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## Limit Semigroups of Stancu-Mühlbach Operators Associated with Positive Projections*

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### Introduction

In [2] Altomare has introduced a general definition of the sequence of Bernstein-Schnabl operators associated with a positive projection and has studied the limit behaviour of this sequence and of its iterates; moreover, in the same paper, it is established the existence of a (uniquely determined) positive contraction semigroup which has an explicit representation in terms of the Bernstein-Schnabl operators [2, Theorem 2.6].

In [3], we have introduced the definition of the sequence of Stancu-Mühlbach operators associated with a positive projection in the same general setting of [2] and we have studied the asymptotic behaviour of this sequence and its iterates. These results generalize to a wider context that obtained by Felbecker in [5] in the case of Stancu-Mühlbach operators on the compact convex set $\mathcal{M}^1(K)$ of all probability Radon measures on a compact Hausdorff topological space $K$.

In this paper, we are interested to investigate the existence of a positive contraction semigroup represented by Stancu-Mühlbach operators; also in this case the results that we obtain generalize the case $\mathcal{M}^1(K)$ studied in [5] by Felbecker.

Among the properties of this semigroup, we point out that it is mean-ergodic and strongly converges to the initial projection as $t$ tends to $\infty$; moreover, its infinitesimal generator is explicitly determined on a dense subspace of its domain and, in the case of some convex compact subsets $X$ of $\mathbb{R}^p$, the generator is a degenerate elliptic second order differential operator. As a consequence it is possible to obtain the solutions of the associated abstract Cauchy problems in terms of Stancu-Mühlbach operators.

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## 1. - Recalls and preliminary results

We need to recall some preliminary results.

Let $X$ be a compact Hausdorff space and $\mathcal{C}(X,\mathbb{R})$ be the Banach lattice of all real continuous functions on $X$, endowed with the sup-norm and the natural order.

If $T:\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R})$ is a linear positive operator and if $S$ is a subset of $\mathcal{C}(X,\mathbb{R})$, we recall that $S$ is called a *T-Korovkin set* if, for every net $(L_i)_{i\in I}^{\leq}$ of linear positive operators on $\mathcal{C}(X,\mathbb{R})$ such that

$$\lim_{i\in I^{\leq}} L_i(h)=T(h) \quad \text{for every } h\in S,$$

it results

$$\lim_{i\in I^{\leq}} L_i(f)=T(f) \quad \text{for every } f\in \mathcal{C}(X,\mathbb{R}).$$

If $T:\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R})$ is a linear positive projection, that is $T$ is a linear positive operator such that $T^2=T$, we have the following result (cf. [1, Theorem 1.3] ad [2, Prop. 1.2]).

THEOREM 1.1. *Let $X$ be a metrizable compact Hausdorff space and $T:\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R})$ a linear positive projection such that $T(\mathbf{1})=\mathbf{1}$ and the range $H=T(\mathcal{C}(X,\mathbb{R}))$ separates the points of $X$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $H$ which separates the points of $X$ and such that the series $\sum_{n=0}^{\infty} h_n^2$ converges uniformly to a function $\phi\in\mathcal{C}(X,\mathbb{R})$.*

*Then $H\cup\{\phi\}$ (and in particular $H\cup H^2$) is a T-Korovkin set.* ■

REMARK 1.2. As observed in [2], if $X$ is a metrizable compact space and $H$ is a linear subspace of $\mathcal{C}(X,\mathbb{R})$, $H$ is separable and therefore we may consider a dense sequence $(\ell_n)_{n\in\mathbb{N}}$ of elements of $H$; if we put $h_n=\dfrac{\ell_n}{\|\ell_n\|2^{n/2}}$ for every $n\in\mathbb{N}$, we obtain a sequence $(h_n)_{n\in\mathbb{N}}$ in $H$ which separates the points of $X$ and such that the series $\sum_{n=0}^{\infty} h_n^2$ is uniformly convergent on $X$. ■

At this point, we may recall the definition of the $n$-th Stancu-Mühlbach operator introduced in [3]; for simplicity, we consider the Stancu-Mühlbach operators associated with the arithmetic mean Toeplitz matrix (cf. [3, (2.13)]) and a sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$.

Let $X$ be a metrizable convex compact subset of some locally convex Hausdorff space and $T:\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R})$ be a linear positive projection; let $H=T(\mathcal{C}(X,\mathbb{R}))$ be the range of $T$.

Denote by $A(X)$ the space of all continuous affine functions on $X$ and suppose that

$$A(X)\subset H \tag{1.1}$$

(hence $H$ separates the points of $X$ and $T(\mathbf{1}) = \mathbf{1}$), and for every $\overline{x} \in X$, $\lambda \in [0,1]$ and $h \in H$

$$\text{(1.2)} \qquad \textit{the function } x \in X \mapsto h((1-\lambda)\overline{x} + \lambda x) \textit{ belongs to } H.$$

For every $x \in X$ we shall denote by $\mu_x \in \mathcal{M}^1(X)$ the probability Radon measure on $X$ defined by putting

$$\text{(1.3)} \qquad \mu_x(f) = T(f)(x) \quad \text{for every } f \in \mathcal{C}(X,\mathbb{R}).$$

Let $n \in \mathbb{N}$, $n \geq 1$; according to [5] and [6] we denote by $p_n : \mathbb{R} \to \mathbb{R}$ the real function defined by putting, for each $a \in \mathbb{R}$,

$$\text{(1.4)} \qquad p_n(a) = \prod_{j=0}^{n-1}(1 + ja);$$

if $k = 1, \ldots, n$, we put

$$\text{(1.5)} \qquad V(n,k) = \left\{ (v_1, \ldots, v_k) \in \mathbb{N}^k \mid v_1, \ldots, v_k \geq 1 \text{ and } \sum_{i=1}^{k} v_i = n \right\};$$

for simplicity we write $|v|_k = n$ instead of $v = (v_1, \ldots, v_k) \in V(n,k)$.

If we denote by $s(n,k)$ the coefficient of $a^{n-k}$ of the polynomial $p_n(a)$, we have

$$\text{(1.6)} \qquad p_n(a) = \sum_{k=1}^{n} s(n,k)\; a^{n-k}$$

and further (cf. [5, (1.1.8), pp. 14-16] and [4, II, pp. 49-50])

$$\text{(1.7)} \qquad s(n,k) = \frac{n!}{k!} \sum_{|v|_k = n} \frac{1}{v_1 \ldots v_k},$$

$$\text{(1.8)} \qquad p_{n+1}(a) = p_2(a) \sum_{k=1}^{n} \frac{(n-1)!}{k!}\, a^{n-k} \sum_{|v|_k=n} \frac{v_1^2 + \ldots + v_k^2}{v_1 \ldots v_k}.$$

Finally, for each $(v_1, \ldots, v_k) \in V(n,k)$ we consider the function $\pi_{v_1,\ldots,v_k} : X^k \to X$ defined by putting, for each $(x_1, \ldots, x_k) \in X^k$,

$$\text{(1.9)} \qquad \pi_{v_1,\ldots,v_k}(x_1, \ldots, x_k) = \frac{v_1 x_1 + \cdots + v_k x_k}{n}.$$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers; for each $n \in \mathbb{N}$, $n \geq 1$, the $n$-th Stancu-Mühlbach operator $Q_{n,a_n} : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ with respect to the projection $T$, is defined by putting, for each $f \in \mathcal{C}(X,\mathbb{R})$ and $x \in X$,

$$(1.10)\qquad Q_{n,a_n}(f)(x) = \frac{1}{p_n(a_n)} \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} \sum_{|v|_k=n} \frac{1}{v_1 \dots v_k} \int_{X^k} f \circ \pi_{v_1,\dots,v_k} \mathrm{d}\left(\bigotimes_{i=1}^{k} \mu_{x,i}\right)$$

$$\left(= \frac{1}{p_n(a_n)} \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} \sum_{|v|_k=n} \frac{1}{v_1 \dots v_k} \int_X \dots \int_X f\left(\frac{v_1x_1 + \dots + v_kx_k}{n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_k\right)$$

where $\mu_{x,i} = \mu_x$ for every $i = 1, \dots, k$.

If $a_n = 0$ the $n$-th Stancu-Mühlbach operator coincides with the $n$-th Bernstein-Schnabl operator (cf. [2, (2.4)]).

The iterates of the Stancu-Mühlbach operators are defined by putting

$$(1.11)\qquad Q_{n,a_n}^0 = I \quad \text{and} \quad Q_{n,a_n}^m = Q_{n,a_n} \circ Q_{n,a_n}^{m-1} \quad \text{for } n \geq 1,\ m \geq 1.$$

By utilizing (1.6-8), we have the following formulas, established in [3, (2.15-19)]; for each $n \in \mathbb{N}$, $n \geq 1$, and for each $h \in H$

$$(1.12)\qquad Q_{n,a_n}(h) = h;$$

moreover, if $m \in \mathbb{N}$, $m \geq 1$ and $h \in A(X)$

$$(1.13)\qquad Q_{n,a_n}^m(h^2) = \left(\frac{n-1}{n} \frac{1}{1+a_n}\right)^m h^2 + \left(1 - \left(\frac{n-1}{n} \frac{1}{1+a_n}\right)^m\right) T(h^2).$$

## 2. - Limit semigroup of Stancu-Mühlbach operators

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers.

In order to study some convergence properties in the case where the sequence $(na_n)_{n\in\mathbb{N}}$ converges to a real number $b$, we assume the following notations; for every $m \geq 1$, we put

$$(2.1)\qquad A_m = \textit{the linear subspace generated by } \left\{\prod_{i=1}^{m} h_i \,\middle|\, h_i \in A(X),\ i = 1, \dots, m\right\};$$

$(A_m)_{m\geq 1}$ is an increasing sequence of linear subspaces of $\mathcal{C}(X,\mathbb{R})$ and further, the subspace

$$(2.2)\qquad A_\infty = \bigcup_{m=1}^{\infty} A_m$$

is a subalgebra of $C(X,\mathbb{R})$ which separates the points of $X$ and so is dense in $C(X,\mathbb{R})$ by Stone-Weierstrass theorem.

Moreover, we consider the linear operator $L_0 : A_\infty \to A_\infty$ defined by putting, for each $m \in \mathbb{N}$ and $h_1, \dots, h_m \in A(X)$,

$$L_0\left(\prod_{i=1}^{m} h_i\right) = \begin{cases} 0 & m=1 \\ T(h_1h_2) - h_1h_2 & m=2 \\ \sum\limits_{1\le i<j\le m} (T(h_ih_j) - h_ih_j) \prod\limits_{\substack{r=1 \\ r\neq i,j}}^{m} h_r & m \ge 3. \end{cases} \tag{2.3}$$

The following lemma is contained in [5, (3.5.3), (3.5.4)], but for the sake of completeness, we prefer to state the proof.

LEMMA 2.1. *Let $n \ge 1$, $k = 1, \dots, n$, and for each $\ell \ge 1$ put*

$$N(\ell) = \{(i_1, \dots, i_l) \in \{1, \dots, k\}^l \mid i_r \neq i_s \text{ for } r \neq s\}. \tag{2.4}$$

*If $(v_1, \dots, v_k) \in V(n,k)$ we have*

$$\sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2} \dots v_{i_l} = n^{l-1} \sum_{i=1}^{k} v_i^2 + U_n(v_1, \dots, v_k; \ell) \tag{2.5}$$

*with*

$$|U_n(v_1, \dots, v_k; \ell)| \le u_{1l}\, n^{l-2} \sum_{i=1}^{k} v_i^3 + u_{2l}\, n^{l-3} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2$$

*and where $u_{1l}$ and $u_{2l}$ are real constants depending on $\ell$.*

*Further, for each $\ell \ge 2$, it results*

$$\sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1} \dots v_{i_l} = n^l n^{l-2} \frac{l(l-1)}{2} \sum_{i=1}^{k} v_i^2 + W_n(v_1, \dots, v_k; \ell) \tag{2.6}$$

*with*

$$|W_n(v_1, \dots, v_k; \ell)| \le w_{1l}\, n^{l-3} \sum_{i=1}^{k} v_i^3 + w_{2l}\, n^{l-4} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2$$

*and where $w_{1l}$ and $w_{2l}$ are real constants depending on $\ell$.*

PROOF. If $\ell = 1$, (2.5) holds with $u_{11} = u_{12} = 0$.

By induction, if (2.5) holds for $\ell \in \mathbb{N}$, one has

$$
\begin{aligned}
& n \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2}\dots v_{i_l} - \sum_{(i_1,\dots,i_{l+1})\in N(\ell+1)} v_{i_1}^2 v_{i_2}\dots v_{i_{l+1}} \\
&= n \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2}\dots v_{i_l} - \sum_{(i_1,\dots,i_l)\in N(\ell)} \sum_{\substack{i=1\\ i\neq i_1,\dots,i_l}}^{k} v_{i_1}^2 v_{i_2}\dots v_{i_l} v_i \\
&= \sum_{(i_1,\dots,i_l)\in N(\ell)} \left( n - \sum_{\substack{i=1\\ i\neq i_1,\dots,i_l}}^{k} v_i \right) v_{i_1}^2 v_{i_2}\dots v_{i_l} \\
&= \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^3 v_{i_2}\dots v_{i_l} + (\ell-1) \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2}^2 v_{i_3}\dots v_{i_l}
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \sum_{(i_1,\dots,i_{l+1})\in N(\ell+1)} v_{i_1}^2 v_{i_2}\dots v_{i_{l+1}} = n \left( n^{l-1} \sum_{i=1}^{k} v_i^2 + U_n(v_1,\dots,v_k;\ell) \right) \\
& - \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^3 v_{i_2}\dots v_{i_l} - (\ell-1) \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2}^2 v_{i_3}\dots v_{i_l} \\
& = n^l \sum_{i=1}^{k} v_i^2 + U_n(v_1,\dots,v_k;\ell+1)
\end{aligned}
$$

with

$$
\begin{aligned}
& |U_n(v_1,\dots,v_k;\ell+1)| \le n \left( u_{1l}\, n^{l-2} \sum_{i=1}^{k} v_i^3 + u_{2l}\, n^{l-3} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2 \right) \\
& + n^{l-1} \sum_{i=1}^{k} v_i^3 + (\ell-1)\, n^{l-2} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2 .
\end{aligned}
$$

Then (2.5) holds for $\ell+1$ with $u_{1,l+1} = u_{1l}+1$ and $u_{2,l+1} = u_{2l}+\ell-1$.

Now, if $\ell = 1$, (2.6) holds with $w_{11} = w_{12} = 0$. By induction, if (2.6) holds for $\ell \in \mathbb{N}$, one has

$$n \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1} v_{i_2} \dots v_{i_l} - \sum_{(i_1,\dots,i_{l+1})\in N(\ell+1)} v_{i_1} v_{i_2} \dots v_{i_{l+1}}$$

$$= \sum_{(i_1,\dots,i_l)\in N(\ell)} \left( n - \sum_{\substack{i=1 \\ i\neq i_1,\dots,i_l}}^{k} v_i \right) v_{i_1} v_{i_2} \dots v_{i_l}$$

$$= \sum_{(i_1,\dots,i_l)\in N(\ell)} (v_{i_1} + v_{i_2} + \dots + v_{i_l})\, v_{i_1} v_{i_2} \dots v_{i_l} = \ell \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2} \dots v_{i_l}$$

and hence (cf. (2.5))

$$\sum_{(i_1,\dots,i_{l+1})\in N(\ell+1)} v_{i_1} v_{i_2} \dots v_{i_{l+1}}$$

$$= n \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1} v_{i_2} \dots v_{i_l} - \ell \sum_{(i_1,\dots,i_l)\in N(\ell)} v_{i_1}^2 v_{i_2} \dots v_{i_{l+1}}$$

$$= n \left( n^l - n^{l-2} \frac{\ell(\ell-1)}{2} \sum_{i=1}^{k} v_i^2 + W_n(v_1,\dots,v_k;\ell) \right)$$

$$- \ell \left( n^{l-1} \sum_{i=1}^{k} v_i^2 + U_n(v_1,\dots,v_k;\ell) \right)$$

$$= n^{l+1} - n^{l-1} \frac{\ell(\ell+1)}{2} \sum_{i=1}^{k} v_i^2 + W_n(v_1,\dots,v_k;\ell+1)$$

with

$$|W_n(v_1,\dots,v_k;\ell+1)| \le n \left( w_{1l}\, n^{l-3} \sum_{i=1}^{k} v_i^3 + w_{2l}\, n^{l-4} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2 \right)$$

$$+ \ell \left( u_{1l}\, n^{l-2} \sum_{i=1}^{k} v_i^3 + u_{2l}\, n^{l-3} \sum_{(i_1,i_2)\in N(2)} v_{i_1}^2 v_{i_2}^2 \right).$$

Then (2.6) holds for $\ell+1$ with $w_{1,l+1} = w_{1l} + \ell u_{1l}$ and $w_{2,l+1} = w_{2l} + \ell w_{2l}$ and this completes the proof. ∎

THEOREM 2.2. *Suppose that conditions* (1.1) *and* (1.2) *are satisfied and suppose that* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers such that the sequence* $(n \cdot a_n)_{n\in\mathbb{N}}$ *converges to* $b \in \mathbb{R}$.

*Then for every $f \in A_\infty$, we have*

$$\lim_{n\to\infty} n \cdot (Q_{n,a_n}(f) - f) = (1+b)\cdot L_0(f) \quad \textit{uniformly on } X.$$

PROOF. We utilize the same arguments of [5, pp. 85-94].

Let $f \in A_\infty$ and let $m \geq 1$ and $h_1, \dots, h_m \in A(X)$ such that $f = \prod_{j=1}^{m} h_j$; for every $(x_1, \dots, x_k) \in X^k$, it results (cf. (2.4))

$$f \circ \pi_{v_1,\dots,v_k}(x_1,\dots,x_k) = \prod_{j=1}^{m} h_j \circ \pi_{v_1,\dots,v_k}(x_1,\dots,x_k)$$

$$= \prod_{j=1}^{m} \frac{1}{n} \sum_{i=1}^{k} v_i h_j(x_i) = \frac{1}{n^m} \sum_{i_1=1}^{k} \dots \sum_{i_m=1}^{k} v_{i_1} \dots v_{i_m} h_1(x_{i_1}) \dots h_m(x_{i_m})$$

$$= \frac{1}{n^m} \Bigg( \sum_{i\in N(1)} v_i^m \, h_1 \dots h_m(x_i)$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, h_1 \dots h_{m-1}(x_{i_1}) \, h_m(x_{i_2})$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, h_1 \dots h_{m-2} h_m(x_{i_1}) \, h_{m-1}(x_{i_2}) + \dots$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, h_2 \dots h_m(x_{i_1}) \, h_1(x_{i_2}) + \dots$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-2} v_{i_2}^2 \, h_1 \dots h_{m-2}(x_{i_1}) \, h_{m-1} h_m(x_{i_2}) + \dots$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-2} v_{i_2}^2 \, h_3 \dots h_m(x_{i_1}) \, h_1 h_2(x_{i_2}) + \dots$$

$$+ \sum_{(i_1,\dots,i_{m-1})\in N(m-1)} v_{i_1}^2 v_{i_2} \dots v_{i_{m-1}} \, h_1 h_2(x_{i_1}) \, h_3(x_{i_2}) \dots h_m(x_{i_{m-1}}) + \dots$$

$$+ \sum_{(i_1,\dots,i_{m-1})\in N(m-1)} v_{i_1}^2 v_{i_2} \dots v_{i_{m-1}} \, h_{m-1} h_m(x_{i_1}) \, h_1(x_{i_2}) \dots h_{m-2}(x_{i_{m-1}})$$

$$+ \sum_{(i_1,\dots,i_m)\in N(m)} v_{i_1} \dots v_{i_m} \, h_1(x_{i_1}) \dots h_m(x_{i_m}) \Bigg)$$

and therefore, for each $x \in X$,

$$\int_{X^k} f \circ \pi_{v_1,\ldots,v_k} \, \mathrm{d}\left(\bigotimes_{i=1}^{k} \mu_{x,i}\right) = \int_X \mathrm{d}\mu_x \ldots \int_X f \circ \pi_{v_1,\ldots,v_k} \, \mathrm{d}\mu_x$$

$$= \frac{1}{n^m}\left(\sum_{i=1}^{k} v_i^m \, T(h_1 \ldots h_m)(x)\right.$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, T(h_1 \ldots h_{m-1})(x)\, T(h_m)(x)$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, T(h_1 \ldots h_{m-2} h_m)(x)\, T(h_{m-1})(x) + \ldots$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-1} v_{i_2} \, T(h_2 \ldots h_m)(x)\, T(h_1)(x)$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-2} v_{i_2}^2 \, h_1 \ldots h_{m-2}(x_{i_1})\, h_{m-1} h_m(x_{i_2}) + \ldots$$

$$+ \sum_{(i_1,i_2)\in N(2)} v_{i_1}^{m-2} v_{i_2}^2 \, h_3 \ldots h_m(x_{i_1})\, h_1 h_2(x_{i_2}) + \ldots$$

$$+ \left(\sum_{(i_1,\ldots,i_{m-1})\in N(m-1)} v_{i_1}^2 v_{i_2} \ldots v_{i_{m-1}}\right) \sum_{1\le i<j\le m} T(h_i h_j)(x) \prod_{\substack{r=1\\ r\neq 1,j}}^{m} T(h_r)(x)$$

$$\left. + \left(\sum_{(i_1,\ldots,i_m)\in N(m)} v_{i_1} \ldots v_{i_m}\right) T(h_1)(x) \ldots T(h_m)(x)\right).$$

By utilizing (2.5) and (2.6) we obtain

$$\int_{X^k} f \circ \pi_{v_1,\ldots,v_k} \, \mathrm{d}\left(\bigotimes_{i=1}^{k} \mu_{x,i}\right)$$

$$= \frac{1}{n^m}\left(\sum_{i=1}^{k} v_i^m \, T(h_1 \ldots h_m)(x) + \ldots\right.$$

$$+ \left(n^{m-2} \sum_{i=1}^{k} v_i^2 + U_n(v_1, \ldots, v_k; m-1)\right)$$

$$\cdot \sum_{1\le i<j\le m} T(h_i h_j)(x) \prod_{\substack{r=1\\ r\neq i,j}}^{m} T(h_r)(x)$$

$$+\left(n^m - n^{m-2}\,\frac{m(m-1)}{2}\sum_{i=1}^{k} v_i^2\right.$$

$$\left.\left. + W_n(v_1,\dots,v_k;m)\right)\,T(h_1)(x)\dots T(h_m)(x)\right)$$

$$= \left(h_1\dots h_m + \frac{1}{n^2}\left(\sum_{i=1}^{k} v_i^2\right)\sum_{1\le i<j\le m}(T(h_ih_j)-h_ih_j)(x)\prod_{\substack{r=1\\ r\neq i,j}}^{m} h_r(x)\right.$$

$$\left. + \sum_{i=1}^{s(m)} R_i(v_1,\dots,v_k)\,B_i(h_1\dots h_m)(x)\right),$$

where $s(m)$ is a natural number depending on $m$ and for each $i = 1,\dots,s(m)$,

$$|R_i(v_1,\dots,v_k)| \le \frac{1}{n^3}\,c_i\sum_{j=1}^{k} v_j^3 + n^{-4}\,d_i\sum_{j\in N(2)} v_{j_1}^2 v_{j_2}^2$$

($c_i$ and $d_i$ are real constants depending on $i$) and $B_i(h_1\dots h_m)$ belongs to the linear subspace generated by

$$\{h_1\dots h_m,\ T(h_1h_2)\,h_3\dots h_m,\dots,\ T(h_1h_2h_3)\,h_4\dots h_m,\dots,\ T(h_1\dots h_m)\}\,.$$

Let $n\in\mathbb{N}$; by (2.3), (1.6) and (1.7), we have

$$(2.7)\qquad Q_{n,a_n}(f) = \frac{1}{p_n(a_n)}\sum_{k=1}^{n}\frac{(n-1)!}{k!}\,a_n^{n-k}$$

$$\cdot\left(\sum_{|v|_k=n}\frac{n}{v_1\dots v_k}\,h_1\dots h_m + \sum_{|v|_k=n}\frac{v_1^2+\dots+v_k^2}{v_1\dots v_k}\,\frac{1}{n}\,L_0(h_1\dots h_m)\right.$$

$$\left. + \sum_{|v|_k=n}\frac{n}{v_1\dots v_k}\sum_{i=1}^{s(m)} R_i(v_1,\dots,v_k)\,B_i(h_1\dots h_m)\right)$$

$$= h_1\dots h_m + \frac{1}{n}\,\frac{1+na_n}{1+a_n}\,L_0(h_1\dots h_m)$$

$$+ \frac{1}{p_n(a_n)}\sum_{k=1}^{n}\frac{(n-1)!}{k!}\,a_n^{n-k}\sum_{|v|_k=n}\frac{n}{v_1\dots v_k}\sum_{i=1}^{s(m)}$$

$$R_i(v_1,\dots,v_k)\,B_i(h_1\dots h_m).$$

By (1.7-9), (2.7) and by the formulas

$$\sum_{k=1}^{n} \frac{(n-1)!}{k!} \sum_{|v|_k=n} \frac{v_1^3+\ldots+v_k^3}{v_1\ldots v_k}\, a_n^{n-k} = (1+2na_n)\,\frac{p_{n+1}(a_n)}{p_3(a_n)}, \tag{2.8}$$

$$\sum_{k=1}^{n} \frac{(n-1)!}{k!} \sum_{|v|_k=n} \frac{1}{v_1\ldots v_k} \sum_{\substack{i,j=1\\ i\neq j}}^{k} v_i^2 v_j^2\, a_n^{n-k} = (n-1)\,\frac{p_{n+2}(a_n)}{p_4(a_n)} \tag{2.9}$$

$\left(\text{with the convention } \sum_{\substack{i,j=1\\ i\neq j}}^{k} v_i^2 v_j^2 = 0 \text{ if } k=1\right)$ established in [5, (1.1.3- 4) and (1.1.11-12)], we finally obtain

$$\begin{aligned}
&\|n(Q_{n,a_n}(f)-f)-(1+b)\,L_0(f)\| \\
&\le \left\| n(Q_{n,a_n}(f)-f) - \frac{1+na_n}{1+a_n}\,L_0(f)\right\| + \left|\frac{1+na_n}{1+a_n}-(1+b)\right| \, \|L_0(f)\| \\
&\le \sum_{i=1}^{s(m)} \frac{1}{p_n(a_n)} \sum_{k=1}^{n} \frac{(n-1)!}{k!}\, a_n^{n-k} \left(\frac{1}{n}\, c_i \sum_{|v|_k=n} \frac{v_1^3+\ldots+v_k^3}{v_1\ldots v_k}\right. \\
&\quad \left. + \frac{1}{n^2}\, d_i \sum_{|v|_k=n} \frac{1}{v_1\ldots v_k} \sum_{\substack{i,j=1\\ i\neq j}}^{k} v_i^2 v_j^2\, a_n^{n-k}\right) \|B_i(h_1\ldots h_m)\| \\
&\quad + \left|\frac{1+na_n}{1+a_n}-(1+b)\right| \, \|L_0(h_1\ldots h_m)\| \\
&\le \sum_{i=1}^{s(m)} \frac{1}{p_n(a_n)} \left(\frac{1}{n}\, c_i(1+2na_n)\,\frac{p_{n+1}(a_n)}{p_3(a_n)}\right. \\
&\quad \left. + \frac{1}{n^2}\, d_i(n-1)\,\frac{p_{n+2}(a_n)}{p_4(a_n)}\right) \|B_i(h_1\ldots h_m)\| \\
&\quad + \left|\frac{1+na_n}{1+a_n}-(1+b)\right| \, \|L_0(h_1\ldots h_m)\| \\
&\le \frac{1}{n} \sum_{i=1}^{s(m)} \left( c_i\, \frac{(1+2na_n)(1+na_n)}{(1+a_n)(1+2a_n)}\right. \\
&\quad \left. + d_i\, \frac{(n-1)(1+na_n)(1+(n+1)a_n)}{n(1+a_n)(1+2a_n)(1+3a_n)}\right) \|B_i(h_1\ldots h_m)\| \\
&\quad + \left|\frac{1+na_n}{1+a_n}-(1+b)\right| \, \|L_0(h_1\ldots h_m)\|.
\end{aligned}$$

Since $\lim_{n\to\infty} n \cdot a_n = b \in \mathbb{R}$, we can conclude that

$$\lim_{n\to\infty} \|n(Q_{n,a_n}(f) - f) - (1+b)\, L_0(f)\| = 0. \qquad \blacksquare$$

REMARK 2.3. In the case $X = \mathcal{M}^1(K)$, Theorem 2.2 has been obtained by Felbecker [5, (3.5.2)]; if $a_n = 0$ for each $n \geq 1$, Theorem 2.2 has been proved by Schnabl [12] in the case $X = \mathcal{M}^1(K)$ and Altomare [2] in the general context.

Moreover, as observed in [5, (3.5.5)], if $X$ is the compact real interval $[0,1]$, the space $A_\infty$ is just the space $\mathcal{P}([0,1])$ of all polynomials on $[0,1]$ and the operator $L_0 : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ is defined by putting $L_0(f)(x) = \frac{1}{2}$ $x(1-x)\, f''(x)$ for each polynomial $f$ and $x \in [0,1]$; then Theorem 2.2 and (1-3) yield

$$\lim_{n\to\infty} n \left( \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} \frac{\prod_{j=0}^{k-1}(x + ja_n) \prod_{j=0}^{n-k-1}(1 - x + ja_n)}{\prod_{j=0}^{n-1}(1 + ja_n)} - f(x) \right)$$

$$= \lim_{n\to\infty} n\,(Q_{n,a_n}(f) - f)(x) = \frac{1}{2}(1+b)\, x\,(1-x)\, f''(x)$$

for each polynomial $f$ and $x \in [0,1]$.

In the case $a_n = 0$ for each $n \geq 1$, the preceding formula has been obtained by Voronovskaja (cf. [8, p. 22]). $\blacksquare$

Now we want to study the asymptotic behaviour of the sequence $(Q_{n,a_n}^{k(n)})_{n\in\mathbb{N}}$ in the case where $\lim_{n\to\infty} \frac{k(n)}{n} = t > 0$.

THEOREM 2.4. *Suppose that conditions* (1.1) *and* (1.2) *are satisfied and suppose that* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers such that the sequence* $(n \cdot a_n)_{n\in\mathbb{N}}$ *converges to* $b \in \mathbb{R}$.

*Consider the sequence* $(Q_{n,a_n})_{n\in\mathbb{N}}$ *of Stancu-Mühlbach operators associated with* $T$ (*cf.* (1.10)) *and suppose that*

(i) $T(A_2) \subset A(X)$

*or, alternatively,*

(i)′ $A(X)$ *is finite dimensional and* $T(A_m) \subset A_m$ *for every* $m \geq 1$.

*Then there exists a strongly continuous positive contraction semigroup* $(Q(t))_{t\geq 0}$ *on* $\mathcal{C}(X,\mathbb{R})$ *such that, for every* $t \geq 0$ *and for every sequence* $(k(n))_{n\in\mathbb{N}}$ *of positive integers such that* $\lim_{n\to\infty} \frac{k(n)}{n} = t$, *one has*

$$\lim_{n\to\infty} Q_{n,a_n}^{k(n)} = Q(t) \quad \textit{strongly on } \mathcal{C}(X,\mathbb{R}).$$

*Moreover,*

$$\lim_{t\to\infty} Q(t) = T \quad \textit{strongly on } \mathcal{C}(X,\mathbb{R})$$

*and the generator of the semigroup* $(Q(t))_{t\geq 0}$ *is the closure of the linear operator* $A : D(A) \to \mathcal{C}(X,\mathbb{R})$ *defined by putting*

$$A(f) = \lim_{n\to\infty} n(Q_{n,a_n}(f) - f) \tag{2.10}$$

*for every* $f \in D(A)$, *where*

$$D(A) = \{f \in \mathcal{C}(X,\mathbb{R}) \mid \lim_{n\to\infty} n(Q_{n,a_n}(f) - f) \textit{ exists in } \mathcal{C}(X,\mathbb{R})\}.$$

*Finally* $A_\infty \subset D(A)$ *and for every* $m \in \mathbb{N}$, $m \geq 1$ *and* $h_1, \dots, h_m \in A(X)$, *it results* (*cf.* (2.3))

$$A\left(\prod_{i=1}^{m} h_i\right) = (1+b)\cdot L_0\left(\prod_{i=1}^{m} h_i\right). \tag{2.11}$$

PROOF. Let $A : D(A) \to \mathcal{C}(X,\mathbb{R})$ be the linear operator defined in (2.10). By Theorem 2.2, we have $A_\infty \subset D(A)$ and therefore $D(A)$ is dense in $\mathcal{C}(X,\mathbb{R})$.

Suppose that condition (i) holds. We show that for every $\lambda > 0$ the range $R(\lambda I - A)$ is dense in $\mathcal{C}(X,\mathbb{R})$, where $I$ denotes the identity operator on $\mathcal{C}(X,\mathbb{R})$. In fact, fix $\lambda > 0$ and consider $\mu \in \mathcal{C}(X,\mathbb{R})'$ such that $\mu(g) = 0$ for every $g \in R(\lambda I - A)$, i.e. $\mu(f) = \frac{1}{\lambda}\mu(A(f))$ for every $f \in D(A)$. So, for every $f \in A_1$, we have (cf. Theorem 2.2 and (2.3)) $\mu(f) = \frac{1}{\lambda}\mu(A(f)) = 0$. Moreover, according to Theorem 2.2 and (2.3), for every $f \in A_2$ we have $\mu(f) = \frac{1}{\lambda}\mu(A(f)) = \frac{1}{\lambda}\mu(T(f)) - \frac{1}{\lambda}\mu(f) = \frac{1}{\lambda}\mu(f)$ and so again $\mu(f) = 0$.

Suppose now that $\mu = 0$ on $A_m$ with $m \geq 2$ and let $f = \prod_{i=1}^{m+1} h_i$, with $h_i \in A(X)$, for every $i = 1, \dots, m+1$. Then

$$\mu(f) = \frac{1}{\lambda}\mu(A(f)) = \frac{1}{\lambda}\mu\left(\sum_{1\leq i<j\leq m+1} T(h_i h_j)\prod_{r\neq i,j} h_r - \binom{m+1}{2} f\right)$$
$$= -\frac{1}{\lambda}\,\frac{m(m-1)}{2}\,\mu(f)$$

since $T(h_i h_j)\prod_{r\neq i,j} h_r \in A_m$ for every $i, j = 1, \dots, m+1$, by virtue of (i). Consequently $\mu(f) = 0$. This implies that $\mu = 0$ on $A_{m+1}$; hence by induction on $m$, we have $\mu = 0$ on $A_\infty$ and so $\mu = 0$.

Thus, we have proved that $R(\lambda I - A)$ is dense in $\mathcal{C}(X,\mathbb{R})$ for every $\lambda > 0$. Using a theorem of Trotter [14, Theorem 5.3], we infer that the closure of $A$ is the infinitesimal generator of a contraction semigroup $(Q(t))_{t\geq 0}$ and

$$Q(t) = \lim_{n\to\infty} Q_{n,a_n}^{[nt]} \qquad \text{strongly on } \mathcal{C}(X,\mathbb{R})$$

for all $t \geq 0$, where $[nt]$ denotes the integer part of $nt$.

In particular, every $Q(t)$ is positive. Consider now a sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} \frac{k(n)}{n} = t \geq 0$. Then for every

$$f \in A_\infty, \ \lim_{n\to\infty} k(n)(Q_{n,a_n}(f) - f) = \lim_{n\to\infty} \frac{k(n)}{n} n(Q_{n,a_n}(f) - f) = t \cdot A(f).$$

Again according to Trotter's theorem, the closure of $tA$ is the infinitesimal generator of a semigroup $(S(u))_{u\geq 0}$ of contractions and for every $u \geq 0$

$$S(u) = \lim_{n\to\infty} Q_{n,a_n}^{[k(n)u]} \qquad \text{strongly on } \mathcal{C}(X,\mathbb{R}).$$

Since the closure of $tA$ is also generated by $(Q(tu))_{u\geq 0}$, we conclude that $S(u) = Q(tu)$ for all $u \geq 0$ and $t \geq 0$ and so

$$Q(t) = S(1) = \lim_{n\to\infty} Q_{n,a_n}^{k(n)} \qquad \text{strongly on } \mathcal{C}(X,\mathbb{R}).$$

If, alternatively, condition (i)$'$ is satisfied, then for every $m \in \mathbb{N}$, $A_m$ is finite dimensional and, by virtue of (2.7), it is invariant under $Q_{n,a_n}$ for every $n \in \mathbb{N}$. So, the existence of the semigroup $(Q(t))_{t\geq 0}$ which satisfies the properties indicated in Theorem 2.4, directly follows from a result of Schnabl [13, Satz 4] (see also a result of Nishishiraho [10, Theorem 1]).

Let $t \geq 0$; since $\lim_{n\to\infty} \frac{[nt]}{n} = t$, for each $h \in H$, we have (cf. (1.12))

$$Q(t)(h) = \lim_{n\to\infty} Q_{n,a_n}^{[nt]}(h) = h = T(h)$$

and for each $h \in A(X)$ (cf. (1.13))

$$\begin{aligned}
Q(t)(h^2) &= \lim_{n\to\infty} Q_{n,a_n}^{[nt]}(h^2) \\
&= \lim_{n\to\infty} \left(\frac{n-1}{n(1+a_n)}\right)^{[nt]} h^2 + \left(1 - \left(\frac{n-1}{n(1+a_n)}\right)^{[nt]}\right) T(h^2) \\
&= T(h^2) + \lim_{n\to\infty} \left(\frac{n-1}{n}\frac{1}{1+a_n}\right)^{[nt]} (h^2 - T(h^2)) \\
&= T(h^2) + e^{-t(1+b)}(h^2 - T(h^2));
\end{aligned}$$

hence for each $h \in H$, $\lim_{t\to\infty} Q(t)(h) = T(h)$ and for each $h \in A(X)$,

$$\lim_{t\to\infty} Q(t)(h^2) = T(h^2);$$

by Remark 1.2, we may consider a sequence $(h_n)_{n\in\mathbb{N}}$ in $A(X)$ which separates the points of $X$ and such that the series $\sum_{n=0}^{\infty} h_n^2$ converges uniformly to a function $\phi$; since $Q(t)$ is a contraction for every $t \geq 0$, we have $\lim_{t\to\infty} Q(t)(\phi) = T(\phi)$ and by Theorem 1.1, we obtain $\lim_{t\to\infty} Q(t) = T$ strongly on $\mathcal{C}(X,\mathbb{R})$. Finally, for each $f \in A_\infty$ and $t \geq 0$, by (2.10) and Theorem 2.2, we have $A(f) = \lim_{n\to\infty} n \cdot (Q_{n,a_n}(f) - f) = (1+b)\cdot L_0(f)$ and this completes the proof.■

REMARK 2.5.

1. In the context of metrizable Bauer simplexes (cf. Ex. 1.) clearly condition (i) of Theorem 2.4 (and also condition (i)′) is satisfied.

2. In the case $X = \mathcal{M}^1(K)$, Theorem 2.4 has been obtained by Felbecker [5]; further, Theorem 2.4 has been proved for Bernstein-Schnabl polynomials by Altomare in [2] in the general case and by Nishishiraho in [10, pp. 79-80], in the context of metrizable Bauer simplexes (see also Schnabl [12], [13]). For the classical Bernstein operators on $[0,1]$, Theorem 2.4 is substantially known (cf. Karlin-Ziegler [7] and Micchelli [9]). In these articles a detailed analysis of the properties of the semigroup $(Q(t))_{t\geq 0}$ can be found.

3. Other results on the convergence of iterates of positive operators to semigroups can be found in [5] and [11]. ■

Finally we give an application of Theorem 2.4 in the case where $X = B(x_0, r)$ is the ball in $\mathbb{R}^p$ $(p \geq 1)$ of center $x_0$ and radius $r$ (other examples may be obtained in a similar manner in the case where $X$ is the standard simplex of $\mathbb{R}^p$ or the hypercube of $\mathbb{R}^p$ (cf. [2, 3.1-2] and [3, ex. 1-2]). In this case, the $n$-th Stancu-Mühlbach operator $Q_{n,a_n}$ associated with the arithmetic mean Toeplitz matrix is defined by putting, for each $f \in \mathcal{C}(X,\mathbb{R})$ and $x \in X$ (cf. [3, 2., ex. 2.] and [3, (2.13)])

$$Q_{n,a_n}(f)(x) = \begin{cases} \dfrac{1}{p_n(a_n)} \displaystyle\sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} \left(\frac{r^2 - \|x_0 - x\|^2}{r\,\sigma_p}\right)^k \sum_{|v|_k = n} \frac{1}{v_1 \dots v_k} \\ \cdot \displaystyle\int_{\partial X} \dots \int_{\partial X} \frac{f\left(\dfrac{v_1 x_1 + \dots + v_k x_k}{n}\right)}{\|x_1 - x\|^p \dots \|x_k - x\|^p} \, d\sigma(x_1) \dots d\sigma(x_k) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \|x - x_0\| < r, \\ f(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } \|x - x_0\| = r, \end{cases}$$

where $\sigma_p$ denotes the surface area of the unit sphere and $\sigma$ is the surface measure on the boundary $\partial X$ of $X$.

Moreover, the positive projection $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ is defined by putting for each $f \in \mathcal{C}(X,\mathbb{R})$ and $x \in X$ (cf. [2, (3.7)])

$$T(f)(x) = \begin{cases} \dfrac{r^2 - \|x_0 - x\|^2}{r\,\sigma_p} \displaystyle\int_{\partial X} \frac{f(z)}{\|z - x\|^p}\, \mathrm{d}\sigma(z) & \text{if } \|x - x_0\| < r, \\ f(x) & \text{if } \|x - x_0\| = r; \end{cases}$$

for every $i, j = 1, \dots, p$, it results (cf. [2, (3.8)])

$$T(pr_i pr_j) = \begin{cases} pr_i pr_j & \text{if } i \neq j, \\ \dfrac{1}{p}\left( r^2 - \displaystyle\sum_{\lambda \neq i} (pr_\lambda - pr_\lambda(x_0))^2 + (p-1)(pr_i - pr_i(x_0))^2 \right) & \\ +2pr_i(x_0)\, pr_i - pr_i^2(x_0) & \text{if } i = j, \end{cases}$$

and therefore the projection $T$ satisfies the condition (i)$'$ of Theorem 2.4 (cf. [2, (3.8)]).

If $A$ denotes the operator defined by (2.10), then, by the preceding formula and (2.11), we may easily deduce that the operator $A$ agrees on $A_\infty$ with the degenerate elliptic second order differential operator

$$W(f)(x) = (1 + b)\, \frac{r^2 - \|x - x_0\|^2}{2p}\, \Delta f(x),$$

and therefore, the function

$$u(t, x) = \lim_{n\to\infty} (Q_n^{[nt]}(u_0))(x) \quad t \geq 0, \;\; x \in X,$$

is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = C\, u(t, x) & \\ u(0, x) = u_0(x) & x \in X, \;\; u_0 \in D(C), \end{cases}$$

where $C$ is the closure of $W$. ■

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