Thorin theorem [32, 1.18.7], we obtain from Lemma 2.2 the following result:

If  $1 < q < n$ ,  $1 < r < n$  and if  $n \leq q < \infty$ ,  $n \leq r < \infty$ , then

$$(2.2) \quad \left[ \hat{H}_0^{1,q}(\Omega), \hat{H}_0^{1,r}(\Omega) \right]_\theta = \hat{H}_0^{1,s}(\Omega),$$

where  $1/s = (1 - \theta)/q + \theta/r$ ,  $0 \leq \theta \leq 1$ .

In the whole space  $\mathbb{R}^n$ , we shall prove the corresponding result without restriction on  $q$  and  $r$ .

LEMMA 2.3. *Let  $n \geq 2$  and  $1 < q < \infty$ ,  $1 < r < \infty$ . Then we have*

$$\left[ \hat{H}_0^{1,q}(\mathbb{R}^n), \hat{H}_0^{1,r}(\mathbb{R}^n) \right]_\theta = \hat{H}_0^{1,s}(\mathbb{R}^n),$$

where  $1/s = (1 - \theta)/q + \theta/r$ ,  $0 \leq \theta \leq 1$ .

PROOF. Let  $E_q \equiv \{\nabla u \in L^q(\mathbb{R}^n)^n; u \in \hat{H}_0^{1,q}(\mathbb{R}^n)\}$ . Then we may regard  $E_q$  as a closed subspace of  $L^q(\mathbb{R}^n)^n$ . Hence  $E_q$  is a Banach space with the norm

$\|\nabla u\|_{E_q} := \|\nabla u\|_{q, \mathbb{R}^n}$  for  $\nabla u \in E_q$ , and isometric to  $\hat{H}_0^{1,q}(\mathbb{R}^n)$ . Now it suffices to show that

$$(2.3) \quad [E_q, E_r]_\theta = E_s \text{ for } q, r, s \text{ and } \theta \text{ as above.}$$

To this end, we need to solve the equation  $\Delta \chi = \operatorname{div} u$  in  $\mathbb{R}^n$  in the following weak sense:

For every  $u \in L^q(\mathbb{R}^n)^n$ , there is a unique  $\chi \in \hat{H}_0^{1,q}(\mathbb{R}^n)$  such that

$$(2.4) \quad (\nabla \chi, \nabla \phi)_{\mathbb{R}^n} = (u, \nabla \phi)_{\mathbb{R}^n} \text{ for all } \phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n).$$

Based on (2.1), we see as in the proof of Lemma 2.1 that the map  $B_q : \nabla u \in E_q \rightarrow B_q(\nabla u) \in E_q^*$  defined by  $\langle B_q(\nabla u), \nabla v \rangle := (\nabla u, \nabla v)_{\mathbb{R}^n}$  for  $\nabla v \in E_{q'}$  is a bijective operator. Here  $\langle \cdot, \cdot \rangle$  denotes the duality between  $E_q^*$  and  $E_{q'}$ . Since the map  $\nabla \phi \in E_{q'} \rightarrow (u, \nabla \phi)_{\mathbb{R}^n} \in \mathbb{R}$  is a continuous functional on  $E_{q'}$ , we can solve (2.4) uniquely for every given  $u \in L^q(\mathbb{R}^n)^n$ .

Now, it is easy to see that the map  $Q : u \rightarrow \nabla \chi$  defined by the relation (2.4) is a projection operator from  $L^q(\mathbb{R}^n)^n$  onto  $E_q$ . Then (2.3) follows from Bergh-Löfström [4, Theorem 6.4.2].  $\square$

We need further the following two lemmas.

LEMMA 2.4. *Let  $1 < q < \infty$  and  $h \in L^q(\mathbb{R}^n)$ . If*

$$\sup \left\{ \frac{|(h, \Delta \phi)_{\mathbb{R}^n}|}{\|\Delta \phi\|_{r', \mathbb{R}^n}}; \phi \in C_0^\infty(\mathbb{R}^n), \phi \neq 0 \right\} < \infty$$

*for some  $1 < r < \infty$ , then we have also  $h \in L^r(\mathbb{R}^n)$ .*

PROOF. Here we follow Simader-Sohr [29]. Since the space  $H \equiv \{\Delta \phi; \phi \in C^\infty(\mathbb{R}^n)\}$  is a dense subspace in  $L^{r'}(\mathbb{R}^n)$ , we see by the assumption that the map  $\Delta \phi \in H \rightarrow (h, \Delta \phi)_{\mathbb{R}^n} \in \mathbb{R}$  is uniquely extended as a continuous functional on  $L^{r'}(\mathbb{R}^n)$ . Hence there is a unique  $\eta \in L^r(\mathbb{R}^n)$  such that  $(\eta, \Delta \phi)_{\mathbb{R}^n} = (h, \Delta \phi)_{\mathbb{R}^n}$  holds for all  $\phi \in C_0^\infty(\mathbb{R}^n)$ . Since  $w := h - \eta \in L_{\text{loc}}^1(\mathbb{R}^n)$ , Weyl's lemma states that the function  $w$  is of class  $C^\infty$  and harmonic in  $\mathbb{R}^n$  in the classical sense. Applying the mean value property to  $w$  on the ball  $B_{|x|}(x)$  centered at  $x \neq 0$  with radius  $|x|$ , and then using the Hölder inequality, we obtain the estimate

$$|w(x)| \leq C \left( \|h\|_{q, \mathbb{R}^n} |x|^{-n/q} + \|\eta\|_{r, \mathbb{R}^n} |x|^{-n/r} \right),$$

where  $C = C(n, q, r)$ . Then it follows from the Liouville theorem that  $w \equiv 0$  in  $\mathbb{R}^n$  and hence  $h \in L^r(\mathbb{R}^n)$ .  $\square$

LEMMA 2.5 (Embedding argument). *Let  $\Omega_0$  be a subdomain of  $\Omega$  with closure  $\overline{\Omega}_0$  contained in  $\Omega$ . Then for each  $1 < q < \infty$ , there is a constant  $C = C(\Omega, \Omega_0, n, q)$  such that*

$$\|f\|_{-1, q, \mathbb{R}^n} \leq C \|f\|_{-1, q, \Omega}$$

*holds for all  $f \in \hat{H}^{-1, q}(\mathbb{R}^n)$  with  $\operatorname{supp} f \subset \overline{\Omega}_0$ .*

PROOF. (i) Case  $1 < q \leq n'$ . Then we have  $n \leq q'$ . Let us take a subdomain  $\Omega_1$  of  $\Omega$  so that  $\bar{\Omega}_0 \subset \Omega_1$  and so that  $D \equiv \Omega/\bar{\Omega}_1$  is a bounded domain in  $\mathbb{R}^n$ . We show first that the space

$$S_D \equiv \left\{ \phi \in C_0^\infty(\mathbb{R}^n); \int_D \phi(x) \, dx = 0 \right\}$$

is dense in  $\hat{H}_0^{1,q'}(\mathbb{R}^n)$ . Indeed, taking the sequence  $\{\zeta_k\}_{k=1}^\infty$  of  $n$ -dimensional cut-off functions as in the proof of Lemma 2.2, we see  $\zeta_k(x) = 1$  for  $|x| \leq k$  and  $\|\nabla \zeta_k\|_{q',\mathbb{R}^n} \leq Ck^{-1+n/q'}$  with  $C$  independent of  $k$ . Letting  $\phi \in C_0^\infty(\mathbb{R}^n)$ , we set  $\phi_k(x) = \phi(x) - (\text{vol } D)^{-1} \left( \int_D \phi(y) \, dy \right) \cdot \zeta_k(x)$ , ( $k = 1, 2, \dots$ ). For large  $k$ , we have  $\phi_k \in S_D$ , so we may assume that  $\phi_k \in S_D$  for all  $k \geq 1$ . Since  $\|\nabla \phi_k - \nabla \phi\|_{q',\mathbb{R}^n} \leq C\|\phi\|_{L^\infty(\mathbb{R}^n)} \cdot k^{-1+n/q'}$  and since  $C_0^\infty(\mathbb{R}^n)$  is dense in  $\hat{H}_0^{1,q'}(\mathbb{R}^n)$ , we see that  $S_D$  is dense in  $\hat{H}_0^{1,q'}(\mathbb{R}^n)$  if  $q' > n$  i.e., if  $1 < q < n'$ . In case  $q' = n$  i.e., in case  $q = n'$ , again by Mazur's theorem, we can choose a sequence  $\{\bar{\phi}_k\}_{k=1}^\infty$  of convex combinations of  $\phi'_k$ s so that  $\nabla \bar{\phi}_k \rightarrow \nabla \phi$  in  $L^n(\mathbb{R}^n)^n$  as  $k \rightarrow \infty$ , and we see that  $S_D$  is also dense in  $\hat{H}_0^{1,n}(\mathbb{R}^n)$ .

Let  $f \in \hat{H}^{-1,q}(\mathbb{R}^n)$  with  $\text{supp } f \subset \bar{\Omega}_0$ . Taking a function  $\eta \in C^\infty(\mathbb{R}^n)$  satisfying  $0 \leq \eta \leq 1$ ,  $\eta(x) = 1$  for  $x \in \bar{\Omega}_1$  and  $\eta(x) = 0$  for  $x \in \mathbb{R}^n/\Omega$ , we have

$$(2.5) \quad |(f, \phi)_{\mathbb{R}^n}| = |(f, \eta\phi)_\Omega| \leq C\|f\|_{-1,q} (\|(\nabla\eta)\phi\|_{q'} + \|\nabla\phi\|_{q',\mathbb{R}^n})$$

for all  $\phi \in S_D$  with  $C$  independent of  $\phi$ . Since  $\text{supp } \nabla\eta \subset D$  and since  $\int_D \phi(x) \, dx = 0$ , we have by the Poincaré inequality on  $D$  that  $\|(\nabla\eta)\phi\|_{q'} \leq C\|\nabla\phi\|_{q',D}$ . Hence from (2.5) it holds

$$|(f, \phi)_{\mathbb{R}^n}| \leq C\|f\|_{-1,q} \|\nabla\phi\|_{q',\mathbb{R}^n} \text{ for all } \phi \in S_D.$$

Since  $S_D$  is dense in  $\hat{H}_0^{1,q'}(\mathbb{R}^n)$ , the above inequality holds for all  $\phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$ , from which we get the desired result in case  $1 < q \leq n'$ .

(ii) Case  $n' < q < \infty$ . Since  $1 < q' < n$ , we can take  $r \in (q', \infty)$  so that  $1/r = 1/q' - 1/n$ . Then it follows from the Sobolev inequality in  $\mathbb{R}^n$  that

$$\|(\nabla\eta)\phi\|_{q'} \leq C\|\phi\|_{q',D} \leq C\|\phi\|_{r,\mathbb{R}^n} \leq C\|\nabla\phi\|_{q',\mathbb{R}^n}$$

for all  $\phi \in C_0^\infty(\mathbb{R}^n)$ . Now we get the desired result by making use of (2.5) with  $\phi \in S_D$  replaced by  $\phi \in C_0^\infty(\mathbb{R}^n)$ .  $\square$

## 2.2. Stokes equations in bounded domains.

In this subsection, we recall the  $L^q$ -theory for the Stokes equations in bounded domains due to Cattabriga [10].

**THEOREM 2.6** (Cattabriga). *Let  $n \geq 2$  and  $G \subset \mathbb{R}^n$  be a bounded domain with boundary  $\partial G$  of class  $C^{2+\mu}$  ( $0 < \mu < 1$ ). Let  $1 < q < \infty$ . Then for every  $f \in \hat{H}^{-1,q}(G)^n$  and  $g \in L^q(G)$  with  $\int_G g(x) dx = 0$ , there is a unique pair  $\{u, p\} \in H_0^{1,q}(G)^n \times L^q(G)$  with  $\int_G p(x) dx = 0$  such that*

$$(2.6) \quad -\Delta u + \nabla p = f, \quad \operatorname{div} u = g \text{ in } G$$

*in the sense of distributions. Such  $\{u, p\}$  is subject to the inequality*

$$(2.7) \quad \|\nabla u\|_{q,G} + \|p\|_{q,G} \leq C (\|f\|_{-1,q,G} + \|g\|_{q,G}),$$

*where  $C = C(G, n, q)$ .*

**REMARK.** Since  $G$  is bounded, we have  $\hat{H}^{-1,q}(G) = H_0^{1,q'}(G)^*$ . Cattabriga [10] gave the above result for  $n = 3$  under the weaker assumption that  $\partial G$  is of class  $C^2$ . Galdi-Simader [18] extended Cattabriga's result for  $n \geq 2$ . Another proof was given by Kozono-Sohr [22] (see also Borchers-Miyakawa [7]).

The following corollary is an immediate consequence of Theorem 2.6.

**COROLLARY 2.7** (Regularity in bounded domains). *Under the same assumption on  $G$ ,  $q$ ,  $f$  and  $g$  as in Theorem 2.6, suppose that  $\{u, p\} \in H_0^{1,q}(G)^n \times L^q(G)$  satisfies (2.6) in the sense of distributions. If, in addition,  $f \in \hat{H}^{-1,r}(G)^n$  and  $g \in L^r(G)$  for some  $1 < r < \infty$ , then we have also  $u \in H_0^{1,r}(G)^n$  and  $p \in L^r(G)$ .*

### 2.3. Stokes equations in $\mathbb{R}^n$ .

In this subsection, we shall give a result on  $\mathbb{R}^n$  corresponding to that of the preceding subsection.

**LEMMA 2.8** (Regularity theory in  $\mathbb{R}^n$ ). *Let  $n \geq 2$ ,  $1 < q < \infty$  and let  $f \in \hat{H}^{-1,q}(\mathbb{R}^n)^n$ ,  $g \in L^q(\mathbb{R}^n)$ . Suppose that  $\{u, p\} \in L_{\text{loc}}^q(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$  with  $\nabla u \in L^q(\mathbb{R}^n)^{n^2}$  satisfy*

$$(2.8) \quad -\Delta u + \nabla p = f, \quad \operatorname{div} u = g \text{ in } \mathbb{R}^n$$

*in the sense of distributions. If, in addition,  $f \in \hat{H}^{-1,r}(\mathbb{R}^n)^n$  and  $g \in L^r(\mathbb{R}^n)$  for some  $1 < r < \infty$ , then we have also  $\nabla u \in L^r(\mathbb{R}^n)^{n^2}$  and  $p \in L^r(\mathbb{R}^n)$ .*

**PROOF.** By Lemma 2.1, there are sequences  $\{u_j\}_{j=1}^\infty$  in  $C_0^\infty(\mathbb{R}^n)^n$  and  $\{p_j\}_{j=1}^\infty$  in  $C_0^\infty(\mathbb{R}^n)$  such that

$$(2.9) \quad \nabla u_j \rightarrow \nabla u \text{ in } L^q(\mathbb{R}^n)^{n^2}, \quad p_j \rightarrow p \text{ in } L^q(\mathbb{R}^n).$$

Set  $f_j := -\Delta u_j + \nabla p_j$  and  $g_j := \operatorname{div} u_j$  ( $j = 1, 2, \dots$ ). Then we have by (2.8) and (2.9)

$$(2.10) \quad (f_j, \Phi)_{\mathbb{R}^n} \rightarrow (f, \Phi)_{\mathbb{R}^n}, \quad (g_j, \phi)_{\mathbb{R}^n} \rightarrow (g, \phi)$$

for all  $\Phi \in C_0^\infty(\mathbb{R}^n)^n$  and all  $\phi \in C_0^\infty(\mathbb{R}^n)$ , respectively. On the other hand, using the fundamental solution  $F_n$  of  $-\Delta$  in  $\mathbb{R}^n$ , we can represent  $u_j$  and  $p_j$  as

$$u_j = F_n * f_j - F_n * \nabla p_j, \quad p_j = -\operatorname{div} F_n * (f_j + \nabla g_j),$$

where  $*$  denotes the convolution. Then it follows that

$$(p_j, \Delta \phi)_{\mathbb{R}^n} = -(f_j, \nabla \phi)_{\mathbb{R}^n} + (g_j, \Delta \phi)_{\mathbb{R}^n} \text{ for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

Letting  $j \rightarrow \infty$  and then using the Calderon-Zygmund inequality, we have by (2.9)-(2.10) that

$$|(p, \Delta \phi)_{\mathbb{R}^n}| \leq C (\|f\|_{-1, r, \mathbb{R}^n} + \|g\|_{r, \mathbb{R}^n}) \|\Delta \phi\|_{r', \mathbb{R}^n}$$

for all  $\phi \in C_0^\infty(\mathbb{R}^n)$ . Hence Lemma 2.4 states that  $p \in L^r(\mathbb{R}^n)$ . Concerning the regularity of  $\nabla u$ , we have similarly

$$(\partial_k u_j, \Delta \Phi)_{\mathbb{R}^n} = (f_j, \partial_k \Phi)_{\mathbb{R}^n} + (p_j, \operatorname{div} (\partial_k \Phi))_{\mathbb{R}^n}, \quad (k = 1, \dots, n),$$

for all  $\Phi \in C_0^\infty(\mathbb{R}^n)^n$ . Since  $p \in L^r(\mathbb{R}^n)$ , the same argument as above yields

$$|(\partial_k u, \Delta \Phi)_{\mathbb{R}^n}| \leq C (\|f\|_{-1, r, \mathbb{R}^n} + \|p\|_{r, \mathbb{R}^n}) \|\Delta \Phi\|_{r', \mathbb{R}^n}, \quad (k = 1, \dots, n),$$

for all  $\Phi \in C_0^\infty(\mathbb{R}^n)^n$ . Again from Lemma 2.4, we get  $\partial_k u \in L^r(\mathbb{R}^n)^n$ , ( $k = 1, \dots, n$ ).  $\square$

Concerning the existence and uniqueness of solutions in the class  $\hat{H}_0^{1, q}(\mathbb{R}^n)$ , we have

**LEMMA 2.9** (A priori estimate in  $\mathbb{R}^n$ ). *Let  $n \geq 2$  and  $1 < q < \infty$ . Then for every  $f \in \hat{H}^{-1, q}(\mathbb{R}^n)^n$  and  $g \in L^q(\mathbb{R}^n)$ , there is a unique  $\{u, p\} \in \hat{H}_0^{1, q}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$  such that (2.8) holds in the sense of distributions. Such  $\{u, p\}$  is subject to the inequality*

$$\|\nabla u\|_{q, \mathbb{R}^n} + \|p\|_{q, \mathbb{R}^n} \leq C (\|f\|_{-1, q, \mathbb{R}^n} + \|g\|_{q, \mathbb{R}^n}),$$

where  $C = C(n, q)$ .

**PROOF.** By the definition of the space  $\hat{H}_0^{1, q'}(\mathbb{R}^n)$ , we see that the operator  $-\nabla : \hat{H}_0^{1, q'}(\mathbb{R}^n) \rightarrow L^{q'}(\mathbb{R}^n)^n$  is injective and has a closed range. Hence by the closed range theorem, the adjoint operator  $\operatorname{div} = (-\nabla)^* : L^q(\mathbb{R}^n)^n \rightarrow \hat{H}^{-1, q}(\mathbb{R}^n)$  is surjective. Since the null space  $\operatorname{Ker}(\operatorname{div})$  of  $\operatorname{div}$  is a closed subspace

in  $L^q(\mathbb{R}^n)^n$ , for each  $h \in \hat{H}^{-1,q}(\mathbb{R}^n)$ , there is at least one  $u \in L^q(\mathbb{R}^n)^n$  such that  $-(u, \nabla \phi)_{\mathbb{R}^n} = (h, \phi)_{\mathbb{R}^n}$  holds for all  $\phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$  and that  $\|u\|_{q,\mathbb{R}^n} \leq C\|h\|_{-1,q,\mathbb{R}^n}$  with  $C$  independent of  $h$ . Let us now use the properties of the space  $E_q$  and the bijective operator  $B_q : E_q \rightarrow E_q^*$  in the proof of Lemma 2.3. Since  $u \in L^q(\mathbb{R}^n)^n$ , the map  $\nabla \phi \in E_{q'} \rightarrow -(u, \nabla \phi)_{\mathbb{R}^n} \in \mathbb{R}$  is an element in  $E_{q'}^*$ , so we can choose  $\pi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$  so that

$$(2.11) \quad (\nabla \pi, \nabla \phi)_{\mathbb{R}^n} = -(u, \nabla \phi)_{\mathbb{R}^n} = (h, \phi)_{\mathbb{R}^n} \text{ for all } \phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n).$$

By (2.1) such a  $\pi$  is uniquely determined by  $h$  and so (2.11) defines a bounded linear operator  $J_q : h \in \hat{H}^{-1,q}(\mathbb{R}^n) \rightarrow \pi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$ . A direct calculation shows that

$$(\operatorname{div} J_q(\nabla \psi) + \psi, \Delta \phi)_{\mathbb{R}^n} = 0 \text{ for all } \psi \in L^q(\mathbb{R}^n), \phi \in C_0^\infty(\mathbb{R}^n).$$

Since the space  $H = \{\Delta \phi; \phi \in C_0^\infty(\mathbb{R}^n)\}$  is dense in  $L^{q'}(\mathbb{R}^n)$ , the above identity yields that  $\operatorname{div} J_q(\nabla \psi) = -\psi$  for all  $\psi \in L^q(\mathbb{R}^n)$ . Then we see that the pair  $\{u, p\}$  defined by

$$u = J_q f + J_q(\nabla \operatorname{div} J_q(f + \nabla g)), \quad p = -\operatorname{div} J_q(f + \nabla g)$$

has the desired property.

Now it remains to show the uniqueness. Let  $\{u', p'\} \in \hat{H}_0^{1,q}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$  satisfy (2.8) in the sense of distributions. Then  $\bar{u} = u - u'$ ,  $\bar{p} = p - p'$  satisfies (2.8) with  $f = 0$ ,  $g = 0$ . Applying the operator  $\operatorname{div}$  to both sides of the first equation, we get  $\Delta \bar{p} = 0$  in the sense of distributions in  $\mathbb{R}^n$ . Since  $\bar{p} \in L^q(\mathbb{R}^n)$ , it follows from the Liouville theorem that  $\bar{p} \equiv 0$  in  $\mathbb{R}^n$ . Therefore  $\Delta \bar{u} \equiv 0$  in  $\mathbb{R}^n$ . Since  $\bar{u} \in \hat{H}_0^{1,q}(\mathbb{R}^n)^n$ , we have by (2.1) that  $\bar{u} \equiv 0$  in  $\mathbb{R}^n$ .  $\square$

REMARK. There have been several results related to Lemma 2.9 (Kozono-Sohr [22, Proposition 2.9], Borchers-Miyakawa [7, Proposition 3.7], Galdi-Simader [18, Theorem 3.1]). Our proof seems to be rather simple: we used only the variational inequality (2.1).

### 3. - Stokes equations in the class $(D_q)$

In this section we shall give some results in  $\hat{H}_0^{1,q}(\Omega)$  analogous to those of subsection 2.3. In exterior domains, because of the boundary condition, we have restrictions on  $q$  and  $r$ .

**THEOREM 3.1** (Regularity theory in  $\Omega$ ). *Let  $n \geq 2$ ,  $1 < q < \infty$  and  $r > n'(= n/(n-1))$  and let  $f \in \hat{H}^{-1,q}(\Omega)^n \cap \hat{H}^{-1,r}(\Omega)^n$  and  $g \in L^q(\Omega) \cap L^r(\Omega)$ . Suppose that  $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  is a generalized solution of (S). Then*

we have  $\nabla u \in L^r(\Omega)^{n^2}$  and  $p \in L^r(\Omega)$ . In case  $r \geq n$  ( $n \geq 3$ ) and  $r > 2$  ( $n = 2$ ), we have  $u \in \hat{H}_0^{1,r}(\Omega)^n$ , and in case  $1 < q < n$ , we have also  $u \in \hat{H}_0^{1,r}(\Omega)^n$ .

PROOF. We use the cut-off procedure. Take a ball  $B_R \equiv \{x \in \mathbb{R}^n; |x| < R\}$  so that  $\partial\Omega \subset B_R$  and take a function  $\psi_1 \in C_0^\infty(\mathbb{R}^n)$  satisfying  $0 \leq \psi_1 \leq 1$ ,  $\psi_1(x) = 1$  for  $x \in \mathbb{R}^n/\Omega$ ,  $\psi_1(x) = 0$  for  $|x| \geq R$ , and set  $\psi_2 = 1 - \psi_1$ . Then from (S) it follows

$$(S_i) \quad -\Delta(\psi_i u) + \nabla(\psi_i p) = f_i, \quad \operatorname{div}(\psi_i u) = g_i \quad (i = 1, 2),$$

where  $f_i = \psi_i f - 2\nabla\psi_i \nabla u - (\Delta\psi_i)u + (\nabla\psi_i)p$ ,  $g_i = \psi_i g + (\nabla\psi_i)u$  ( $i = 1, 2$ ). We may regard  $(S_1)$  and  $(S_2)$  as equations in  $\Omega_R \equiv \Omega \cap B_R$  and in  $\mathbb{R}^n$ , respectively. Set  $\Omega_1 \equiv \Omega_R$  and  $\Omega_2 \equiv \mathbb{R}^n$ .

Let us first assume that  $1/q - 1/n \leq 1/r < 1/n' = 1 - 1/n$ . Taking  $s \in (1, \infty)$  so that  $1/s = 1/r + 1/n$ , we have  $s \leq q$  and  $1/s' = 1/r' - 1/n$ . Since  $q' \leq s'$ , it follows from the Sobolev embedding  $H_0^{1,r'}(\Omega_i) \subset L^{s'}(\Omega_i)$  that

$$\|\phi_i\|_{q', \Omega_R} \leq C\|\phi_i\|_{s', \Omega_R} \leq C\|\nabla\phi_i\|_{r', \Omega_i} \quad \text{for all } \phi_i \in C_0^\infty(\Omega_i) \quad (i = 1, 2).$$

Since  $\operatorname{supp} \nabla\psi_i$  and  $\operatorname{supp} \Delta\psi_i$  are contained in  $\Omega_R$ , we have by assumption and the above inequality that  $f_i \in \hat{H}^{-1,r}(\Omega_i)^n$  ( $i = 1, 2$ ). By the Sobolev embedding  $H^{1,q}(\Omega_R) \subset L^r(\Omega_R)$ , we get easily  $g_i \in L^r(\Omega_i)$  ( $i = 1, 2$ ), and also

$$\int_{\Omega_1} g_1 \, dx = \int_{\Omega_R} \operatorname{div}(\psi_1 u) \, dx = \int_{\partial\Omega} u \cdot \nu \, dS = 0.$$

Now applying Corollary 2.7 and Lemma 2.8 to  $\{\psi_1 u, \psi_1 p\}$  and  $\{\psi_2 u, \psi_2 p\}$ , respectively, we obtain

$$(3.1) \quad \nabla(\psi_i u) \in L^r(\Omega_i)^{n^2}, \quad \psi_i p \in L^r(\Omega_i) \quad (i = 1, 2).$$

We next consider the case  $1/q - 2/n \leq 1/r < 1/q - 1/n$ . Taking  $\bar{q} = (1/q - 1/n)^{-1}$ , we have by (3.1) that  $\nabla u \in L^{\bar{q}}(\Omega)^{n^2}$  and  $p \in L^{\bar{q}}(\Omega)$ . Now, taking  $\bar{q}$  instead of  $q$  in the above, we get (3.1) also for  $r > n'$  with  $1/r \geq 1/q - 2/n$ . Proceeding in the same way to the case  $1/r < 1/q - 2/n$ , by the bootstrap argument, we get (3.1) for all  $r > n'$  and hence  $\nabla u \in L^r(\Omega)^{n^2}$ ,  $p \in L^r(\Omega)$  for all  $r > n'$ .

It remains to show that  $u \in \hat{H}_0^{1,r}(\Omega)^n$  in case  $r \geq n$  ( $n \geq 3$ ),  $r > 2$  ( $n = 2$ ), and in case  $1 < q < n$ . To this end, we may show  $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$  in (3.1). Consider first the case when  $r \geq n$  ( $n \geq 3$ ) and  $r > 2$  ( $n = 2$ ). Since  $\nabla(\psi_2 u) \in L^r(\mathbb{R}^n)^{n^2}$  and since  $\psi_2 u$  vanishes in a neighbourhood of  $\partial\Omega$ , we get by Lemma 2.2(ii) that  $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$ . We next consider the case when  $1 < q < n$ ,  $n' < r < n$  ( $n \geq 3$ ). Since  $f_2 \in \hat{H}^{-1,r}(\mathbb{R}^n)^n$  and  $g_2 \in L^r(\mathbb{R}^n)$ , it follows from Lemma 2.9 that there is a unique pair  $\{v, \chi\} \in \hat{H}_0^{1,r}(\mathbb{R}^n)^n \times L^r(\mathbb{R}^n)$  such that  $-\Delta v + \nabla\chi = f_2$ ,  $\operatorname{div} v = g_2$  in the sense of distributions in  $\mathbb{R}^n$ . Taking



$w = v - \psi_2 u$  and  $\eta = \chi - \psi_2 p$ , we see that  $\{w, \eta\}$  satisfies (2.8) with  $f = 0$  and  $g = 0$ ; applying  $\operatorname{div}$  to both sides of the first equation, we obtain that  $\eta$  is harmonic in  $\mathbb{R}^n$ . Since  $\eta \in L^r(\mathbb{R}^n)$ , the Liouville theorem yields that  $\eta \equiv 0$  in  $\mathbb{R}^n$ ; hence  $w$  is also harmonic in  $\mathbb{R}^n$ . Moreover, by the Sobolev embedding theorem, we obtain  $w \in L^{\bar{q}}(\mathbb{R}^n) + L^{\bar{r}}(\mathbb{R}^n)$ , where  $1/\bar{q} = 1/q - 1/n$  and  $1/\bar{r} = 1/r - 1/n$ . Using the same argument as in the proof of Lemma 2.4, we get  $w \equiv 0$  in  $\mathbb{R}^n$ , from which  $\psi_2 u \in \hat{H}_0^{1,r}(\mathbb{R}^n)^n$  follows. Now, again by Lemma 2.2(i), we have  $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$ .  $\square$

REMARK. The restriction  $n' < r$  is a critical condition; we *cannot* take  $1 < r \leq n'$  in Theorem 3.1. Indeed, let us assume the main results in Section 1. Taking some  $n < q < \infty$  and  $A = 0$ ,  $a \neq 0$  in Theorem B(ii), we get such  $\{u, p\} \in \mathbb{N}_q^0$  as  $\lim_{x \rightarrow \infty} u(x) = a$  in case  $n \geq 3$  and as  $\int_{\Omega} |\nabla u(x) - \nabla E(x)a|^2 dx < \infty$  in case  $n = 2$ , and by Lemma 2.2(ii), we have  $u \in \hat{H}_0^{1,q}(\Omega)^n$ . Suppose now that Theorem 3.1 is true for  $1 < r \leq n'$ . Then it follows that  $\nabla u \in L^r(\Omega)^{n^2}$  for some  $1 < r \leq n'$ . Thus by Theorem A, we get  $u \equiv 0$  in  $\Omega$ , which contradicts  $a \neq 0$ . Note that  $\int_{\Omega} |\nabla E(x)|^2 dx = \infty$  in case  $n = 2$ .

We shall next give an a priori estimate in the class  $(D_q)$ . For this purpose we need:

LEMMA 3.2. *Let  $n \geq 2$  and  $1 < q < \infty$  and let  $\{f, g\} \in \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$ . Suppose that  $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  is a generalized solution of (S). Then we have*

$$(3.2) \quad \begin{aligned} & \|\nabla u\|_q + \|p\|_q \\ & \leq C \left( \|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \left| \int_{\Omega_R} \psi_1(x)p(x) \, dx \right| \right), \end{aligned}$$

where  $\Omega_R = \Omega \cap B_R$  and  $\psi_1$  are the same as in the proof of Theorem 3.1 and where  $C$  is a constant independent of  $u$  and  $p$ .

PROOF. We use again the cut-off method. Recalling the equations  $(S_i)$  ( $i = 1, 2$ ) in the proof of Theorem 3.1, we first consider  $(S_1)$  in  $\Omega_R$ . Since  $\operatorname{supp} \nabla \psi_1, \operatorname{supp} \Delta \psi_1 \subset \Omega_R$ , we obtain

$$\begin{aligned} \|f_1\|_{-1,q,\Omega_R} & \leq C (\|f\|_{-1,q} + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \\ \|g_1\|_{q,\Omega_R} & \leq C (\|g\|_q + \|u\|_{q,\Omega_R}). \end{aligned}$$

Applying Theorem 2.6 to  $\{\psi_1 u, \psi_1 p\}$  in  $(S_1)$  and then using the above inequa-

lities, we get

$$(3.3) \quad \begin{aligned} & \|\nabla(\psi_1 u)\|_{q,\Omega_R} + \|\psi_1 p\|_{q,\Omega_R} \\ & \leq C \left( \|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \left| \int_{\Omega_R} \psi_1(x) p(x) \, dx \right| \right) \end{aligned}$$

with  $C$  independent of  $u$  and  $p$ .

We next consider  $(S_2)$  in  $\mathbb{R}^n$ . Since  $\text{supp } f_2 \subset \text{supp } \psi_2$ , it follows from Lemma 2.5 that

$$(3.4) \quad \|f_2\|_{-1,q,\mathbb{R}^n} \leq C \|f_2\|_{-1,q} \text{ with } C = C(n, q).$$

Since the inequality  $\|\phi\|_{q',\Omega_R} \leq C \|\nabla \phi\|_{q'}$  holds for all  $\phi \in C_0^\infty(\Omega)$  and since  $\text{supp } \nabla \phi_2, \text{supp } \Delta \phi_2 \subset \Omega_R$ , we obtain

$$(3.5) \quad \begin{aligned} \|f_2\|_{-1,q} & \leq C (\|f\|_{-1,q} + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \\ \|g_2\|_q & \leq C (\|g\|_q + \|u\|_{q,\Omega_R}). \end{aligned}$$

Now applying Lemma 2.9 to  $\{\psi_2 u, \psi_2 p\}$  in  $(S_2)$  and then using (3.4)-(3.5), we obtain

$$(3.6) \quad \begin{aligned} & \|\nabla(\psi_2 u)\|_{q,\mathbb{R}^n} + \|\psi_2 p\|_{q,\mathbb{R}^n} \\ & \leq C (\|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}). \end{aligned}$$

Then the desired result follows from (3.3) and (3.6).  $\square$

Now we introduce the weak Stokes operator  $S_q$ . Let  $X_q \equiv \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  and  $Y_q \equiv \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$ . We define two bounded linear operators  $S_q$  and  $T_q$  by

$$\begin{aligned} S_q : \{u, p\} & \in X_q \rightarrow \{-\Delta u + \nabla p, \text{div } u\} \in Y_q; \\ T_q : \{v, \chi\} & \in X_q \rightarrow \{-\Delta v - \nabla \chi, -\text{div } v\} \in Y_q, \end{aligned}$$

respectively. It is easy to see that

$$(3.7) \quad S_q^* \text{ (adjoint operator of } S_q) = T_{q'} \text{ for all } 1 < q < \infty.$$

Then Lemma 3.2 enables us to apply such a standard argument as Lions-Magenes [24, p. 153, Lemma 5.1], so we see that

$$(3.8) \quad \begin{aligned} & \text{Ker } S_q \text{ (the kernel of } S_q) \text{ is of finite dimension and} \\ & R(S_q) \text{ (the range of } S_q) \text{ closed in } Y_q. \end{aligned}$$

More precisely we have

**THEOREM 3.3** (A priori estimate). *Let  $n \geq 2$ ,  $1 < q < n$  for  $n \geq 3$  and  $1 < q \leq 2$  for  $n = 2$ , and let  $\{f, g\} \in \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$ . Suppose that  $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  is a generalized solution of (S). Then it holds*

$$(3.9) \quad \|\nabla u\|_q + \|p\|_q \leq C (\|f\|_{-1,q} + \|g\|_q)$$

where  $C = C(\Omega, q, n)$ .

**PROOF.** We show first that  $\text{Ker } S_q = \{0, 0\}$  for such  $q$  as in the theorem. Let  $\{u, p\} \in \text{Ker } S_q$ . Then it is enough to show that  $\{u, p\} \in \hat{H}_0^{1,2}(\Omega)^n \times L^2(\Omega)$ , because we can insert  $\Phi = u$  as a test function in the definition of the generalized solution and hence  $\|\nabla u\|_2^2 = 0$ ,  $\nabla p = 0$  follows. Then we get  $u \equiv 0$ ,  $p \equiv 0$ . If  $n \geq 3$ , then we can take  $r = 2 > n'$  in Theorem 3.1 and get  $\{u, p\} \in \hat{H}_0^{1,2}(\Omega)^n \times L^2(\Omega)$ . If  $n = 2$ , then we get by Theorem 3.1 and the interpolation property that  $\nabla u \in L^r(\Omega)^{n^2}$ ,  $p \in L^r(\Omega)$  for all finite  $r \geq q$ . Since  $n = 2$ , it follows from Lemma 2.2(ii) that  $u \in \hat{H}_0^{1,2}(\Omega)^2$ .

Now we prove (3.9) by contradiction. Suppose the contrary. Then there is a sequence  $\{u_k, p_k\}_{k=1}^\infty$  in  $\hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  such that  $\|\nabla u_k\|_q + \|p_k\|_q = 1$  and that  $-\Delta u_k + \nabla p_k \rightarrow 0$  in  $\hat{H}^{-1,q}(\Omega)^n$ ,  $\text{div } u_k \rightarrow 0$  in  $L^q(\Omega)$  as  $k \rightarrow \infty$ . A well known compactness argument yields that there is a subsequence, which we denote by  $\{u_k, p_k\}_{k=1}^\infty$  for simplicity, such that  $\{u_k\}_{k=1}^\infty$ ,  $\{p_k\}_{k=1}^\infty$  and  $\left\{ \int_{\Omega_R} \psi_1(x) p_k(x) \, dx \right\}_{k=1}^\infty$  converge strongly in  $L^q(\Omega_R)^n$ ,  $\hat{H}^{-1,q}(\Omega_R)$  and  $\mathbb{R}$ , respectively. Then, applying Lemma 3.2 to  $\{u_k - u_{k'}, p_k - p_{k'}\}_{k,k'=1}^\infty$ , we see that  $\{u_k\}_{k=1}^\infty$  and  $\{p_k\}_{k=1}^\infty$  are Cauchy sequences in  $\hat{H}_0^{1,q}(\Omega)^n$  and in  $L^q(\Omega)$ , respectively. Thus, there is a pair  $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  such that  $u_k \rightarrow u$  in  $\hat{H}_0^{1,q}(\Omega)^n$  and  $p_k \rightarrow p$  in  $L^q(\Omega)$ . Moreover, we have  $\{u, p\} \in \text{Ker } S_q$  and that  $\|\nabla u\|_q + \|p\|_q = 1$ , but this contradicts  $\text{Ker } S_q = \{0, 0\}$ .  $\square$

Using (3.7)-(3.8) and a closed range theorem, we have by Theorem 3.3 the following corollary.

**COROLLARY 3.4.** (i) *Let  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Then we have  $\text{Ker } S_q = \{0, 0\}$  and  $R(S_q) = (\text{Ker } T_q)^\perp$ .*

(ii) *Let  $n' < q < n$  for  $n \geq 3$  and  $q = 2$  for  $n = 2$ . Then we have  $\text{Ker } S_q = \{0, 0\}$  and  $R(S_q) = Y_q$ .*

(iii) *Let  $n \leq q < \infty$  for  $n \geq 3$  and  $2 < q < \infty$  for  $n = 2$ . Then we have  $\text{Ker } S_q = R(T_q)^\perp$  and  $R(S_q) = Y_q$ .*

*Here  $W^\perp$  denotes the annihilator of the subspace  $W$ .*

**REMARK.** Theorem 3.3 was first proved by Kozono-Sohr [22] in case  $n \geq 3$  and  $n' < q < n$ . Borchers-Miyakawa [7] extended the result to the case when  $n \geq 3$  and  $1 < q \leq n'$ . Recently Galdi-Simader [18] gave a similar result for  $n \geq 2$ , but with a different method from ours.

#### 4. - Proof of the main results

##### 4.1. Stokes paradox; Proof of Theorems A and A'.

Let us first give the following auxiliary lemma due to Bogovskii [5, 6].

LEMMA 4.1 (Bogovskii). (i) Let  $1 < q < \infty$ . Suppose that  $w \in \hat{H}^{-1,q}(\Omega)^n$  satisfies  $(w, \Phi) = 0$  for all  $\Phi \in C_0^\infty(\Omega)^n$  with  $\operatorname{div} \Phi = 0$ . Then there is a unique  $p \in L^q(\Omega)$  such that  $w = \nabla p$ , i.e.,  $(w, \Psi) = -(p, \operatorname{div} \Psi)$  for all  $\Psi \in \hat{H}_0^{1,q'}(\Omega)^n$ .

(ii) Let  $1 < r < n$  and let  $u \in L_{\operatorname{loc}}^1(\bar{\Omega})$  with  $\nabla u \in L^r(\Omega)^n$ . Then there is a constant  $C = C(u, n, r)$  such that  $u + C \in L^q(\Omega)$  with  $1/q = 1/r - 1/n$ .

For the proof, see also Giga-Sohr [19, Corollary 2.2] and Borchers-Sohr [9, Lemma 4.1].

In the forthcoming argument, we use the linear extension operator  $\Gamma : C^2(\partial\Omega)^n \rightarrow C_0^2(B_R)^n$  satisfying

$$\Gamma\phi = \phi \text{ on } \partial\Omega, \quad \|\Gamma\phi\|_{H^{m,q}(B_R)} \leq C\|\phi\|_{H^{m-1/q,q}(\partial\Omega)}, \quad (m = 1, 2)$$

for all  $\phi \in C^2(\partial\Omega)^n$  with  $C = C(\partial\Omega, R, n, m, q)$ . Here  $B_R = \{x \in \mathbb{R}^n; |x| < R\}$  is a ball containing  $\partial\Omega$ .

PROOF OF THEOREM A. If  $n = n' = q = 2$ , then the desired result follows from Corollary 3.4(ii) and Lemma 2.2(ii), so we may prove only the case  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Since  $\nabla u \in L^q(\Omega)^{n^2}$ , it follows from Lemma 4.1(ii) that there is a constant vector  $a = a(u, q) \in \mathbb{R}^n$  such that  $u - a \in L^{nq/(n-q)}(\Omega)^n$ . Set  $w = \Gamma a \in C_0^2(B_R)$  and define  $\hat{u} = u - a + w$ . Then we see by Lemma 2.2(i) that  $\hat{u} \in \hat{H}_0^{1,q}(\Omega)^n$ , and by assumption we get

$$(4.1) \quad -\Delta\hat{u} + \nabla p = -\Delta w, \quad \operatorname{div} \hat{u} = \operatorname{div} w \text{ in } \Omega$$

in the sense of distributions. Since  $\{-\Delta w, \operatorname{div} w\} \in \hat{H}^{-1,\gamma}(\Omega)^n \times L^\gamma(\Omega)$  for all  $\gamma > 1$ , it follows from Theorem 3.1 that  $\{\hat{u}, p\} \in \hat{H}_0^{1,r}(\Omega)^n \times L^r(\Omega)$  for all  $r > n'$ . Moreover, since  $q' \geq n$  for  $n \geq 3$  and  $q' > 2$  for  $n = 2$ , we obtain from Lemma 2.2(ii) that  $\{u, -p\} \in \operatorname{Ker} T_{q'}$ . Therefore it follows from (4.1) and Corollary 3.4(i) that

$$\begin{aligned} 0 &= (-\Delta w, u) + (\operatorname{div} w, -p) \\ &= - \int_{\Omega} |\nabla u(x)|^2 dx + (\nabla u, \nabla \hat{u}) - (p, \operatorname{div} \hat{u}). \end{aligned}$$

Since  $\{\hat{u}, p\} \in \hat{H}_0^{1,q'}(\Omega)^n \times L^{q'}(\Omega)$  and since  $C_0^\infty(\Omega)^n \times C_0^\infty(\Omega)$  is dense in  $\hat{H}_0^{1,q'}(\Omega)^n \times L^{q'}(\Omega)$ , it follows from the assumption on  $\{u, p\}$  that  $(\nabla u, \nabla \hat{u}) - (p, \operatorname{div} \hat{u}) = 0$ . So we get  $u \equiv 0$  in  $\Omega$ , then  $\nabla p = 0$  in  $\Omega$ . Since  $p \in L^q(\Omega)$ , we have also  $p \equiv 0$  in  $\Omega$ .  $\square$

PROOF OF THEOREM A'. Since  $\nabla u \in L^q(\Omega)^{n^2}$ , we have  $-\Delta u \in \hat{H}^{-1,q}(\Omega)^n$  and, by assumption,  $(-\Delta u, \Phi) = 0$  for all  $\Phi \in C_0^\infty(\Omega)^n$  with  $\operatorname{div} \Phi = 0$ . Then it follows from Lemma 4.1(i) that there is a scalar function  $p \in L^q(\Omega)$  such that

$$(\nabla u, \nabla \Psi) - (p, \operatorname{div} \Psi) = 0 \text{ for all } \Psi \in C_0^\infty(\Omega)^n.$$

Now Theorem A yields that  $u \equiv 0$  in  $\Omega$ . □

#### 4.2. Characterization of the null space; Proof of Theorem B.

We shall first consider the cases (i) and (ii), i.e.,  $n \geq 2$ ,  $1 < q < \infty$  except for  $n = q = 2$ . The proof will be done by three lemmas. Let us define the vector spaces  $V$  and  $\hat{V}_q$  for  $1 < q \leq n'$  ( $n \geq 3$ ) and  $1 < q < 2$  ( $n = 2$ ) as follows:

$$\begin{aligned} V &= \{ \{A, a\} \in \mathbb{R}^{n^2} \times \mathbb{R}^n; \operatorname{Tr} A = 0 \}, \\ \hat{V}_q &= \left\{ \{A, a\} \in V; \int_{\partial\Omega} \left[ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right] dS = 0 \right. \\ &\quad \left. \text{for all } \{v, \chi\} \in \mathbb{N}_{q'}^0 \right\}. \end{aligned}$$

Then the existence of a generalized solution with (1.2) and (1.3-3') in Theorem B(i)-(ii) is guaranteed by the following lemma.

LEMMA 4.2 (Existence). (i) Let  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Then there is a linear operator  $K_q : \{A, a\} \rightarrow \{u, p\}$  from  $\hat{V}_q$  to  $\bigcap_{r \geq q} \mathbb{N}_r$  such that  $\nabla u - A \in L^r(\Omega)^{n^2}$  for all  $r \geq q$  and such that (1.3) holds.

(ii) There is a linear operator  $L : \{A, a\} \rightarrow \{u, p\}$  from  $V$  to  $\bigcap_{r > n'} \mathbb{N}_r$  such that:  $\nabla u - A \in L^r(\Omega)^{n^2}$  for all  $r > n'$ , (1.3) holds if  $n \geq 3$ , and

$$(4.2) \quad \int_{\Omega} |\nabla[u(x) - Ax - E(x)a]|^s dx < \infty$$

holds for all  $s \geq 2$  if  $n = 2$ .

PROOF. (i) Since  $q' \geq n$  for  $n \geq 3$  and  $q' > 2$  for  $n = 2$ , we have by Lemma 2.2(ii) that  $\{v, -\chi\} \in \operatorname{Ker} T_{q'}$  for  $\{v, \chi\} \in \mathbb{N}_{q'}^0$ . Taking  $w = \Gamma(A \cdot + a)$  for  $\{A, a\} \in \hat{V}_q$ , where  $\Gamma$  is the extension operator defined above and where  $A \cdot + a$  is a function on  $\partial\Omega$  defined by  $x \in \partial\Omega \rightarrow Ax + a \in \mathbb{R}^n$ , a direct calculation shows that

$$\begin{aligned} &(-\Delta w, v) + (\operatorname{div} w, -\chi) \\ &= \int_{\partial\Omega} \left\{ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right\} dS = 0 \end{aligned}$$

for all  $\{v, \chi\} \in \mathbb{N}_q^0$ . Hence it follows from Corollary 3.4(i) that there is a unique generalized solution  $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  of (4.1). Moreover, by Theorem 3.1 and the interpolation inequality,  $\nabla \hat{u} \in L^r(\Omega)^{n^2}$  and  $p \in L^r(\Omega)$  for all  $r \geq q$ . If  $n \geq 3$ , we have again by Theorem 3.1 that  $\hat{u} \in \hat{H}_0^{1,\gamma}(\Omega)^n$  for all  $\gamma > n'$ . For  $\gamma \in (n/2, n)$ ,  $\hat{H}_0^{1,\gamma}(\Omega)$  is continuously embedded into  $L^{n\gamma/(n-\gamma)}(\Omega)$  (see Lemma 2.2(i)). For such  $\gamma$ , we have  $n\gamma/(n-\gamma) > n$  and hence, in particular,  $\hat{u} \in H^{1,s}(\Omega)^n$  for  $s > n$ . By the Sobolev embedding theorem,  $\hat{u} \in C^0(\bar{\Omega})^n$  and  $\lim_{x \rightarrow \infty} |\hat{u}(x)| = 0$ . If  $n = 2$ , we have by Lemma 2.2(i) that  $\hat{u} \in H_0^{1,2q/(2-q)}(\Omega)^2$ . Since  $2q/(2-q) > 2$ ,  $\hat{u}$  has the same properties as above. Now, setting  $u = \hat{u} + Ax + a - w$  and then defining  $K_q\{A, a\} = \{u, p\}$ , we obtain the operator  $K_q$ .

(ii) We first consider the case  $n \geq 3$ . Let  $\{A, a\} \in V$ . We set  $w = \Gamma(A \cdot + a)$ . Then it follows from Corollary 3.4(ii) and Theorem 3.1 that there is a unique generalized solution  $\{\hat{u}, p\}$  of (4.1) such that  $\{\hat{u}, p\} \in \hat{H}_0^{1,r}(\Omega)^n \times L^r(\Omega)$  for all  $r > n'$ . Then in the same way as above, we can show that  $\hat{u} \in C^0(\bar{\Omega})^n$ ,  $\lim_{x \rightarrow \infty} |\hat{u}(x)| = 0$  and that the map  $L : \{A, a\} \rightarrow \{u, p\}$  with  $u = \hat{u} + Ax + a - w$  satisfies the required conditions.

We next construct  $L$  for  $n = 2$ . Without loss of generality, we may assume that  $0 \in \mathbb{R}^n/\bar{\Omega}$ . Set  $w = \Gamma(A \cdot + Ea)$ , where  $A \cdot + Ea$  is the function on  $\partial\Omega$  defined by  $x \in \partial\Omega \rightarrow Ax + E(x)a \in \mathbb{R}^2$ . Then it follows from Corollary 3.4(ii) and Theorem 3.1 that there is a unique generalized solution  $\{\hat{u}, p\}$  of (4.1) belonging to  $\hat{H}_0^{1,r}(\Omega)^2 \times L^r(\Omega)$  for all  $r \geq 2$ . Setting  $u = \hat{u} + Ax + Ea - w$ , we see that the map  $L : \{A, a\} \rightarrow \{u, p\}$  enjoys the desired properties.  $\square$

We next show the uniqueness of generalized solutions.

LEMMA 4.3 (Uniqueness). (i) *Let  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . Then for every  $\{A, a\} \in \hat{V}_q$ , there is a unique  $\{u, p\} \in \mathbb{N}_q$  with properties (1.2) and (1.3).*

(ii) *Let  $n' < q < \infty$ ,  $n \geq 2$ . Then for every  $\{A, a\} \in V$ , there exists a unique  $\{u, p\} \in \mathbb{N}_q$  with properties (1.2)-(1.3) if  $n \geq 3$ , and (1.2)-(1.3') if  $n = 2$ .*

PROOF. The proof of existence is contained in Lemma 4.2 so we may only prove uniqueness.

(i) Suppose that  $\{\bar{u}, \bar{p}\} \in \mathbb{N}_q$  satisfies (1.2) and (1.3) with  $u$  replaced by  $\bar{u}$ . Set  $\bar{u} = u\bar{u}$  and  $\bar{p} = p - \bar{p}$  (note that  $p, \bar{p}$  and  $\bar{p}$  do not denote integral exponents but functions of the pressure). Then we have  $\nabla \bar{u} \in L^q(\Omega)^{n^2}$ ,  $\bar{u} \in C^0(\bar{\Omega})^n$  and from Theorem 3.1 that there is a unique generalized solution  $\lim_{x \rightarrow \infty} |\bar{u}(x)| = 0$ . On the other hand, by Lemma 4.1(ii), there is a constant vector  $\bar{C} \in \mathbb{R}^n$  such that  $\bar{u} + \bar{C} \in L^{nq/(n-q)}(\Omega)^n$ . Since  $\lim_{x \rightarrow \infty} |\bar{u}(x)| = 0$ , we have  $\bar{C} = 0$  and hence  $\bar{u} \in L^{nq/(n-q)}(\Omega)^n$ . Then from Lemma 2.2(i) we obtain  $\bar{u} \in \hat{H}_0^{1,q}(\Omega)^n$  and so  $\{\bar{u}, \bar{p}\} \in \text{Ker } S_q$ . Now applying Corollary 3.4(i), we get  $\bar{u} \equiv 0$ ,  $\bar{p} \equiv 0$  and the assertion on uniqueness follows.

(ii) Let us first assume  $n \geq 3$ . Let  $\{\bar{u}, \bar{p}\} \in \mathbb{N}_q$  and  $\{\bar{u}, \bar{p}\}$  as above. If  $n' < q < n$ , we can argue in the same way as above and get  $\{\bar{u}, \bar{p}\} \in \text{Ker } S_q$ .

Then it follows from Corollary 3.4(ii) that  $\bar{u} \equiv 0$ ,  $\bar{p} \equiv 0$ . If  $n \leq q < \infty$ , we see by Lemma 2.2(ii) that  $\bar{u} \in \hat{H}_0^{1,q}(\Omega)^n$ . From Theorem 3.1, we obtain  $\nabla \bar{u} \in L^r(\Omega)^{n^2}$ ,  $\bar{p} \in L^r(\Omega)$  for all  $r > n'$ . In the same way as in (i), we get  $\{\bar{u}, \bar{p}\} \in \text{Ker } S_\gamma$  for some  $\gamma$  with  $n' < \gamma < n$  and Corollary 3.4(ii) yields  $\bar{u} \equiv 0$ ,  $\bar{p} \equiv 0$ . In case  $n = 2$ , we see by Lemma 2.2(ii) and (1.3') that  $\{\bar{u}, \bar{p}\} \in \text{Ker } S_2 = \{0, 0\}$ .  $\square$

Now it remains to give the dimensions of  $\mathbb{N}_q$  and  $\mathbb{N}_q^0$ . To this end, we shall make use of the operators  $K_q$  and  $L$  constructed in Lemma 4.2.

LEMMA 4.4. (i) For each  $q$  with  $1 < q \leq n'$  ( $n \geq 3$ ) and with  $1 < q < 2$  ( $n = 2$ ),  $K_q$  defines a bijection from  $\hat{V}_q$  onto  $\mathbb{N}_q$ .

(ii) For each  $q$  with  $n' < q < \infty$  ( $n \geq 2$ ),  $L$  defines a bijection from  $V$  onto  $\mathbb{N}_q$ .

PROOF. (i) *Injectivity.* Let  $K_q\{A, a\} = \{0, 0\}$  for  $\{A, a\} \in \hat{V}_q$ . Then by (1.3),  $|Ax + a| \rightarrow 0$  as  $x \rightarrow \infty$ ; hence we get  $A = 0$  and  $a = 0$ .

*Surjectivity.* Suppose that  $\{u, p\} \in \mathbb{N}_q$ . Then  $\nabla u - A \in L^q(\Omega)^{n^2}$  for some  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$ . By Lemma 4.1(ii), there is a constant vector  $a \in \mathbb{R}^n$  such that  $u - Ax - a \in L^{nq/(n-q)}(\Omega)^n$ . Introducing  $w = \Gamma(A \cdot + a)$  as in the proof of Lemma 4.2 and then defining  $\hat{u} = u - Ax - a + w$ , we see by Lemma 2.2(i) that  $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  and that  $\{\hat{u}, p\}$  satisfies (4.1) in the sense of distributions. Moreover it follows from Theorem 3.1 that  $\hat{u} \in \hat{H}_0^{1,r}(\Omega)^n$  for all  $r > n'$ . Now using the same argument as in the proof of Lemma 4.2, we get  $\hat{u} \in C^0(\bar{\Omega})^n$ ,  $\lim_{x \rightarrow \infty} |\hat{u}(x)| = 0$  and hence  $u$  satisfies (1.3). Then the uniqueness stated in Lemma 4.3(i) yields that  $\{u, p\} = K_q\{A, a\}$ .

(ii) *Injectivity.* Let  $L\{A, a\} = \{0, 0\}$  for  $\{A, a\} \in V$ . If  $n \geq 3$ , we get in the same way as above  $A = 0$ ,  $a = 0$ . If  $n = 2$ , we obtain by (4.2) that  $A - \nabla E a \in L^s(\Omega)^{2^2}$  for all  $s \geq 2$ . The explicit expression of  $E$  shows that  $\nabla E$  is not in  $L^2(\Omega)^{2^3}$ , but in  $L^r(\Omega)^{2^3}$  for all  $r > 2$ . Hence  $A = 0$ ,  $a = 0$ .

*Surjectivity.* Let us first assume that  $n \geq 3$ . The proof for  $q$  with  $n' < q < n$  is parallel to that of case (i), so we may only show it for  $n \leq q < \infty$ . Suppose that  $\{u, p\} \in \mathbb{N}_q$  ( $q \geq n$ ). Then  $\nabla u - A \in L^q(\Omega)^{n^2}$  for some  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A = 0$ , and taking  $w = \Gamma(A \cdot)$  and  $\hat{u} = u - Ax + w$ , we see by Lemma 2.2(ii) that  $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  and that  $\{\hat{u}, p\}$  is a generalized solution of (4.1). Moreover, by Theorem 3.1,  $\nabla \hat{u} \in L^r(\Omega)^{n^2}$  for all  $r > n'$ , and in particular, we have  $\nabla \hat{u} \in L^\gamma(\Omega)^{n^2}$  for  $\gamma$  with  $n/2 < \gamma < n$ . By Lemma 4.1(ii), there is a constant vector  $a \in \mathbb{R}^n$  such that  $\hat{u} - a \in L^\sigma(\Omega)^n$  with  $1/\sigma = 1/\gamma - 1/n$ . Since  $\sigma > n$ , we have  $\hat{u} - a \in H^{1,\sigma}(\Omega)^n$  and hence by the Sobolev embedding theorem  $\hat{u} - a \in C^0(\bar{\Omega})^n$  and  $\lim_{x \rightarrow \infty} |\hat{u}(x) - a| = 0$ , from which (1.3) follows. Now the uniqueness result of Lemma 4.3(ii) yields that  $\{u, p\} = L\{A, a\}$ .

We next consider the case  $n = 2$  and  $q > 2$ . Let  $\{u, p\} \in \mathbb{N}_q$ . Then  $\nabla u - A \in L^q(\Omega)^{2^2}$  for some  $A \in \mathbb{R}^{2^2}$  with  $\text{Tr } A = 0$ . Since  $u - Ax - \Gamma(A \cdot) \in \hat{H}_0^{1,q}(\Omega)^2$ , we obtain by Lemma 2.2(ii) that  $u(x) - Ax = O(|x|^{1-2/q})$  as  $|x| \rightarrow \infty$ .

Applying the regularity theorem of Finn-Smith [13, Theorem 5.11-12], we see  $u - Ax \in C^2(\Omega) \cap C^1(\bar{\Omega})$  and  $p \in C^1(\Omega) \cap C^0(\bar{\Omega})$ . Then it follows from the representation formula of Chang-Finn [11, Theorem 1] that

$$\begin{aligned} u(x) - Ax - E(x)a &= u^\infty - \int_{\partial\Omega} A\xi \cdot TE(x - \xi) \nu_\xi dS_\xi \\ &\quad - \int_{\partial\Omega} (E(x - \xi) - E(x)) T(u(\xi)A\xi) \nu_\xi dS_\xi, \end{aligned}$$

where  $a = - \int_{\partial\Omega} T(u(\xi) - A\xi) \nu_\xi dS_\xi$ ,  $u^\infty \in \mathbb{R}^2$  and  $T$  denotes the stress tensor. Using the explicit expression of  $E$ , we see that

$$\sup_{\xi \in \partial\Omega} |\nabla_x TE(x - \xi)| = O(|x|^{-2}), \quad \sup_{\xi \in \partial\Omega} |\nabla_x E(x - \xi) - \nabla_x E(x)| = O(|x|^{-2})$$

as  $x \rightarrow \infty$ . Therefore  $\nabla(u - Ax - Ea) \in L^2(\Omega)^{2^2}$  and it follows from the uniqueness proved in Lemma 4.2(ii) that  $\{u, p\} = L\{A, a\}$ .  $\square$

*Properties of  $\dim \mathbb{N}_q$  and  $\dim \mathbb{N}_q^0$ .*

Let us first consider the case  $n' < q < \infty$  ( $n \geq 2$ ). Then by Lemma 4.4(ii) and the definition of  $V$ , we obtain

$$(4.3) \quad \dim \mathbb{N}_q = \dim V = n^2 + n - 1, \quad \dim \mathbb{N}_q^0 = n, \quad n' < q < \infty \quad (n \geq 2).$$

Hence Theorem B(ii) follows from (4.3) and Lemmas 4.2-3(ii).

We next consider the case  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ . By Theorem A and the definition of  $\mathbb{N}_q^0$ , we have

$$(4.4) \quad \dim \mathbb{N}_q^0 = 0, \quad 1 < q \leq n' \quad (n \geq 3), \quad 1 < q < 2 \quad (n = 2).$$

Moreover, it follows from Lemma 4.4(i) that  $\mathbb{N}_q^0$  is isometric to the subspace  $W_q$  of  $\hat{V}_q$ :

$$W_q \equiv \left\{ a \in \mathbb{R}^n; \int_{\partial\Omega} \left\{ a \cdot \frac{\partial v}{\partial \nu} - \chi a \cdot \nu \right\} dS = 0 \text{ for all } \{v, \chi\} \in \mathbb{N}_{q'}^0 \right\}.$$

Hence  $W_q = \{0\}$ . On the other hand, by (4.3), we see  $\dim \mathbb{N}_{q'}^0 = n$ . Therefore it follows that  $\dim \hat{V}_q = n^2 + n - 1 - \dim \mathbb{N}_{q'}^0 = n^2 - 1$ . Now, Lemma 4.4(i) yields

$$(4.5) \quad \dim \mathbb{N}_q = \dim \hat{V}_q = n^2 - 1, \quad 1 < q \leq n' \quad (n \geq 3), \quad 1 < q < 2 \quad (n = 2).$$

Hence Theorem B(i) follows from (4.4-5) and Lemmas 4.2-3(i).



(iii) *Case  $n = q = 2$ .* In the same way as in Lemmas 4.2-4.3, we can construct a bijective operator  $L' : A \rightarrow \{u, p\}$  from  $V' \equiv \{A \in \mathbb{R}^{2^2}; \text{Tr } A = 0\}$  onto  $\mathbb{N}_2$  such that  $u$  satisfies (4.2) with  $a = 0$ . Hence we get  $\dim \mathbb{N}_2 = 3$ . By Lemma 2.2(ii) and Corollary 3.4(ii), we have  $\mathbb{N}_2^0 = \text{Ker } S_2 = \{0, 0\}$ . And therefore existence and uniqueness derive from the same argument as before, so we may omit the details.  $\square$

### 4.3. Inhomogeneous equations; Proof of Theorem C.

Recall the function  $A \cdot +a : x \in \partial\Omega \rightarrow Ax + a \in \mathbb{R}^n$  and set  $w = \Gamma(A \cdot +a)$ . Taking  $\hat{u} = u - Ax - a + w$ , we get from (S)

$$(4.6) \quad \begin{aligned} -\Delta \hat{u} + \nabla p &= f - \Delta w, \quad \text{div } \hat{u} = g \text{Tr } A + \text{div } w \text{ in } \Omega, \\ \hat{u} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

In order to solve (S), we shall make use of (4.6).

PROOF OF THEOREM C. (i) *Case  $1 < q \leq n'$  for  $n \geq 3$  and  $1 < q < 2$  for  $n = 2$ .* As we have seen in the proof of Theorem B, (1.5) is equivalent to the identity

$$(f - \Delta w, v) + (g - \text{Tr } A + \text{div } w, -\chi) = 0 \text{ for all } \{v, \chi\} \in \mathbb{N}_q^0.$$

This implies that  $\{f - \Delta w, g - \text{Tr } A + \text{div } w\} \in (\text{Ker } T_{q'})^\perp$ . Hence by Corollary 3.4(i), there is a unique generalized solution  $\{\hat{u}, p\} \in \dot{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  of (4.6). By Lemma 2.2(i) we have also  $\hat{u} \in L^{nq/(n-q)}(\Omega)^n$ . Moreover, from Theorem 3.3 and the continuity of the extension operator  $\Gamma$ , we obtain

$$\begin{aligned} \|\nabla \hat{u}\|_q + \|p\|_q &\leq C (\|f - \Delta w\|_{-1,q} + \|g - \text{Tr } A + \text{div } w\|_q) \\ &\leq C (\|f\|_{-1,q} + \|g - \text{Tr } A\|_q + |A| + |a|), \end{aligned}$$

where  $C = C(\Omega, n, q)$ . Taking  $u = \hat{u} + Ax + a - w$ , we see that  $\{u, p\}$  is the desired generalized solution of (S). The uniqueness follows from the fact that  $\mathbb{N}_q^0 = \{0, 0\}$ . Conversely, suppose that  $\{u, p\} \in H_{\text{loc}}^{1,q}(\overline{\Omega})^n \times L^q(\Omega)$  is a generalized solution of (S) satisfying (1.2) and (1.4) for some  $A \in \mathbb{R}^{n^2}$  with  $\text{Tr } A - g \in L^q(\Omega)$  and  $a \in \mathbb{R}^n$ . Taking  $w = \Gamma(A \cdot +a)$ , we see by Lemma 2.2(i) that  $\hat{u} = u - Ax - a + w \in \dot{H}_0^{1,q}(\Omega)^n$  and that  $\{\hat{u}, p\}$  is a generalized solution of (4.6). Hence it follows from Corollary 3.4(i) that

$$\{f - \Delta w, g - \text{Tr } A + \text{div } w\} \in R(S_q) = (\text{Ker } T_{q'})^\perp,$$

from which we get (1.5).

(ii) *Case  $n' < q < n$  for  $n \geq 3$ .* Take  $w = \Gamma(A \cdot +a)$  and consider (4.6) for  $\{\hat{u}, p\}$ . Then we have by Corollary 3.4(ii) that (4.6) is uniquely solvable in  $\dot{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  for all  $f, g, A$  and  $a$  as given in the assumptions. Then the

proof of existence and uniqueness is quite the same as in the case (i) above. Suppose in addition that  $f \in \hat{H}^{-1,r}(\Omega)^n$  and  $g - \text{Tr } A \in L^r(\Omega)$  for some  $r > n$ . Since  $1 < r' < q' < n$ , it follows from (2.2) and an interpolation argument (see, e.g., Triebel [32, 1.11.2]) that  $f \in \hat{H}^{-1,\gamma}(\Omega)^n$  and  $g - \text{Tr } A \in L^\gamma(\Omega)$  for all  $q \leq \gamma \leq r$ . Hence we have by Theorem 3.1 that  $\{\hat{u}, p\} \in \hat{H}_0^{1,\gamma}(\Omega)^n \times L^\gamma(\Omega)$  for all  $q \leq \gamma \leq r$  and that, in particular,  $\hat{u} \in H^{1,s}(\Omega)^n$  for some  $s > n$ . By the Sobolev embedding theorem, we obtain  $\hat{u} \in C^0(\bar{\Omega})^n$  and  $\lim_{x \rightarrow \infty} |\hat{u}(x)| = 0$ . Now it is easy to see that  $u = \hat{u} + Ax + a - w$  satisfies (1.3).

(iii) *Case  $n \leq q < \infty$  for  $n \geq 3$  and  $2 < q < \infty$  for  $n = 2$ .* Taking  $w = \Gamma(A \cdot)$  in (4.6), we have by Corollary 3.4(iii) that there is at least one generalized solution  $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$  of (4.6). On the other hand, we have by Lemma 2.2(ii) that  $\mathbb{N}_q^0 = \text{Ker } S_q$  and that  $R(S_q)$  is isometric to the quotient space  $X_q/\mathbb{N}_q^0$ . Therefore it follows that

$$\begin{aligned} & \inf \{ \|\nabla \hat{u} - \nabla v\|_q + \|p - \chi\|_q; \{v, \chi\} \in \mathbb{N}_q^0 \} \\ & \leq C (\|f - \Delta w\|_{-1,q} + \|g - \text{Tr } A + \text{div } w\|_q) \\ & \leq C (\|f\|_{-1,q} + \|g - \text{Tr } A\|_q + |A|), \end{aligned}$$

where  $C = C(\Omega, n, q)$ . Taking  $u = \hat{u} + Ax - w$ , we see that  $\{u, p\} \in H_{\text{loc}}^{1,q}(\bar{\Omega})^n \times L^q(\Omega)$  has the desired property.

Suppose that  $\{\tilde{u}, \tilde{p}\} \in H_{\text{loc}}^{1,q}(\bar{\Omega})^n \times L^q(\Omega)$  is another generalized solution of (S) with (1.2). Set  $u' = u - \tilde{u}$  and  $p' = p - \tilde{p}$ . Then we get  $\{u', p'\} \in \mathbb{N}_q^0$ . Hence

$$\begin{aligned} & \inf \{ \|\nabla \tilde{u} - A - \nabla v\|_q + \|\tilde{p} - \chi\|_q; \{v, \chi\} \in \mathbb{N}_q^0 \} \\ & = \inf \{ \|\nabla u - A - \nabla v\|_q + \|p - \chi\|_q; \{v, \chi\} \in \mathbb{N}_q^0 \} \end{aligned}$$

so uniqueness and (1.7) follow.

(iv) *Case  $n = q = 2$ .* By Theorem B(iii), we see that  $\mathbb{N}_2^0 = \{0, 0\}$ ; so the proof is quite similar to that of the case (ii) above.  $\square$

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