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On Regularity of Solutions of Nonlinear Parabolic Systems

JINDŘICH NEČAS - VLADIMÍR ŠVERÁK

1. - Introduction

It is well-known that the full regularity of the elliptic systems

$$D_\alpha A_i^\alpha(Du) = 0$$

in two dimensions can (under standard assumptions) be proved by using $W^{1,2+\delta}$ -estimates for linear elliptic systems with L^∞ coefficients. (See, for example, M. Giaquinta [2]). The purpose of this paper is to show that a similar method can be used when dealing with nonlinear parabolic systems

$$\frac{\partial u_i}{\partial t} = D_\alpha A_i^\alpha(Du).$$

The idea is to show that $\frac{\partial u}{\partial t}$ is bounded in $L^\infty(-T, 0; L^{2+\delta}(\Omega))$ and then apply the theory of elliptic systems. The required estimate is obtained by using estimates for solutions of linear parabolic systems with L^∞ -coefficients. (See Lemma 1). In the two-dimensional case we get full regularity.

2. - Preliminaries

Let $n \geq 2$, $N \geq 1$. We shall be dealing with open sets $Q = \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$, where Ω is a bounded domain in $\mathbb{R}^n$ and $T > 0$. A typical point of $\mathbb{R}^{n+1}$ is denoted by $z = (x, t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

For $\delta > 0$ we let

$$\Omega_\delta = \{x \in \Omega, \text{ dist } (x, \partial\Omega) > \delta\}$$

and

$$Q_\delta = \Omega_\delta \times (-T + \delta, 0).$$

For $x \in \mathbb{R}^n$ and $\rho > 0$ we define

$$B_{x,\rho} = \{y \in \mathbb{R}^n, |x - y| < \rho\}.$$

If $a, b \in \mathbb{R}$, we denote by $a \wedge b$ the minimum of the two numbers.

The Sobolev spaces $W_p^k, \mathring{W}_p^k$ are defined in the standard way.

The space $L^2(-T, 0; W_2^1(\Omega))$ is denoted by $W_2^{1,0}(Q)$. The norm $[\cdot]_{2,Q}$ on $W_2^{1,0}(Q)$ is defined by

$$[u]_{2,Q} = \left\{ \int_Q (|u|^2 + \sum_{i=1}^n |D_i u|^2) \right\}^{\frac{1}{2}}.$$

The spaces $L^\infty(-T, 0; L^p(\Omega))$, $p \geq 1$ will be denoted by $L^{p,\infty}(Q)$ and the corresponding norm is denoted by $\|\cdot\|_{p,\infty,Q}$.

The usual L^p -norm is denoted by $\|\cdot\|_{p,Q}$.

Let us consider the nonlinear parabolic system

$$(1) \quad \frac{\partial u_i}{\partial t} - D_\alpha A_i^\alpha(Du) = 0. \quad (i = 1, 2, \dots, N)$$

where $u = (u_1, \dots, u_N)$, $Du = (D_\alpha u_i)_{1 \leq i \leq N, 1 \leq \alpha \leq n} = (\frac{\partial u_i}{\partial x_\alpha})_{1 \leq i \leq N, 1 \leq \alpha \leq n}$ is the gradient matrix of u and the summation over repeated indexes is understood.

We shall suppose that the functions A_i^α have continuous derivatives satisfying

$$(2) \quad \left\{ \sum_{i,j} \sum_{\alpha,\beta} \left| \frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi) \right|^2 \right\}^{\frac{1}{2}} \leq M$$

and

$$\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi) \pi_\alpha^i \pi_\beta^j \geq \nu |\pi|^2, \quad \nu > 0,$$

for every $\xi, \pi \in \mathbb{R}^{nN}$.

(Of course, for higher regularity results we have to assume higher smoothness of A_i^α).

By a weak solution of (1) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$\int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - A_i^\alpha(Du) D_\alpha \varphi_i \right) dz = 0$$

for every $\varphi \in \mathring{W}_2^1(Q)$.

We shall also be dealing with linear strongly parabolic systems

$$(4) \quad \frac{\partial u_i}{\partial t} - D_\alpha a_{ij}^{\alpha\beta} D_\beta u_j = 0 \quad (i = 1, \dots, N)$$

where $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(z)$ are L^∞ -functions in Q satisfying for almost every $z \in \Omega$ the conditions

$$(2') \quad \left\{ \sum_{i,j} \sum_{\alpha,\beta} |a_{ij}^{\alpha\beta}|^2 \right\}^{\frac{1}{2}} \leq M$$

and

$$(3') \quad a_{ij}^{\alpha\beta} \xi_\beta^j \xi_\alpha^i \geq \nu |\xi|^2$$

for every $\xi \in \mathbb{R}^{nN}$. By a weak solution of (4) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$(*) \quad \int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - a_{ij}^{\alpha\beta} D_\beta u_j D_\alpha \varphi_i \right) dz = 0$$

for every $\varphi \in \overset{\circ}{W}_2^1(Q)$.

We shall use the following well-known results.

(i) If u is a weak solution of (1) or (4), then u is continuous in time with respect to the L^2 -norm. More precisely, if $\Omega' \subset\subset \Omega$, then the map $t \rightarrow u(\cdot, t)$ from $(-T, 0)$ into $L^2(\Omega')$ is continuous. (See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3, Lemma 4.3).

(ii) We have the imbedding

$$L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)$$

where

$$q_0 = \begin{cases} \frac{2(n+2)}{n}, & \text{if } n > 2 \\ \text{is any number} \in [1, 4) & \text{if } n = 2. \end{cases}$$

(See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 2).

We denote by c_i various constants. The value of these constants can depend on ν, M, Ω, T, n and N . The dependence on additional parameters will be indicated.

3. - L^p -estimates

The first statement of the following Lemma is well known (see, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3).

The second statement will be used for the $L^\infty(-T, 0; L^{2+\delta}(\Omega))$ -estimate mentioned in the introduction.

LEMMA 1. *Let u be a weak solution of the linear system (4). Then, for any $\delta > 0$,*

(i)

$$u \in L^{2,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$$

and

$$\|u\|_{2,\infty,Q_\delta} + [u]_{2,Q_\delta} \leq c_1(\delta)\|u\|_{2,Q}.$$

(ii) *For every $p \in [2, (2 + \frac{\nu}{NM}) \wedge q_0)$ the function u belongs to $L^{p,\infty}(Q_\delta)$ and*

$$\|u\|_{p,\infty,Q_\delta} \leq c_2(\delta, p)\|u\|_{2,Q}.$$

PROOF. Let $\gamma \geq 1$ and let $k > 0$ be such that $\text{meas}\{z \in Q, |u(z)| = k\} = 0$. Define $g_k : [0, \infty) \rightarrow \mathbb{R}$ by

$$g_k(t) = \begin{cases} t^\gamma, & \text{if } 0 \leq t \leq k, \\ k^\gamma + \gamma k^{\gamma-1}(t - k), & \text{if } t \geq k. \end{cases}$$

Clearly $g'_k(t) = \gamma(t \wedge k)^{\gamma-1}$ and

$$g''_k(t) = \begin{cases} \gamma(\gamma - 1)t^{\gamma-2}, & \text{if } 0 < t < k, \\ 0, & \text{if } t > k. \end{cases}$$

Define also the function $\eta^k : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\eta^k(u) = g_k(|u|^2)$$

We have

$$\eta^k_{u_i}(u) = \frac{\partial \eta^k}{\partial u_i}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}u_i$$

$$\eta^k_{u_i u_j}(u) = \frac{\partial^2 \eta^k}{\partial u_i \partial u_j}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}(\delta_{ij} + 2(\gamma - 1)d_{ij}(u)).$$

In the second formula we assume $|u|^2 \neq k$ and

$$d_{ij}(u) = \begin{cases} 0, & \text{if } |u|^2 > k, \\ \frac{u_i u_j}{|u|^2}, & \text{if } |u|^2 < k. \end{cases}$$

Let ω_ϵ be a family of symmetric mollifying functions satisfying
 $\omega_\epsilon \in \mathcal{D}(\mathbb{R})$, $\omega_\epsilon(t) \geq 0$,
 $\omega_\epsilon(t) = \omega_\epsilon(-t)$,
support $\omega_\epsilon \subset (\epsilon, -\epsilon)$,
 $\int_{\mathbb{R}} \omega_\epsilon = 1$.

For $f \in L^1(Q)$ let us denote by $(f)_\epsilon$ the function defined a.e. in Q by

$$(f)_\epsilon(x, t) = \int_{\mathbb{R}} f(x, t - s) ds.$$

(We extend f by zero outside Q). Let $\psi \in \mathring{W}_2^1(\Omega \times (-T + \epsilon, -\epsilon))$. Following E. Giusti, M. Giaquinta [3] we set $\varphi = (\psi)_\epsilon$ in (*) and we see that

$$(5) \quad \int_Q \frac{\partial(u_i)_\epsilon}{\partial t} \psi_i dz = - \int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon D_\alpha \psi_i dz.$$

Let $\theta \in \mathcal{D}(\Omega)$ with $0 \leq \theta \leq 1$ and $\theta = 1$ on Ω_δ and let $\rho \in \mathcal{D}(-T, 0)$, $\rho \geq 0$. For ϵ sufficiently small we can use (5) with

$$\psi_i = \eta_{u_i}^k(u_\epsilon) \theta^2 \rho^2$$

to get

$$\begin{aligned} \int_Q \left(\frac{\partial}{\partial t} \eta^k(u_\epsilon) \right) \theta^2 \rho^2 dz &= - \int_Q \left(a_{ij}^{\alpha\beta} D_\beta u_j \right)_\epsilon D_\alpha (\eta_{u_i}^k(u_\epsilon) \theta^2 \rho^2) dz \\ &\quad - \int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon \eta_{u_i}^k(u_\epsilon) 2\theta D_\alpha \theta \rho^2 dz. \end{aligned}$$

Integrating by parts on the left-hand side, letting $\epsilon \rightarrow 0$ and then using the chain rule for the derivative $D_\alpha(\eta_{u_i}^k(u))$ (which is legal) we see that

$$\begin{aligned} (6) \quad \int_Q \eta^k(u) \theta^2 (\rho^2)' dz &= - \int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i, u_i}^k(u) D_\alpha u_i \theta^2 \rho^2 dz \\ &\quad - \int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i}^k(u) 2\theta D_\alpha \theta \rho^2 dz. \end{aligned}$$

Since $|d_{ij}| \leq 1$ we see that if $0 \leq 2(\gamma - 1) < \frac{\nu}{NM}$ then the matrix $\tilde{a}_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(\delta_{il} + 2(\gamma - 1)d_{il}(u))$ satisfies the condition (2') with ν replaced by $\nu_1 = \nu - 2(\gamma - 1)NM$.

We can estimate the right-hand side of (6) by

$$\begin{aligned}
& \nu_1 \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \\
& \quad + 2M \left\{ \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \right\}^{\frac{1}{2}} \\
& \quad \times \left\{ \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz \right\}^{\frac{1}{2}} \\
& \leq -\frac{\nu_1}{2} \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \\
& \quad + \frac{4M^2}{\nu_1} \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz.
\end{aligned}$$

Let $t_1 \in (-T + \delta, 0)$. As we have remarked in Section 2, under our assumptions the function $t \rightarrow u(\cdot, t)$ is continuous mapping of $(-T, 0)$ into $L^2(\Omega_\delta)$. Hence we can use (6) with ρ defined by

$$\rho^2(t) = \begin{cases} 0 & \text{if } t \in (-T, -T + \frac{\delta}{2}) \\ \frac{2}{\delta}(t + T - \frac{\delta}{2}) & \text{if } t \in (-T + \frac{\delta}{2}, -T + \delta) \\ 1 & \text{if } t \in (-T + \delta, t_1) \\ 0 & \text{if } t \in (t_1, 0). \end{cases}$$

We get

$$\begin{aligned}
& \int_Q \eta^k(u(x, t_1)) \theta(x) dx + \frac{\nu_1}{2} \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \\
& \leq c_3 \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz + \int_{\Omega \times (-T, t_1)} (\rho^2)' \eta^k(u) \theta^2 dz.
\end{aligned}$$

Letting $\gamma = 1$ we get (i).

We can use (i) and the imbedding

$$L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)$$

to infer that $\|u\|_{q_0, Q_\delta} \leq c_4(\delta) \|u\|_{2, Q}$. Using this and letting $k \rightarrow \infty$ in (7) we get (ii) with $p = 2\gamma$.

LEMMA 2. *Let u be a weak solution of the nonlinear system (1). Then $u \in W_2^1(Q_\delta)$, the derivatives $D_i u$, $i = 1, \dots, n$ and $D_{n+1} u = \frac{\partial u}{\partial t}$ belong to the space $L^{p,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$ and for each $i = 1, \dots, n, n+1$*

$$\|D_i u\|_{p,\infty,Q_\delta} + [D_i u]_{2,Q_\delta} \leq [u]_{2,Q}.$$

PROOF. As above, we denote by Du the vector $(D_1 u, \dots, D_n u) \in \mathbb{R}^{nN}$. let us fix an index r , $1 \leq r \leq n+1$ and let $e_r \in \mathbb{R}^n \times \mathbb{R}$ be the r -th vector of the canonical basis. Let $\delta' > 0$. For $0 < h < \delta'$ let

$$u_h(z) = h^{-1}[u(z) - u(z - h e_r)].$$

Define the functions $a_{hij}^{\alpha\beta} \in L^\infty(Q_{\delta'})$ for a.e. $z \in Q_{\delta'}$ by

$$a_{hij}^{\alpha\beta}(z) = \int_0^1 A_{i,\xi_\beta}^\alpha(Du(z) - hDu_h(z) + \tau hDu_h(z))d\tau.$$

It is not difficult to see that u_h is the weak solution of the linear system

$$\frac{\partial u_{hi}}{\partial t} - D_\alpha a_{hij}^{\alpha\beta} D_\beta u_{hj} = 0$$

in $Q_{\delta'}$. The functions $a_{hij}^{\alpha\beta}$ clearly satisfy the conditions (2') and (3'). Hence, by Lemma 1

$$(8) \quad \begin{aligned} \|u_h\|_{p,\infty,Q_{2\delta'}} &\leq c_6(\delta', p) \|u_h\|_{2,Q_{\delta'}} \\ \|Du_h\|_{2,Q_{2\delta'}} &\leq c_7(\delta') \|u_h\|_{2,Q_{\delta'}}. \end{aligned}$$

Suppose first $1 \leq r \leq n$. In this case the difference is taken in the direction of the space variables. Since $u \in W_2^{1,0}(Q)$, we have

$$(9) \quad \|u_h\|_{2,Q_{\delta'}} \leq \|D_r u\|_{2,Q}.$$

Using Nirenberg's Lemma we see from (8) that $Du \in W_2^{1,0}(Q_{\delta'})$ and

$$(10) \quad \begin{aligned} \|D_r u\|_{p,\infty,Q_{2\delta'}} &\leq c_6(\delta', p) \|D_r u\|_{2,Q_{\delta'}} \\ \|DD_r u\|_{2,Q_{2\delta'}} &\leq c_7(\delta') \|D_r u\|_{2,Q_{\delta'}} \end{aligned}$$

for every $1 \leq r \leq n$. Now let $r = n+1$. Following S. Campanato [1] we notice that we can use equation (1) and the L^2 -estimate of $D_\alpha D_\beta u$ obtained above to infer that $\frac{\partial u}{\partial t} \in L^2(Q_{2\delta'})$ and

$$(11) \quad \left\| \frac{\partial u}{\partial t} \right\|_{2,Q_{2\delta'}} \leq c_8(\delta') \|Du\|_{2,Q}.$$

Now we can use (8) with Q replaced by $Q_{2\delta'}$ and using (11) we get by the same argument as above

$$\begin{aligned} \left\| \frac{\partial u}{\partial t} \right\|_{p, \infty, Q_{4\delta'}} &\leq c_9(\delta', p) \|Du\|_{2, Q} \\ \left\| D \frac{\partial u}{\partial t} \right\|_{2, Q_{4\delta'}} &\leq c_{9'}(\delta') \|Du\|_{2, Q}. \end{aligned}$$

The proof is finished.

THEOREM 1. *Let u be a weak solution of the system (1) and let p be the exponent from Lemma 1. The for each $\delta > 0$*

$$\frac{\partial u}{\partial t} \in L^{p, \infty}(Q_\delta)$$

and

$$u \in L^\infty(-T + \delta, 0; W_q^2(\Omega_\delta))$$

for some $q = q(\nu, M, p, \delta)$ with $2 < q < p$. Moreover

$$\|u\|_{L^\infty(-T+\delta, 0; W_q^2(\Omega_\delta))} + \left\| \frac{\partial u}{\partial t} \right\|_{p, \infty, Q_\delta} \leq c_{10}(\delta, p, q) \|u\|_{2, Q}.$$

PROOF. Let $\delta' > 0$. We notice that u can be considered as a weak solution of the linear system (4) with

$$a_{ij}^{\alpha\beta}(z) = \int_0^1 A_{i, \xi_\beta}^\alpha(\tau Du(z)) d\tau.$$

(See, for example S. Campanato [1]). Using this and Lemma 1 we get estimates for the norms $\|u\|_{p, \infty, Q_{\delta'}}$ and $[u]_{2, Q_{\delta'}}$. Now we can use Lemma 2 to get estimates of the norms $\|Du\|_{p, \infty, Q_{2\delta'}}$, $\left\| \frac{\partial u}{\partial t} \right\|_{p, \infty, Q_{2\delta'}}$. Lemma 2 also implies $D_\alpha D_\beta u \in L^2(Q_{2\delta'})$, ($0 \leq \alpha, \beta \leq n$). We see that equation (1) is satisfied pointwise almost everywhere in $Q_{2\delta'}$ and that for almost every $t \in (-T + 2\delta', 0)$ the function $u(\cdot, t)$ belongs to $W_2^2(\Omega_{2\delta'})$ and is the weak solution of the elliptic system

$$D_\alpha A_i^\alpha(Dv) = \frac{\partial u_i}{\partial t}$$

in $\Omega_{2\delta'}$. We can now use well-known L^p -estimates for elliptic systems (see Lemma 3 below). The proof is finished.

LEMMA 3. *Let $p > 2$ and let $g \in L^p(\Omega)$. Let $u \in W_2^1(\Omega)$ be a weak solution of the elliptic system*

$$(12) \quad D_\alpha A_i^\alpha(Du) = g_i \quad i = 1, \dots, n$$

Then there exists $q = q(\nu, M, p) > 2$ such that $u \in W_{q,loc}^2(\Omega)$. Moreover, for every $\delta > 0$

$$\|u\|_{W_q^2(\Omega_\delta)} \leq c_{11}(\nu, M, p, q, \delta)(\|u\|_{W_2^1(\Omega)} + \|g\|_{p,\Omega}).$$

PROOF. Using the standard difference quotient technique, it is not difficult to verify that the following computations are legal.

Let $1 \leq s \leq n$. We let $v = D_s u$ and take the s -th derivative of (12). We get

$$(13) \quad D_\alpha a_{ij}^{\alpha\beta} D_\beta v_j = D_s g_i$$

where $a_{ij}^{\alpha\beta}(z) = A_{i,\xi_j^\beta}^\alpha(Du(x))$.

This implies

$$(14) \quad \frac{\nu}{2} \int_\Omega \zeta^2 |Dv|^2 dx \leq c_{12}(\nu, M) \int_\Omega (|v|^2 |D\zeta|^2 + |g|^2) dx$$

for every $\zeta \in \mathcal{D}(\Omega)$ (Cacciopoli's inequality). The required estimate can now be obtained by using the technique of reverse Hölder inequalities. (See, for example, M. Giaquinta [2], Chap. 5, Theorem 2.2). The proof is finished.

COROLLARY. *Let the assumptions of Theorem 1 be satisfied.*

- (i) *If $n \leq 4$, then u is Hölder continuous in Q .*
- (ii) *If $n \leq 2$, then Du is Hölder continuous in Q .*
- (iii) *If $n \leq 2$ and the functions A_i^α are smooth, then the solution u is smooth.*

REMARK. If $n \geq 3$, then Du may not be continuous. Examples are provided by nonregular solutions of elliptic systems. These can be found in J. Nečas [5].

PROOF OF THE COROLLARY. Let $\delta > 0$.

- (i) Since $W_q^2(\Omega_{\delta/2}) \hookrightarrow C^{0,\alpha}(\Omega_\delta)$ with $\alpha = (2 - \frac{n}{q}) \wedge 1$, we have $u \in L^\infty(-T + \delta, 0; C^{0,\alpha}(\Omega_\delta))$.

Since we have also $\frac{\partial u}{\partial t} \in L^2(Q_\delta)$, u is Hölder continuous by Lemma 4 below.

- (ii) In this case we have $W_q^2(\Omega_{\delta/2}) \hookrightarrow C^{1,\beta}(\Omega_\delta)$ $\beta = 1 - \frac{n}{q}$.

Hence $Du \in L^\infty(-T + \delta, 0; C^{0,\beta}(\Omega_\delta))$. Using the Hölder continuity of u it is easy to see that in fact $Du(\cdot, t) \in C^{0,\beta}(\Omega_\delta)$ for every $t \in (-T + \delta, 0)$, the $C^{0,\beta}$ -norm being bounded independently of t .

Now we can use Lemma 3.1, Chap. 2 from O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4] to infer that Du is Hölder continuous in Q_δ .

- (iii) The higher regularity follows in the standard way from the theory of linear equations.

LEMMA 4. *Let $\alpha > 0$, $q > 1$, $\delta > 0$ and suppose*

$$u \in L^\infty(-T, 0; C^{0,\alpha}(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^q(Q).$$

Denote $K_1 = \|u\|_{L^\infty(-T, 0; C^{0,\alpha}(\Omega))}$, $K_2 = \|\frac{\partial u}{\partial t}\|_{q,Q}$. Then there exists $K = K(K_1, K_2, \delta)$ such that

$$|\tilde{u}(x, t_1) - \tilde{u}(x, t_2)| \leq K|t_1 - t_2|^\beta$$

for every $x \in \Omega_\delta$ and every $t_1, t_2 \in (-T, 0)$, where $\beta = \frac{\alpha/q'}{\alpha + n/q}$, $q' = \frac{q}{q-1}$ and $\tilde{u}$ is a suitable representative of u .

PROOF. Suppose first that u is continuous. Let $x \in \Omega_\delta$ and let $0 < \rho < \delta$. Define

$$w_\rho(t) = \frac{1}{|B_{x,\rho}|} \int_{B_{x,\rho}} u(y, t) dy.$$

It is easy to see that w'_ρ is bounded in $L^q(-T, 0)$ by $c_{13}\rho^{-\frac{n}{q}}K_2$.

Let $t_1, t_2 \in (-T, 0)$. We can write

$$\begin{aligned} |u(x, t_1) - u(x, t_2)| &\leq |u(x, t_1) - w_\rho(t_1) + w_\rho(t_1) - w_\rho(t_2) + w_\rho(t_2) - u(x, t_2)| \\ &\leq 2K_1\rho^\alpha + c_{12}K_2\rho^{-\frac{n}{q}}|t_1 - t_2|^{\frac{1}{q}}. \end{aligned}$$

The proof is easily finished by using this inequality with $\rho = |t_1 - t_2|^{\frac{\beta}{\alpha}}$.

REMARK. It is not difficult to see that if the boundary of Ω is sufficiently regular (say, lipshitzian), then K can be chosen independent of δ .

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REFERENCES

- [1] S. CAMPANATO, *On the Nonlinear Parabolic Systems in Divergence Form. Hölder Continuity and Partial Hölder Continuity of the Solutions*. Ann. Mat. Pura Appl., **137** (1984).
- [2] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, 1983.

- [3] M. GIAQUINTA - E. GIUSTI, *Partial Regularity for the Solutions to Nonlinear Parabolic Systems*, Ann. Mat. Pura Appl., **97** (1973).
- [4] O.A. LADYZENSKAJA - V.A. SOLONNIKOV - N.N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, Moscow, 1966.
- [5] J. NEČAS, *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner-Texte zur Math. **52**, Leipzig, 1983.

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