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Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the n -Laplacian

ADIMURTHI

1. - Introduction

Let Ω be a bounded open set in \mathbb{R}^n with smooth boundary. We are looking for a solution of the following problem:

Let $1 < p \leq n$, find $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$(1.1) \quad \begin{aligned} \Delta_p u &= f(x, u)|u|^{p-2} && \text{in } \Omega \\ u &\geq 0, \end{aligned}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian and $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function with $f(x, 0) = 0$, $f(x, t) \geq 0$ for $t \geq 0$ and of critical growth.

For $p = 2$ and $n \geq 3$, Brézis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when f has critical growth of the form $u^{(n+2)/(n-2)} + \lambda u$. A generalization of this result, on the same lines, for the p -Laplacian with $p \leq n$ and $p^2 \leq n$, has been studied by Garcia Azorero-Peral Alonso [7]. When $p = n$, in view of the Trudinger [13] imbedding, a critical growth function $f(x, u)$ behaves like $\exp(b|u|^{n/(n-1)})$ for some $b > 0$. In this context, when $p = n = 2$ and Ω is a ball in \mathbb{R}^2 , existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brézis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let $f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)})$ be a function of critical growth and $F(x, t)$ be its primitive (see definition (2.1)). For $u \in W_0^{1,n}(\Omega)$, let

$$(1.2) \quad J(u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \int_{\Omega} F(x, u) \, dx$$

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$$(1.3) \quad \lambda_1(u) = \inf \left\{ \int_{\Omega} |\nabla u|^n dx; u \in W_0^{1,n}(\Omega), \int_{\Omega} |u|^n dx = 1 \right\}$$

$$(1.4) \quad \alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{Volume of } S^{n-1}.$$

THEOREM Let $f(x, t) = h(x, t) \exp(bt|t|^{n/(n-1)})$ be a function of critical growth on Ω . Then

1) $J : W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ satisfies the Palais-Smale Condition on the interval $\left(-\infty, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1}\right)$;

2) Let $f'(x, t) = \frac{\partial}{\partial t} f(x, t)$ and further assume that

$$(1.5) \quad \sup_{x \in \Omega} f'(x, 0) < \lambda_1(\Omega)$$

$$(1.6) \quad \overline{\lim}_{t \rightarrow \infty} \inf_{x \in \Omega} h(x, t)t^{n-1} = \infty,$$

then there exists some $u_0 \in W_0^{1,n}(\Omega) \setminus \{0\}$ such that

$$(1.7) \quad \begin{aligned} \Delta_n u_0 &= f(x, u_0)u_0^{n-2} && \text{in } \Omega \\ u_0 &\geq 0 \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adapt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

$$(1.8) \quad \frac{a(\Omega, f)^n}{n} = \inf \left\{ J(u); \int_{\Omega} |\nabla u|^n dx = \int_{\Omega} f(x, u)u^{n-1} dx, u \neq 0 \right\},$$

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that α_n is the best constant appearing in Moser's [10] result about the Trudinger's imbedding of $W_0^{1,n}(\Omega)$. In view of this, one expects that J should satisfy the Palais-Smale Condition on $\left(-\infty, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1}\right)$. Therefore, in order to get a minimizer of (1.8), the question remains to show that

$$(1.9) \quad a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$$

and this has been achieved by showing the following relation

$$(1.10) \quad \sup_{\int_{\Omega} |\nabla w|^n dx \leq 1} \int_{\Omega} f(x, a(\Omega, f)w)w^{n-1} dx \leq a(\Omega, f).$$

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when $n = 2$.

2. - Preliminaries

Let Ω be a bounded domain with smooth boundary. In view of the Trudinger-Moser [13,10] imbedding, we have the following definition of functions of critical growth.

DEFINITION 2.1. Let $h : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -function and $b > 0$. Let $f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)})$. We say that f is a function of critical growth on Ω if the following holds:

There exist constants $M > 0, \sigma \in [0, 1)$ such that, for every $\epsilon > 0$, and for every $(x, t) \in \overline{\Omega} \times (0, \infty)$,

$$(H_1) \quad f(x, 0) = 0, \quad f(x, t) > 0, \quad f(x, t)t^{n-1} = f(x, -t)(-t)^{n-1};$$

$$(H_2) \quad f'(x, t) > \frac{f(x, t)}{t}, \quad \text{where } f'(x, t) = \frac{\partial f}{\partial t}(x, t);$$

$$(H_3) \quad F(x, t) \leq M(1 + f(x, t)t^{n-2+\sigma}), \quad \text{where}$$

$$F(x, t) = \int_0^t f(x, s)s^{n-2} ds$$

is the primitive of f ;

$$(H_4) \quad \limsup_{t \rightarrow \infty} \sup_{x \in \overline{\Omega}} h(x, t) \exp(-\epsilon t^{n/(n-1)}) = 0,$$

$$\liminf_{t \rightarrow \infty} \inf_{x \in \overline{\Omega}} h(x, t) \exp(\epsilon t^{n/(n-1)}) = \infty.$$

Let $A(\Omega)$ denote the set of all functions of critical growth on Ω .

EXAMPLES. In view of (H_1) , it is enough to define f on $\overline{\Omega} \times (0, \infty)$.

- 1) For $m \geq 1, b > 0, \beta \geq 0$ and $0 \leq \alpha < \frac{n}{n-1}$, $f(x, t) = t^m \exp(\beta t^\alpha) \exp(bt^{n/(n-1)})$ is in $A(\Omega)$.
- 2) $f(x, t) = t^2 e^{-t} \exp(t^{n/(n-1)})$ is in $A(\Omega)$.
- 3) Let $f(x, t) = h(x, t) \exp(bt^{n/(n-1)})$, satisfying (H_1) and (H_4) .

Further assume that $h'(x, t) \geq \frac{h(x, t)}{t}$ for $(x, t) \in \bar{\Omega} \times (0, \infty)$. Then f is in $A(\Omega)$.

For

$$\frac{f'(x, t)}{f(x, t)} = \frac{h'(x, t)}{h(x, t)} + \frac{nb}{n-1} t^{1/(n-1)} > \frac{1}{t}$$

and hence f satisfy (H_2) .

Let $\epsilon > 0$, and $\sigma = \frac{1}{n-1}$

$$\begin{aligned} F(x, t) - F(x, \epsilon) &= \frac{n-1}{nb} \int_{\epsilon}^t h(x, s) s^{n-1-\sigma} \frac{d}{ds} \exp\left(bs^{n/(n-1)}\right) ds \\ &\leq \frac{n-1}{nb} [f(x, t)t^{n-2-\sigma} - f(x, \epsilon)\epsilon^{n-2-\sigma}]. \end{aligned}$$

This implies that there exists a constant $M > 0$ such that $F(x, t) \leq M[1 + f(x, t)t^{n-2-\sigma}]$ for $(x, t) \in \bar{\Omega} \times (0, \infty)$. This shows that f satisfy (H_3) and hence $f \in A(\Omega)$.

Let $W_0^{1,n}(\Omega)$ be the usual Sobolev space and $f(x, t) = h(x, t) \exp(bt^{n/(n-1)})$ be in $A(\Omega)$. For $u \in W_0^{1,n}(\Omega)$, define

$$(2.1) \quad \|u\|^n = \int_{\Omega} |\nabla u|^n dx$$

$$(2.2) \quad J(u) = \frac{1}{n} \|u\|^n - \int_{\Omega} F(x, u) dx$$

$$(2.3) \quad I(u) = \frac{1}{n} \int_{\Omega} f(x, u) u^{n-1} dx - \int_{\Omega} F(x, u) dx$$

$$(2.4) \quad \partial B(\Omega, f) = \left\{ u \in W_0^{1,n}(\Omega) \setminus \{0\}; \|u\|^n = \int_{\Omega} f(x, u) u^{n-1} dx \right\}$$

$$(2.5) \quad \frac{a(\Omega, f)^n}{n} = \inf \{ J(u); u \in \partial B(\Omega, f) \}$$

$$(2.6) \quad \lambda_1(\Omega) = \inf \left\{ \|u\|^n; \int_{\Omega} |u|^n dx = 1 \right\}$$

$$\alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{Volume of } S^{n-1}.$$

DEFINITION OF MOSER FUNCTIONS. Let $x_0 \in \Omega$ and $R \leq d(x_0, \partial\Omega)$, where d denotes the distance from x_0 to $\partial\Omega$. For $0 < \ell < R$, define

$$m_{\ell,R}(x, x_0) = \frac{1}{\omega_n^{1/n}} \begin{cases} (\log \frac{R}{\ell})^{1-\frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\ \frac{\log \frac{R}{r}}{(\log \frac{R}{\ell})^{1/n}} & \text{if } \ell \leq r = |x - x_0| \leq R \\ 0 & \text{if } |x - x_0| \geq R. \end{cases}$$

Then it is easy to see that $m_{\ell,R} \in W_0^{1,n}(\Omega)$ and $\|m_{\ell,R}\| = 1$.

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

THEOREM 2.1 (Moser). 1) Let $u \in W_0^{1,n}(\Omega)$, and $p < \infty$, then $\exp(|u|^{n/(n-1)}) \in L^p(\Omega)$.

$$2) \left(\frac{\alpha_n}{b}\right)^{n-1} = \max \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} \exp(bc^{n/(n-1)}|w|^{n/(n-1)}) dx < \infty \right\}.$$

THEOREM 2.2 (P.L. Lions). Let $\{u_k; \|u_k\| = 1\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly to a non-zero function u . Then, for every $p < (1 - \|u\|^n)^{-1/(n-1)}$,

$$\sup_k \int_{\Omega} \exp(p\alpha_n|u_k|^{n/(n-1)}) dx < \infty.$$

3. - Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

LEMMA 3.1. Let $f \in A(\Omega)$. Then we have

1) If $u \in W_0^{1,n}(\Omega)$, then $f(x, u) \in L^p(\Omega)$ for all $p \geq 0$.

$$2) \left(\frac{\alpha_n}{b}\right)^{n-1} = \sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} dx < \infty \right\}.$$

3) Let $\{u_k\}$ and $\{v_k\}$ be bounded sequences in $W_0^{1,n}(\Omega)$ converging weakly and for almost every x in Ω to u and v respectively. Further assume that

$$\overline{\lim}_{k \rightarrow \infty} \|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

Then, for every integer $\ell \geq 0$,

$$\lim_{k \rightarrow \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^{\ell} dx = \int_{\Omega} \frac{f(x, u)}{u} v^{\ell} dx.$$

4) Let $\{u_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly and for almost every x in Ω to u , such that

$$\sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} dx < \infty.$$

Then, for any $0 \leq \tau < 1$,

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, |u_k|) |u_k|^{n-2+\tau} dx = \int_{\Omega} f(x, |u|) |u|^{n-2+\tau} dx$$

$$\lim_{k \rightarrow \infty} \int_{\Omega} F(x, u_k) dx = \int_{\Omega} F(x, u) dx.$$

5) $I(u) \geq 0$ for all u and $I(u) = 0$ iff $u \equiv 0$. Further, there exists a constant $M_1 > 0$ such that, for all $u \in W_0^{1,n}(\Omega)$,

$$\int_{\Omega} f(x, u) u^{n-1} dx \leq M_1(1 + I(u)).$$

LEMMA 3.2. Let $f = h \exp(b|t|^{n/(n-1)}) \in A(\Omega)$ and define

$$h_0(t) = \inf_{x \in \bar{\Omega}} h(x, t), \quad M_0 = \sup_{t \geq 0} h_0(t) t^{n-1}, \quad R_0 = \sup_{x \in \bar{\Omega}} d(x, \partial\Omega),$$

and

$$k_0 = \begin{cases} \left(\frac{n}{R_0}\right)^{n/(n-1)} M_0^{-1/(n-1)} & \text{if } M_0 < \infty \\ 0 & \text{if } M_0 = \infty. \end{cases}$$

Let $a \geq 0$ be such that

$$\sup_{\|w\| \leq 1} \int_{\Omega} f(x, aw) w^{n-1} dx \leq a.$$

If $\frac{k_0}{b} < 1$, then $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

PROOF. From 2) of lemma 3.1, we have $a^n \leq \left(\frac{\alpha_n}{b}\right)^{n-1}$. Suppose $a^n = \left(\frac{\alpha_n}{b}\right)^{n-1}$. Let $x_0 \in \Omega$ such that $d(x_0, \partial\Omega) = R_0$ and $0 < \ell < R_0$. Let

$$m_{\ell}(x) = m_{\ell, R_0}(x, x_0).$$

be the Moser functions and

$$t = a\omega_n^{-1/n} \left(\log \frac{R_0}{\ell} \right)^{(n-1)/n},$$

then from (3.1) we have

$$\begin{aligned} a &\geq \int_{\Omega} f(x, am_{\ell})m_{\ell}^{n-1} dx \\ &\geq \int_{B(x_0, \ell)} h_0(am_{\ell})m_{\ell}^{n-1} \exp\left(ba^{n/(n-1)}m_{\ell}^{n/(n-1)}\right) dx \\ &= \frac{h_0(t)t^{n-1}\omega_n R_0^n}{na^{n-1}}. \end{aligned}$$

This implies that

$$\left(\frac{\alpha_n}{b}\right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1}\omega_n R_0^n}{n}.$$

That is, for all $t \in (0, \infty)$,

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)} (h_0(t)t^{n-1})^{-1/(n-1)}$$

and hence

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)} \inf_{t \geq 0} (h_0(t)t^{n-1})^{-1/(n-1)} \leq k_0$$

which contradicts the hypothesis $b > k_0$. Hence $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$ and this proves the lemma.

LEMMA 3.3. (Compactness Lemma). *Let f be in $A(\Omega)$ and $\{u_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly and for almost every x in Ω to a non-zero function u . Further, assume that*

- (i) *There exists $C \in \left(0, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1}\right]$ such that $\lim_{k \rightarrow \infty} J(u_k) = C$;*
- (ii) $\|u\|^n \geq \int_{\Omega} f(x, u)u^{n-1} dx$;
- (iii) $\sup_k \int_{\Omega} f(x, u_k)u_k^{n-1} dx < \infty$;

then

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} dx = \int_{\Omega} f(x, u)u^{n-1} dx.$$

PROOF. From 5) of lemma 3.1, $I(u) > 0$. Therefore, from (ii) we have $J(u) \geq I(u) > 0$ and $J(u) \leq \liminf_{k \rightarrow \infty} J(u_k) = C$. Hence we can choose an $\epsilon > 0$ such that

$$(3.2) \quad (C - J(u)) (1 + \epsilon)^{n-1} < \frac{1}{n} \left(\frac{\alpha_n}{b} \right)^{n-1}.$$

Let $\beta = \int_{\Omega} F(x, u) dx$. Then, from (iii) and 4) of lemma 3.1, we have

$$(3.3) \quad \begin{aligned} \lim_{k \rightarrow \infty} \|u_k\|^n &= n \lim_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) dx \right\} \\ &= n(C + \beta). \end{aligned}$$

From (3.2) and (3.3) we can choose a $k_0 > 0$ such that, for all $k \geq k_0$,

$$(3.4) \quad (1 + \epsilon)^{n-1} \left(\frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n < \frac{C + \beta}{C - J(u)} = \left(1 - \frac{\|u\|^n}{n(C + \beta)} \right)^{-1}.$$

Now choose p such that

$$(3.5) \quad (1 + \epsilon)^{n-1} \left(\frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n \leq p^{n-1} < \frac{C + \beta}{C - J(u)}.$$

Applying theorem 2.2 to the sequence $\frac{u_k}{\|u_k\|}$ and using (3.3) and (3.5), we have

$$(3.6) \quad \sup_k \int_{\Omega} \exp \left[p \alpha_n \left(\frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] dx < \infty.$$

From (3.5) and (3.6), we have

$$(3.7) \quad \begin{aligned} \sup_k \int_{\Omega} \exp \left((1 + \epsilon)^{n-1} b |u_k|^{n/(n-1)} \right) dx \\ \leq \sup_k \int_{\Omega} \exp \left[p \alpha_n \left(\frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] dx < \infty. \end{aligned}$$

Let

$$M_1 = \sup_{(x,t) \in \bar{\Omega} \times \mathbb{R}} |h(x, t) t^{n-1}| \exp \left(-\epsilon \frac{b}{2} |t|^{n/(n-1)} \right)$$

and $N > 0$. Then from (3.7) we have

$$\begin{aligned}
 (3.8) \quad \int_{|u_k| \geq N} f(x, u_k) u_k^{n-1} dx &= \int_{|u_k| \geq N} h(x, u_k) u_k^{n-1} \exp\left(b|u_k|^{n/(n-1)}\right) dx \\
 &\leq M_1 \int_{|u_k| \geq N} \exp\left(-\epsilon \frac{b}{2}|u_k|^{n/(n-1)}\right) \exp\left[(1+\epsilon)b|u_k|^{n/(n-1)}\right] dx \\
 &= O\left(\exp\left(-\epsilon \frac{b}{2} N^{n/(n-1)}\right)\right).
 \end{aligned}$$

Hence

$$\int_{\Omega} f(x, u_k) u_k^{n-1} dx = \int_{|u_k| \leq N} f(x, u_k) u_k^{n-1} dx + O\left(\exp\left(-\epsilon \frac{b}{2} N^{n/(n-1)}\right)\right).$$

Now letting $k \rightarrow \infty$, and $N \rightarrow \infty$ in the above equation, we obtain

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} dx = \int_{\Omega} f(x, u) u^{n-1} dx.$$

This proves the lemma.

LEMMA 3.4. Let $f \in A(\Omega)$ and assume that

(i) $\overline{\lim}_{t \rightarrow \infty} h_0(t) t^{n-1} = \infty$,

where $h_0(t) = \inf_{x \in \overline{\Omega}} h(x, t)$;

(ii) $\sup_{x \in \overline{\Omega}} f'(x, 0) < \lambda_1(\Omega)$;

then

$$0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

PROOF. The lemma is proved in several steps.

STEP 1. $a(\Omega, f) > 0$.

Suppose $a(\Omega, f) = 0$. Then there exists a sequence $\{u_k\}$ in $\partial B(\Omega, f)$ such that $J(u_k) \rightarrow 0$ as $k \rightarrow \infty$. Since $J(u_k) = I(u_k)$, hence from 5) of lemma 3.1

$$(3.9) \quad \sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} dx < \infty$$

$$(3.10) \quad \sup_k \|u_k\|^n < \infty.$$

Then, by extracting a subsequence, we can assume that $\{u_k\}$ converges weakly and for almost every x in Ω to a function u . Now by Fatou's lemma,

$$0 \leq I(u) \leq \liminf_{k \rightarrow \infty} I(u_k) = \liminf_{k \rightarrow \infty} J(u_k) = 0.$$

Hence $u \equiv 0$. From (3.9) and 4) of lemma 3.1, we have

$$(3.12) \quad \lim_{k \rightarrow \infty} \|u_k\|^n = n \lim_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) dx \right\} = 0.$$

Let $v_k = \frac{u_k}{\|u_k\|}$ and converging weakly to v . Using $u_k \in \partial B(\Omega, f)$, (3.12), 3) of lemma 3.1 and (ii), we have

$$\begin{aligned} 1 &= \lim_{k \rightarrow \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^n dx \\ &= \int_{\Omega} f'(x, 0) v^n dx < \lambda_1(\Omega) \int_{\Omega} v^n dx \leq 1, \end{aligned}$$

which is a contradiction. This prove step 1.

STEP 2. For every $u \in W_0^{1,n}(\Omega) \setminus \{0\}$, there exists a constant $\gamma > 0$ such that $\gamma u \in \partial B(\Omega, f)$. Moreover, if

$$(3.13) \quad \|u\|^n \leq \int_{\Omega} f(x, u) u^{n-1} dx,$$

then $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$.

For $\gamma > 0$, define

$$\psi(\gamma) = \frac{1}{\gamma} \int_{\Omega} f(x, \gamma u) u^{n-1} dx.$$

Then, from 3) of lemma 3.1 and (ii), we have

$$\lim_{\gamma \rightarrow 0} \psi(\gamma) = \int_{\Omega} f'(x, 0) u^n dx < \|u\|^n,$$

$$\lim_{\gamma \rightarrow \infty} \psi(\gamma) = \infty.$$

Hence there exists $\gamma > 0$ such that $\psi(\gamma) = \|u\|^n$; this implies that $\gamma u \in \partial B(\Omega, f)$. From (H_1) and (H_2) , it follows that $\frac{f(x, \gamma u)}{\gamma} u^{n-1}$ is an

increasing function for $t > 0$. Hence, if u satisfies (3.13), it follows that $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$. This proves step 2.

STEP 3. $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

Let $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$. From step 2, we can choose a $\gamma > 0$ such that $\gamma w \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma w) \leq \frac{\gamma^n}{n} \|w\|^n = \frac{\gamma^n}{n};$$

this implies that $a(\Omega, f) \leq \gamma$. Using again the fact that $\frac{f(x, tw)}{t} w^{n-1}$ is an increasing function of t in $(0, \infty)$ and $\gamma w \in \partial B(\Omega, f)$, we have

$$\int_{\Omega} \frac{f(x, a(\Omega, f)w)}{a(\Omega, f)} w^{n-1} dx \leq \int_{\Omega} \frac{f(x, \gamma w)}{\gamma} w^{n-1} dx = 1.$$

This implies that

$$(3.14) \quad \sup_{\|w\| \leq 1} \int_{\Omega} f(x, a(\Omega, f)w) w^{n-1} dx \leq a(\Omega, f).$$

Now from (i), (3.14) and lemma 3.2 we have $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. This proves the lemma.

LEMMA 3.5. Let $f \in A(\Omega)$ and $u_0 \in \partial B(\Omega, f)$ such that $J'(u_0) \neq 0$ ($J'(u)$ denote the derivative of J at u). Then

$$J(u_0) > \inf\{J(u); u \in \partial B(\Omega, f)\}.$$

PROOF. Choose $h_0 \in W_0^{1,n}(\Omega)$ such that $\langle J'(u_0), h_0 \rangle = 1$ and, for $\alpha, t \in \mathbb{R}$, define $\sigma_t(\alpha) = \alpha u_0 - t h_0$. Then

$$\lim_{\alpha \rightarrow 1} \frac{d}{dt} J(\sigma_t(\alpha)) = -\langle J'(u_0), h_0 \rangle = -1$$

and hence we can choose $\epsilon > 0, \delta > 0$ such that, for all $\alpha \in [1 - \epsilon, 1 + \epsilon]$ and $0 < t \leq \delta$,

$$(3.15) \quad J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0).$$

Let

$$\rho_t(\alpha) = \|\sigma_t(\alpha)\|^n - \int_{\Omega} f(x, \sigma_t(\alpha)) \sigma_t(\alpha)^{n-1} dx.$$

Since $\frac{f(x, \alpha u_0)}{\alpha} u_0^{n-1}$ is an increasing function of α and using $u_0 \in \partial B(\Omega, f)$, by shrinking ϵ and δ if necessary, we have, for $0 < t \leq \delta$, $\rho_t(1-\epsilon) > 0$ and $\rho_t(1+\epsilon) < 0$. Hence there exists α_t such that $\rho_t(\alpha_t) = 0$. Therefore $\sigma_t(\alpha_t)$ is in $\partial B(\Omega, f)$. Hence from (3.15) we have

$$\begin{aligned} \inf\{J(u); u \in \partial B(\Omega, f)\} &\leq J(\sigma_t(\alpha_t)) \\ &< J(\alpha_t u_0) \leq \sup_{t \in \mathbb{R}} J(tu_0) = J(u_0). \end{aligned}$$

This proves the lemma.

PROOF OF THE THEOREM.

1) *Palais-Smale Condition.* Let $C \in \left(-\infty, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1}\right)$ and $\{u_k\}$ be a sequence such that

$$(3.16) \quad \begin{aligned} \lim_{k \rightarrow \infty} J(u_k) &= C \\ \lim_{k \rightarrow \infty} J'(u_k) &= 0. \end{aligned}$$

Let $h \in W_0^{1,n}(\Omega)$, then we have

$$(3.18) \quad \langle J'(u_k), h \rangle = \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla h \, dx - \int_{\Omega} f(x, u_k) u_k^{n-2} h \, dx.$$

Hence we have

$$(3.19) \quad J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle = I(u_k).$$

CLAIM 1.

$$(3.20) \quad \sup_k \|u_k\| + \sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty.$$

Since $\{J(u_k)\}$ and $\{J'(u_k)\}$ are bounded and hence from (3.19), $I(u_k) = O(\|u_k\|)$. Now from 5) of lemma 3.1, we have $\int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = O(\|u_k\|)$.

Now from (H_3) it follows that

$$\int_{\Omega} F(x, u_k) \, dx = O(\|u_k\|)$$

and, by using the boundedness of $J(u_k)$, we obtain $\|u_k\|^n = O(\|u_k\|)$. This implies (3.20) and hence the claim.

By extracting a subsequence, we can assume that

$$(3.21) \quad u_k \rightarrow u_0 \text{ weakly and for almost all } x \text{ in } \Omega.$$

CASE (I). $C \leq 0$.

From Fatou's lemma and 5) of lemma 3.1, we have

$$\begin{aligned} 0 \leq I(u_0) &\leq \liminf_{k \rightarrow \infty} I(u_k) \\ &= \liminf_{k \rightarrow \infty} \left\{ J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} \\ &= C. \end{aligned}$$

Hence $u_0 \equiv 0$. If $C < 0$, no Palais-Smale sequence exists. If $C = 0$, then from (3.20) and 4) of lemma 3.1 we have

$$\lim_{k \rightarrow \infty} \|u_k\|^n = n \lim_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.$$

This proves that $u_k \rightarrow 0$ strongly.

CASE (II). $C \in \left(0, \frac{1}{n} \left(\frac{\alpha_n}{b} \right)^{n-1} \right)$.

CLAIM 2. $u_0 \not\equiv 0$ and $u_0 \in \partial B(\Omega, f)$.

Suppose $u_0 \equiv 0$. Then, from (3.20) and 4) of lemma 3.1, we have

$$(3.22) \quad \begin{aligned} \lim_{k \rightarrow \infty} \|u_k\|^n &= n \lim_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} \\ &= nC < \left(\frac{\alpha_n}{b} \right)^{n-1}. \end{aligned}$$

Hence, from 3) of lemma 3.1 and (3.22), we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx = 0.$$

This implies that $\lim_{k \rightarrow \infty} I(u_k) = 0$ and hence from (3.19)

$$0 < C = \lim_{k \rightarrow \infty} J(u_k) = \lim_{k \rightarrow \infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} = 0$$

which is a contradiction. Hence $u_0 \neq 0$. From (3.20) and 4) of lemma 3.1, taking $h \in C_0^\infty(\Omega)$ and letting $k \rightarrow \infty$ in (3.19), we obtain

$$\int_{\Omega} |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_{\Omega} f(x, u_0) u_0^{n-2} h \, dx.$$

By density, the above equation holds for all $h \in W_0^{1,n}(\Omega)$. Hence, by taking $h = u_0$, we obtain

$$(3.23) \quad \|u_0\|^n = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx.$$

Hence $u_0 \in \partial B(\Omega, f)$ and this proves the claim.

Now from (3.20) and claim 2, $\{u_k, u_0\}$ satisfy all the hypotheses of the compactness lemma 3.3. Hence we have

$$\begin{aligned} \|u_0\|^n &\leq \liminf_{k \rightarrow \infty} \|u_k\|^n \\ &= n \liminf_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} \\ &= n \liminf_{k \rightarrow \infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle + \int_{\Omega} F(x, u_k) \, dx \right\} \\ &= \liminf_{k \rightarrow \infty} \left\{ \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx + \langle J'(u_k), u_k \rangle \right\} \\ &= \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx = \|u_0\|^n. \end{aligned}$$

This implies that u_k converges to u_0 strongly. This proves the Palais-Smale condition.

2) *Existence of Positive Solution.* Since the critical points of J are the solutions of the equation (1.7) and $J(u) = J(|u|)$ for all u in $\partial B(\Omega, f)$ and hence in view of lemma 3.5, it is enough to prove that there exists $u_0 \neq 0$ such that

$$(3.24) \quad \frac{a(\Omega, f)^n}{n} = J(u_0).$$

Let $u_k \in \partial B(\Omega, f)$ such that

$$\lim_{k \rightarrow \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.$$

Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

$$(3.25) \quad \sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} dx < \infty,$$

$$(3.26) \quad \sup_k \|u_k\| < \infty.$$

Hence we can extract a subsequence such that

$$u_k \rightarrow u_0 \text{ weakly and for almost all } x \text{ in } \Omega.$$

CLAIM 3. $u_0 \not\equiv 0$ and

$$(3.28) \quad \|u_0\|^n \leq \int_{\Omega} f(x, u_0) u_0^{n-1} dx.$$

Suppose $u_0 \equiv 0$, then from (3.25) and 4) of lemma 3.1

$$(3.29) \quad \begin{aligned} \lim_{k \rightarrow \infty} \|u_k\|^n &= n \lim_{k \rightarrow \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) dx \right\} \\ &= a(\Omega, f)^n. \end{aligned}$$

From lemma 3.4, we have $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} dx = 0.$$

This implies that

$$0 < \frac{a(\Omega, f)^n}{n} = \lim_{k \rightarrow \infty} J(u_k) = \lim_{k \rightarrow \infty} I(u_k) = 0,$$

which is a contradiction. This proves $u_0 \not\equiv 0$. Suppose (3.28) is false, then

$$(3.30) \quad \|u_0\|^n > \int_{\Omega} f(x, u_0) u_0^{n-1} dx.$$

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, $\{u_k, u_0\}$ satisfy all the hypotheses of lemma 3.3. Hence

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} dx = \int_{\Omega} f(x, u_0) u_0^{n-1} dx.$$

This implies that

$$\begin{aligned} \|u_0\|^n &\leq \liminf_{k \rightarrow \infty} \|u_k\|^n = \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} dx \\ &= \int_{\Omega} f(x, u_0) u_0^{n-1} dx \end{aligned}$$

contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists $0 < \gamma \leq 1$ such that $\gamma u_0 \in \partial B(\Omega, f)$. Hence

$$\begin{aligned} \frac{a(\Omega, f)^n}{n} &\leq J(\gamma u_0) = I(\gamma u_0) \\ &\leq I(u_0) \leq \liminf_{k \rightarrow \infty} I(u_k) \\ &= \liminf_{k \rightarrow \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}. \end{aligned}$$

This implies that $\gamma = 1$ and $u_0 \in \partial B(\Omega, f)$. Hence $J(u_0) = \frac{a(\Omega, f)^n}{n}$ and this proves the Theorem.

4. Concluding Remarks

REMARK 4.1. (Regularity). From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in $C^{1,\alpha}(\Omega)$ if $n \geq 3$ and in $C^{2,\alpha}(\bar{\Omega})$ if $n = 2$.

REMARK 4.2. Let $f \in A(\Omega)$ and $f'(x, 0) < \lambda_1(\Omega)$ for all $x \in \bar{\Omega}$. We prove the existence of a solution for (1.7) under the assumption that

$$(4.1) \quad \overline{\lim}_{t \rightarrow \infty} \inf_{x \in \bar{\Omega}} h(x, t) t^{n-1} = \infty.$$

The only place where it is used is to show that $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. But, from lemma 3.2, this inequality holds if

$$(4.2) \quad \frac{k_0}{b} < 1.$$

Hence the theorem is true under the less restrictive condition (4.2).

Now the question is what happens if $\frac{k_0}{b} \geq 1$ or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions $f \in A(\Omega)$ such that

$$\liminf_{t \rightarrow \infty} \inf_{x \in \bar{\Omega}} h(x, t)t^{n-1} = 0$$

for which no solution to problem (1.7) exists if Ω is a ball of sufficiently small radius. In this context, we raise the following question:

Open Problem. Let Ω be a ball and $f \in A(\Omega)$ such that $\sup_{x \in \bar{\Omega}} f'(x, 0) < \lambda_1(\Omega)$.

Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case $n = 2$, this question is related to the question of Brézis [3]: “where is the border line between the existence and non-existence of a solution of (1.7)?”.

REMARK 4.3. Let $\beta \geq 0$, then by using the norm

$$\left(\int_{\Omega} |\nabla u|^n dx + \beta \int_{\Omega} |u|^n dx \right)^{1/n}$$

in $W_0^{1,n}(\Omega)$, the Theorem still holds if we replace $-\Delta_n u$ by $-\Delta_n u + \beta|u|^{n-2}u$ in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

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5. - Appendix

PROOF OF THE LEMMA 3.1.

1) Let $f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)}) \in A(\Omega)$. From (H_4) , for every $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that

$$|f(x, t)| \leq C(\epsilon) \exp((b + \epsilon)|t|^{n/(n-1)})$$

and hence, from theorem 2.1, $f(x, u) \in L^p(\Omega)$ for every $p < \infty$.

2) From (H_4) , for every $\epsilon > 0$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$(5.1) \quad |f(x, t)t^{n-1}| \leq C_1(\epsilon) \exp(b(1 + \epsilon)|t|^{n/(n-1)})$$

$$(5.2) \quad |f(x, t)t^{n-1}| \geq C_2(\epsilon) \exp\left(b(1-\epsilon)|t|^{n/(n-1)}\right) \text{ for } |t| \geq 1.$$

Hence, if $c > 0$ such that

$$\sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} dx < \infty,$$

it implies that, for every $\epsilon > 0$,

$$\sup_{\|w\| \leq 1} \int_{\Omega} \exp\left(b(1-\epsilon)c^{n/(n-1)}|w|^{n/(n-1)}\right) dx < \infty.$$

Therefore, from Theorem 2.1, we have

$$(1-\epsilon)^{n-1}c^n \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

This implies that

$$\sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} dx < \infty \right\} \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

On the other hand, if $c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, then by choosing $\epsilon > 0$ such that $(1+\epsilon)^{2n-1}c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from Theorem 2.1 and from (5.1), we have

$$\begin{aligned} & \sup_{\|w\| \leq 1} \int_{\Omega} f(x, (1+\epsilon)cw)w^{n-1} dx \\ & \leq C_1(\epsilon) \sup_{\|w\| \leq 1} \int_{\Omega} \exp\left[b((1+\epsilon)c|w|)^{n/(n-1)}\right] dx < \infty \end{aligned}$$

this proves

$$\sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} dx < \infty \right\} = \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

3) Since $\overline{\lim}_{k \rightarrow \infty} \|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from 2) we can choose a $p > 1$ such that

$$c_1^p = \sup_k \int_{\Omega} |f(x, u_k)|^p dx < \infty.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$c_2^q = \sup_k \int_{\Omega} |v_k|^{\ell q} dx.$$

Then, for any $N > 0$ and by Holder's inequality,

$$\left| \int_{|u_k| > N} \frac{f(x, u_k)}{u_k} v_k^{\ell} dx \right| \leq \frac{1}{N} \int_{\Omega} |f(x, u_k)| |v_k^{\ell}| dx \leq \frac{c_1 c_2}{N}.$$

Hence

$$\int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^{\ell} dx = \int_{|u_k| \leq N} \frac{f(x, u_k)}{u_k} v_k^{\ell} dx + O(1/N).$$

By dominated convergence theorem, letting $k \rightarrow \infty$ and then $N \rightarrow \infty$ in the above equation, it implies that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^{\ell} dx = \int_{\Omega} \frac{f(x, u)}{u} v^{\ell} dx.$$

4) Let $N > 0$, then

$$\begin{aligned} \int_{|u_k| > N} f(x, |u_k|) |u_k|^{n-2+\tau} dx &\leq \frac{1}{N^{1-\tau}} \int_{\Omega} f(x, |u_k|) |u_k|^{n-1} dx \\ &= \frac{1}{N^{1-\tau}} \int_{\Omega} f(x, u_k) u_k^{n-1} dx = O\left(\frac{1}{N^{1-\tau}}\right). \end{aligned}$$

Hence

$$\int_{\Omega} f(x, |u_k|) |u_k|^{n-2+\tau} dx = \int_{|u_k| \leq N} f(x, |u_k|) |u_k|^{n-2+\tau} dx + O\left(\frac{1}{N^{1-\tau}}\right).$$

By dominated convergence theorem, letting $k \rightarrow \infty$ and $N \rightarrow \infty$ in the above equation, we obtain

$$(5.3) \quad \lim_{k \rightarrow \infty} \int_{\Omega} f(x, |u_k|) |u_k|^{n-2+\tau} dx = \int_{\Omega} f(x, |u|) |u|^{n-2+\tau} dx.$$

Now from (H_3) ,

$$|F(x, t)| \leq M(1 + |f(x, t)| |t|^{n-2+\sigma})$$

for some $\sigma \in [0, 1)$. Hence, from (5.3) and the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.$$

5) From (H_2) we have, for $t > 0$,

$$(5.4) \quad \frac{\partial}{\partial t} [f(x, t)t^{n-1} - nF(x, t)] = \left[f'(x, t) - \frac{f(x, t)}{t} \right] t^{n-1} > 0.$$

Therefore from (H_1) and (5.4), $f(x, t)t^{n-1} - nF(x, t)$ is an even positive function and increasing for $t > 0$. This implies that $I(u) \geq 0$ and $I(u) = 0$ iff $u \equiv 0$. From (H_3) we have

$$\begin{aligned} nI(u) &= \int_{\Omega} [f(x, u)u^{n-1} - nF(x, u)] \, dx \\ &\geq \int_{\Omega} [f(x, u)u^{n-1} - nM(1 + |f(x, u)| |u|^{n-2+\sigma})] \, dx \\ &\geq C_1 + \frac{1}{2} \int_{|u| \geq C_2} f(x, u)u^{n-1} \, dx \end{aligned}$$

for some constants C_1 and $C_2 > 0$. This implies that there exists a constant $M_1 > 0$ such that

$$\int_{\Omega} f(x, u)u^{n-1} \, dx \leq M(1 + I(u)).$$

This proves the lemma 3.1.

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