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On the Evolution Equations of Viscous Gaseous Stars

PAOLO SECCHI

1. - Introduction

In this paper we study the non-stationary motion of a star regarded as a compressible viscous fluid with self gravitation, bounded by a free surface. The star is supposed to occupy a given bounded domain Ω_0 of $\mathbb{R}^3$ at the initial time $t = 0$, while for each subsequent instant t it occupies the domain Ω_t , not known a priori. The equations governing the motion, obtained by the three laws of conservation (momentum, mass and energy), are the following (see for instance Serrin [7] and for detailed discussions in astrophysical context Ledoux-Walraven [1]):

$$(1.1) \quad \bar{\rho}[\dot{u} + (u \cdot \nabla)u + \nabla \bar{\phi} - \bar{b}] = -\nabla p + \sum_j \mu D_j(D_j u + \nabla u_j) + \left(\zeta - \frac{2}{3}\mu\right) \nabla \operatorname{div} u$$

in $D_T \equiv \{(t, y) \in]0, T[\times \mathbb{R}^3 \mid y \in \Omega_t\}$,

$$(1.2) \quad \dot{\bar{\rho}} + \operatorname{div}(\bar{\rho}u) = 0 \quad \text{in } D_T,$$

$$(1.3) \quad \begin{aligned} c_v \bar{\rho}(\dot{\bar{\theta}} + u \cdot \nabla \bar{\theta}) &= -\bar{\theta} \frac{\partial p}{\partial \bar{\theta}} \operatorname{div} u + \chi \Delta \bar{\theta} + \bar{\rho} \bar{r} + \frac{\mu}{2} \sum_{i,j} (D_i u_j + D_j u_i)^2 \\ &\quad + \left(\zeta - \frac{2}{3}\mu\right) (\operatorname{div} u)^2 \end{aligned}$$

in D_T .

The unknowns are the density $\bar{\rho} = \bar{\rho}(t, y)$, the fluid velocity $u = u(t, y) = {}^t(u_1, u_2, u_3)$, the temperature $\bar{\theta} = \bar{\theta}(t, y)$ and the domain Ω_t . Here $\dot{u}$ represents the time derivative. The external force field per unit mass $\bar{b}$ and the heat supply

per unit mass per unit time $\bar{\tau}$ are known functions defined in $]0, T_0[\times \mathbb{R}^3$. The pressure $p = p(\bar{\rho}, \bar{\theta})$ and the specific heat at constant volume $c_v = c_v(\bar{\rho}, \bar{\theta})$ are given functions depending on the density and the temperature; the viscosity coefficients μ and ζ and the coefficient of heat conductivity χ are assumed to be constant and to satisfy $\mu > 0$, $\zeta > 0$, $\chi > 0$. Moreover $\bar{\phi}$ represents the Newtonian gravitational potential given by

$$(1.4) \quad \bar{\phi}(t, y) = -\kappa \int_{\mathbb{R}^3} \frac{\bar{\rho}(t, z)}{|y - z|} dz,$$

κ standing for the constant of gravitation. We consider the following boundary conditions. The velocity satisfies a dynamical condition expressing the continuity of stress across the free boundary:

$$(1.5) \quad -pn_i^t + \mu \sum_j (D_i u_j + D_j u_i) n_j^t + \left(\zeta - \frac{2}{3} \mu \right) \operatorname{div} u n_i^t = -\bar{p} n_i^t$$

on $S_T \equiv \{(t, y) \in]0, T[\times \mathbb{R}^3 \mid y \in \partial\Omega_t\}$.

Here $\bar{p}$ means the external pressure, a known function defined in $]0, T_0[\times \mathbb{R}^3$; $n^t = n^t(y)$ is the unit outward normal vector to $\partial\Omega_t$ at the point $y \in \partial\Omega_t$. The free boundary $\partial\Omega_t$ must be subjected to another kinematic condition, namely

(1.6) at each instant t of time it consists of the very same particles.

For a discussion on the above two boundary conditions see, for instance, Wehausen-Laitone [9]. We consider also the following boundary condition for the temperature:

$$(1.7) \quad \chi \frac{\partial \bar{\theta}}{\partial n} = h(\hat{\theta} - \bar{\theta}) \quad \text{on } S_T,$$

where the external temperature $\hat{\theta}$ is a known function defined in $]0, T_0[\times \mathbb{R}^3$ and h is a given positive constant. Finally we consider the following initial conditions:

$$(1.8) \quad u(0, y) = u_0(y) \quad \text{in } \Omega_0,$$

$$(1.9) \quad \bar{\rho}(0, y) = \rho_0(y) \quad \text{in } \Omega_0,$$

$$(1.10) \quad \bar{\theta}(0, y) = \theta_0(y) \quad \text{in } \Omega_0.$$

The free boundary problem for compressible Navier-Stokes equations (without considering self-gravitation) has been studied by P. Secchi and A. Valli [6] and by A. Tani [8]. T. Makino [4] investigated the Cauchy problem

for the equations describing the evolution of a star regarded as an isentropic ideal gas with self gravitation (i.e., equations (1.2), (1.1) without the viscous terms and without considering (1.3); moreover, it is assumed that $p = K\rho^\gamma$, γ being the adiabatic exponent). In the present paper we find a solution of the above problem in a space of Sobolev type for short time. The proof of existence is obtained by linearization and by a fixed point argument as in [6]; to simplify the proof, the solution is found to exist in a Sobolev space with less regularity. In particular, if we neglect in (1.1) the term with the gravitational potential and do not consider (1.4) (i.e., we set $\kappa = 0$), we obtain a simpler proof of the result in [6].

As usual in free boundary problems, it is convenient to write the problem in the Lagrangian formulation, so that the domain of the unknowns becomes fixed in time.

Let $\eta(t, \cdot) : \bar{\Omega}_0 \rightarrow \mathbb{R}^3$ be the solution of

$$(1.6)' \quad \begin{aligned} \dot{\eta}(t, x) &= u(t, \eta(t, x)) && \text{in }]0, T[\times \bar{\Omega}_0, \\ \eta(0, x) &= x && \text{in } \bar{\Omega}_0, \end{aligned}$$

so that $(t, y) = (t, \eta(t, x))$ for a suitable $x \in \Omega_0$. Then $\Omega_t = \eta(t, \Omega_0)$ and, if η is an homeomorphism, $\eta(t, \partial\Omega_0) = \partial[\eta(t, \Omega_0)]$. Hence condition (1.6) can be substituted by (1.6)'. If we set $v(t, x) = u(t, \eta(t, x))$, $\rho(t, x) = \bar{\rho}(t, \eta(t, x))$, $\theta(t, x) = \bar{\theta}(t, \eta(t, x))$, $\phi(t, x) = \bar{\phi}(t, \eta(t, x))$, $b(t, x) = \bar{b}(t, \eta(t, x))$, $r(t, x) = \bar{r}(t, \eta(t, x))$, $p'(t, x) = \bar{p}(t, \eta(t, x))$, $\theta'(t, x) = \bar{\theta}(t, \eta(t, x))$, problem (1.1)-(1.10) becomes

$$(1.11) \quad \begin{aligned} \rho(\dot{v}_i + a_{ki} D_k \phi - b_i) &= -a_{ki} D_k p + \mu a_{kj} D_k (a_{sj} D_s v_i + a_{si} D_s v_j) \\ &+ \left(\zeta - \frac{2}{3}\mu\right) a_{ki} D_k (a_{sj} D_s v_j) \quad \text{in } Q_T \equiv]0, T[\times \Omega_0, \end{aligned}$$

$$(1.12) \quad \dot{\rho} + \rho a_{ki} D_k v_i = 0 \quad \text{in } Q_T,$$

$$(1.13) \quad \begin{aligned} c_v(\rho, \theta) \rho \dot{\theta} &= -\theta \frac{\partial p}{\partial \theta}(\rho, \theta) a_{ki} D_k v_i + \chi a_{ki} D_k (a_{si} D_s \theta) + \rho r \\ &+ \frac{\mu}{2} \sum_{i,j} (a_{kj} D_k v_i + a_{ki} D_k v_j)^2 + \left(\zeta - \frac{2}{3}\mu\right) (a_{ki} D_k v_i)^2 \quad \text{in } Q_T, \end{aligned}$$

$$(1.14) \quad \dot{\eta} = v \quad \text{in } Q_T,$$

$$(1.15) \quad \phi(t, x) = -\kappa \int_{\Omega_0} \frac{\rho(t, \xi)}{|\eta(t, x) - \eta(t, \xi)|} |\det[D\eta]| d\xi \quad \text{in } Q_T,$$

$$(1.16) \quad -pN_i + \mu(a_{ki}D_k v_j + a_{kj}D_k v_i)N_j + \left(\zeta - \frac{2}{3}\mu\right) a_{kj}D_k v_j N_i = -p'N_i$$

on $\Sigma_T \equiv]0, T[\times \partial\Omega_0$,

$$(1.17) \quad \chi a_{ki}D_k \theta N_i = h(\theta' - \theta) \quad \text{on } \Sigma_T,$$

$$(1.18) \quad v(0) = u_0 \quad \text{in } \Omega_0,$$

$$(1.19) \quad \rho(0) = \rho_0 \quad \text{in } \Omega_0,$$

$$(1.20) \quad \theta(0) = \theta_0 \quad \text{in } \Omega_0,$$

$$(1.21) \quad \eta(0) = \text{Id} \quad \text{in } \Omega_0 \text{ (Id identity function in } \Omega_0).$$

All indices run through 1, 2, 3; here and in the sequel, we adopt the Einstein convention about summation over repeated indices. The coefficients $a_{ki}(t, x)$ are the entries (k, i) of the Jacobian matrix $[D\eta]^{-1}$ (where $D\eta$ has the term $D_k \eta_i$ in the i -th row, k -th column) and $N(t, x)$ is the normal to $\partial[\eta(t, \Omega_0)]$ calculated in $\eta(t, x)$, i.e., $N(t, x) = n^t(\eta(t, x))$. When problem (1.11)-(1.21) is solved, we can find a solution to the original problem (1.1)-(1.10) if η is a regular enough homeomorphism. We shall see in Theorems A and B the precise results.

Set $B_R \equiv \{x \in \mathbb{R}^3 \mid |x| < R\}$. Let us denote with $C^0(\overline{\Omega}_0)$ the space of continuous (and bounded) functions on $\overline{\Omega}_0$ and with $C^k(\overline{\Omega}_0)$ (k positive integer) the space of functions with derivatives up to order k in $C^0(\overline{\Omega}_0)$. Moreover, if m is a positive integer, $H^m(\Omega_0)$ is the Sobolev space of functions with m derivatives in $L^2(\Omega_0)$; we shall denote its norm by $\|\cdot\|_m$. For the definitions of $H^s(\Omega_0)$ and $H^s(\partial\Omega_0)$ (s not integer) see [2]. If X is a Banach space, $L^2(0, T; X)$, $L^\infty(0, T; X)$, $H^m(0, T; X)$, $H^s(0, T; X)$ are the spaces of X -valued functions in L^2 , L^∞ , and H^m , H^s respectively. $C^\alpha([0, T]; X)$ is the space of X -valued Hölder continuous functions with exponent α . We shall denote by $\|\cdot\|_{p,m,T}$ the norm of $L^p(0, T; H^m(\Omega_0))$, $1 \leq p \leq +\infty$, by $\|\cdot\|_{m,m/2,Q_T}$ the norm in the space

$$H^{m,m/2}(Q_T) \equiv L^2(0, T; H^m(\Omega_0)) \cap H^{m/2}(0, T; L^2(\Omega_0)),$$

by $\|\cdot\|_{s,s/2,\Sigma_T}$ the norm in the space

$$H^{s,s/2}(\Sigma_T) \equiv L^2(0, T; H^s(\partial\Omega_0)) \cap H^{s/2}(0, T; L^2(\partial\Omega_0)).$$

The norm in $H^s(0, T; X)$ (s not integer) is defined in this way:

$$\|w\|_{H^s(0,T;X)}^2 \equiv \|w\|_{H^{[s]}(0,T;X)}^2 + \int_0^T \int_0^T \frac{\|w(t) - w(\tau)\|_X^2}{|t - \tau|^{1+2(s-[s])}} dt d\tau.$$

We shall prove the following results.

THEOREM A. *Let Ω_0 be a bounded connected open subset of $\mathbb{R}^3$, locally situated on one side of its boundary $\partial\Omega_0$; we assume $\partial\Omega_0 \in C^3$. Suppose that*

$$\bar{b} \in L^2(0, T_0; H^1(B_R)) \cap L^2(0, T_0; C^0(\bar{B}_R)),$$

$$\bar{r} \in L^2(0, T_0; H^1(B_R)) \cap L^2(0, T_0; C^0(\bar{B}_R))$$

for each $R > 0$, $p \in C^2$, $c_v \in C^2$, $c_v > 0$,

$$\bar{p} \in H^{3/4}(0, T_0; H^1(B_R)) \cap L^\infty(0, T_0; H^2(B_R)),$$

$$\hat{\theta} \in H^{3/4}(0, T_0; H^1(B_R)) \cap L^\infty(0, T_0; H^2(B_R))$$

for each $R > 0$, $u_0 \in H^2(\Omega_0)$, $\rho_0 \in H^2(\Omega_0)$ with $\min_{x \in \Omega_0} \rho_0(x) \equiv m > 0$, $\theta_0 \in H^2(\Omega_0)$.

Assume that the (necessary) compatibility conditions

$$(1.22) \quad \begin{aligned} & \mu[D_i(u_0)_j + D_j(u_0)_i]n_j^0 + \left(\zeta - \frac{2}{3}\mu\right) \operatorname{div} u_0 n_i^0 \\ & = [p(\rho_0, \theta_0) - \bar{p}(0)]n_i^0 \quad \text{on } \partial\Omega_0, \end{aligned}$$

for $i = 1, 2, 3$,

$$(1.23) \quad \chi \frac{\partial \theta_0}{\partial n} = h(\hat{\theta}(0) - \theta_0) \quad \text{on } \partial\Omega_0,$$

are satisfied.

Then there exist $T' \in]0, T_0]$,

$$v \in L^2(0, T'; H^3(\Omega_0)) \cap H^1(0, T'; H^1(\Omega_0)),$$

$\rho \in H^1(0, T'; H^2(\Omega_0))$ with $\dot{\rho} \in L^\infty(0, T'; H^1(\Omega_0))$ such that $\rho > 0$ in $\bar{Q}_{T'}$,

$$\theta \in L^2(0, T'; H^3(\Omega_0)) \cap H^1(0, T'; H^1(\Omega_0))$$

and a diffeomorphism

$$\eta \in H^1(0, T'; H^3(\Omega_0)) \cap H^2(0, T'; H^1(\Omega_0))$$

such that (v, ρ, θ, η) is a solution of (1.11)-(1.21).

A direct consequence of Theorem A is the following result (see [6] for a sketch of proof).

THEOREM B. *If the hypotheses of Theorem A hold, then there exists $T' \in]0, T_0]$, and for each $t \in [0, T']$ there exist a diffeomorphism $x \rightarrow \eta(t, x)$, a*

domain $\Omega_t = \eta(t, \Omega_0)$, a velocity field $u(t, \cdot)$, a temperature $\bar{\theta}(t, \cdot)$ and a density $\bar{\rho}(t, \cdot)$ which are solution of (1.1)-(1.10). Moreover $\partial\Omega_t$ is of class C^1 , and we have

$$\eta \in H^1(0, T'; H^3(\Omega_0)) \cap H^2(0, T'; H^1(\Omega_0)),$$

$$D^k u \in L^2(D_{T'}), \quad D^k \bar{\theta} \in L^2(D_{T'}) \quad \text{for } k = 0, 1, 2, 3,$$

$$D^k \dot{u} \in L^2(D_{T'}), \quad D^k \dot{\bar{\theta}} \in L^2(D_{T'}) \quad \text{for } k = 0, 1,$$

$$D^k \bar{\rho} \in L^2(D_{T'}), \quad D^k \dot{\bar{\rho}} \in L^2(D_{T'}) \quad \text{for } k = 0, 1, 2,$$

$$\|D^k \dot{\bar{\rho}}\|_{L^2(\Omega_t)} \in L^\infty(0, T') \quad \text{for } k = 0, 1,$$

with $\bar{\rho} > 0$ in $\bar{D}_{T'}$.

2. - Proof of Theorem A

We prove the existence of a solution to (1.11)-(1.21) by a fixed point argument, following the approach of [6]. From now on each constant $c, c_i, c'_i, C_i, C'_i, T_i, T'$ will depend at most on the data of the problem $\Omega_0, T_0, \mu, \zeta, \chi, u_0, \rho_0, \theta_0, p, c_v, \bar{b}, \bar{p}, \bar{r}$. Moreover we shall assume the outward unit normal n^0 to $\partial\Omega_0$ extended in a regular way, i.e., $n^0 \in C^2(\bar{\Omega}_0)$. Define the operators

$$(2.1) \quad A_i(x, D)w = -\frac{1}{\rho_0} \left\{ \mu D_j (D_j w_i + D_i w_j) + \left(\zeta - \frac{2}{3} \mu \right) D_i \operatorname{div} w \right\},$$

$$(2.2) \quad L(x, D)\Theta = -\frac{\chi}{\rho_0 c_0} \Delta \Theta,$$

and the boundary operator

$$(2.3) \quad B_i(x, D)w = \mu (D_i w_j + D_j w_i) n_j^0 + \left(\zeta - \frac{2}{3} \mu \right) \operatorname{div} w n_i^0,$$

where $c_0(x) = c_v(\rho_0(x), \theta_0(x))$. First we consider the linear problem

$$(2.4) \quad \begin{cases} \dot{w} + Aw = F & \text{in } Q_T \\ Bw = G & \text{on } \Sigma_T \\ w(0) = u_0 & \text{in } \Omega_0. \end{cases}$$

Define the operator A in $H^1(\Omega_0)$ setting

$$D(A) = \{w \in H^3(\Omega_0) \mid Bw = 0 \text{ on } \partial\Omega_0\}.$$

To solve problem (2.4), we want to apply Theorem 3.2, chap. 4 of Lions-Magenes [3] for $H = H^1(\Omega_0)$. Hence, some estimates are needed for the solution $w \in D(A)$ of the problem

$$(2.5) \quad Aw + \lambda w = f$$

where $f \in H^1(\Omega_0)$, $\lambda \in \mathbb{C}$. We introduce the bilinear form

$$a_\lambda(w, u) = \frac{\mu}{2} \int (D_i w_j + D_j w_i)(D_i \bar{u}_j + D_j \bar{u}_i) + \left(\zeta - \frac{2}{3} \mu \right) \int \operatorname{div} w \operatorname{div} \bar{u} + \lambda \int \rho_0 w \bar{u}.$$

Here and in the sequel $\int$ denotes integration over Ω_0 . Then the weak formulation of (2.5) is given by

$$a_\lambda(w, u) = \int \rho_0 f \bar{u} \quad \text{for any } u \in H^1(\Omega_0).$$

As in [6], we have

LEMMA 2.1. *If $\operatorname{Re} \lambda \geq \lambda_0 \equiv \frac{1}{m} \min \left(\frac{\mu}{2}, \frac{3}{4} \zeta \right)$, a_λ is a bounded and coercive form in $H^1(\Omega_0)$. The coerciveness constant is independent of λ .*

LEMMA 2.2. *For any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \lambda_0$, $\lambda + A$ is an isomorphism from $D(A)$ (endowed with the graph norm) into $H^1(\Omega_0)$. Moreover, for any solution $w \in D(A)$ and for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \lambda_0 + 1$,*

$$(2.6) \quad \|w\|_0 \leq \frac{c}{|\lambda|} \|f\|_0,$$

where c does not depend on λ .

LEMMA 2.3. *If $\operatorname{Re} \lambda > \lambda_0 + 1$, for any solution $w \in D(A)$ we have*

$$(2.7) \quad \|w\|_1 \leq \frac{c}{|\lambda| + 1} \|f\|_1,$$

where c does not depend on λ .

PROOF. From Lemma 2.1 one has

$$\|w\|_1^2 \leq c |a_\lambda(w, w)| \leq c \|\rho_0\|_2 \|f\|_0 \|w\|_0$$

which gives

$$(2.8) \quad \|w\|_1 \leq c \|f\|_1,$$

with c independent of λ . Taking the scalar product in $L^2(\Omega_0)$, of (2.5) by $\rho_0 A \bar{w}$, gives

$$(2.9) \quad \|\sqrt{\rho_0} A w\|_0^2 + \lambda \int \rho_0 w_i A_i \bar{w} = \int \rho_0 f_i A_i \bar{w}.$$

Integrating by parts yields

$$\begin{aligned} \int \rho_0 w_i A_i \bar{w} &= \frac{\mu}{2} \sum_{i,j} \int |D_i w_j + D_j w_i|^2 + \left(\zeta - \frac{2}{3} \mu \right) \int |\operatorname{div} w|^2 \\ &\geq m \lambda_0 \sum_{i,j} \int |D_i w_j + D_j w_i|^2 \end{aligned}$$

(see (2.9), (2.10) and the proof of Lemma 3.1 in [6]). From Korn's inequality

$$(2.10) \quad \|w\|_1^2 \leq K \left[\sum_{i,j} \int |D_i w_j + D_j w_i|^2 + \int |w|^2 \right]$$

for any $w \in H^1(\Omega_0)$, we obtain

$$(2.11) \quad \int \rho_0 w_i A_i \bar{w} \geq m \lambda_0 \left\{ \frac{1}{K} \|\nabla w\|_0^2 - \left(1 - \frac{1}{K} \right) \|w\|_0^2 \right\}.$$

Moreover

$$\begin{aligned} \left| \int \rho_0 f_i A_i \bar{w} \right| &= \left| \frac{\mu}{2} \sum_{i,j} \int [D_i f_j + D_j f_i] [D_i \bar{w}_j + D_j \bar{w}_i] \right. \\ (2.12) \quad &\quad \left. + \left(\zeta - \frac{2}{3} \mu \right) \int \operatorname{div} f \operatorname{div} \bar{w} \right| \leq c \|w\|_1 \|f\|_1. \end{aligned}$$

Observing that $\int \rho_0 w_i A_i \bar{w} \geq 0$, $\operatorname{Re} \lambda > 0$, from (2.9)-(2.12) we obtain

$$\begin{aligned} m \lambda_0 |\lambda| \left\{ \frac{1}{K} \|\nabla w\|_0^2 - \left(1 - \frac{1}{K} \right) \|w\|_0^2 \right\} &\leq \left| \|\sqrt{\rho_0} A w\|_0^2 + \lambda \int \rho_0 w_i A_i \bar{w} \right| \\ &\leq c \|w\|_1 \|f\|_1, \end{aligned}$$

from which it follows

$$(2.13) \quad |\lambda| \|w\|_1 \leq c [\|f\|_1 + |\lambda| \|w\|_0].$$

From (2.6), (2.8), (2.13) we obtain (2.7). □

LEMMA 2.4. *Let $F \in L^2(0, T; H^1(\Omega_0))$, $G \in H^{3/2, 3/4}(\Sigma_T)$, $u_0 \in H^2(\Omega_0)$ and the compatibility conditions*

$$(2.14) \quad \mu[D_i(u_0)_j + D_j(u_0)_i]n_j^0 + \left(\zeta - \frac{2}{3}\mu\right) \operatorname{div} u_0 n_i^0 = G_i(0) \quad \text{on } \partial\Omega_0$$

be satisfied. Then there exists a unique solution

$$w \in L^2(0, T; H^3(\Omega_0)) \cap H^1(0, T; H^1(\Omega_0))$$

of (2.4). Moreover

$$(2.15) \quad |w|_{2,3,T} + |\dot{w}|_{2,1,T} \leq C_0[|F|_{2,1,T} + \|G\|_{3/2,3/4,\Sigma_T} + \|G(0)\|_1 + \|u_0\|_2]$$

where the constant C_0 does not depend on T .

PROOF. The trace $G(0)$ on $\partial\Omega_0$ belongs to $H^{1/2}(\partial\Omega_0)$ so that it is possible to find a function $\Phi \in H^{3/2, 3/4}([0, +\infty[\times \partial\Omega_0)$ such that $\Phi(0) = G(0)$ and

$$\|\Phi\|_{3/2,3/4,\Sigma_{+\infty}} \leq c\|G(0)\|_{1/2,\partial\Omega_0},$$

where the constant c does not depend on T and $\|\cdot\|_{s,\partial\Omega_0}$ is the norm in $H^s(\partial\Omega_0)$.

Now, we can extend $G - \Phi$ from $[0, T] \times \partial\Omega_0$ to $\mathbb{R} \times \partial\Omega_0$ in such a way that the extension $P(G - \Phi) \in H^{3/2, 3/4}(\mathbb{R} \times \partial\Omega_0)$, $P(G - \Phi) = 0$ for $t < 0$ and

$$\|P(G - \Phi)\|_{3/2,3/4,\Sigma_{+\infty}} \leq c\|G - \Phi\|_{3/2,3/4,\Sigma_T},$$

where the constant c does not depend on T (extension by reflection around $t = T$: see Lions-Magenes [2], Theorem 2.2, chap. 1 and Theorem 11.3, chap. 1). Hence we have extended G to $\overline{G} = P(G - \Phi) + \Phi$ and $\overline{G} \in H^{3/2, 3/4}([0, +\infty[\times \partial\Omega_0)$ with

$$(2.16) \quad \|\overline{G}\|_{3/2,3/4,\Sigma_{+\infty}} \leq c[\|G\|_{3/2,3/4,\Sigma_T} + \|G(0)\|_{1/2,\partial\Omega_0}].$$

Now, compatibility conditions (2.14) are necessary and sufficient to find a function $W \in H^{3, 3/2}([0, +\infty[\times \Omega_0)$ such that

$$\begin{cases} BW = \overline{G} & \text{on }]0, +\infty[\times \partial\Omega_0 \\ W(0) = u_0 & \text{in } \Omega_0 \end{cases}$$

and satisfying the estimate

$$(2.17) \quad \|W\|_{3,3/2,+\infty} \leq c[\|\overline{G}\|_{3/2,3/4,\Sigma_{+\infty}} + \|u_0\|_2],$$

where the constant c does not depend on T (see [6]). Let us consider the problem

$$\begin{cases} \dot{V} + AV = F - \dot{W} - AW & \text{in } Q_T \\ BV = 0 & \text{on } \Sigma_T \\ V(0) = 0 & \text{in } \Omega_0. \end{cases}$$

Since Lemma 2.2 and Lemma 2.3 hold, we can apply Theorem 3.2, chap. 4 of [3], and find a solution

$$V \in L^2(0, T; H^3(\Omega_0)) \cap H^1(0, T; H^1(\Omega_0))$$

such that

$$(2.18) \quad |V|_{2,3,T} + |\dot{V}|_{2,1,T} \leq c[|F|_{2,1,T} + \|W\|_{3,3/2,+\infty}],$$

where the constant c does not depend on T . The function $w = V + W$ is the solution of (2.4); from (2.16)-(2.18) we have

$$|w|_{2,3,T} + |\dot{w}|_{2,1,T} \leq c[|F|_{2,1,T} + \|G\|_{3/2,3/4,\Sigma_T} + \|G(0)\|_{1/2,\partial\Omega_0} + \|u_0\|_2].$$

Finally,

$$\|G(0)\|_{1/2,\partial\Omega_0} \leq c\|G(0)\|_1$$

gives (2.15). □

In a similar way we solve the problem

$$(2.19) \quad \begin{cases} \dot{\Theta} + L\Theta = H & \text{in } Q_T; \\ \chi \frac{\partial \Theta}{\partial n} = K & \text{on } \Sigma_T; \\ \Theta(0) = \theta_0 & \text{in } \Omega_0. \end{cases}$$

We obtain

LEMMA 2.5. *Let $H \in L^2(0, T; H^1(\Omega_0))$, $K \in H^{3/2,3/4}(\Sigma_T)$, $\theta_0 \in H^2(\Omega_0)$ and the compatibility condition*

$$\chi \frac{\partial \theta_0}{\partial n} = K(0) \quad \text{on } \partial\Omega_0$$

be satisfied. Then there exists a unique solution

$$\Theta \in L^2(0, T; H^3(\Omega_0)) \cap H^1(0, T; H^1(\Omega_0))$$

of (2.19). Moreover

$$(2.20) \quad |\Theta|_{2,3,T} + |\dot{\Theta}|_{2,1,T} \leq C'_0[|H|_{2,1,T} + \|K\|_{3/2,3/4,\Sigma_T} + \|K(0)\|_1 + \|\theta_0\|_2]$$

where the constant C'_0 does not depend on T .

We state a proposition which will be useful in the sequel (see [6]).

PROPOSITION 2.6. Let $w \in L^2(0, T; H^k(\Omega_0))$ with

$$\frac{\partial^s w}{\partial t^s} \equiv w^{(s)} \in L^2(0, T; H^r(\Omega_0)).$$

Then, for each $0 \leq j \leq s$, $j \in \mathbb{R}$,

$$(2.21) \quad \sup_{t \in [0, T]} \|w(t)\|_\beta + \sum_{0 \leq j \leq s} |w^{(j)}|_{2, \beta_j, T} + [w]_{j, \beta_j, T} \\ \leq c \left[|w|_{2, k, T} + |w^{(s)}|_{2, r, T} + \sum_{0 \leq j < s-1/2} \|w^{(j)}(0)\|_{\gamma_j} \right]$$

where $\beta = \left(1 - \frac{1}{2s}\right)k + \frac{r}{2s}$, $\beta_j = \left(1 - \frac{j}{s}\right)k + \frac{j}{s}r$, $\gamma_j = \beta + \beta_j - k$, and

$$[w]_{j, \beta_j, T} \equiv \left(\int_0^T \int_0^T \frac{\|w^{[j]}(t) - w^{[j]}(\tau)\|_{\beta_j}^2}{|t - \tau|^{1+2(j-[j])}} dt d\tau \right)^{1/2}.$$

The constant c does not depend on T .

Set now

$$(2.22) \quad \frac{E}{2} \equiv \|\bar{p}(0)n^0\|_1 + \|p(\rho_0, \theta_0)n^0\|_1 + \|h\hat{\theta}(0)\|_1 + \|h\theta_0\|_1 + \|u_0\|_2 + \|\rho_0\|_2 \\ + \|\theta_0\|_2 + \|\text{Id}\|_1.$$

Let η be the solution of

$$\begin{aligned} \dot{\eta} &= v && \text{in } Q_T, \\ \eta(0) &= \text{Id} && \text{in } \Omega_0; \end{aligned}$$

it is easily verified that there exist constants C_1 and C_2 such that if, for an arbitrary $T \leq T_0$, v , ρ and η satisfy $|v|_{2,3,T} + |\dot{v}|_{2,1,T} \leq C_0 E$, $\sup_{t \in [0, T]} \|\rho(t)\|_2 \leq E$,

$\det[D\eta(t, x)] \geq \frac{1}{2}$ in $\bar{Q}_T$, $v(0) = u_0$, then

$$(2.23) \quad \left(\int_0^T \|\rho(t)a_{ki}(t)D_k v_i(t)\|_2^2 dt \right)^{1/2} \leq C_1,$$

$$(2.24) \quad \sup_{t \in [0, T]} \|\rho(t)a_{ki}(t)D_k v_i(t)\|_1 \leq C_2,$$

where $a_{ki} = ([D\eta]^{-1})_{ki}$. The constants C_1 and C_2 do not depend, as usual, on T . Set now

$$R_T \equiv \left\{ (v, \rho, \theta) \mid |v|_{2,3,T} + |\dot{v}|_{2,1,T} \leq C_0 E, \ v(0) = u_0, \right. \\ \left. \sup_{t \in [0,T]} \|\rho(t)\|_2 \leq E, \ |\dot{\rho}|_{2,2,T} \leq C_1, \ \sup_{t \in [0,T]} \|\dot{\rho}(t)\|_1 \leq C_2, \right. \\ \left. \rho(0) = \rho_0, \ \rho(t, x) \geq \frac{m}{2} \text{ in } \overline{Q}_T, \ |\theta|_{2,3,T} + |\dot{\theta}|_{2,1,T} \leq C'_0 E, \ \theta(0) = \theta_0 \right\}.$$

First, we show that $R_T \neq \emptyset$ for each T . Proceeding as in the proof of Lemma 2.4, we see that there exist two functions v', θ' , in

$$L^2(0, +\infty; H^3(\Omega_0)) \cap H^1(0, +\infty; H^1(\Omega_0))$$

such that $v'(0) = u_0, \theta'(0) = \theta_0$. Moreover

$$|v'|_{2,3,+\infty} + |\dot{v}'|_{2,1,+\infty} \leq c_1 [\|u_0\|_2 + \|Bu_0\|_1], \\ |\theta'|_{2,3,+\infty} + |\dot{\theta}'|_{2,1,+\infty} \leq c'_1 \left[\|\theta_0\|_2 + \left\| \chi \frac{\partial \theta_0}{\partial n} \right\|_1 \right].$$

The constants c_1, c'_1 are easily seen to be less than C_0 and C'_0 , respectively. Using the compatibility conditions (1.22), (1.23) we have

$$|v'|_{2,3,+\infty} + |\dot{v}'|_{2,1,+\infty} \leq C_0 [\|u_0\|_2 + \|\bar{p}(0)n^0\|_1 + \|p(\rho_0, \theta_0)n^0\|_1] \leq C_0 \frac{E}{2}, \\ |\theta'|_{2,3,+\infty} + |\dot{\theta}'|_{2,1,+\infty} \leq C'_0 [\|\theta_0\|_2 + \|h\hat{\theta}(0)\|_1 + \|h\theta_0\|_1] \leq C'_0 \frac{E}{2}.$$

Hence $(v', \rho_0, \theta') \in R_T$ for any T . We can now construct a map Λ defined in R_T . We shall show that it has a fixed point, namely a solution of our problem. Take $(v^*, \rho^*, \theta^*) \in R_T$. Let η^* be the solution of

$$\begin{aligned} \eta^* &= v^* && \text{in } Q_T, \\ \eta^*(0) &= \text{Id} && \text{in } \Omega_0, \end{aligned}$$

that is $\eta^*(t, x) = x + \int_0^t v^*(s, x) ds$. Moreover, if $T \leq T_0$,

$$\|\eta^*(t)\|_3 \leq \|\text{Id}\|_1 + C_0 E t^{1/2} \leq \frac{E}{2} + C_0 E T_0^{1/2} \quad \text{for } t \in [0, T].$$

Hence, for each $T \leq T_0$, $(v^*, \rho^*, \theta^*) \in R_T$ yields

$$\sup_{t \in [0,T]} \|\eta^*(t)\|_3 \leq \frac{E}{2} + C_0 E T_0^{1/2}.$$

Furthermore, there exists a constant $C_3 \geq 1$ such that, for an arbitrary $T \leq T_0$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$\begin{aligned}\|\mathbf{D}\eta^*\|_{C^{1/2}([0,T];C^0(\bar{\Omega}_0))} &\leq C_3, \\ \|\det \mathbf{D}\eta^*\|_{C^{1/2}([0,T];C^0(\bar{\Omega}_0))} &\leq 6C_3^3.\end{aligned}$$

Since

$$\begin{aligned}|\det[\mathbf{D}\eta^*(t, x)] - 1| &= |\det[\mathbf{D}\eta^*(t, x)] - \det[\mathbf{D}\eta^*(0, x)]| \\ &\leq \|\det \mathbf{D}\eta^*\|_{C^{1/2}([0,T];C^0(\bar{\Omega}_0))} t^{1/2},\end{aligned}$$

and for any pair of orthonormal vectors $\tau_1(x)$, $\tau_2(x)$ we have

$$\begin{aligned}(2.25) \quad &| |\mathbf{D}\eta^*(t, x)\tau_1(x) \wedge \mathbf{D}\eta^*(t, x)\tau_2(x)| - 1 | \\ &= | |\mathbf{D}\eta^*(t, x)\tau_1(x) \wedge \mathbf{D}\eta^*(t, x)\tau_2(x)| \\ &\quad - |\mathbf{D}\eta^*(0, x)\tau_1(x) \wedge \mathbf{D}\eta^*(0, x)\tau_2(x)| | \\ &\leq 2\|\mathbf{D}\eta^*\|_{C^{1/2}([0,T];C^0(\bar{\Omega}_0))}^2 t^{1/2},\end{aligned}$$

we can find $T_1 \in]0, T_0]$ such that

$$2C_3^2 T_1^{1/2} \leq 6C_3^3 T_1^{1/2} \leq \frac{1}{2};$$

hence, for $T \leq T_1$, $(v^*, \rho^*, \theta^*) \in R_T$ yields

$$(2.26) \quad \det[\mathbf{D}\eta^*(t, x)] \geq \frac{1}{2} \quad \text{in } \bar{Q}_T,$$

$$(2.27) \quad |\mathbf{D}\eta^*(t, x)\tau_1(x) \wedge \mathbf{D}\eta^*(t, x)\tau_2(x)| \geq \frac{1}{2} \quad \text{in } \bar{Q}_T,$$

for any pair of orthonormal vectors $\tau_1(x)$, $\tau_2(x)$. Finally we have

$$\eta^*(t, x) - \eta^*(t, y) = x - y + \int_0^t [v^*(s, x) - v^*(s, y)] ds$$

and consequently

$$|\eta^*(t, x) - \eta^*(t, y)| \geq |x - y| - \int_0^t |v^*(s, x) - v^*(s, y)| ds.$$

Since

$$\begin{aligned} \int_0^t |v^*(s, x) - v^*(s, y)| ds &\leq C_4 \int_0^t \|v^*(s)\|_3 |x - y| ds \\ &\leq C_4 \left(\int_0^t \|v^*(s)\|_3^2 ds \right)^{1/2} |x - y| t^{1/2}, \end{aligned}$$

we can find an instant $T_2 \in]0, T_1]$ such that, for $T \leq T_2$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$(2.28) \quad |\eta^*(t, x) - \eta^*(t, y)| \geq \frac{1}{2} |x - y| \quad \forall t \in [0, T], \quad \forall x, y \in \overline{\Omega}_0,$$

i.e., $\eta^*(t, \cdot)$ is injective for any $t \in [0, T]$. Define now the operators

$$\begin{aligned} A_i^*(t, x, D)w &\equiv -\frac{1}{\rho^*} \left\{ \mu a_{kj}^* D_k (a_{sj}^* D_s w_i + a_{si}^* D_s w_j) \right. \\ &\quad \left. + \left(\zeta - \frac{2}{3} \mu \right) a_{ki}^* D_k (a_{sj}^* D_s w_j) \right\}, \\ L^*(t, x, D)\Theta &\equiv -\frac{\chi}{\rho^* c^*} a_{kj}^* D_k (a_{sj}^* D_s \Theta), \end{aligned}$$

and the boundary operator

$$B_i^*(t, x, D)w \equiv \mu (a_{kj}^* D_k w_i + a_{ki}^* D_k w_j) N_j^* + \left(\zeta - \frac{2}{3} \mu \right) a_{kj}^* D_k w_j N_i^*,$$

where $a_{kj}^* = a_{kj}^*(t, x)$ is the entry (k, j) of the matrix $[D\eta^*]^{-1}$, $N^* = N^*(t, x)$ is the unit outward normal to $\eta^*(t, \partial\Omega_0)$ in $\eta^*(t, x)$, $c^*(t, x) \equiv c_v(\rho^*(t, x), \theta^*(t, x))$. Since (v^*, ρ^*, θ^*) satisfy the initial conditions (1.18)-(1.21), we have

$$\begin{aligned} A^*(0, x, D) &= A(x, D), \quad L^*(0, x, D) = L(x, D), \quad B^*(0, x, D) = B(x, D), \\ a_{ki}^*(0, x) &= \delta_{ki}, \quad N^*(0, x) = n^0(x). \end{aligned}$$

Consider the following problems

$$(2.29) \quad \begin{cases} \dot{v}_i + A_i v = A_i v^* - A_i^* v^* - a_{ki}^* D_k \phi^* - \frac{1}{\rho^*} a_{ki}^* D_k p^* + \bar{b}_i^* \equiv F_i^* & \text{in } Q_T; \\ B_i v = B_i v^* - B_i^* v^* + p^* N_i^* - \bar{p}^* N_i^* \equiv G_i^* & \text{on } \Sigma_T; \\ v(0) = u_0 & \text{in } \Omega_0; \end{cases}$$

$$(2.30) \quad \begin{cases} \dot{\theta} + L\theta = L\theta^* - L^*\theta^* - \frac{\theta^*}{\rho^*c^*} \bar{D}_2 p^* a_{kj}^* D_k v_j^* + \frac{\bar{r}^*}{c^*} \\ \quad + \frac{\mu}{2\rho^*c^*} \sum_{ij} (a_{kj}^* D_k v_i^* + a_{ki}^* D_k v_j^*)^2 \\ \quad + \frac{1}{\rho^*c^*} \left(\zeta - \frac{2}{3}\mu \right) (a_{ki}^* D_k v_i^*)^2 \equiv H^* & \text{in } Q_T; \\ \chi \frac{\partial \theta}{\partial n} = \chi \frac{\partial \theta^*}{\partial n} - \chi a_{ki}^* D_k \theta^* N_i^* + h(\hat{\theta}^* - \theta^*) \equiv K^* & \text{on } \Sigma_T; \\ \theta(0) = \theta_0 & \text{in } \Omega_0; \end{cases}$$

$$(2.31) \quad \begin{cases} \dot{\rho} = -\rho^* a_{ki}^* D_k v_i^* & \text{in } Q_T; \\ \rho(0) = \rho_0 & \text{in } \Omega_0; \end{cases}$$

where

$$(2.32) \quad \phi^*(t, x) \equiv -\kappa \int_{\Omega_0} \frac{\rho^*(t, \xi)}{|\eta^*(t, x) - \eta^*(t, \xi)|} |\det[D\eta^*(t, \xi)]| d\xi, \quad \forall (t, x) \in Q_T,$$

$$p^*(t, x) \equiv p(\rho^*(t, x), \theta^*(t, x)), \quad \bar{b}^*(t, x) \equiv \bar{b}(t, \eta^*(t, x)),$$

$$\bar{r}^*(t, x) \equiv \bar{r}^*(t, \eta^*(t, x)), \quad \bar{D}_2 p^*(t, x) \equiv \frac{\partial p}{\partial \theta}(\rho^*(t, x), \theta^*(t, x)),$$

$$\hat{\theta}^*(t, x) \equiv \hat{\theta}(t, \eta^*(t, x)), \quad \bar{p}^*(t, x) \equiv \bar{p}(t, \eta^*(t, x)).$$

First, we note that, if $(v^*, \rho^*, \theta^*) \in R_T$, then

$$\eta^* \in H^1(0, T; H^3(\Omega_0)) \cap H^2(0, T; H^1(\Omega_0))$$

and $|\eta^*|_{2,3,T} + |\ddot{\eta}^*|_{2,1,T} \leq C_0 E$. Hence $\det D\eta^* \in H^1(0, T; H^2(\Omega_0))$ and, since for $T \leq T_2$ (2.26) holds, we also have $a_{ki}^* \in H^1(0, T; H^2(\Omega_0))$ with $\dot{a}_{ki}^* \in H^1(0, T; L^2(\Omega_0))$; by interpolation $\dot{a}_{ki}^* \in C^0([0, T]; H^1(\Omega_0))$. The norms of all these functions are bounded by some constants depending on the data of the problem but, from Proposition 2.6, independent of T . Assume $T \leq T_2$; we want now to solve (2.29). Since by (2.26), (2.28) η^* is a diffeomorphism, instead of (2.32) we can consider

$$(2.33) \quad \bar{\phi}^*(t, z) \equiv \phi^*(t, (\eta^*)^{-1}(t, z)) = -\kappa \int_{\Omega_0^*} \frac{\rho^*(t, (\eta^*)^{-1}(t, y))}{|z - y|} dy,$$

for each $t \in [0, T]$, $z \in \Omega_0^* \equiv \eta^*(t, \Omega_0)$. For any $t \in [0, T]$, extend $\rho^*(t, \cdot)$ to $\mathbb{R}^3$ by 0 out of Ω_0 . The changement of variables $z \rightarrow x = (\eta^*)^{-1}(t, z)$ shows that the function $(t, z) \rightarrow \rho^*(t, (\eta^*)^{-1}(t, z))$ belongs to $L^\infty(0, T; L^p(\mathbb{R}^3))$ for any $p > 1$. Each norm is bounded by a constant depending on the data but independent of T . By potential theoretic estimates $\nabla \bar{\phi}^* \in L^\infty(0, T; W^{1,p}(\mathbb{R}^3))$ for any $p > 3/2$.

Hence we obtain $\nabla\phi^* \in L^\infty(0, T; W^{1,p}(\Omega_0))$ for any $p > 3/2$, with each norm bounded by a constant independent of T . In particular,

$$\nabla\phi^* \in L^\infty(0, T; H^1(\Omega_0)) \cap L^\infty(Q_T).$$

Now we can estimate F^* in $L^2(0, T; H^1(\Omega_0))$. We have

$$\begin{aligned} |F^*|_{2,1,T} \leq c \left\{ \left| \frac{\rho^* - \rho_0}{\rho^* \rho_0} \right|_{\infty,2,T} |\rho^* A^* v^*|_{2,1,T} + \left\| \frac{1}{\rho_0} \right\|_2 |(\rho^* A^* - \rho_0 A) v^*|_{2,1,T} \right. \\ \left. + |a_{ki}^* D_k \phi^*|_{2,1,T} + \left| \frac{1}{\rho^*} a_{ki}^* D_k p^* \right|_{2,1,T} + |\bar{b}|_{2,1,T,R} \right\} \end{aligned}$$

where $|\bar{b}|_{2,1,T,R}$ is the norm in $L^2(0, T; H^1(B_R))$ and R is such that

$$\|\eta^*\|_{L^\infty(Q_T)} \leq c \left(\frac{E}{2} + C_0 E T_0^{1/2} \right) \leq R.$$

Recalling that, if $f \in H^1(0, T; X)$, where X is a Banach space, we have

$$\|f(t) - f(0)\|_X \leq |f|_{2,X,T} t^{1/2}$$

and that, if $f \in L^\infty(0, T; X)$, we have

$$|f|_{2,X,T} \leq |f|_{\infty,X,T} T^{1/2},$$

we obtain

$$(2.34) \quad |F^*|_{2,1,T} \leq c \left[T^{1/2} + |\bar{b}|_{2,1,T,R} \right].$$

Next we estimate the boundary term G^* in $H^{3/2,3/4}(\Sigma_T)$. We have

$$\|G^*\|_{3/2,3/4,\Sigma_T} \leq |G^*|_{2,3/2,\Sigma_T} + [G^*]_{3/4,0,\Sigma_T}$$

where

$$[G^*]_{3/4,0,\Sigma_T} \equiv \left(\int_0^T \int_0^T \frac{\|G^*(t) - G^*(\tau)\|_{0,\partial\Omega_0}^2}{|t - \tau|^{5/2}} dt d\tau \right)^{1/2}.$$

Since by conditions (2.26), (2.28) $\eta^*(t, \cdot)$ is a diffeomorphism, for each $t \in [0, T]$, in each local chart $\psi = \psi(\xi_1, \xi_2)$ of $\partial\Omega_0$ the unit vector N^* can be written as

$$N^*(t, x) = \frac{D\eta^*(t, x)\tau_1(x) \wedge D\eta^*(t, x)\tau_2(x)}{|D\eta^*(t, x)\tau_1(x) \wedge D\eta^*(t, x)\tau_2(x)|},$$

where

$$\begin{aligned}\tau_1(x) &\equiv \frac{\frac{\partial \psi}{\partial \xi_1}(\psi^{-1}(x))}{\left| \frac{\partial \psi}{\partial \xi_1}(\psi^{-1}(x)) \right|}, \\ \tau_2(x) &\equiv \frac{\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) - \left[\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) \tau_1(x) \right] \tau_1(x)}{\left| \frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) - \left[\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) \tau_1(x) \right] \tau_1(x) \right|}.\end{aligned}$$

From the estimates on v^* , η^* , we obtain

$$(2.35) \quad \|N^*\|_{\infty, 3/2, \Sigma_T} \leq c,$$

$$(2.36) \quad \|N^*(t) - N^*(s)\|_{3/2, \partial\Omega_0} \leq c|t - s|^{1/2} \quad \forall t, s \in [0, T].$$

Hence we can proceed as for F^* obtaining

$$(2.37) \quad \|G^*\|_{2, 3/2, \Sigma_T} \leq cT^{1/2}.$$

To estimate the seminorm $[G^*]_{3/4, 0, \Sigma_T}$ is more complicated; we observe that

$$[N^*]_{3/4, 1/2, \Sigma_T} \leq cT^{3/4},$$

that $\dot{a}_{ki}^*$ bounded in $L^\infty(0, T; H^1(\Omega_0))$ gives

$$[a_{ki}^*(\cdot) - a_{ki}^*(0)]_{3/4, 1, \Sigma_T} \leq cT^{3/4},$$

and that Dv^* is bounded in $H^{3/4}([0, T]; L^2(\partial\Omega_0))$. Using also (2.36) we thus obtain

$$(2.38) \quad [Bv^* - B^*v^*]_{3/4, 0, \Sigma_T} \leq cT^{1/2}.$$

Next we observe that $\dot{\rho}^*$ bounded in $L^\infty([0, T]; H^1(\Omega_0))$ yields

$$[\rho^*]_{3/4, 0, \Sigma_T} \leq cT^{3/4},$$

moreover, by using Proposition 2.6, $\theta^* \in H^1(0, T; H^1(\Omega_0))$ gives

$$[\theta^*]_{3/4, 0, \Sigma_T} \leq \sqrt{2}T^{1/4-\varepsilon}[\theta^*]_{1-\varepsilon, 0, \Sigma_T} \leq cT^{1/4-\varepsilon},$$

where $0 < \varepsilon < 1/4$. Hence we obtain

$$(2.39) \quad [p^*N^*]_{3/4, 0, \Sigma_T} \leq cT^{1/4-\varepsilon}.$$

For the last term in G^* we have

$$(2.40) \quad [\bar{p}^*N^*]_{3/4, 0, \Sigma_T} \leq c[\bar{p}]_{3/4, 1, T, R} + c|\bar{p}|_{\infty, 2, T_0, R}T^{3/4}.$$

Hence from (2.38)-(2.40) we obtain

$$[G^*]_{3/4,0,\Sigma_T} \leq c \left[T^{1/4-\varepsilon} + [\bar{p}]_{3/4,1,T,R} \right].$$

Lemma 2.4 yields that there exists the solution v of (2.29) such that

$$\begin{aligned} |v|_{2,3,T} + |\dot{v}|_{2,1,T} &\leq C_0 \left[c(T^{1/2} + |\bar{b}|_{2,1,T,R}) + cT^{1/2} \right. \\ &\quad \left. + c(T^{1/4-\varepsilon} + [\bar{p}]_{3/4,1,T,R}) + \frac{E}{2} \right]. \end{aligned}$$

Hence we can find $T_3 \in]0, T_2]$ such that for an arbitrary $T \leq T_3$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$|v|_{2,3,T} + |\dot{v}|_{2,1,T} \leq C_0 E.$$

Now we proceed with the estimates for θ . First we consider the estimate of H^* in $L^2(0, T; H^1(\Omega_0))$. Observe that $H^{3/2+\varepsilon}(\Omega_0) \subset L^\infty(\Omega_0)$, $0 < \varepsilon < 1/2$, so that if $f \in H^{3/2+\varepsilon}(\Omega_0)$, $g \in H^1(\Omega_0)$ then $fg \in H^1(\Omega_0)$. We have

$$\begin{aligned} |H^*|_{2,1,T} &\leq c \left\{ \left| \frac{\rho^* - \rho_0}{\rho^* \rho_0} \right|_{\infty,2,T} |\rho_0 L \theta^*|_{2,1,T} \right. \\ &\quad + \left| \frac{1}{\rho^*} \right|_{\infty,2,T} \left| \frac{c^* - c_0}{c^* c_0} \right|_{\infty,3/2+\varepsilon,T} |\rho_0 c_0 L \theta^*|_{2,1,T} \\ &\quad + \left| \frac{1}{\rho^* c^*} \right|_{\infty,2,T} |(\rho_0 c_0 L - \rho^* c^* L^*) \theta^*|_{2,1,T} \\ &\quad + \left| \frac{\theta^*}{\rho^* c^*} \bar{D}_2 p^* a_{kj}^* D_k v_j^* \right|_{2,1,T} + \left| \frac{\bar{r}^*}{c^*} \right|_{2,1,T} \\ &\quad + \left| \frac{\mu}{2 \rho^* c^*} \sum_{ij} (a_{kj}^* D_k v_i^* + a_{ki}^* D_k v_j^*)^2 \right|_{2,1,T} \\ &\quad \left. + \left| \frac{1}{\rho^* c^*} \left(\zeta - \frac{2}{3} \mu \right) (a_{ki}^* D_k v_i^*)^2 \right|_{2,1,T} \right\}. \end{aligned}$$

Observe that, by interpolation,

$$\theta^* \in H^{3/4-\varepsilon/2}(0, T; H^{3/2+\varepsilon}(\Omega_0)) \subset C^{1/4-\varepsilon/2}([0, T]; H^{3/2+\varepsilon}(\Omega_0)),$$

$0 < \varepsilon < 1/2$, so that

$$\left| \frac{c^* - c_0}{c^* c_0} \right|_{\infty,3/2+\varepsilon,T} \leq c \left[|\rho^* - \rho_0|_{\infty,2,T} + |\theta^* - \theta_0|_{\infty,3/2+\varepsilon,T} \right] \leq c T^{1/4-\varepsilon/2}.$$

Concerning the quadratic terms in Dv^* , we observe that by interpolation Dv^* is bounded in $H^{1/4-\varepsilon/2}(0, T; H^{3/2+\varepsilon}(\Omega_0)) \subset L^{4/(1+2\varepsilon)}(0, T; H^{3/2+\varepsilon}(\Omega_0))$, where $0 < \varepsilon < 1/2$; hence

$$|D_i v_j^* D_k v_h^*|_{2,1,T} \leq c |Dv^*|_{\infty,1,T} |Dv^*|_{4/(1+2\varepsilon),3/2+\varepsilon,T} T^{1/4-\varepsilon/2} \leq c T^{1/4-\varepsilon/2};$$

as usual the constants do not depend on T . In a similar way we estimate the term containing $\bar{D}_2 p^*$. The other terms can be treated in a straightforward manner. Thus we obtain

$$|H^*|_{2,1,T} \leq c [T^{1/4-\varepsilon/2} + |\bar{r}|_{2,1,T,R}].$$

The estimate of K^* in $H^{3/2,3/4}(\Sigma_T)$ is similar to the one of G^* . We obtain

$$\|K^*\|_{3/2,3/4,\Sigma_T} \leq c [T^{1/4-\varepsilon} + [\hat{\theta}]_{3/4,1,T,R}],$$

where $0 < \varepsilon < 1/4$. Hence, from Lemma 2.5 there exists the solution θ of (2.30) such that

$$|\theta|_{2,3,T} + |\dot{\theta}|_{2,1,T} \leq C'_0 \left[c(T^{1/4-\varepsilon/2} + |\bar{r}|_{2,1,T,R}) + c(T^{1/4-\varepsilon} + [\hat{\theta}]_{3/4,1,T,R}) + \frac{E}{2} \right],$$

where we can take one same ε , $0 < \varepsilon < 1/4$. Then there exists an instant $T_4 \in]0, T_3]$ such that for an arbitrary $T \leq T_4$,

$$|\theta|_{2,3,T} + |\dot{\theta}|_{2,1,T} \leq C'_0 E.$$

Now we consider problem (2.31). Let $T \leq T_4$. From (2.23) we have

$$\left(\int_0^T \|\rho^*(t) a_{ki}^*(t) D_k v_i^*(t)\|_2^2 dt \right)^{1/2} \leq C_1$$

so that

$$\left(\int_0^T \|\rho^*(t) a_{ki}^*(t) D_k v_i^*(t)\|_{L^\infty(\Omega_0)}^2 dt \right)^{1/2} \leq C'_1.$$

Let $T' \in]0, T_4]$ such that

$$(C_1 + C'_1)(T')^{1/2} \leq \min \left(\frac{m}{2}, \frac{E}{2} \right).$$

The solution of (2.31) is given by

$$\rho(t, x) = \rho_0(x) - \int_0^t \rho^*(s, x) a_{ki}^*(s, x) D_k v_i^*(s, x) ds;$$

hence $(v^*, \rho^*, \theta^*) \in R_{T'}$ implies

$$\begin{aligned}\rho(t, x) &\geq \rho_0(x) - \int_0^{T'} \|\rho^*(s) a_{ki}^*(s) D_k v_i^*(s)\|_{L^\infty(\Omega_0)} ds \\ &\geq m - C_1'(T')^{1/2} \geq \frac{m}{2} \text{ for each } (t, x) \in \overline{Q}_{T'}.\end{aligned}$$

Moreover

$$\sup_{t \in [0, T']} \|\rho(t)\|_2 \leq \|\rho_0\|_2 + \int_0^{T'} \|\rho^*(s) a_{ki}^*(s) D_k v_i^*(s)\|_2 ds \leq \frac{E}{2} + C_1(T')^{1/2} \leq E.$$

From (2.23), (2.24), we have

$$|\dot{\rho}|_{2,2,T'} \leq C_1, \quad \sup_{t \in [0, T']} \|\dot{\rho}(t)\|_1 \leq C_2.$$

Hence we have proved that the map $\Lambda : (v^*, \rho^*, \theta^*) \rightarrow (v, \rho, \theta)$ satisfies $\Lambda(R_{T'}) \subseteq R_{T'}$. Let us introduce the space

$$X \equiv \{(v, \rho, \theta) | v, \rho, \theta \in C^0([0, T']; H^{2-\varepsilon}(\Omega_0))\},$$

where ε is a fixed small positive parameter, say $0 < \varepsilon < 1/2$.

LEMMA 2.7. $R_{T'}$ is a convex and compact subset of X .

PROOF. $R_{T'}$ is obviously convex and bounded in $Y \times Y \times H^1(0, T'; H^2(\Omega_0))$, where

$$Y \equiv L^2(0, T'; H^3(\Omega_0)) \cap H^1(0, T'; H^1(\Omega_0)).$$

The space Y is continuously embedded in

$$C^0([0, T']; H^2(\Omega_0)) \cap C^{\varepsilon/2}([0, T']; H^{2-\varepsilon}(\Omega_0)),$$

which is, from Ascoli-Arzelà's and Rellich's theorems, compactly embedded in $C^0([0, T']; H^{2-\varepsilon}(\Omega_0))$. Analogously, $H^1(0, T'; H^2(\Omega_0))$ is compactly embedded in $C^0([0, T']; H^{2-\varepsilon}(\Omega_0))$. Hence $R_{T'}$ is relatively compact in X . Finally, it is easily verified that $R_{T'}$ is closed in X . $\square$

LEMMA 2.8 The map Λ is continuous from the topology of X into the topology of $C^0([0, T']; L^2(\Omega_0))$.

PROOF. Suppose that $(v_n^*, \rho_n^*, \theta_n^*) \in R_{T'}$ converge in X to (v^*, ρ^*, θ^*) and let $(v_n, \rho_n, \theta_n) \equiv \Lambda(v_n^*, \rho_n^*, \theta_n^*)$, $(v, \rho, \theta) \equiv \Lambda(v^*, \rho^*, \theta^*)$. We observe that $\eta_n^* \rightarrow \eta^*$

in $C^1([0, T']; H^{2-\varepsilon}(\Omega_0))$ (with obvious notation). Assume that E is an extension operator from Ω_0 to $\mathbb{R}^3$, bounded in $H^{1-\varepsilon}$ and H^2 , i.e.,

$$E \in \mathcal{L}(H^{1-\varepsilon}(\Omega_0), H^{1-\varepsilon}(\mathbb{R}^3)) \cap \mathcal{L}(H^2(\Omega_0), H^2(\mathbb{R}^3)).$$

If $f \in H^{1-\varepsilon}(\Omega_0)$, $g \in H^2(\Omega_0)$, we have (see [5])

$$\|fg\|_{1-\varepsilon} \leq \|Ef\|_{H^{1-\varepsilon}(\mathbb{R}^3)} \|Eg\|_{H^2(\mathbb{R}^3)} \leq c\|f\|_{1-\varepsilon} \|g\|_2,$$

so that $fg \in H^{1-\varepsilon}(\Omega_0)$. Since $D\eta_n^* \rightarrow D\eta^*$ in $C^1([0, T']; H^{1-\varepsilon}(\Omega_0))$ and $D\eta_n^*$ is bounded in $H^1(0, T'; H^2(\Omega_0))$, then $a_{ki}^{*(n)} \rightarrow a_{ki}^*$ in $H^1(0, T'; H^{1-\varepsilon}(\Omega_0))$. Observe also that

$$\|N_n^* - N^*\|_{L^2(\partial\Omega_0)} \leq c\|\eta_n^* - \eta^*\|_{2-\varepsilon}.$$

Take the difference between the equations for (v_n, ρ_n, θ_n) and (v, ρ, θ) , multiply by $\rho_0(v_n - v)$, $(\rho_n - \rho)$, $\rho_0 c_0(\theta_n - \theta)$ respectively, and integrate over Ω_0 . Integrating by parts and using Korn's inequality (2.10), from the equations for the velocity we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_0}(v_n - v)\|_0^2 + m\lambda_0 \left[\frac{1}{K} \|\nabla(v_n - v)\|_0^2 - \left(1 - \frac{1}{K}\right) \|v_n - v\|_0^2 \right] \\ \leq \int_{\partial\Omega_0} [G_n^* - G^* - B(v_n^* - v^*)](v_n - v) + \int \rho_0(F_n^* - F^*)(v_n - v), \end{aligned}$$

with obvious notations. Estimating the right-hand side, after long but straightforward calculations, gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_0}(v_n - v)\|_0^2 + \frac{m\lambda_0}{K} \|v_n - v\|_1^2 \\ \leq \varepsilon_0 \|v_n - v\|_1^2 + C(\varepsilon_0) \|v_n - v\|_0^2 \\ + C(\varepsilon_0) \left\{ |v_n^* - v^*|_{\infty, 2-\varepsilon}^2 + |\rho_n^* - \rho^*|_{\infty, 2-\varepsilon}^2 + |\theta_n^* - \theta^*|_{\infty, 2-\varepsilon}^2 \right. \\ + |\eta_n^* - \eta^*|_{\infty, 2-\varepsilon}^2 + (1 + \|v_n^*\|_3^2) |a_n^* - a^*|_{\infty, 1-\varepsilon}^2 \\ \left. + \|\nabla(\phi_n^* - \phi^*)\|_0^2 + \|\bar{b}_n^* - \bar{b}^*\|_0^2 + \|\bar{p}_n^* - \bar{p}^*\|_{L^2(\partial\Omega_0)}^2 \right\}, \end{aligned}$$

where ε_0 is a small positive parameter. Choosing $\varepsilon_0 = \frac{m\lambda_0}{2K}$, we obtain by Gronwall's lemma

$$\begin{aligned} \|(v_n - v)(t)\|_0^2 \leq c \{ & |v_n^* - v^*|_{\infty, 2-\varepsilon}^2 + |\rho_n^* - \rho^*|_{\infty, 2-\varepsilon}^2 \\ & + |\theta_n^* - \theta^*|_{\infty, 2-\varepsilon}^2 + |\eta_n^* - \eta^*|_{\infty, 2-\varepsilon}^2 + |a_n^* - a^*|_{\infty, 1-\varepsilon}^2 \\ & + \|\nabla(\phi_n^* - \phi^*)\|_{2,0,T'}^2 + \|\bar{b}_n^* - \bar{b}^*\|_{2,0,T'}^2 + \|\bar{p}_n^* - \bar{p}^*\|_{2,0,\Sigma_{T'}}^2 \}, \end{aligned} \quad (2.41)$$

for any $t \in]0, T']$, where

$$\phi_n^*(t, x) \equiv -\kappa \int_{\Omega_0} \frac{\rho_n^*(t, \xi)}{|\eta_n^*(t, x) - \eta_n^*(t, \xi)|} |\det D\eta_n^*(t, \xi)| d\xi,$$

$\bar{b}_n^*(t, x) \equiv \bar{b}(t, \eta_n^*(t, x))$, $\bar{p}_n^*(t, x) \equiv \bar{p}(t, \eta_n^*(t, x))$. Fix an arbitrary parameter $\varepsilon_1 > 0$. We observe that $\eta_n^*(t, x) \rightarrow \eta^*(t, x)$ for each $(t, x) \in Q_{T'}$. Since $\bar{b} \in L^2(0, T_0; C^0(\bar{B}_R))$, $\bar{p} \in L^\infty(0, T_0; C^0(\bar{B}_R))$, by Lebesgue's theorem we can find $n_1 > 0$ such that

$$(2.42) \quad [|\bar{b}_n^* - \bar{b}^*|_{2,0,T'}^2 + |\bar{p}_n^* - \bar{p}^*|_{2,0,\Sigma_{T'}}^2] < \varepsilon_1$$

for each $n > n_1$. Consider now the term with the gravitational potential. Recalling the boundedness of η_n^* , η^* , we obtain

$$(2.43) \quad \begin{aligned} |\nabla(\phi_n^* - \phi^*)|_{2,0,T'} &\leq c [\|\nabla \bar{\phi}_n^*\|_{L^\infty(0,T'; W^{1,p}(\mathbb{R}^3))} |\eta_n^* - \eta^*|_{\infty, 2-\varepsilon} \\ &\quad + \|\nabla(\bar{\phi}_n^* - \bar{\phi}^*)\|_{L^2([0,T'] \times \mathbb{R}^3)} \\ &\quad + |\nabla(\bar{\phi}^*(\cdot, \eta_n^*) - \bar{\phi}^*(\cdot, \eta^*))|_{2,0,T'}], \end{aligned}$$

where $p > 3$ so that $W^{1,p}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, $\bar{\phi}_n^*(t, z) \equiv \phi_n^*(t, (\eta_n^*)^{-1}(t, z))$, $\bar{\phi}^*(t, z) \equiv \phi^*(t, (\eta^*)^{-1}(t, z))$. Recall that $\nabla \bar{\phi}_n^*$, $\nabla \bar{\phi}^* \in L^\infty(0, T'; W^{1,p}(\mathbb{R}^3))$ for any $p > 3/2$; here and in the right hand side of (2.43) the gradients of $\bar{\phi}_n^*$, $\bar{\phi}^*$ are with respect to the variable z . Consider the second term in the right-hand side of (2.43). $(\bar{\phi}_n^* - \bar{\phi}^*)(t, z)$ satisfies the equation

$$(2.44) \quad \Delta(\bar{\phi}_n^* - \bar{\phi}^*)(t, z) = 4\pi\kappa(\bar{\rho}_n^* - \bar{\rho}^*)(t, z)$$

for each $t \in [0, T']$, $z \in \mathbb{R}^3$, where $\bar{\rho}_n^*(t, z) \equiv \rho_n^*(t, (\eta_n^*)^{-1}(t, z))$ and similarly for $\bar{\rho}^*$. We multiply (2.44) by $\bar{\phi}_n^* - \bar{\phi}^*$, integrate over $\mathbb{R}^3$, integrate by parts. Using the estimate $\|f\|_{L^6(\mathbb{R}^3)} \leq c \|\nabla f\|_{L^2(\mathbb{R}^3)}$ we obtain

$$(2.45) \quad \begin{aligned} &\|\nabla(\bar{\phi}_n^* - \bar{\phi}^*)(t)\|_{L^2(\mathbb{R}^3)} \\ &\leq c \|(\bar{\rho}_n^* - \bar{\rho}^*)(t)\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq c \{ |\rho_n^* - \rho^*|_{\infty, 2-\varepsilon} \\ &\quad + \|\rho^*(t, (\eta_n^*)^{-1}(t, \cdot)) - \rho^*(t, (\eta^*)^{-1}(t, \cdot))\|_{L^{6/5}(\mathbb{R}^3)} \}. \end{aligned}$$

Here ρ^* is considered 0 out of Ω_0 , so that we define

$$\begin{aligned} \rho^*(t, (\eta_n^*)^{-1}(t, z)) &\equiv 0 & \text{if } z \notin \eta_n^*(t, \Omega_0), \\ \rho^*(t, (\eta^*)^{-1}(t, z)) &\equiv 0 & \text{if } z \notin \eta^*(t, \Omega_0). \end{aligned}$$

Let $z \in \eta_n^*(t, \Omega_0) \cap \eta^*(t, \Omega_0)$, define $x_n \equiv (\eta_n^*)^{-1}(t, z)$, $x \equiv (\eta^*)^{-1}(t, z)$, i.e., $z = \eta_n^*(t, x_n) = \eta^*(t, x)$. Then

$$\begin{aligned} |x_n - x| &\leq \int_0^t |v_n^*(s, x_n) - v^*(s, x)| ds \\ &\leq c|v_n^* - v^*|_{\infty, 2-\varepsilon} + C_4 \left(\int_0^t \|v^*(s)\|_3^2 ds \right)^{1/2} |x_n - x|^{1/2} \\ &\leq c|v_n^* - v^*|_{\infty, 2-\varepsilon} + \frac{1}{2}|x_n - x|, \end{aligned}$$

from which we obtain

$$|x_n - x| \leq c|v_n^* - v^*|_{\infty, 2-\varepsilon}$$

(see the calculations which lead to (2.28)). On the other hand, the Lebesgue measure of $\eta_n^*(t, \Omega_0) \Delta \eta^*(t, \Omega_0)$ (here Δ stands for the symmetric difference of the two sets) goes to zero as $n \rightarrow +\infty$, uniformly for $t \in [0, T']$. Then, from the continuity of ρ^* , we can find $n_2 \geq n_1$ such that

$$(2.46) \quad \|\nabla(\bar{\phi}_n^* - \bar{\phi}^*)(t)\|_{L^2([0, T'] \times \mathbb{R}^3)}^2 < \varepsilon_1$$

for each $n > n_2$. Finally, since $\nabla \bar{\phi}^* \in L^\infty(0, T'; C^0(\bar{B}_R))$, by Lebesgue's theorem we can find $n_3 > n_2$ such that

$$(2.47) \quad |\nabla(\bar{\phi}^*(\cdot, \eta_n^*) - \bar{\phi}^*(\cdot, \eta^*))|_{2,0,T'} < \varepsilon_1$$

for each $n > n_3$. Hence, from (2.41)-(2.43), (2.45)-(2.47) we obtain the convergence of v_n to v in $C^0([0, T']; L^2(\Omega_0))$. In an analogous way we obtain the convergence of θ_n to θ . The convergence of ρ_n to ρ is obtained by a direct computation. $\square$

By a compactness argument, Λ is continuous from the topology of X into the topology of X . We can finally apply the Schauder's fixed point theorem, and find a fixed point $(v, \rho, \theta) = \Lambda(v, \rho, \theta)$, that is a solution of our problem. The proof of theorem A is complete.

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