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0. - Introduction

Consider the semigroup of positive contractions in $L^p(\gamma_d)$, $\{T_t : t > 0\}$, defined by

$$(T_t f)(y) = \frac{1}{\pi^{d/2}\left(1-e^{-2t}\right)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|z-e^{-t}y|^2}{(1-e^{-2t})}\right) f(z)\ \mathrm{d}z.$$

This is called the *Ornstein-Uhlenbeck semigroup*.

The infinitesimal generator of this semigroup is called the *Ornstein-Unlenbeck operator*, denoted by L, which have as explicit representation $L = \Delta - 2I_{\mathbb{R}^d} \cdot \nabla$. This operator has as eigenfunctions the Hermite polynomials.

The operator L plays, with respect to the Gaussian measure γ_d, a similar role that the Laplacian Δ plays with respect to the Lebesgue measure m_d, as we are going to try to explain now.

Using the Bochner subordination formula, one can define a second semigroup $\{Q_t : t > 0\}$ as

$$(Q_t f)(y) = \int_0^\infty (T_s f)(y) \frac{t}{\sqrt{\pi}} e^{-t^2/2s} s^{-3/2}\ \mathrm{d}s;$$

it is easy to see that this semigroup has, as infinitesimal generator, $(-L)^{1/2}$ which is the square root of L.

Now if $\int_{\mathbb{R}^d} f(x) e^{-|x|^2}\ \mathrm{d}x = 0$, then it can be proved that the "Riesz potentials" for L, $(-L)^{-n/2}$, can be represented as

$$(-L)^{-n/2} f(y) = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} (Q_t f)(y)\ \mathrm{d}t$$

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for $n \geq 1$.

From these two last expressions we can obtain, after the change of parameter $r = e^{-t}$,

$$(-L)^{-n/2} f(y) = C \int_{\mathbb{R}^d} \left[\int_0^1 \frac{(-\log r)^{(n-2)/2}}{\pi^{(d+1)/2} r} \frac{\exp\left(-\frac{|z-ry|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \right] \mathrm{d}r f(z) \, \mathrm{d}z.$$

In the classical case of the Laplacian, the Riesz Transform R_i can be defined in $\mathbb{R}^d$ as $R_i = -\mathrm{D}_i(-\Delta)^{-1/2}$, $i = 1, \cdots, d$. Thus the analogous singular operators for L are defined as $\mathrm{D}_i(-L)^{-1/2}$, $i = 1, \cdots, d$, and are called *the Riesz Transforms associated to the Ornstein-Uhlenbeck operator*. We also define, for any multi-index α, the operator $\mathrm{D}^\alpha(-L)^{-|\alpha|/2}$: *the Riesz Transform of order α associated to the Ornstein-Uhlenbeck operator*. It can be seen from the previous formula that

$$\begin{aligned} &\mathrm{D}^\alpha(-L)^{-|\alpha|/2} f(y) \\ &= C_{d,\alpha} \int_{\mathbb{R}^d} \int_0^1 r^{|\alpha|-1} \left(\frac{-\log r}{1-r^2} \right)^{(|\alpha|-2)/2} \frac{1}{1-r^2} h_\alpha \left(\frac{z-ry}{(1-r^2)^{1/2}} \right) (1-r^2)^{-d/2} \\ &\qquad \cdot \exp\left(-\frac{|z-ry|^2}{1-r^2} \right) \mathrm{d}r f(z) \mathrm{d}z \end{aligned}$$

where h_α is the Hermite polynomial in d variables of order α.

The main result of this work is to prove, using analytic tools and the explicit representation, that these singular operators are $L^p(\gamma_d)$-continuous, that is:

THEOREM 7. *Let $f \in L^p(\gamma_d)$, $1 < p < \infty$. Then $\mathrm{D}^\alpha(-L)^{-|\alpha|/2} f \in L^p(\gamma_d)$ and moreover*

$$||\mathrm{D}^\alpha(-L)^{-|\alpha|/2} f||_{L^p(\gamma_d)} \leq C_{p,d} ||f||_{L^p(\gamma_d)}.$$

A consequence of the techniques used to prove this result is that, if we define the operator

$$\begin{aligned} &(Kf)(y) \\ &= C_{d,n} \int_{\mathbb{R}^d} \int_0^1 r^{(n-1)} \left(\frac{-\log r}{1-r^2} \right)^{(n-2)/2} \frac{1}{1-r^2} P \left(\frac{z-ry}{(1-r^2)^{1/2}} \right) (1-r^2)^{-d/2} \\ &\qquad \cdot \exp\left(-\frac{|z-ry|^2}{1-r^2} \right) \mathrm{d}r f(z) \mathrm{d}z \end{aligned}$$

where $n > 1$ and $P \in C^1$ is such that it and its derivatives have at most polynomial growth order and satisfies the condition

$$\int_{\mathbb{R}^d} P(z)e^{-|z|^2}\mathrm{d}z = 0,$$

then we have:

THEOREM 8. *Let* $f \in L^p(\gamma_d)$, $1 < p < \infty$. *Then* $Kf \in L^p(\gamma_d)$ *and*

$$||Kf||_{L^p(\gamma_d)} \leq C_{p,d}||f||_{L^p(\gamma_d)}.$$

Theorem 7 was proved in the case $d = 1$ and $\alpha = 1$ by B. Muckenhoupt [Mu-2] although with a different motivation. The techniques that we are going to use here are an outgrowth of his ideas. The result has been proved using probabilistic methods by P.A. Meyer and later by R. Gundy. The big advantage of their methods is that their estimates are independent of the dimension, something that we do not obtain in our proof. This is important since the independence of dimension allows immediately a generalization to infinite dimensions which is the natural context where the Malliavin Calculus is developed and where the Ornstein-Uhlenbeck operator plays a central role; for more details see [Wa-1], [Wa-2] and [Str].

Recently we have learned that G. Pisier [Pi] has obtained a different analytic proof for the Riesz Transform that can be extended to any Riesz Transform of odd order (i.e. $|\alpha|$ is odd). His proof also gives independence of dimension using the Transference Method due to A.P. Calderón.

Even though our method does not gives independence of dimension, it gives a more broad class of singular operators in $\mathbb{R}^d$ (see Theorem 8) than what Pisier can obtain with his proof.

We have organized this paper as follows. In §1 we give the notation needed throughout this work. In §2 we introduce some definitions and some results used for the proof of the main results and finally in §3 we prove the Theorems 7 and 8. This work was my PhD Thesis at the University of Minnesota. I am grateful to my advisor, Prof Eugene Fabes, for his invaluable help, support and encouragement.

1. - Notations

C will always denote a constant, not necessarily the same in each occurrence.

By $x = (x_1, \cdots, x_d)$ we will denote a point of the d-dimensional Euclidean space $\mathbb{R}^d$ and $|x|$ will denote its Euclidean norm, i.e.

$$|x| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}.$$

We will also use y and z to denote elements of $\mathbb{R}^d$.

By x_i we will always denote the i-th coordinate of $x \in \mathbb{R}^d$ and by x^i we will denote the point of the $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$ defined as

$$x^i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d).$$

Similarly $x^{ij} \in \mathbb{R}^{d-2}$ is defined as

$$x^{ij} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_d)$$

if $j > i$ and so on.

χ_E will denote the characteristic function of the set E, a subset of $\mathbb{R}^d$.

γ_d will denote the non-standard Gaussian measure in $\mathbb{R}^d$ defined by

$$\gamma_d(\mathrm{d}x) = e^{-|x|^2}\mathrm{d}x = \exp\left(-\left(\sum_{i=1}^{d} x_1^2\right)^{1/2}\right)\mathrm{d}x.$$

$L^p(\gamma_d)$, $1 < p < \infty$, will denote the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p e^{-|x|^2}\mathrm{d}x < \infty$$

and we define its norm as

$$||f||_{L^p(\gamma_d)} = \left(\int_{\mathbb{R}^d} |f(x)^p e^{-|x|^2}\mathrm{d}x\right)^{1/p}.$$

m_d will denote the Lebesgue measure in $\mathbb{R}^d$.

$L^p(\mathrm{m}_d)$, $1 < p < \infty$, will denote the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p\mathrm{d}x < \infty$$

and we define its norm as

$$||f||_{L^p(\mathrm{m}_d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p\mathrm{d}x\right)^{1/p}.$$

h_n will denote the Hermite polynomial of degree n in $\mathbb{R}$, i.e.

$$h_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$$

or more explicity

$$h_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}.$$

h_n° will denote the polynomial of degree n defined by

$$h_n^\circ(x) = n! \sum_{k=0}^{[n/2]} \frac{(2|x|)^{n-2k}}{k!(n-2k)!};$$

we will call h_n° the *majorant* of h_n since trivially $|h_n(x)| \leq h_n^\circ(x)$.

$\alpha = (\alpha_1, \cdots, \alpha_d)$ will denote a multi-index, i.e. α_i is a non-negative integer $i = 1, \cdots, d$,

$$|\alpha| = \sum_{i=1}^{d} \alpha_i$$

and as before

$$\alpha^{\hat{i}} = (\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_d).$$

h_α will denote the Hermite polynomial of order α in $\mathbb{R}^d$, i.e.

$$h_\alpha(x) = \prod_{i=1}^{d} h_{\alpha_i}(x_i).$$

2. - Preliminary definitions and results

2-1. *Maximal functions*

For the proof of the main result we are going to use two very specific Maximal functions, so let us study them briefly.

DEFINITION 1. Let $f \in L^p(\gamma_1)$, $1 \leq p \leq \infty$, and consider the function $M_*^{[1]} f$ defined as:

$$\left(M_*^{[1]} f\right)(y) = \sup_{y \neq z} \frac{\left[\int_y^z |f(u)| e^{-u^2} du\right]}{\left[\int_y^z e^{-u^2} du\right]}.$$

This function is called *the one-dimensional one-sided Hardy-Littlewood Maximal function for the one dimensional Gaussian measure* γ_1 *of* f.

For this operation we have the following result.

LEMMA 1. *If $f \in L^1(\gamma_1)$, then $M_*^{[1]}f$ is finite almost everywhere and moreover*

$$\gamma_1\left(\{y|\left(M_*^{[1]}f\right)(y) > \lambda\}\right) \leq \frac{C}{\lambda}\|f\|_{L^1(\gamma_1)},$$

and if in addition $f \in L^p(\gamma_1)$, $1 < p \leq \infty$, then $M_*^{[1]}f \in L^p(\gamma_1)$ and

$$\|M_*^{[1]}f\|_{L^p(\gamma_1)} \leq C_p\|f\|_{L^p(\gamma_1)}.$$

PROOF. The proof of this result follows the classical scheme. For details see [Ste] or [Ca]. □

The second maximal function that we are going to need is the following:

DEFINITION 2. Let $f \in L^p(\gamma_1)$, $1 \leq p \leq \infty$, and consider the function $M_T^{[1]}f$ defined as

$$\left(M_T^{[1]}f\right)(y) = \sup_{0<t\leq 1\wedge\frac{1}{|y|}} \frac{1}{2t}\left[\int_{y-t}^{y+t} |f(z)|\mathrm{d}z\right].$$

This function is called *the one dimensional truncated Hardy-Littlewood Maximal function of f.*

The interesting thing about this maximal function is that, even though it is defined with respect to the Lebesgue measure, the truncation makes it $L^p(\gamma_1)$-continuous as the next lemma establishes.

LEMMA 2. *If $f \in L^p(\gamma_1)$, $1 < p \leq \infty$, then $M_T^{[1]}f \in L^p(\gamma_1)$ and*

$$\left\|M_T^{[1]}f\right\|_{L^p(\gamma_1)} \leq C_p\|f\|_{L^p(\gamma_1)}.$$

PROOF. The proof is essentially simple, the idea is that, in the set $|z-y| < 1 \wedge \frac{1}{|y|}$, all the values of $e^{-|z|^2}$ are equivalent.

There is also a *d-dimensional truncated Hardy-Littlewood Maximal function* which is defined as

$$\left(M_T^{[d]}f\right)(y) = \sup_{0<t\leq 1\wedge\frac{1}{|y|}} \frac{1}{2t^d}\left[\int_{|y-z|<t} |f(z)|\mathrm{d}z\right].$$

Using the Method of Rotation and the $L^p(\gamma_1)$-continuity ($1 < p \leq \infty$) of $M_*^{[1]}$, one can prove the $L^p(\gamma_d)$-continuity of $M_T^{[d]}$, but since we are not going to use this fact we will skip the details.

2-2. *Natanson Lemma*

Following Muckenhoupt [Mu-1] we will use a nice result due to I.P. Natanson [Na] in the particular case of the Gaussian measure γ_1.

THEOREM 1. *If $f, g \in L^1(\gamma_1)$ and g is non-negative, monotone increasing until y and monotone decreasing after it, then*

$$\left| \int_{-\infty}^{\infty} f(z) f(z) e^{-z^2} \mathrm{d}z \right| \leq \|g\|_{L^1(\gamma_1)} \left(M_*^{[1]} f\right)(y).$$

PROOF. For the details of this proof see [Mu-1]. □

This leads to the following Corollary that is the form in which Theorem 1 will be used.

COROLLARY 1.1. *Let $L(y,z)$ be a non-negative function, monotone increasing in z for $z \leq y$, monotone decreasing in z for $z \geq y$ and $\int_{-\infty}^{\infty} L(y,z) e^{-z^2} \mathrm{d}z \leq B$, where B is independent of y, then*

$$\left| \int_{-\infty}^{\infty} L(y,z) f(z) e^{-|z|^2} \mathrm{d}z \right| \leq B \left(M_*^{[1]} f\right)(y)$$

for any $f \in L^1(\gamma_1)$ and furthermore

$$\left\| \int_{-\infty}^{\infty} L(y,z) f(z) e^{-|z|^2} \mathrm{d}z \right\|_{L^p(\gamma_1)} \leq B C_p \|f\|_{L^p(\gamma_1)}$$

for any $f \in L^p(\gamma_1)$, $1 < p \leq \infty$.

Moreover, the same holds for any kernel K such that $|K(y,z)| \leq L(y,z)$ where L is as above.

PROOF. Immediate. □

We will give a name for the functions that satisfy the properties of L in the Corollary 1.1.

DEFINITION 3. If $L(y,z)$ is a non-negative function, monotone increasing in z for $z \leq y$, monotone decreasing in z for $z \geq y$ and such that $\int_{-\infty}^{\infty} L(y,z) e^{-z^2} \mathrm{d}z \leq B$, where B is independent of y, then L is called a *Natanson kernel* (with respect to γ_1).

Since we are going to use heavily Corollary 1.1 for specific subintervals on the real line, let us describe each case in detail:

COROLLARY 1.2. *The following kernels are bounded, in absolute value, by Natanson kernels and therefore the conclusions of Corollary 1.1 hold for them:*

i) $$K_1(y,z) = \begin{cases} \frac{1}{y} & \text{if } y > 1 \text{ and } z \leq 0 \\ 0 & \text{otherwise;} \end{cases}$$

ii) $$K_2(y,z) = \begin{cases} \frac{e^{z^2}}{y} & \text{if } y > 1 \text{ and } 0 < z < \frac{y}{2} \\ 0 & \text{otherwise;} \end{cases}$$

iii) $$K_3(y,z) = \begin{cases} ye^{y^2} & \text{if } y > 1 \text{ and } y + \frac{1}{y} \leq z < 2y \\ 0 & \text{otherwise;} \end{cases}$$

iv) $$K_4(y,z) = \begin{cases} e^{y^2} & \text{if } y > 0 \text{ and } [2y \vee (y+1)] \leq z \\ 0 & \text{otherwise;} \end{cases}$$

v) $$K_5(y,z) = \begin{cases} \frac{e^{z^2}}{y^{1/2}(y-z)^{3/2}} & \text{if } y > \sqrt{2} \text{ and } \frac{y}{2} < z < y - \frac{1}{y} \\ 0 & \text{otherwise;} \end{cases}$$

vi) $$K_6(y,z) = \begin{cases} 1 & \text{if } 0 < y < 1 \text{ and } |z-y| > 1 \\ 0 & \text{otherwise;} \end{cases}$$

vii) $$K_7(y,z) = \begin{cases} ye^{y^2}[1 - \log(y|z-y|)] & \text{if } y > 0 \text{ and } |z-y| < 1 \wedge \frac{1}{y} \\ 0 & \text{otherwise;} \end{cases}$$

viii) $$K_8(y,z) = \begin{cases} \left(\frac{y}{|z-y|}\right)^{1/2} e^{y^2} & \text{if } y > 0 \text{ and } |z-y| < 1 \wedge \frac{1}{y} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Immediate. □

2-3. *The Poisson-Hermite integral and its generalizations*

DEFINITION 4. If $f \in L^1(\gamma_d)$, then its *Poisson-Hermite integral* is defined, for $0 \leq r < 1$, by:

$$\left(P_r^{[d]} f\right)(y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} \cdot \int_{\mathbb{R}^d} \exp\left(\frac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right) f(z)e^{-|z|^2}dz;$$

and the maximal operation associated to $P_r^{[d]}$ is defined by:

$$\left(P_*^{[d]}f\right)(y) = \sup_{0\leq r<1} \left|\left(P_r^{[d]}f\right)(y)\right|.$$

We observe that $\left\{P_r^{[d]} : 0 \leq r < 1\right\}$ corresponds to the Ornstein-Uhlenbeck semigroup with parameter - log r. Also observe that $P_r^{[d]}$ can be rewritten as:

$$\left(P_r^{[d]}f\right)(y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) f(z)\mathrm{d}z.$$

These operations are $L^p(\gamma_d)$-continuous:

THEOREM 2. *The following inequalities hold:*

i) $\left\|P_r^{[d]}f\right\|_{L^p(\gamma_d)} \leq \|f\|_{L^p(\gamma_d)}$, *for* $1 \leq p \leq \infty$, $0 \leq r < 1$, *and*

ii) $\left\|P_*^{[d]}f\right\|_{L^p(\gamma_d)} \leq C_d\|f\|_{L^p(\gamma_d)}$, *for* $1 < p \leq \infty$.

PROOF. For the details of this proof see [Mu-1] and [Ca]. □

Now we are going to define generalizations of these operations. Let us start with the one dimensional ones.

DEFINITION 5. Let $f \in L^1(\gamma_1)$, define the operation

$$\left(Q_{r,m}^{[1]}f\right)(y) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{|z-ry|^m}{(1-r^2)^{(m+1)/2}} \cdot \exp\left(\frac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right) f(z)e^{-|z|^2}\mathrm{d}z$$

that can be rewritten as

$$\left(Q_{r,m}^{[1]}f\right)(y) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{|z-ry|^m}{(1-r^2)^{(m+1)/2}} \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) f(z)\mathrm{d}z.$$

Also we define the maximal operation associated to this operator by

$$\left(Q_{*,m}^{[1]}f\right)(y) = \sup_{0\leq r<1} \left|\left(Q_{r,m}^{[1]}f\right)(y)\right|.$$

Observe that

$$Q_{r,0}^{[1]} = P_r^{[1]}.$$

We will prove now that these operations are $L^p(\gamma_i)$-continuous.

THEOREM 3. *Let* $f \in L^p(\gamma_1)$, $1 < p < \infty$, *then*

i) $\|Q^{[1]}_{r,m} f\|_{L^p(\gamma_1)} \leq C_p \|f\|_{L^p(\gamma_1)}$, *for* $0 \leq r < 1$ *and*

ii) $\|Q^{[1]}_{*,m} f\|_{L^p(\gamma_1)} \leq C_p \|f\|_{L^p(\gamma_1)}$.

PROOF. i) Let us take $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\left|\left(Q^{[1]}_{r,m} f\right)(y)\right| \leq C \left[\int_{-\infty}^{\infty} \frac{|z - ry|^{mq}}{(1 - r^2)^{[(m+1)/21/2p]q}} \exp\left(-\frac{|z - ry|^2}{1 - r^2}\right) \mathrm{d}z \right]^{1/q}$$

$$\cdot \left[\int_{-\infty}^{\infty} \frac{1}{(1 - r^2)^{1/2}} \exp\left(-\frac{|z - ry|^2}{1 - r^2}\right) |f(z)|^p \mathrm{d}z \right]^{1/p}$$

$$= 2C \left[\int_0^{\infty} u^{mq} e^{-u^2} \mathrm{d}u \right]^{1/q} \left[\left(P^{[1]}_r |f|^p\right)(y)\right]^{1/p}$$

$$= C \left[\left(P^{[1]}_r |f|^p\right)(y)\right]^{1/p}.$$

Thus, taking $L^p(\gamma_1)$-norm, we get, by Theorem 2:

$$\left[\int_{-\infty}^{\infty} \left|\left(Q^{[1]}_{r,m} f\right)(y)\right|^p e^{-y^2} \mathrm{d}y \right]^{1/p} \leq C_p \left[\int_{-\infty}^{\infty} \left(P^{[1]}_r |f|^p\right)(y) e^{-y^2} \mathrm{d}y \right]^{1/p}$$

$$\leq C_p \left[\int_{-\infty}^{\infty} |f|^p(y) e^{-y^2} \mathrm{d}y \right]^{1/p}.$$

ii) Let us take $1 < p_0 < p$ and q_0 such that $\frac{1}{p_0} + \frac{1}{q_0} = 1$, then by the same trick done in i) we obtain

$$\left|\left(Q^{[1]}_{r,m} f\right)(y)\right| \leq C_{p_0} \left[\left(P^{[1]}_r |f|^{p_0}\right)(y)\right]^{1/p_0},$$

and therefore

$$\left(Q^{[1]}_{*,m} f\right)(y) \leq C_{p_0} \left[\left(P^{[1]}_* |f|^{p_0}\right)(y)\right]^{1/p_0}.$$

Now, taking $L_p(\gamma_1)$-norm, we have

$$\left[\int_{-\infty}^{\infty} \left[\left(Q^{[1]}_{*,m} f\right)(y)\right]^p e^{-y^2} \mathrm{d}y\right]^{1/p} \leq C^p_{p_0} \left[\int_{-\infty}^{\infty} \left[\left(P^{[1]}_{*} |f|^{p_0}\right)(y)\right]^{p/p_0} e^{-y^2} \mathrm{d}y\right]^{1/p}$$

$$\leq C_p \left[\int_{-\infty}^{\infty} |f|^p e^{-y^2} \mathrm{d}y\right]^{1/p}$$

by Theorem 2 ii). □

DEFINITION 6. Let $f \in L^1(\gamma_1)$, h_n the Hermite polynomial of order n and h°_n its majorant, we define the operator

$$\left(H^{[1]}_{r,n} f\right)(y) = \frac{1}{\pi^{1/2}(1-r^2)^{1/2}} \int_{-\infty}^{\infty} h^\circ_n \left(\frac{|z-ry|}{(1-r^2)^{1/2}}\right) \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) f(z) \mathrm{d}z$$

and its associated Maximal operator:

$$\left(H^{[1]}_{*,n} f\right)(y) = \sup_{0 \leq r < 1} \left|\left(H^{[1]}_{r,n} f\right)(y)\right|.$$

COROLLARY 3.1. *If* $f \in L^p(\gamma_1)$, $1 < p < \infty$, *then*

i) $\left\|H^{[1]}_{r,n} f\right\|_{L^p(\gamma_1)} \leq C_p \|f\|_{L^p(\gamma_1)}$, *for* $0 \leq r < 1$, *and*

ii) $\left\|H^{[1]}_{*,n} f\right\|_{L^p(\gamma_1)} \leq C_p \|f\|_{L^p(\gamma_1)}$.

PROOF. Immediate, as a consequence of Theorem 3. □

Observe that

$$H^{[1]}_{r,0} = P^{[1]}_r.$$

The d-dimensional version of these operators are defined as:

DEFINITION 7. Let $f \in L^1(\gamma_d)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ a multi-index, we define the operator

$$\left(H^{[1]}_{r,\alpha} f\right)(y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} \int_{\mathbb{R}^d} \left[\prod_{i=1}^{d} h^\circ_{\alpha_i} \left(\frac{|z_i - ry_i|}{(1-r^2)^{1/2}}\right) \cdot \exp\left(-\frac{|z_i - ry_i|^2}{1-r^2}\right)\right] f(z) \mathrm{d}z$$

and its Maximal operator as

$$\left(H^{[d]}_{*,\alpha} f\right)(y) = \sup_{0 \leq r < 1} \left|\left(H^{[d]}_{r,\alpha} f\right)(y)\right|.$$

COROLLARY 3.2. *Let* $f \in L^p(\gamma_d)$, $1 < p < \infty$, *then*

i) $\|H^{[d]}_{r,\alpha} f\|_{L^p(\gamma_d)} \le C_p \|f\|_{L^p(\gamma_d)}$, *for* $0 \le r < 1$, *and*

ii) $\|H^{[d]}_{*,\alpha} f\|_{L^p(\gamma_d)} \le C_p \|f\|_{L^p(\gamma_d)}$.

PROOF. This is an immediate application of the previous result and Minkowski integral inequality. □

2-4. *Generalized Muckenhoupt lemmas*

We are going to use extensively the following generalizations of Lemmas 2 and 4 of [Mu-2]. Lemma 2 will be generalized in two different senses, the first one for $L^1(m_1)$ kernels, that is a sort of Young's inequality for the Gaussian measure γ_1 (see Theorem 4); the second generalization will be for singular integral operators (see Theorem 5). Lemma 4 will be generalized for any power m using a pure absolute value argument (see Theorem 6).

THEOREM 4. *If* $f \in L^p(\gamma_1)$, $1 < p < \infty$, *and* $k \in L^1(m_1)$, *then*

$$\left[\int_{-\infty}^{\infty} \left| \int_{|z-y|<1\wedge\frac{1}{|y|}} k(y-z) f(z) \mathrm{d}z \right|^p e^{-y^2} \mathrm{d}y \right]^{1/p} \le C_p \|k\|_{L^1(m_1)} \|f\|_{L^p(\gamma_1)}.$$

PROOF. Let $\{I_n : I_n = [x_n, \ x_{(n+1)})\}$ be a partition of $\mathbb{R}$ as follows.

Consider the interval $[0, 2]$, the intervals of length 2^{-n+1} between 2^n and 2^{n+1}, $n \in \mathbb{N}$, and the mirror images of these intervals for the negative numbers.

By construction, this partition has the following properties:

i) A compact subset of $\mathbb{R}$ intersects a finite number of the subintervals I_n;

ii) An interval of the partition is not more than twice as long as the adjacent intervals. Furthermore if $y \in I_n$ then $1 \wedge \frac{1}{|y|}$ is not greater than half of the length of the interval;

iii) The ratio $\dfrac{\sup_{I_n} e^{-y^2}}{\inf_{I_n} e^{-y^2}}$ is not more than e^{12}.

Now using this partition we have

$$\int_{-\infty}^{\infty} \left| \int_{|z-y|<1\wedge\frac{1}{|y|}} k(y-z) f(z) \mathrm{d}z \right|^p e^{-y^2} \mathrm{d}y$$

$$\leq C \sum_n e^{-x_n^2} \int_{I_n} \left| \int_{|z-y|<1\wedge\frac{1}{|y|}} k(y-z) f(z) \mathrm{d}z \right|^p \mathrm{d}y,$$

and, by properties of I_n,

$$\int_{I_n} \left| \int_{|z-y|<1\wedge\frac{1}{|y|}} k(y-z) f(z) \mathrm{d}z \right|^p \mathrm{d}y$$

will be the same if f were 0 outside of $J_n = I_{n-1} \cup I_n \cup I_{n+1}$. Then, by Young's inequality and by property iii) of the partition, we get that right hand side of the previous inequality is bounded by

$$C \sum_n e^{-x_n^2} \|k\|^p_{L^1(m_1)} \left(\int_{J_n} |f(z)|^p \mathrm{d}z \right)$$

$$\leq C \sum_n \|k\|^p_{L^1(m_1)} \left(\int_{J_n} |f(z)|^p e^{-z^2} \mathrm{d}z \right),$$

and the left hand side is then bounded by

$$C \|k\|^p_{L^1(m_1)} \left(\int_{-\infty}^{\infty} |f(z)|^p e^{-z^2} \mathrm{d}z \right),$$

and this ends the proof. □

COROLLARY 4.1. *Let $f \in L^p(\gamma_1)$, $1 < p < \infty$, then for any $m \geq 0$*

$$\left[\int_{-\infty}^{\infty} \left| \int_{|z-y|<1\wedge\frac{1}{|y|}} \left(\frac{|z-y|}{\alpha^{1/2}} \right)^m \exp\left(-\frac{|z-y|^2}{\alpha} \right) f(z) \mathrm{d}z \right|^p e^{-y^2} \mathrm{d}y \right]^{1/p}$$

$$\leq C_p \alpha^{1/2} \|f\|_{L^p(\gamma_1)},$$

for any constant $\alpha > 0$.

PROOF. Immediate. □

DEFINITION 8. For $f \in L^p(\gamma_d)$, $1 < p < \infty$, and k is a Calderón-Zygmund kernel, that is

i) k is homogeneous of degree -n, i.e. $k(\lambda x) = \lambda^{-n}k(x), \lambda > 0$ and $x \neq 0$,

ii) k belongs to $C^1(\mathbb{R}^d - \{0\})$, and

iv) k has mean value zero over the unit sphere:

$$\int_{S^{d-1}} k(\sigma)\mathrm{d}\sigma = 0,$$

define the operator

$$\left(K^T f\right)(y) = \int_A k(y-z)f(z)\mathrm{d}z,$$

where $A = \left\{z = (z_1, \cdots, z_d) : |z_i - y_i| < 1 \wedge \frac{1}{|y_i|},\ i = 1, \cdots, d\right\}$.

We have that K^T is an $L^p(\gamma_d)$-continuous operation, that is

THEOREM 5. *If $f \in L^p(\gamma_d)$, $1 < p < \infty$, then*

$$\|K^T f\|_{L^p(\gamma_d)} \leq C_p \|f\|_{L^p(\gamma_d)},$$

therefore, any Calderón-Zygmund singular operator truncated over A is strongly $L^p(\gamma_d)$-continuous for $1 < p < \infty$.

PROOF. Using the same partition of the previous Theorem in each variable

$$\left\{I_n^i : I_n^i = [x_{i,n}, x_{i,(n+1)}]\right\},\ i = 1, 2, \cdots, d,$$

we can write

$$\int_{\mathbb{R}^d} \left|(K^T f)(y)\right|^p e^{-|y|^2}\mathrm{d}y$$

$$= \sum_{\alpha \in Z_+^d} \int_{\prod_{i=1}^d I_{\alpha_i}^i} \left|\int_A k(y-z)f(z)\mathrm{d}z\right|^p e^{-|y|^2}\mathrm{d}y$$

$$\leq C \sum_{\alpha \in Z_+^d} \exp\left(-\sum_{i=1}^d x_{i,\alpha_i}^2\right) \int_{\prod_{i=1}^d I_{\alpha_i}^i} \left|\int_A k(y-z)f(z)\mathrm{d}z\right|^p \mathrm{d}y$$

and the last integral would be the same if f were zero outside of

$$\prod_{i=1}^d J_{\alpha_i}^i, \text{ where } J_{\alpha_i}^i = I_{\alpha_{i-1}}^i \cup I_{\alpha_i}^i \cup I_{\alpha_{i+1}}^i.$$

Now using the $L^d(m_d)$-continuity of the singular operation, the last expression is bounded by

$$C \sum_{\alpha \in Z_+^d} \exp\left(-\sum_{i=1}^{d} x_{i,\alpha_i}^2\right) \int_{\prod_{i=1}^{d} J_{\alpha_i}^i} |f(y)|^p \mathrm{d}y$$

and, by the properties of the partitions $\{I_n^i\}$, this is bounded by

$$C_{d,p} \sum_{\alpha \in Z_+^d} \int_{\prod_{i=1}^{d} J_{\alpha_i}^i} |f(y)|^p e^{-|y|^2} \mathrm{d}y \leq C_{d,p} \left[\int_{\mathbb{R}^d} |f(y)|^p e^{-|y|^2} \mathrm{d}y \right]. \qquad \square$$

Finally let us see the generalization of the Lemma 4 in [Mu-2].

DEFINITION 9. Let $f \in L^p(\gamma_1)$, $1 < p < \infty$, we define the operator

$$(L_m f)(y) = \int_{|z-y|>1 \wedge \frac{1}{|y|}} \int_0^1 \varphi(r) \frac{|z-ry|^m}{(1-r^2)^{(m+3)/2}} \cdot \exp\left(\frac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right) \mathrm{d}r\, f(z) e^{-z^2} \mathrm{d}z,$$

where φ is a bounded function on [0,1] and $m \geq 0$.

Observe that this can be written also as

$$(L_m f)(y) = \int_{|z-y|>1 \wedge \frac{1}{|y|}} \int_0^1 \varphi(r) \frac{|z-ry|^m}{(1-r^2)^{(m+3)/2}} \cdot \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) \mathrm{d}r\, f(z) \mathrm{d}z.$$

THEOREM 6. *If $f \in L^p(\gamma_1)$, $1 < p < \infty$, then*

$$||L_m f||_{L_p(\gamma_1)} \leq C_p ||f||_{L^p(\gamma_1)}.$$

PROOF. By the form of the kernel, we may assume without loss of generality that $y > 0$; and for a further simplification, we will work, most of the time, only with the integral over r.

Let us divide the proof in five cases depending upon where z is located with respect to y.

Case 1. $z < [0 \wedge (y-1)]$.

$m = 0$:

in this case we split the integral in r into the sum of integrals over $[0, 1/2]$ and $[1/2, 1]$.

The first one of these integrals is trivially bounded.

For the second integral let us consider two cases:

If $y > 1/2$, the integral is bounded in absolute value by

$$C \int_{1/2}^{1} \frac{1}{(1-r)^{3/2}} \exp\left(-\frac{y^2}{8(1-r)}\right) dr \leq \frac{C}{y} \int_{1/2}^{\infty} \frac{e^{-u}}{u^{1/2}} du = \frac{C}{y} \leq C.$$

If $0 < y < 1/2$, the integral is bounded in absolute value by

$$C \int_{1/2}^{1} \frac{1}{(1-r)^{3/2}} \exp\left(-\frac{z^2}{8(1-r)}\right) dr \leq \frac{C}{(-z)} \int_{1/2}^{\infty} \frac{e^{-u}}{u^{1/2}} du = \frac{C}{(-z)} \leq C,$$

since $-z \geq 1/2$.

Therefore

$$\int_{0}^{1} \varphi(r) \frac{1}{(1-r^2)^{3/2}} \exp\left(\frac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right) dr$$

is bounded using Corollary 1.2 i) and vi), in absolute value, by a Natanson kernel.

$m \geq 1$: here we consider two cases again:

$y > 1/2$.

Since $(|z-ry|)^m = ((-z)+ry)^m \leq 2^m[(-z)^m + (ry)^m]$ then in this case the integral is bounded, in absolute value, by

$$C_m \int_{0}^{1} \frac{(-z)^m}{(1-r)^{(m+3)/2}} \exp\left(\frac{ryz}{1-r}\right) dr + C_m \int_{0}^{1} \frac{(ry)^m}{(1-r)^{(m+3)/2}} \cdot \exp\left(-\frac{r^2y^2}{2(1-r)}\right) dr.$$

The second of these integrals is easy, since it is bounded by

$$\frac{C_m}{y} \int_{0}^{1} \frac{(ry)^{m-1}}{(1-r)^{(m-1)/2}} \left[\frac{(2-r)ry^2}{(1-r)^2} \exp\left(-\frac{r^2y^2}{4(1-r)}\right)\right] \exp\left(-\frac{r^2y^2}{4(1-r)}\right) dr$$

and using that $x^{m-1}e^{-x^2} \leq C_m$, for $x > 0$, and integrating in r, we get that this is bounded by $\frac{C_m}{y}$. Therefore, by Corollary 1.2 i), we conclude that the second integral is bounded by a Natanson kernel.

The first integral is split into the sum of integrals over $[0, 1/2]$ and $[1/2, 1]$. Now for the integral over $[0, 1/2]$, $1/2 \leq 1 - r \leq 1$, and then replacing it, as required, and replacing $(1 + r)$ by 1 or 2, as needed, we get the upper bound $C_m(-z)^m$. Therefore, as the part of the operator corresponding to this case is

$$C_m \int_{-\infty}^{0\wedge(y-1)} \int_0^{1/2} \frac{(-z)^m}{(1-r)^{(m+3)/2}} \exp\left(\frac{ryz}{1-r}\right) dr f(z) e^{-z^2} dz,$$

then it is bounded, using Hölder's inequality, by

$$C_m \left(\int_{-\infty}^{\infty} |z|^{mq} e^{-z^2} dz \right)^{1/q} \|f\|_{L^p(\gamma_1)}.$$

The integral over $[1/2, 1]$ can be written as

$$\frac{(-z)^{m-1}}{y} \int_{1/2}^{1} \left[\frac{(-z)y}{(1-r)^2} \exp\left(\frac{ryz}{2(1-r)}\right) \right] \frac{1}{(1-r)^{(m-1)/2}} \exp\left(\frac{ryz}{2(1-r)}\right) dr,$$

which is, by using the inequality $x^{m-1}e^{-x} \leq C_m$, for $x > 0$, $m > 1$, and integrating in r, bounded by $C_m \exp\left(\frac{yz}{2}\right)$, as $y > 1/2, z < 0$ and $1/2 \leq r \leq 1$; but this is bounded by a constant and therefore bounded by a Natanson kernel by Corollary 1.2 i) and iv).

$0 < y < 1/2$.

Then in this case $z < -1/2, |z - ry| < 2(-z)(2 - r)$, and the integral in r is bounded by the sum

$$C_m \int_0^{1/2} \frac{(-2z)^m (2-r)^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{r^2 z^2}{2(1-r)}\right) dr$$

$$+C_m \int_{1/2}^{1} \frac{(-2z)^m (2-r)^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{r^2 z^2}{2(1-r)}\right) dr.$$

The first integral is trivially bounded by $C_m(-z)^m$, so again, by Hölder's inequality, the part of the operator corresponding to this case is bounded by

$$C_{m,p} \|f\|_{L^p(\gamma_1)}.$$

The second integral is bounded by

$$\frac{C_m}{(-z)}\int_{1/2}^{1}\frac{(-z)^{m-1}(2-r)^{m-1}}{(1-r)^{(m-1)/2}}\;\frac{1}{r}\left[\frac{z^2r(2-r)}{(1-r)^2}\exp\left(-\frac{r^2z^2}{4(1-r)}\right)\right]$$

$$\cdot\exp\left(-\frac{r^2z^2}{4(1-r)}\right)dr$$

and this is bounded by $\frac{C_m}{-z}$ by using the inequality that $x^{m-1}e^{-x}\leq C_m$, for $x>0,\ m>1$ and integrating on r. This is bounded by a constant in this range and therefore bounded by a Natanson kernel by Corollary 1.2 vi).

Case 2. $0<z<\left[\left(y-\frac{1}{y}\right)\wedge\frac{y}{2}\right]$.

In this case $y>1$ and we will work the two cases $m=0$ and $m\geq 1$ simultaneously. Let us use the second representation of the operator; replace $1+r$ by 1 or 2 as required and split the integral into the sum of integrals over $\left[0,1-\frac{y-z}{2y}\right]$ and $\left[1-\frac{y-z}{2y},1\right]$.

For the first integral, since $z<\frac{y}{2}$ then $r\leq\frac{3}{4}$ and therefore $\frac{1}{4}\leq 1-r\leq 1$. Using this to replace $1-r$ and the change of variables $u=z-ry$, it is easy to see that the integral is bounded, in absolute value, by

$$C\frac{1}{y-z}\left(\int_{-\infty}^{\infty}|u|^m e^{-u^2/2}du\right)$$

and therefore the corresponding part of the operator, for this case, is bounded by

$$C_m\frac{1}{y}\int_0^{\left[\left(y-\frac{1}{y}\right)\wedge\frac{y}{2}\right]}f(z)dz$$

which is bounded by $\left(M_*^{[1]}f\right)(y)$, by Corollary 1.2 ii).

For the second integral, using the fact that $z-ry=(z-y)+(1-r)y$, we get that $\frac{y-z}{2}\leq|z-ry|\leq 2(y-z)$ and, using this and the change of variable $u=\frac{|z-y|^2}{8(1-r)}$, we have the second integral bounded by

$$\frac{C_m}{y}\left[\int_{1/8}^{\infty}u^{(m-1)/2}e^{-u^2}du\right]$$

and we repeat the argument given for the above integral.

Case 3. $y + \frac{1}{y} < z < 2y$.

In this case $y > 1$, again we will work $m = 0$ and $m \geq 1$ simultaneously. Here we can write the corresponding part of the operator as

$$\int_{y+\frac{1}{y}}^{2y} e^{y^2} \int_0^1 \varphi(r) \frac{(z - ry)^m}{(1 - r^2)^{(m+3)/2}} \exp\left(-\frac{|y - rz|^2}{1 - r^2}\right) \mathrm{d}r f(z) e^{-z^2} \mathrm{d}z.$$

Now split the integral in r into the sum of integrals over $\left[0, 1 - \frac{2(z - y)}{z}\right]$, $\left[1 - \frac{2(z - y)}{z}, 1 - \frac{z - y}{2z}\right]$ and $\left[1 - \frac{z - y}{2z}, 1\right]$.

For the first one of these integrals we have $|z - ry| \leq 3y(1 - r)$ and $|y - rz| \geq \frac{z(1 - r)}{2}$ and using this we get that this integral is bounded, in absolute value, by

$$\int_0^{1-\frac{2(z-y)}{z}} y^m (1 - r)^{(m-3)/2} \exp\left(-\frac{z^2(1 - r)}{8}\right) \mathrm{d}r,$$

and now, taking the change of variable $u = \frac{z^2(1 - r)}{8}$, we obtain the bound

$$C_m \left(\frac{y}{z}\right)^m z \int_{1/4}^{\infty} u^{(m-3)/2} e^{-u} \mathrm{d}u,$$

and therefore the first integral in r is bounded by $C_m y$, since z is equivalent to y.

This means we have

$$\int_{y+\frac{1}{y}}^{2y} e^{y^2} \int_0^{1-\frac{2(z-y)}{z}} \varphi(r) \frac{(z - ry)^m}{(1 - r)^{(m+3)/2}}$$

$$\exp\left(-\frac{|y - rz|^2}{8(1 - r)}\right) \mathrm{d}r f(z) e^{-z^2} \mathrm{d}z \leq C_m \int_{\frac{y+1}{y}}^{2y} e^{y^2} y e^{-z^2} f(z) \mathrm{d}z.$$

To this expression we can apply now the Corollary 1.2 iii).

For the second integral in r, using that $|z - ry| \leq 3(z - y)$ holds and that trivially $|y - rz| \geq 0$, we obtain the upper bound

$$\int_{1-\frac{2(z-y)}{z}}^{1-\frac{z-y}{2z}} \frac{1}{(1-r)^{(m+3)/2}}(z-y)^m \mathrm{d}r,$$

which gives immediately that the integrated less than

$$C_m z^{(m+1)/2}(z-y)^{(m-1)/2}.$$

For the cases $m = 0$ or 1, using that z is equivalent to y, we can use immediately Corollary 1.2 iii). But, for the case $m > 1$, we need to work with the whole expression and use a little trick. In this case the expression

$$C_m y^{(m+1)/2} \int_{y+\frac{1}{y}}^{2y} (z-y)^{(m-1)/2} e^{-(z^2-y^2)} f(z) \mathrm{d}z$$

is bounded, from Hölder's inequality for $1 < p_0 < p$, $\frac{1}{p_0} + \frac{1}{q_0} = 1$, and the change of variable $u = (z - y)$, by

$$C_m y^{(m+1)/2} \left[e^{y^2} \int_{y+\frac{1}{y}}^{2y} |f(z)|^{p_0} e^{-z^2} \mathrm{d}z \right]^{1/p_0} \left[\int_{\frac{1}{y}}^{y} e^{-2yu} u^{q_0(m-1)/2} \mathrm{d}u \right]^{1/q_0}.$$

Now using the change of variables $v = yu$ we get the bound

$$C_m \left[y e^{y^2} \int_{y+\frac{1}{y}}^{2y} |d(z)|^{p_0} e^{-z^2} \mathrm{d}z \right]^{1/p_0}$$

and, by Corollary 1.2 iii), this is bounded by $C_m \left[\left(M_*^{[1]} f^{p_0} \right)(y) \right]^{1/p_0}$. When we take $L^p(\gamma_1)$-norm in y, this will give us $C_m \|f\|_{L^p(\gamma_1)}$.

Finally for the third integral we have $|z - ry| \leq 2(z-y)$ and $|y - rz| \geq \frac{z-y}{2}$. Then the integral in r is bounded in absolute value by

$$C_m \int_{1-\frac{z-y}{2z}}^{1} \frac{(z-y)^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{(z-y)^2}{8(1-r)} \right) \mathrm{d}r$$

and now the change of variables $u = \frac{(z-y)^2}{8(1-r)}$ gives us

$$C_m \left[\int_{1/4}^{\infty} u^{(m-1)/2} e^{-u} du \right] \frac{1}{(z-y)}.$$

Thus we can use again Corollary 1.2 iii) in this case.

Case 4. $z \geq [2y \vee (y+1)]$.

Again in this case we will work $m = 0$ and $m \geq 1$ simultaneously. As in the previous case, we write the corresponding part of the operator as

$$\int_{[2y \vee (y+1)]}^{\infty} e^{y^2} \int_0^1 \varphi(r) \frac{(z-ry)^m}{(1-r^2)^{(m+3)/2}} \exp\left(-\frac{|y-rz|^2}{1-r^2}\right) dr f(z) e^{-z^2} dz.$$

Let us split the integral in r into the sum of integrals over $\left[0, 1 - \frac{(z-y)}{2z}\right]$ and $\left[1 - \frac{(z-y)}{2z}, 1\right]$.

In the first one of the integrals with respect to r, as $z \geq 2y$, $r \leq \frac{3}{4}$ and so $\frac{1}{4} \leq 1 - r \leq 1$ and $|z - ry| \leq z$. Thus replacing $1 - r$ by the appropriate bound, replacing $(1+r)$ by 1 or 2 as needed and using the change of variable $u = y - rz$, we obtain that this integral is bounded, in absolute value, by $C_m z^{(m-1)}$. Hence, for the cases $m = 0$ or $m = 1$, we can use immediately Corollary 1.2 iv) (as $z > 1$). But, for the case $m > 1$, we need to work with the whole expression and use again the little trick. Thus using Hölder's inequality, with $1 < p_0 < p$, $\frac{1}{p_0} + \frac{1}{q_0} = 1$, we get the upper bound

$$C_m \left[e^{y^2} \int_{[2y \vee (y+1)]}^{\infty} z^{(m-1)q_0} e^{-z^2} dz \right]^{1/q_0} \left[e^{y^2} \int_{[2y \vee (y+1)]}^{\infty} |f(z)|^{p_0} e^{-z^2} dz \right]^{1/p_0}$$

and this is bounded by $C_m \left[\left(M_*^{[1]} f^{p_0} \right)(y) \right]^{1/p_0}$, by the Corollary 1.2 iv).

Now for the second integral we have that $|z - ry| < \frac{3}{2}(z - y)$ and also $|y - rz| > \frac{z-y}{2}$, so using this in the integral in r and replacing $(1+r)$ by 1 or 2 as needed we have that its absolute value is less than

$$C_m \int_{1-\frac{z-y}{2z}}^{1} \frac{(z-y)^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{|z-y|^2}{8(1-r)}\right) dr.$$

Using the change of variable $u = \frac{|z-y|^2}{8(1-r)}$, we get that this is less than

$$\frac{z}{(z-y)}\left[\int_{1/4}^{\infty} u^{(m-1)/2} e^{-u} du\right]$$

and as $(z-y) > 1$ and $\frac{z}{z-y} < 2$ we can apply again Corollary 1.2 iv).

Case 5. $\frac{y}{2} \le z \le y - \frac{1}{y}$.

In this case $y \ge \sqrt{2}$, and again we will work $m = 0$ and $m \ge 1$ simultaneously. We use the following representation of the corresponding part of the operator:

$$\int_{\frac{y}{2}}^{y-\frac{1}{y}} \int_0^1 \varphi(r) \frac{(z-ry)^m}{(1-r^2)^{(m+3)/2}} \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) dr f(z) dz.$$

Let us split the integral in r into the sum of integrals over $\left[0, 1-\frac{3(y-z)}{2y}\right]$, $\left[1-\frac{3(y-z)}{2y},\ 1-\frac{y-z}{2y}\right]$ and $\left[1-\frac{y-z}{2y}, 1\right]$.

In the first integral with respect to r, use that

$$y\frac{1-r}{3} \le |z-ry| \le 2(1-r)y$$

and replace $(1+r)$ by 1 or 2 as needed to bound this integral by

$$C_m y^m \int_0^{1-\frac{3(y-z)}{2y}} (1-r)^{(m-3)/2} \exp\left(-\frac{y^2}{18}(1-r)\right) dr.$$

Making the substitution $s = y^2(1-r)$, the above integral becomes

$$C_m y \int_{3(y-z)y}^{y^2} s^{(m+2-5)/2} e^{s/18} ds$$

and using the inequality $x^{(m+2)/2} e^{-x} \le C_m$, for $x > 0$, $m \ge 0$, this is bounded by

$$C_m \frac{1}{y^{1/2}(y-z)^{3/2}}.$$

Now we can use Corollary 1.2 v).

For the third integral we use similar arguments. In this case we have that $\frac{y-z}{2} \leq |z-ry| \leq 2(y-z)$ and we use this to substitute $|z-ry|$, again replace $(1+r)$ by 1 or 2 as convenient and we obtain the following upper bound for the integral in r:

$$\int_{1-\frac{y-z}{2y}}^{1} \frac{|z-y|^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{|z-y|^2}{8(1-r)}\right) dr;$$

then using the change of variable $u = \frac{|z-y|^2}{8(1-r)}$ and the inequality $x^{(m+4)/2}e^{-x} \leq C_m$, for $x > 0$, $m \geq 0$, give us the bound $\frac{C_m}{y^{1/2}(y-z)^{3/2}}$ and therefore we can apply again the Corollary 1.2 v).

Finally the second integral is a trickly one. We will need to work with the whole expression corresponding to this case, which is

$$\int_{\frac{y}{2}}^{y-\frac{1}{y}} e^{z^2} \int_{1-\frac{3(y-z)}{2y}}^{1-\frac{y-2}{2y}} \varphi(r) \frac{(z-ry)^m}{(1-r)^{(m+3)/2}} \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) dr f(z) e^{z^2} dz.$$

What we want is to compute the $L^p(\gamma_1)$ norm of this expression and for that let us consider $g \in L^q(\gamma_1)$ with $\|g\|_{L^1(\gamma_1)} \leq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let us look at

$$\left| \int_{\sqrt{2}}^{\infty} g(y) \int_{\frac{y}{2}}^{y-\frac{1}{y}} e^{z^2} \int_{1-\frac{3(y-z)}{2y}}^{1-\frac{y-z}{2y}} \varphi(r) \frac{(z-ry)^m}{(1-r)^{(m+3)/2}} \right.$$

$$\left. \cdot \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) dr f(z) e^{-z^2} dz e^{-y^2} dy \right|.$$

Then taking the absolute value inside and using Fubini's theorem, it is easy to see that this is bounded by

$$C \int_{-\infty}^{\infty} e^{z^2} \int_{z+\frac{1}{2z}}^{2z} |g(y)| \int_{1-\frac{3(y-z)}{2y}}^{1-\frac{y-z}{2y}} \frac{|z-ry|^m}{(1-r)^{(m+3)/2}}$$

$$\cdot \exp\left(-\frac{|z-ry|^2}{1-r^2}\right) dr e^{-y^2} dy |f(z)| e^{-z^2} dz,$$

since $\frac{y-z}{2y} \leq 1-r \leq \frac{3(y-z)}{2y}$, replacing $1-r$ by the appropriate bound, and using the change of variable $u = (z-ry)\sqrt{\frac{y}{3(y-z)}}$, we have that the integral in

r can be estimated from above by

$$\frac{C}{y-z}\int_{-\frac{\sqrt{y(y-z)}}{6}}^{\frac{\sqrt{y(y-z)}}{6}} |u|^m e^{-u^2}\mathrm{d}u \leq Cz, \text{ since } \frac{y}{2} < z < y - \frac{1}{y}.$$

The kernel

$$K(z,y) = \begin{cases} ze^{z^2}, & \text{if } z + \frac{1}{2z} \leq y \leq 2z \\ 0, & \text{otherwise} \end{cases}$$

is bounded by a Natanson kernel using the same argument given for K_3 in the Corollary 1.2 iii) (with the roles of y and z interchanged). Therefore, the whole expression is then bounded by

$$C\int_{-\infty}^{\infty} \left(M_*^{[1]}g\right)(z)|f(z)|e^{-z^2}\mathrm{d}z$$

and the Hölder's inequality and the $L^p(\gamma_1)$-continuity of $M_*^{[1]}$ give us an upper bound of $C_m\|f\|_{L^p(\gamma_1)}$.

Now it is easy to observe that the five cases considered include all values of z for which $|z-y| \geq 1 \wedge \frac{1}{y}$. □

DEFINITION 10. If $f \in L^1(\gamma_1)$, h_n is the Hermite polynomial of order n, h_n° its majorant, then we can define the operator

$$(N_n f)(y) = \int_{|z-y|>1\wedge\frac{1}{|y|}} \int_0^1 \frac{\varphi(r)}{(1-r^2)^{3/2}} h_n^\circ\left(\frac{|z-ry|}{(1-r^2)^{1/2}}\right) \cdot \exp\left(\frac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right)\mathrm{d}r\, f(z)e^{-z^2}\mathrm{d}z$$

$$= \int_{|z-y|>1\wedge\frac{1}{|y|}} \int_0^1 \varphi(r)\frac{1}{(1-r^2)^{3/2}} h_n^\circ\left(\frac{z-ry}{(1-r^2)}\right) \cdot \exp\left(-\frac{|z-ry|^2}{1-r^2}\right)\mathrm{d}r\, f(z)\mathrm{d}z,$$

where φ is a bounded function on $[0,1]$.

Then as a consequence of the Theorem 6, we have:

COROLLARY 6.1. *If* $f \in L^p(\gamma_1)$, $1 < p < \infty$, *then*

$$\|N_n f\|_{L^p(\gamma_1)} \leq C_p\|f\|_{L^p(\gamma_1)}.$$

PROOF. Immediate. □

Finally observe that $N_0 = L_0$ and $N_1 = L_1$.

3. - Proof of the main result

We are going to prove Theorem 7, that is the $L^p(\gamma_p)$-continuity of what we call the Riesz Transform of order $\alpha, \mathrm{D}^\alpha(-L)^{-|\alpha|/2}$, associated to the Ornstein-Uhlenbeck operator L, which is the main result of our work. As a consequence of the techniques used in its proof, we can obtain a whole class of singular operators with respect to the Gaussian measure, which is the result given in Theorem 8.

3-1. *Proof of Theorem 7*

Let us remember that the Riesz Transform of order α associated to the Ornstein-Uhlenbeck operator L is defined as:

$$\left(\mathrm{D}^\alpha(-L)^{-|\alpha|/2}f\right)(y)$$
$$= C_{d,\alpha}\int_{\mathbb{R}^d}\int_0^1 r^{(|\alpha|-1)}\left(\frac{-\log r}{1-r^2}\right)^{(|\alpha|-2)/2}\frac{1}{1-r^2}h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)$$
$$\cdot\frac{\exp\left(\dfrac{-r^2(|y|^2+|z|^2)+2ry\cdot z}{1-r^2}\right)}{(1-r^2)^{d/2}}\mathrm{d}r\, f(z)e^{-|z|^2}\mathrm{d}z,$$

where $f\in L^p(\gamma_d)$.

We observe that this can be rewritten as

$$\left(\mathrm{D}^\alpha(-L)^{-|\alpha|/2}f\right)(y)$$
$$= C_{d,\alpha}\int_{\mathbb{R}^d}\int_0^1 r^{(|\alpha|-1)}\left(\frac{-\log r}{1-r^2}\right)^{(|\alpha|-2)/2}\frac{1}{1-r^2}h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)$$
$$\cdot\frac{\exp\left(-\dfrac{|z-ry|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\cdot\mathrm{d}r\, f(z)\mathrm{d}z.$$

First of all observe that if we define, for any $i=1,2,\cdots,d$, the operation

$$(\tau_i f)(z_1,z_2,\cdots,z_d) = f(z_1,\cdots,z_{i-1},-z_i,z_{i+1},\cdots,z_d),$$

then it is easy to see that, due to the properties of the Hermite Polynomials, we have

$$\tau_i(\mathrm{D}^{\alpha}(-L)^{-|\alpha|/2}f) = (-1)^{\sigma}[\mathrm{D}^{\alpha}(-L)^{-|\alpha|/2}(\tau_i f)],$$

where σ depends on α and i; and therefore it is enough to work for

$$y = (y_1, y_2, \cdots, y_d), \text{ where } y_1 \geq 0, y_2 \geq 0, \cdots, y_d \geq 0.$$

A second remark is that, as can be easily proved, for any α multi-index, the function

$$\varphi(r) = r^{(|\alpha|-1)}\left(\frac{-\log r}{1-r^2}\right)^{(|\alpha|-2)/2}$$

is a bounded, increasing function in $[0,1]$ and such that $\int_0^1 \frac{|\varphi(r)-\varphi(1)|}{1-r}\mathrm{d}r < \infty$.

Now for a $y = (y_1, y_2, \cdots, y_d)$ fixed, we can write $\mathbb{R}^d$ as a disjoint union $A \cup (\cup B_k)$, where $A = \left\{z = (z_1, \cdots, z_d) : |z_i - y_i| < 1 \wedge \frac{1}{|y_i|},\ i = 1, \cdots d\right\}$ and the B_k's are such that, for at least one coordinate z_1, we have $|z_i - y_i| \geq 1 \wedge \frac{1}{|y_i|}$.

In order to prove the $L^p(\gamma_d)$-continuity of our operator we are going to divide the proof in two cases.

Case 1.

In this case we are going to work the corresponding part of the operator over a given B_k. This is going to be a pure absolute value argument and for that reason we may assume without loss of generality that there exists one and only one coordinate z_i such that $|z_i - y_i| \geq 1 \wedge \frac{1}{|y_i|}$.

In this case the operator is bounded in absolute value by

$$\int_{|z_i-y_i|\geq 1\wedge\frac{1}{y_i}} \int_0^1 r^{(|\alpha|-1)}\left(\frac{-\log r}{1-r^2}\right)^{(|\alpha|-2)/2} \frac{1}{(1-r^2)^{3/2}} h_{\alpha_i}\left(\frac{|z_i - ry_i|}{(1-r^2)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_i - ry_i|^2}{1-r^2}\right) \int_{\mathbb{R}^{d-1}} \prod_{j\neq i} h^{\circ}_{\alpha_j}\left(\frac{|z_j - ry_j|}{(1-r^2)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_j - ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}|f(z_i, z^{i})|\mathrm{d}z^{i}\mathrm{d}r\mathrm{d}z_i,$$

and this is bounded by

$$\int_{|z_i-y_i|\geq 1\wedge\frac{1}{y_i}} \int_0^1 r^{(|\alpha|-1)}\left(\frac{-\log r}{1-r^2}\right)^{(|\alpha|-2)/2} \frac{1}{(1-r^2)}^{3/2} h^{\circ}_{\alpha_i}\left(\frac{|z_i - ry_i|}{(1-r^2)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_j - ry_j|^2}{1-r^2}\right) \frac{1}{(1-r^2)^{1/2}} \left(H^{[d-1]}_{*,\alpha^i}|f|(z_i,\cdot)\right)(y^i)\mathrm{d}z_i.$$

But this is just $\left[N_{\alpha_i}\left(H^{[d-1]}_{*,\alpha^i}|f(\cdot,y^i)|\right)\right](y_i)$ and then the result in this case follows by Corollaries 3.2 and 6.1 and the Minkowski integral inequality.

Case 2.

In this case we are going to work the corresponding part of the operator over A. We are going to make a series of reductions using standard techniques, mainly the Mean Value Theorem, until we get a singular part.

Without loss of generality we may assume that $\alpha_1 > 0$. Then we start by dividing the integral in r into the sum of integrals over the intersections of $[0,1]$ with the intervals $\left(-\infty, 1 - \frac{|z_1 - y_1|}{2y_1}\right]$ and $\left(1 - \frac{|z_1-y_1|}{2y_1}, 1\right]$ and therefore we decompose the operator into the sum of two corresponding terms.

We can bound the absolute value of the first term by

$$C \int\limits_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int\limits_0^{1-\frac{|z_1-y_1|}{2y_1}} \frac{1}{(1-r^2)^{3/2}} h^{\circ}_{\alpha_1}\left(\frac{|z_1-ry_1|}{(1-r^2)^{1/2}}\right) \exp\left(-\frac{|z_1-ry_1|^2}{1-r^2}\right)$$

$$\cdot \int\limits_{\mathbb{R}^{d-1}} \prod_{j>1} h^{\circ}_{\alpha_j}\left(\frac{|z_j-ry_j|}{(1-r^2)^{1/2}}\right) \exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right) \frac{1}{(1-r^2)^{1/2}} |f(z_1,z^1)|\mathrm{d}z^1\mathrm{d}r\mathrm{d}z_1$$

and this is less than

$$C \int\limits_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int\limits_0^{1-\frac{|z_1-y_1|}{2y_1}} \frac{1}{(1-r^2)^{3/2}} h^{\circ}_{\alpha_1}\left(\frac{|z_1-ry_1|}{(1-r^2)^{1/2}}\right) \exp\left(-\frac{|z_1-ry_1|^2}{1-r^2}\right)\mathrm{d}r$$

$$\cdot \left(H^{[d-1]}_{*,\alpha^1}|f|(z_1,\cdot)\right)(y^1)\mathrm{d}z_1.$$

Since $x^n e^{-x^2} \le C$, if $x>0$, $n>0$, and replacing $1+r$ by 1 or 2 as convenient, this last expression is bounded by

$$C \int\limits_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int\limits_0^{1-\frac{|z_1-y_1|}{2y_1}} \frac{1}{(1-r^2)^{3/2}}\mathrm{d}r \left(H^{[d-1]}_{*,\alpha^1}|f|(z_1,\cdot)\right)(y^1)\mathrm{d}z_1.$$

We integrate in r and use Corollary 1.2 viii) and the Definition 2 to obtain as an upper bound

$$C\left[M^{[1]}_T\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1) + C\left[M^{[1]}_*\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1).$$

Now we have to deal with the second term, which corresponds to the integration in r over $(1-\frac{|z_1-ry_1|}{2y_1},1]$. In this case we cannot use simply an absolute value argument as before, instead we will make a series of reductions. First, we will eliminate the function φ. To do this let us write it as

$$C_{d,\alpha}\varphi(1)\int_A\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\frac{1}{1-r^2}h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)\cdot\exp\left(-\frac{|z-ry|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{d/2}}\mathrm{d}rf(z)\mathrm{d}z$$
$$+C_{d,\alpha}\int_A\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}[\varphi(r)-\varphi(1)]\frac{1}{1-r^2}h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)\cdot\exp\left(-\frac{|z-ry|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{d/2}}\mathrm{d}rf(z)\mathrm{d}z.$$

Now using the properties of φ, we obtain that the second integral is bounded by

$$C\left(H^{[d]}_{*,\alpha}|f|\right)(y).$$

The first integral can be written as

$$C\int_A\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\frac{1}{(1-r^2)^{3/2}}h_{\alpha_1}\left(\frac{z_1-y_1}{\sqrt{2}(1-r)^{1/2}}\right)\cdot\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right)\prod_{j>1}h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}rf(z)\mathrm{d}z$$
$$+C\int_A\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\left[\frac{1}{(1-r^2)^{3/2}}h_{\alpha_1}\left(\frac{z_1-ry_1}{(1-r^2)^{1/2}}\right)\cdot\exp\left(-\frac{|z_1-ry_1|^2}{1-r^2}\right)-\frac{1}{2^{3/2}(1-r)^{3/2}}h_{\alpha_1}\left(\frac{z_1-y_1}{\sqrt{2}(1-r)^{1/2}}\right)\cdot\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right)\right]\prod_{j>1}h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}rf(z)\mathrm{d}z.$$

Using the Mean Value Theorem the integrand in the second term is bounded in absolute value by

$$C\left\{\left[\frac{y_1}{(1-r)}+\frac{|z_1-sy_1|}{(1-r)}\right]h^{\circ}_{\alpha_1-1}\left(\frac{|z_1-sy_1|}{(1-r)^{1/2}}\right)\right.$$
$$\left.+\left[\frac{|z_1-sy_1|y_1}{(1-r)^{3/2}}+\frac{|z_1-sy_1|^2}{(1-r)^{3/2}}+\frac{1}{(1-r)^{1/2}}\right]h^{\circ}_{\alpha_1}\left(\frac{|z_1-sy_1|}{(1-r)^{1/2}}\right)\right\}$$
$$\cdot\exp\left(-\frac{|z_1-sy_1|^2}{2(1-r)}\right)\cdot\prod_{j>1}h^{\circ}_{\alpha_j}\left(\frac{|z_j-ry_j|}{(1-r^2)^{1/2}}\right)$$
$$\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}|f(z)|$$

for some s such that $r\le s\le 1$.
Observe now that, by using the inequality $x^k e^{-x^2/n}\le C,\ x>0$ and $k\ge 0,\ n=1,2$, and the fact that in A we have

$$\exp\left(-\frac{|z_1-sy_1|^2}{2(1-r)}\right)\le C\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right),$$

the above integral is less than the sum

$$C\int_{|z_1-y_1|<1\wedge\frac{1}{y_1}} y_1\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\frac{1}{(1-r)}$$
$$\cdot\exp\left(-\frac{|z_1-y_1|^2}{4(1-r)}\right)\mathrm{d}r\left(H^{[d-1]}_{*,\alpha^1}|f|(z_1,\cdot)\right)(y^1)\mathrm{d}z_1$$
$$+C\int_{|z_1-y_1|<1\wedge\frac{1}{y_1}}\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\frac{1}{(1-r)^{1/2}}\mathrm{d}r\left(H^{[d]}_{*,\alpha^1}|f|(z_1,\cdot)\right)(y^1)\mathrm{d}z_1.$$

Concerning the first of these integrals, we observe that

$$y_1\int_{1-\frac{|z_1-y_1|}{2y_1}}^{1}\frac{1}{1-r}\exp\left(-\frac{|z_1-y_1|^2}{4(1-r)}\right)\mathrm{d}r\le Cy_1\left[1-\log(|y_1-z_1|y_1)\right].$$

Corollary 1.2 viii) then implies that the first integral is bounded by

$$\left[M^{[1]}_{*}\left(H^{[d-1]}_{*,\alpha}|f|(\cdot,y^1)\right)\right](y_1).$$

For the second integral, integrating in r and using the Corollary 1.2 vii) and the Definition 2, we obtain as an upper bound

$$C\left[M^{[1]}_{T}\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1)+C\left[M^{[1]}_{*}\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1).$$

Now the integral

$$C\int_A \int_{1-\frac{|z_1-y_1|}{2y_1}}^{1} \frac{1}{(1-r)^{3/2}} h_{\alpha_1}\left(\frac{z_1-y_1}{\sqrt{2}(1-r)^{1/2}}\right)\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right)$$

$$\cdot\prod_{j>1} h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}r\, f(z)\mathrm{d}z$$

can be rewritten as the difference of two integrals, the first one with the integral in r over $[0,1]$ and the second one with the integral in over $\left[0, 1-\frac{|z_1-y_1|}{2y_1}\right]$. This last integral can be bounded in absolute value by

$$C\int_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int_0^{1-\frac{|z_1-y_1|}{2y_1}} \frac{1}{(1-r)^{3/2}} h^{\circ}_{\alpha_1}\left(\frac{|z_1-y_1|}{\sqrt{2}(1-r)^{1/2}}\right)\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right)\mathrm{d}r$$

$$\cdot\left(H^{[d-1]}_{*,\alpha^1} f(z_1,\cdot)\right)(y^1)\mathrm{d}z_1.$$

Thus using again the inequality $x^n e^{-x^2}\le C$, if $x>0$ and $n\ge 0$, this is bounded by

$$C\int_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int_0^{1-\frac{|z_1-y_1|}{2y_1}} \frac{1}{(1-r)^{3/2}}\mathrm{d}r\left(H^{[d-1]}_{*,\alpha^1}|f|(z_1,\cdot)\right)(y^1)\mathrm{d}z_1,$$

and so we integrate in r and use Corollary 1.2 viii) and again the Definition 2 to obtain the upper bound

$$C\left[M^{[1]}_T\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1)+C\left[M^{[1]}_*\left(H^{[d-1]}_{*,\alpha^1}|f|(\cdot,y^1)\right)\right](y_1).$$

The first term, which has the integral in r over $[0,1]$, can be rewritten as

$$C\int_A\int_0^1 \frac{1}{(1-r)^{3/2}} h_{\alpha_1}\left(\frac{z_1-y_1}{\sqrt{2}(1-r)^{1/2}}\right)\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right) h_{\alpha_2}\left(\frac{z_2-ry_2}{(1-r^2)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_2-ry_2|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\cdot\prod_{j>2} h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}r\, f(z)\mathrm{d}z$$

and for this integral we repeat the argument just done for the variable z_1 but now for the variable z_2; that is, we divide the integral in r into the sum of integrals over the intersections of $[0,1]$ with the intervals $\left(-\infty, 1-\frac{|z_2-y_2|}{2y_2}\right]$ and $\left(1-\frac{|z_2-y_2|}{2y_1}, 1\right]$ and therefore we decompose the integral into the sum of two terms. In this case, nevertheless, the argument is going to be a little different, in it we will use strongly the Corollary 4.1.

The first term is bounded in absolute value by

$$C \int\limits_{|z_1-y_2|<1\wedge\frac{1}{y_2}} \int\limits_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int\limits_0^{1-\frac{|z_2-y_2|}{2y_2}} \frac{1}{(1-r)^2} h^\circ_{\alpha_1}\left(\frac{|z_1-y_1|}{(1-r)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right) \cdot h^\circ_{\alpha_2}\left(\frac{|z_2-ry_2|}{(1-r)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_2-ry_2|^2}{2(1-r)}\right) \mathrm{d}r \left(H^{[d-2]}_{*,\alpha^{1,2}}|f|(z_1,z_2,\cdot)\right)(y^{1,2})\mathrm{d}z_1\mathrm{d}z_2.$$

Using again the inequality $x^n e^{-x^2} \leq C$, if $x>0$ and $n \geq 0$ for z_2, taking $L^p(\gamma_{d-1})$-norm in $y_1, y_3, \cdots, y_d$ and using the Corollaries 3.1 and 4.1, we get the bound

$$C \int\limits_{|z_2-y_2|<1\wedge\frac{1}{y_2}} \left[\int\limits_0^{1-\frac{|z_2-y_2|}{2y_2}} \frac{\mathrm{d}r}{(1-r)^{3/2}}\right] \left[\int\limits_{\mathbb{R}^{d-1}} |f(z_2,y^2)|^p e^{-|y^2|^2} \mathrm{d}y^2\right]^{1/p} \mathrm{d}z_2.$$

Integrating in r, using the Corollary 1.2 viii) and the Definition 2, we get as an upper bound

$$C\left\{M^{[1]}_*\left[\left(\int\limits_{\mathbb{R}^{d-1}} |f(\cdot,y^2)|^p e^{-|y^2|^2}\mathrm{d}y\right)^{1/p}\right]\right\}(y_2)$$

$$+C\left\{M^{[1]}_T\left[\left(\int\limits_{\mathbb{R}^{d-1}} |f(\cdot,y^2)|^p e^{-|y^2|^2}\mathrm{d}y\right)^{1/p}\right]\right\}(y_2).$$

Now for the second term, which corresponds to the integration in r over

$$\left(1-\frac{|z_2-y_2|}{2y_2}, 1\right],$$

as in the previous case, we cannot use simply an absolute value argument. Again, as before, we will make a series of reductions using the Mean Value

Theorem. We write the second term as

$$C\int_A \int_{1-\frac{|z_2-y_2|}{2y_2}}^{1} \frac{1}{(1-r)}\prod_{i=1}^{2} h_{\alpha_i}\left(\frac{z_i-y_i}{\sqrt{2}(1-r)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_i-y_i|^2}{2(1-r)}\right)\frac{1}{\sqrt{2}(1-r)^{1/2}}\cdot\prod_{j>3} h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}rf(z)\mathrm{d}z$$

$$+C\int_A \int_{1-\frac{|z_2-y_2|}{2y_2}}^{1} \frac{1}{(1-r)^{3/2}}h_{\alpha_1}\left(\frac{z_1-y_1}{\sqrt{2}(1-r)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_1-y_1|^2}{2(1-r)}\right)\cdot\left[h_{\alpha_2}\left(\frac{z_2-ry_2}{(1-r^2)^{1/2}}\right)\right.$$

$$\cdot\exp\left(-\frac{|z_2-ry_2|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}-h_{\alpha_2}\left(\frac{z_2-y_2}{\sqrt{2}(1-r)^{1/2}}\right)$$

$$\left.\cdot\exp\left(\frac{-|z_2-y_2|^2}{2(1-r)}\right)\frac{1}{\sqrt{2}(1-r)^{1/2}}\right]\cdot\prod_{j>2} h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}}\mathrm{d}rf(z)\mathrm{d}z.$$

Now using the Mean Value Theorem, the integrand in the second term is bounded in absolute value by

$$C\frac{1}{(1-r)^{1/2}}h^{\circ}_{\alpha_1}\left(\frac{|z_1-y_1|}{(1-r)^{1/2}}\right)$$

$$\cdot\exp\left(-\frac{|z_1-y_1|^2}{1-r}\right)\cdot\left\{\left[\frac{y_2}{(1-r)}+\frac{|z_2-sy_2|}{(1-r)}\right]h^{\circ}_{\alpha_2-1}\left(\frac{|z_2-sy_2|}{(1-r)^{1/2}}\right)\right.$$

$$\left.+\left[\frac{-z_2-sy_2|y_2}{(1-r)^{3/2}}+\frac{|z_2-sy_2|^2}{(1-r)^{3/2}}+\frac{1}{(1-r)^{1/2}}\right]h^{\circ}_{\alpha_2}\left(\frac{|z_2-sy_2|}{(1-r)^{1/2}}\right)\right\}$$

$$\cdot\exp\left(-\frac{|z_2-sy_2|^2}{2(1-r)}\right)\cdot\left(H^{[d-2]}_{*,\alpha^{1,2}}|f|(z_1,z_2,\cdot)\right)(y^{1,2})$$

for some s such that $r\le s\le 1$.
Thus using again the inequality $x^ke^{-x^2/n}\le C,\ x>0,\ k\ge 0$ and $n=1,2$, and the fact that in A we have

$$\exp\left(-\frac{|z_2-sy_2|^2}{2(1-r)}\right)\le C\exp\left(-\frac{|z_2-y_2|^2}{2(1-r)}\right),$$

we bound the $L^p(\gamma_{d-1})$-norm in $y_1, y_3, \cdots, y_d$ of the second integral with the aid of Corollaries 3.2 and 4.1, by the sum

$$C \int_{|z_2-y_2|<1\wedge\frac{1}{y_2}} y_2 \left[\int_{1-\frac{|z_2-y_2|}{2y_2}}^{1} \frac{dr}{(1-r)} \exp\left(-\frac{|z_2-y_2|^2}{4(1-r)}\right)\right] \left[\int_{\mathbb{R}^{d-1}} |f(z_2,y^2)|^p e^{-|y^2|^2} dy^2\right]^{1/p} dz_2$$

$$+C \int_{|z_2-y_2|<1\wedge\frac{1}{y_2}} \left[\int_{1-\frac{|z_2 y_2|}{2y_2}}^{1} \frac{dr}{(1-r)^{1/2}}\right] \left[\int_{\mathbb{R}^{d-1}} |f(z_2,y^2)|^p e^{-|y^2|^2} dy^2\right]^{1/p} dz_2.$$

For the first one of these integrals, we estimate the integral in r and then use the Corollary 1.2 vii) to get that it is bounded by

$$C\left\{M_*^{[1]}\left[\left(\int_{\mathbb{R}^{d-1}} |f(\cdot,y^2)|^p e^{-|y^2|^2} dy^2\right)^{1/p}\right]\right\}(y_2).$$

For the second integral, integrating in r and using the Corollary 1.2 viii) and the Definition 2, we get as upper bound

$$C\left\{M_*^{[1]}\left[\left(\int_{\mathbb{R}^{d-1}} |f(\cdot,y^2)|^p e^{-|y^2|^2} dy^2\right)^{1/p}\right]\right\}(y_2)$$
$$+C\left\{M_T^{[1]}\left[\left(\int_{\mathbb{R}^{d-1}} |f(\cdot,y^2)|^p e^{-|y^2|^2} dy^2\right)^{1/p}\right]\right\}(y_2).$$

Finally we have to deal with the integral

$$C\int_A \int_{1-\frac{|z_2-y_2|}{2y_2}}^{1} \frac{1}{(1-r)}\prod_{i=1}^{2} h_{\alpha_i}\left(\frac{z_i-y_i}{\sqrt{2}(1-r)^{1/2}}\right)\exp\left(-\frac{|z_i-y_i|^2}{2(1-r)}\right)\frac{1}{\sqrt{2}(1-r)^{1/2}}$$
$$\cdot \prod_{j>3} h_{\alpha_j}\left(\frac{z_j-ry_j}{(1-r^2)^{1/2}}\right)\exp\left(-\frac{|z_j-ry_j|^2}{1-r^2}\right)\frac{1}{(1-r^2)^{1/2}} dr\, f(z) dz.$$

This can be rewritten as the difference of two integrals the first one with the integral in r over $[0,1]$ minus one with the integral in r over $\left[0, 1 - \frac{|z_2 - y_2|}{2y_2}\right]$. This last integral can be bounded in absolute value by

$$C \int\limits_{|z_2-y_2|<1\wedge\frac{1}{y_2}} \int\limits_{|z_1-y_1|<1\wedge\frac{1}{y_1}} \int\limits_0^{1-\frac{|z_2-y_2|}{2y_2}} \frac{1}{(1-r)} \prod_{i=1}^{2} h^{\circ}_{\alpha_i}\left(\frac{|z_i - y_i|}{\sqrt{2}(1-r)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_i - y_i|^2}{2(1-r)}\right) \frac{1}{\sqrt{2}(1-r)^{1/2}} dr \left(H^{[d-2]}_{*,\alpha^{1,2}} |f|(z_1, z_2, \cdot)\right)(y^{1,2}) dz_1 dz_2.$$

Again, use the inequality $x^n e^{-x^2} \leq C$, if $x > 0$ for z_2. Taking $L^p(\gamma_{d-1})$-norm in $y_1, y_3, \cdots, y_d$ and using the Corollaries 3.2 and 4.1, we obtain the bound

$$C \int\limits_{|z_2-y_2|<1\wedge\frac{1}{y_2}} \left[\int\limits_0^{1-\frac{|z_2-y_2|}{2y_2}} \frac{dr}{(1-r)^{3/2}}\right] \left[\int\limits_{\mathbb{R}^{d-1}} |f(z_2, y^2)|^p e^{-|y^2|^2} dy^2\right]^{1/p} dz_2.$$

Integrating in r and using the Corollary 1.2 viii) and the Definition 2, we obtain as upper bound

$$C\left\{M^{[1]}_{*}\left[\left(\int\limits_{\mathbb{R}^{d-1}} |f(\cdot, y^2)|^p e^{-|y^2|^2} dy^2\right)^{1/p}\right]\right\}(y_2)$$

$$+C\left\{M^{[1]}_{T}\left[\left(\int\limits_{\mathbb{R}^{d-1}} |f(\cdot, y^2)|^p e^{-|y^2|^2} dy^2\right)^{1/p}\right]\right\}(y_2).$$

Finally it remains to work with the first term which has the integral in r over $[0,1]$. This can be written as

$$C\int\limits_A \int\limits_0^1 \frac{1}{(1-r)} \prod_{i=1}^{2} h_{\alpha_i}\left(\frac{z_i - y_i}{\sqrt{2}(1-r)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_i - y_i|^2}{2(1-r)}\right) \frac{1}{\sqrt{2}(1-r)^{1/2}} \cdot h_{\alpha_3}\left(\frac{z_3 - ry_3}{(1-r^2)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_3 - ry_3|^2}{1-r^2}\right) \frac{1}{(1-r^2)^{1/2}} \cdot \prod_{j>3} h_{\alpha_j}\left(\frac{z_j - ry_j}{(1-r^2)^{1/2}}\right)$$

$$\cdot \exp\left(-\frac{|z_j - ry_j|^2}{1-r^2}\right) \frac{1}{(1-r^2)^{1/2}} dr f(z) dz.$$

It is clear now how the argument follows. We iterate the argument just done for z_2, to the $z_3, z_4, \cdots$ and z_d-variables until we obtain at the end of this process the integral

$$C \int_A \int_0^1 \frac{1}{(1-r)^{d/2+1}} h_\alpha \left(\frac{z-y}{\sqrt{2}(1-r)^{1/2}} \right) \exp\left(-\frac{|z-y|^2}{2(1-r)} \right) \mathrm{d}r f(z) \mathrm{d}z$$

and we can rewrite it as

$$C \int_A \int_0^1 \frac{1}{(1-r)^{d/2+1}} h_\alpha \left(\frac{z-y}{\sqrt{2}(1-r)^{1/2}} \right) \exp\left(-\frac{|z-y|^2}{2(1-r)} \right) \mathrm{d}r f(z) \mathrm{d}z.$$

Now using the change of variables $u = \dfrac{|z-y|}{\sqrt{2}(1-r)^{1/2}}$ gives us

$$C \int_A \frac{1}{|z-y|^d} \left[\int_{\frac{|z-y|}{\sqrt{2}}}^{\infty} u^{d-1} h_\alpha \left(\frac{z-y}{|z-y|} u \right) e^{-u^2} \mathrm{d}u \right] f(z) \mathrm{d}z$$

and this can be written as the difference of the same expression but with the integral in u over $[0,\infty)$ minus the same with integral in u over $\left[0, \dfrac{|z-y|}{\sqrt{2}}\right]$. This last expression can be bounded, using the inequality $x^n e^{-x^2} \leq C$, if $x > 0$ and $n \geq 0$, and integrating in u, by $C \int_A |f(z)| \mathrm{d}z$, and this is immediately bounded by

$$C \left(M_T^{[1]} \left(M_T^{[1]} \cdots \left(M_T^{[1]} f(\cdot) \right) (y_d) \cdots \right) (y_2) \right) (y_1).$$

The first expression is nothing but the singular part of the operator, which can be written as

$$C \int_A \frac{\Omega(y-z)}{|y-z|^d} f(z) \mathrm{d}z,$$

where

$$\Omega(x) = C \int_0^\infty u^{d-1} h_\alpha \left(-\frac{x}{|x|} u \right) e^{-u^2} \mathrm{d}u = C(-1)^d \int_0^\infty u^{d-1} h_\alpha \left(\frac{x}{|x|} u \right) e^{-u^2} \mathrm{d}u.$$

Now Ω is clearly homogeneous of degree zero and by the orthogonality of the Hermite polynomials we have

$$\int_{S^{d-1}} \left[\int_0^\infty u^{d-1} h_\alpha(\sigma u) e^{-u^2} \mathrm{d}u \right] \mathrm{d}\sigma = \int_{\mathbb{R}^d} h_\alpha(z) e^{-|z|^2} \mathrm{d}z = 0$$

as $|\alpha| > 0$.

Thus $\frac{\Omega(x)}{|x|^d}$ is a Calderón-Zygmund kernel and so by Theorem 5 this is $L^p(\gamma_d)$ bounded and this finishes the proof of Theorem 7. □

3-2. *Proof of Theorem 8*

We observe first that if P is a polynomial, the result is immediate since the Hermite polynomials form an algebraic base for the polynomials.

Now for the general case, remember that the fact that P and its first derivatives have at most polynomial growth order means that there exist $C_0, N_0, C_i, N_i, i = 1, \cdots, d$, such that

$$|P(x)| \leq C_0(1 + |x|^{N_0}), \qquad \text{for any } x \in \mathbb{R}^d, \text{ and}$$

$$|D_i P(x)| \leq C_i(1 + |x|^{N_i}), \quad \text{for any } x \in \mathbb{R}^d, i = 1, \cdots, d.$$

As in the proof of Theorem 7, we will consider two cases.

Case 1. There exists one coordinate z_i such that $|z_i - y_1| \geq 1 \wedge \frac{1}{|y_i|}$.

In this case we use the condition over P and essentially repeat the argument given in Case 1 of Theorem 7.

Case 2. For all $i = 1, \cdots, d$, we have $|z_i - y_i| < 1 \wedge \frac{1}{|y_i|}$.

In this case we repeat the argument given in Case 2 of Theorem 7 and the inequalities for $D_i P$. □

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