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Rearrangement and Continuity Properties of $BMO(\phi)$ Functions on Spaces of Homogeneous Type

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In this note we study the behaviour of the non-increasing rearrangement of functions satisfying conditions on their mean oscillation over balls on spaces of homogeneous type. We extend a result of S. Spanne [S] and as a corollary we get extensions of the results of Campanato [C], John and Nirenberg [J-N] and Meyers [M]. The central tool is an extension of A.P. Calderón's proof of John-Nirenberg Lemma [N]. Related results can be found in [M-S].

Let X be a set. A symmetric function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a quasi-distance if $d(x,y) = 0$ iff $x = y$ and there is a constant K such that $d(x,z) \leq K[d(x,y) + d(y,z)]$ for every x, y and z in X. The ball with center $x \in X$ and radius $r > 0$ is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. We shall say that a measure μ defined on a σ-algebra containing the balls satisfies a doubling condition if and only if there is a positive constant A such that

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty,$$

for every $x \in X$ and every $r > 0$. If d is a quasi-distance on X and μ satisfies a doubling condition, then we say, following [C-W], that (X,d,μ) is a space of homogeneous type.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function satisfying the Δ_2 Orlicz's condition $\phi(2r) \leq C\phi(r)$ for some positive constant C and every $r > 0$ (see (K-R)). We say that a locally integrable function $f : X \to \mathbb{R}$ belongs to the class $BMO(\phi)$ if and only if there exists a positive constant D such that the inequality

$$\frac{1}{\mu(B)} \int_B |f - m_B(f)| \mathrm{d}\mu \leq D\phi(r(B)),$$

holds for every ball B in X, where $r(B)$ is the radius of the ball B and $m_B(f) = \mu(B)^{-1} \int_B f \mathrm{d}\mu$.

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Given a ball B and a non-negative measurable function g on B, we write η_B for the distribution function of g on B, i.e.

$$\eta_B(s) = \mu(\{x \in B : g(x) > s\}), \ \ s \geq 0;$$
$$\psi_B(t) = \sup\{\sigma : \eta_B(\sigma) > t\}.$$

The functions ψ_B and g have the same distribution function and, consequently, ψ_B contains the integral properties of g. On the other hand, while g is a function defined on the abstract space X, ψ_B is a function of a real variable. For the basic properties of the rearrangement see (Z).

Given a ball $B = B(x, r)$ we write $\tilde{B}$ for $B(x, 2Kr)$ and ψ_B for the non-increasing rearrangement of the function $|f(x) - m_{\tilde{B}}(f)|$ on B.

The main result in this note is the following theorem.

THEOREM. *Let (X, d, μ) be a space of homogeneous type such that continuous functions are dense in $L^1(X, d, \mu)$. Then a function f belongs to $BMO(\phi)$ if and only if there exist positive constants α, β and γ such that for every ball $B = B(x, r)$ the inequality*

$$\psi_B(t) \leq \beta \int_{\frac{r}{5K^2}\left[\frac{t}{\gamma\mu(B)}\right]^\alpha}^{r} \frac{\phi(\xi)}{\xi} \mathrm{d}\xi$$

holds for every $t \in (0, \gamma\mu(B))$.

COROLLARY. *Let (X, d, μ) be as in the Theorem.*

(a) *A locally integrable function f is of bounded mean oscillation ($\phi \equiv 1$) if and only if there exist positive constants β and γ such that for every ball $B = B(x, r)$ the inequality*

$$\psi_B(t) \leq \beta \ \log\left(\gamma 5K^2 \frac{\mu(B)}{t}\right)$$

holds for every $t \in (0, \gamma\mu(B))$.

(b) *If $\int_0^1 \frac{\phi(\xi)}{\xi} \mathrm{d}\xi < \infty$, then a function f in $BMO(\phi)$ is continuous and*

$$|f(x) - f(y)| \leq C \int_0^{d(x,y)} \frac{\phi(\xi)}{\xi} \mathrm{d}\xi,$$

which in the special case of $\phi(\xi) = \xi^\alpha$ is equivalent to $|f(x) - f(y)| \leq C(\mathrm{d}(x, y))^\alpha$.

The following lemma contains three simple but useful properties of $BMO(\phi)$ functions.

LEMMA.

(1). *If* $f \in BMO(\phi)$, *then* $|f| \in BMO(\phi)$.

(2). *Let* f *be a locally integrable function on* X. *If there is a constant* D *such that for every ball* B *there exists a constant* m_B *satisfying*

$$\frac{1}{\mu(B)} \int_B |f - m_B| \mathrm{d}\mu \leq D\phi(r(B)),$$

then $f \in BMO(\phi)$.

(3). *Let* $\tilde{B}$ *denote the ball with the same center as* B *and twice its radius. If* $f \in BMO(\phi)$, *then there exists a constant* $\tilde{D}$ *such that the inequality*

$$\frac{1}{\mu(B)} \int_B |f - m_{\tilde{B}}(f)| \mathrm{d}\mu \leq \tilde{D}\phi(r(B)),$$

holds for every ball B *in* X.

The next covering Lemma is a slight modification of that in [C-W].

(4). COVERING LEMMA. *Let* (X, d, μ) *be a space of homogeneous type. Let* $\mathcal{B} = \{B_\alpha = B(x_\alpha, r_\alpha) : \alpha \in \Gamma\}$ *be a family of balls in* X *such that* $\bigcup_{\alpha\in\Gamma} B_\alpha$ *is bounded. Then there exists a sequence of disjoint balls* $\{B_i\} \subset \mathcal{B}$ *such that for every* $\alpha \in \Gamma$ *there exists* i *satisfying* $r_\alpha \leq 2r_i$ *and* $B_\alpha \subset B(x_i, 5K^2 r_i)$.

From now on $B_0 = B(x_0, r_0)$ is a given ball in (X, d, μ) and f a function in $BMO(\phi)$ such that $m_{B(x_0, 2Kr_0)}(f) = 0$. Set $M = 5K^2$ and $\lambda_j = \sum_{k=0}^{j-1} \phi\left(\frac{r_0}{M^k}\right)$.

LEMMA. *There is a constant* C_1 *depending only on* K, A, C *and* D *such that the inequality*

$$m_{B(x,r)}(|f|) < C_1 \lambda_i \tag{5}$$

holds for every $x \in B_0$ *and every* $r \in \left[\frac{r_0}{M^{i+1}}, \frac{r_0}{M^i}\right)$.

PROOF. Since, for $r \in \left[\frac{r_0}{M^{i+1}}, \frac{r_0}{M^i}\right)$, we have

$$m_{B(x,r)}(|f|) \leq \frac{\mu\left(B\left(x, \frac{r_0}{M^i}\right)\right)}{\mu\left(B\left(x, \frac{r_0}{M^{i+1}}\right)\right)} m_{B\left(x, \frac{r_0}{M^i}\right)}(|f|),$$

inequality (5) will follow if we prove that there is a constant C_2 depending only on K, A, C and D such that the inequality

$$m_{B\left(x,\frac{r_0}{M^i}\right)}(|f|) \leq C_2\lambda_i \tag{6}$$

holds for every $X \in B_0$ and every $i \in \mathbb{N}$. In order to prove (6), let us first observe that for $x \in B_0$ we have

$$B\left(x, \frac{r_0}{M^i}\right) \subset B\left(x, \frac{r_0}{M^{i-1}}\right) \subset \ldots \subset B\left(x, \frac{r_0}{M}\right) \subset B(x, r_0) \subset B(x_0, 2Kr_0).$$

From (1), it follows that

$$\begin{aligned}
&\left| m_{B\left(x,\frac{r_0}{M^i}\right)}(|f|) - m_{B(x_0,2Kr_0)}(|f|)\right| \\
&\leq \sum_{h=1}^{i} \left| m_{B\left(x,\frac{r_0}{M^h}\right)}(|f|) - m_{B\left(x,\frac{r_0}{M^{h-1}}\right)}(|f|)\right| \\
&\qquad + |m_{B(x,r_0)}(|f|) - m_{B(x_0,2Kr_0)}(|f|)| \\
&\leq \sum_{h=1}^{i} \frac{1}{\mu\left(B\left(x, \frac{r_0}{M^h}\right)\right)} \int\limits_{B\left(x,\frac{r_0}{M^h}\right)} \left| |f| - m_{B\left(x,\frac{r_0}{M^{h-1}}\right)}(|f|)\right| \mathrm{d}\mu \qquad (7) \\
&\qquad + \frac{1}{\mu(B(x,r_0))} \int\limits_{B(x,r_0)} ||f| - m_{B(x_0,2Kr_0)}(|f|)| \,\mathrm{d}\mu \\
&\leq 2D \left\{ \sum_{h=1}^{i} \frac{\mu\left(B\left(x, \frac{r_0}{M^{h-1}}\right)\right)}{\mu\left(B\left(x, \frac{r_0}{M^h}\right)\right)} \phi\left(\frac{r_0}{M^{h-1}}\right) + \frac{\mu(B(x_0, 2Kr_0))}{\mu(B(x,r_0))} \phi(2Kr_0) \right\} \\
&\leq C_3 \sum_{h=0}^{i-1} \phi\left(\frac{r_0}{M^h}\right) = C_3\lambda_i.
\end{aligned}$$

Since $m_{B(x_0,2Kr_0)}(f) = 0$, we have $m_{B(x_0,2Kr_0)}(|f|) \leq D\phi(2Kr_0)$. Now (6) follows from (7) and the last inequality.

Let t be a positive real number. Let us consider the set

$$\Omega_t^j = \left\{x \in B_0 : \text{ there exists } r \in \left(0, \frac{r_0}{M}\right) \text{ such that } m_{B(x,r)}(|f|) > t\lambda_j\right\}$$

and, given $x \in \Omega_t^j$,

$$R_t^j(x) = \left\{r \in \left(0, \frac{r_0}{M}\right) : m_{B(x,r)}(|f|) > t\lambda_j\right\}.$$

(8) LEMMA. $R_t^j(x) \subset \left(0, \frac{r_0}{M^{j+1}}\right)$ *provided that* $t > C_1$.

PROOF. Given $r \in \left[\frac{r_0}{M^{j+1}}, \frac{r_0}{M}\right)$ there is an $h \leq j$ such that

$$\frac{r_0}{M^{j+1}} \leq \frac{r_0}{M^{h+1}} \leq r < \frac{r_0}{M^h} \leq \frac{r_0}{M}.$$

From (5) we have

$$m_{B(x,r)}(|f|) < C_1\lambda_h \leq t\lambda_h \leq t\lambda_j,$$

so that $r \not\in R_t^j(x)$.

(9) LEMMA. *Let n be a given positive integer. For $k = 1, 2, \ldots, n$ there is a function r^k defined on Ω_t^k such that*

(10) $$r^k(x) \in R_t^k(x);$$

(11) $$0 < r^k(x) < \frac{r_0}{M^{k+1}};$$

(12) $$m_{B(x,r^k(x))}(|f|) > t\lambda_k \geq m_{B(x,Mr^k(x))}(|f|);$$

(13) $$r^{k-1}(x) \geq r^k(x), \ \ x \in \Omega_t^k.$$

PROOF. Given $x \in \Omega_t^n$ pick $r^n(x) \in R_t^n(x)$ in such a way that

$$Mr^n(x) \notin R_t^n(x).$$

The second inequality in (11) for $k = n$ follows from Lemma (8). The second inequality in (12) holds since $Mr^n(x) \notin R_t^n(x)$ and $Mr^n \in \left(0, \frac{r_0}{M}\right)$. Let us now define r^{n-1}. Observe that $\Omega_t^n \subset \Omega_t^{n-1}$. If $x \in \Omega_t^{n-1} - \Omega_t^n$ then we get $r^{n-1}(x)$ in the same way as we have got r^n. If $x \in \Omega_t^n$, then pick $r^{n-1}(x) \in R_t^{n-1}(x)$ in such a way that $r^{n-1}(x) \geq r^n(x)$ and $Mr^n(x) \notin R_t^{n-1}(x)$.

Given $k = 1, 2, \ldots, n$, set

$$\mathcal{B}^k = \{B(x, r^k(x)) : x \in \Omega_t^k\}.$$

(14) LEMMA. *For each $k = 1, \ldots, n$, there exists a sequence $\{x_i^k : i \in \mathbb{N}\}$ of points in Ω_t^k such that the following properties hold*

(15) $B(x_i^k, r^k(x_i^k)) \cap B(x_j^k, r^k(x_j^k)) = \emptyset, \ i \neq j;$

(16) *for every $x \in \Omega_t^k$ there exists $i \in \mathbb{N}$ such that $r^k(x) \leq 2r^k(x_i^k)$ and*

$$B(x, r^k(x)) \subset B(x_i^k, 5K^2 r^k(x_i^k));$$

(17) $\Omega_t^k \subset \bigcup_{i=1}^{\infty} B(x_i^k, 5K^2 r^k(x_i^k));$

(18) $r^k(x_i^k) < \dfrac{r_0}{M^{k+1}};$

(19) $m_{B(x_i^k, r^k(x_i^k))}(|f|) > t\lambda_k \geq m_{B(x_i^k, Mr^k(x_i^k))}(|f|);$

(20) *Given $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that*

$$B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, 5K^2 r^k(x_i^k)).$$

(21) *Given $i \in \mathbb{N}$, set*

$$J_i = \{j \in \mathbb{N} : B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, 5K^2 r^k(x_i^k)) \text{ and}$$
$$B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \not\subset B(x_\ell^k, 5K^2 r^k(x_\ell^k))$$
$$\text{for } \ell = 1, 2, \ldots, i-1\}.$$

Then $J_i \cap J_h = \emptyset$ for $i \neq h$ and $\mathbb{N} = \bigcup_{i \in \mathbb{N}} J_i$.

PROOF. Applying the covering lemma (4) to the family $\mathcal{B}^k$, we obtain a sequence $\{x_i^k : i \in \mathbb{N}\}$ satisfying (15) to (19). In order to prove (20), observe that $x_j^{k+1} \in \Omega_t^{k+1} \subset \Omega_t^k$, then $B(x_j^{k+1}, r^k(x_j^{k+1})) \in \mathcal{B}^k$, thus, from (16) there exists $i \in \mathbb{N}$ such that $B(x_j^{k+1}, r^k(x_j^{k+1})) \subset B(x_i^k, r^k(x_i^k))$. Now, since $r^{k+1}(x_j^{k+1}) \leq r^k(x_j^{k+1})$ from (13), we get (20).

PROOF OF THE THEOREM. Let us first prove the "if" part of the theorem. Computing the integral of $|f - m_{\tilde{B}}(f)|$ using its non-increasing rearrangement, we get

$$\frac{1}{\mu(B)}\int_B |f - m_{\tilde{B}}(f)| d\mu \le \frac{1}{\mu(B)}\int_0^{\mu(B)} \psi_B(t)dt$$

$$\le \frac{\beta}{\mu(B)}\int_0^{\gamma\mu(B)/2}\int_{\frac{r}{5K^2}\left[\frac{t}{\gamma\mu(B)}\right]^\alpha}^{r} \frac{\phi(\xi)}{\xi}d\xi dt + (1-\gamma/2)\psi_B\left(\frac{\gamma}{2}\mu(B)\right)$$

$$\le \frac{\beta}{\mu(B)}\left\{\int_{\frac{r}{2^\alpha 5K^2}}^{r}\frac{\phi(\xi)}{\xi}d\xi\int_0^{\frac{\gamma\mu(B)}{2}} dt + \int_0^{\frac{r}{2^\alpha 5K^2}}\frac{\phi(\xi)}{\xi}\int_0^{\gamma\mu(B)\left[\frac{5K^2\xi}{r}\right]^{1/\alpha}} dt d\xi\right\}$$

$$+\left(1-\frac{\gamma}{2}\right)\int_{\frac{r}{2^\alpha 5K^2}}^{r}\frac{\phi(\xi)}{\xi}d\xi.$$

The first and the last terms on the right hand side are bounded by a constant times $\log 2^\alpha 5K^2\phi(r)$. For the second term we have the bound

$$\beta\gamma(5K^2)^{1/\alpha}\frac{1}{r^{1/\alpha}}\left(\int_0^{\frac{r}{2^\alpha 5K^2}} \xi^{1/\alpha-1}d\xi\right)\phi\left(\frac{r}{2^\alpha 5K^2}\right),$$

which, using condition Δ_2, is actually bounded by a constant times $\phi(r)$. The desired result follows now from (2). In order to prove the "only if" part of the theorem let us first assume that $m_{B(x_0,2Kr_0)}(f) = 0$. Applying the first inequality in (19) for $k+1$, (21), the second inequality in (19), (15), the fact that $f \in BMO(\phi)$

and (18), we get the following inequalities

$$t\lambda_{k+1}\sum_{j\in\mathbb{N}}\mu(B(x_j^{k+1},r^{k+1}(x_j^{k+1})))\leq\sum_{i\in\mathbb{N}}\sum_{j\in J_i}\int_{B(x_j^{k+1},r^{k+1}(x_j^{k+1}))}|f|\mathrm{d}\mu$$

$$\leq\sum_{i\in\mathbb{N}}\sum_{j\in J_i}\int_{B(x_j^{k+1},r^{k+1}(x_j^{k+1}))}|f-m_{B(x_i^k,5K^2r^k(x_i^k))}(f)|\mathrm{d}\mu$$

$$+t\lambda_k\sum_j\mu(B(x_j^{k+1},r^{k+1}(x_j^{k+1})))$$

$$\leq DC_4\phi\left(\frac{r_0}{M^k}\right)\sum_i\mu(B(x_i^k,r^k(x_i^k)))$$

$$+t\lambda_k\sum_j\mu(B(x_j^{k+1},r^{k+1}(x_j^{k+1}))),$$

where C_4 depends only on A, K and C. Set $\sum_k=\sum_j\mu(B(x_j^k,r^k(x_j^k)))$. Then

$$t(\lambda_{k+1}-\lambda_k)\sum_{k+1}\leq DC_4\phi\left(\frac{r_0}{M^k}\right)\sum_k.$$

From the definition of the sequence $\{\lambda_k\}$, taking $t=2C_4D$, we get

$$\sum_{k+1}\leq\frac{1}{2}\sum_k\quad\text{for every } k.$$

By iteration

$$\sum_n\leq\frac{1}{2^{n-1}}\sum_1\leq\frac{C_5}{2^n}\mu(B_0),$$

consequently

$$\mu(\Omega_t^n)\leq\frac{C_6}{2^n}\mu(B_0).$$

Since continuous functions are dense in $L^1(X,d)$, Lebesgue theorem on differentiation of integrals holds, so that

$$\{x\in B_0:|f(x)|>t\lambda_n\}\subset\Omega_t^n.$$

Thus

$$\mu(\{x\in B_0:|f(x)|>t\lambda_n\})\leq\frac{C_7}{2^n}\mu(B_0).$$

Given $s \in (0,1)$, take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < s \leq \frac{1}{2^{n-1}}$, then, for the rearrangement ψ_{B_0} of $|f|$ on B_0, we have

$$\psi_{B_0}(sC_8\mu(B_0)) \leq t\lambda_n \leq DC_8 \sum_{k=1}^{n} \phi\left(\frac{r_0}{M^k}\right)$$

$$\leq DC_8 \int_{\frac{r_0}{M^n}}^{r_0} \frac{\phi(\xi)}{\xi} \mathrm{d}\xi \leq DC_9 \int_{\frac{r_0}{M} s^{\alpha}}^{r_0} \frac{\phi(\xi)}{\xi} \mathrm{d}\xi.$$

This finishes the proof of the theorem for the case $m_{B(x_0,2Kr_0)}(f) = 0$. For the general case we use (3) and we apply the previous result to $f - m_{B(x_0,2Kr_0)}(f)$.

PROOF OF PART (b) OF THE COROLLARY. Let x and y be different points in X and take $B = B(x, 2d(x,y))$. Since $|f(x) - m_{\tilde{B}}(f)|$ and ψ_B have the same distribution function and ψ_B is non-increasing, we have

$$|f(x) - f(y)| \leq |f(x) - m_{\tilde{B}}(f)| + |f(y) - m_{\tilde{B}}(f)| \leq 2\psi_B(0),$$

now, applying the theorem we get the desired result.

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