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Uniform Boundary Regularity
of Proper Holomorphic Maps

WILHELM KLINGENBERG

1. - Introduction

According to recent results in [2] and [10], the family of proper holomorphic maps from $D \subset \subset \mathbb{C}^n$ to $G \subset \subset \mathbb{C}^n$ of multiplicity bounded from above by some $m \in \mathbb{N}$ is normal. That is any sequence $f_j$ of such maps either has a convergent subsequence or is compactly divergent.

If $D$ and $G$ are in addition $C^\infty$ smoothly bounded and pseudoconvex of finite type [6], then by [5], [7] the maps $f_j$ are known to extend smoothly up to the boundary of $D$. Here we study the behaviour of this extension as $j \to \infty$. We denote by $\text{Prop}(D, G, m)$ the set of proper holomorphic maps from $D$ to $G$ of multiplicity $m$.

**THEOREM 1.** Let $D, G, \subset \subset \mathbb{C}^n$ be $C^\infty$-smoothly bounded pseudoconvex domains of finite type, and $f_j \in \text{Prop}(D, G, m)$, $f_j \to f : D \to \overline{G}$. Then, if $f \in \text{Prop}(D, G, m_0)$, one has

i) if $m_0 = m$, then $f_j \to f$ in $C^\infty(D)$

ii) if $m_0 < m$, then $\exists j', \{p_i\}_1^{m_0} \subset \partial D$ with $f_{j'} \to f$ in $C^\infty(\overline{D} - \{p_i\})$.

Otherwise, $f$ is a constant map to some $q \in \partial G$, and ii) holds with $m_0 = 0$ and $C^\infty$ replaced by $C^0$.

In [1], Bell gave an analogous result for biholomorphic maps. The points $\{p_i\}$ in ii) are limits of $f_{j_1}^{-1}(w)$ for $w \in G$. The example of a sequence of $m$-fold Blanschke products as maps from the unit disc in $\mathbb{C}$ to itself shows that one cannot expect smooth convergence at these points, see also [1], [2] and [10].

The main ingredients of the proof are: the transformation rule for the Bergman kernel function under proper maps [3]; a Proposition of Bell [1] on the density of span $K_2(\cdot, w)$, $w \in G$, in $A^\infty(G) = A(G) \cap C^\infty(\overline{G})$; $C^\infty(\overline{G} \times \overline{G} - \Delta)$-regularity of the Bergman kernel for pseudoconvex domains of finite type, see [4], [9]. Here, $\Delta$ is the boundary diagonal of $G \times G$, $A(G)$ the holomorphic functions and...
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$K_2$ the Bergman kernel function of $G$. Finally a division Theorem in $A^\infty(D)$, which is of independent interest; we write $| \cdot |_{\ell,D}$ for the $C^\ell$-sup norm on $D$.

**Theorem 2:** Let $D$ be a smoothly bounded domain in $\mathbb{C}^n$. Assume that

- $u_j \in A^\infty(D)$ converge in $C^\infty(\overline{D})$ to $u \in A^\infty(D)$.
- The order of vanishing of the $u_j$, $u$ in $\overline{D}$ is of uniformly bounded order.
- $h_j \in A(D)$ are uniformly bounded: $|h_j(z)| \leq M$ for all $z \in D$, $j \in \mathbb{N}$.
- For all $m \geq 0$, $\{u_j \cdot h_j^m\}_{j \geq 1}$ is bounded in $A^\infty(D)$, that is: $\forall \ell \geq 1, m \geq 0 \exists c_1(\ell, m) \forall j : |u_j \cdot h_j^m|_{\ell,D} \leq c_1(\ell, m)$.

Then $\{h_j\}_{j \geq 1}$ is bounded in $A^\infty(D)$: $\forall \ell \exists c_2(\ell, u, \text{finely many } c_1) : |h_j|_{\ell,D} \leq c_2$.

It is a pleasure to thank my thesis advisor Steven Bell for his advice during this project.

2. - Proof of Theorem 1

If $f \in \text{Prop}(D, G, m)$, then $f^{-1}$ is an $m$-valued holomorphic map or correspondence from $G$ to $D$ or a holomorphic map $f^{-1} : G \rightarrow D^m_{\text{sym}}$, the $m$-fold symmetric product of $D$, see [12].

**Proposition 3.** Assume $D, G \subset \subset \mathbb{C}^n$ and $f_j \in \text{Prop}(D, G, m)$, $f_j \rightarrow f : D \rightarrow G$. Then, if $f \in \text{Prop}(D, G, m_0)$, one has

- if $m_0 = m$, then $f_j^{-1}(w) \rightarrow f^{-1}(w)$ in $D^m_{\text{sym}}$.
- if $1 \leq m_0 < m$, then there is a subsequence $j'$ and an $(m - m_0)$-valued holomorphic map $h : G \rightarrow \partial D$ with $f_{j'}^{-1} \rightarrow (f^{-1} \cup h) : G \rightarrow (\overline{D})^m_{\text{sym}}$.

Otherwise, $f$ is into $\partial G$, and $\exists j'$, $h : G \rightarrow (\partial D)^m_{\text{sym}}$ with $f_{j'}^{-1}(w) \rightarrow h$. If in addition $D$ and $G$ are pseudoconvex of finite type, then the maps $h$ above are constant: $h : G \rightarrow \{p_i\}_{i = 1}^{m - m_0} \subset \partial D$, and in case $f(D) \subset \partial G$, $f$ is constant and $m_0 = 0$.

**Proof.** By [2], [10], either $f \in \text{Prop}(D, G, m_0)$ for some $1 \leq m_0 \leq m$ or $f$ maps $D$ into $\partial G$. We may pass to a subsequence $j'$ such that $f_{j'}^{-1}$ converges to an $m$-valued map $F : G \rightarrow \overline{D}$. If $f(D) \subset \partial G$, then $F(G) \subset \partial D$, and $h = F$. Otherwise, given $K_1 \subset \subset D$ there exists $K_2 \subset \subset G$ such that $f_j(K_1) \subset K_2$ for all $j$. Therefore $f_{j'}^{-1} \circ f_j \rightarrow F \circ f$ as $j' \rightarrow \infty$. Note that we may write $f_{j'}^{-1} \circ f_j = \text{id} \cup g_j$, where $g_j$ is an $(m - 1)$-valued map from $D$ to $D$, therefore $\overline{F} \circ f = \text{id} \cup g$. This implies that $F = f^{-1} \cup h$ for some $(m - m_0)$-valued map $h$. If $m_0 = m$, then $f^{-1}$ is $m$-valued, so $h = \emptyset$, and $F = f^{-1}$. We see that every subsequence of $f_{j'}^{-1}$ has a subsequence that converges to $f^{-1}$. This proves i). In case ii) we
need to show that $h(G) \subset \partial D$. In this case, if $\partial D$ is pseudoconvex of finite type [6], it does not contain any complex varieties, and $h$ must be constant: $h = \{p_i\}_{i=1}^{m-m_0}$ for some $p_i \in \partial D$. Since $f : D \to G$ is proper, given $K_2 \subset G$ there exist $K_1 \subset D$ such that $f^{-1}(K_2) \subset K_1$. Claim: $K_1 \cap F(w) \subset f^{-1}(w)$ for $w \in K_2$. It follows that $h(G) \subset \partial D$. Proof of claim: Let $z_j' \in K_1 \cap f^{-1}(w)$ and $z_j' \to z$. Then $f_j(z_j') = w$, and we may pass to $j' \to \infty$: $f(z) = w$. □

**PROPOSITION 4.** [1, Fact 1]. Let $G \subset \subset \mathbb{C}^n$ be a smooth pseudoconvex domain of finite type. Then $\forall r \in \mathbb{N} \exists \ell \in \mathbb{N}, \{w_k\}_{k=1}^{\ell} \subset G$, $c > 0 \forall h \in A^\infty(G), p \in G \exists \{c_k\}_{i=1}^{\ell} \in \mathbb{C} :$

i) $\sum_{k=1}^{\ell} c_k K_2(z, w_k) = h(z) + O(|z - p|^{r+1}),$

ii) $|c_k| \leq c|h|_{r,G}$

Next consider the transformation formula of the Bergman kernel function under proper maps [3]:

$$u_j(z)K_2(f_j(z), w) = \sum_{i=1}^{m} K_1(z, F_j^{(i)}(w))\overline{U_j^{(i)}(w)}.$$ (1)

Here, $\{F_j^{(i)}(w)\}_{i=1}^{m} = f_j^{-1}(w)$ are the branches of the multi-valued inverses, and $U_j^{(i)} = \det(F_j^{(i)})'$. We follow Bell [1]. Now let $h \in A^\infty(D)$, $q \in D$, $r \in \mathbb{N}$. By Proposition 4 where $p$ is replaced by $f_j(q)$ and by (1) there exist $w_k \in G$, $c_k \in \mathbb{C}$ depending on $j$ with

$$u_j(z)h \circ f_j(z) = \sum_{k=1}^{\ell} \sum_{i=1}^{m} c_k K_1(z, F_j^{(i)}(w_k))\overline{U_j^{(i)}(w_k)} + O(|z - q|^{r+1}).$$

In case i) of the Theorem, $F_j$ and $U_j$ converge uniformly on $\{w_k\}$ as $j \to \infty$ by Proposition 3. Then, since $K_1 \in C^\infty(\overline{D} \times \overline{D} \setminus \Delta)$ (see [4]), and since the $c_k$ are bounded independently of $q$ and $j$, we conclude that $\{u_j \circ h \circ f_j\}$ is bounded in $C^\infty(\overline{D})$. Letting $h = 1$, we conclude that $\{u_j\}$ is bounded in $C^\infty(\overline{D})$ and therefore converges in $C^\infty(\overline{D})$ to $u = \det f'$. By [5], $u$ and $u_j$ vanish at most of order $m \cdot n$ at any point in $\overline{D}$. Letting $h(w) = w_i^m$, $h_j = h \circ f_j$ for $i = 1, \ldots, n$, $m \geq 0$, we finally verify the assumptions c) and d) of Theorem 2. We may then conclude that $\{f_j\}$ is bounded in $C^\infty(\overline{D})$. This proves part i).

In case ii) by Proposition 3 we may pass to a subsequence $j'$ such that $\{u_{j'} \circ h \circ f_{j'}\}$ is bounded in $C^\infty(\overline{D}\setminus\{p_i\})$. Here again the regularity of $K_1$ is used. A local version of Theorem 2 allows to conclude that convergence of $f_j$ takes place in $C^\infty(\overline{D}\setminus\{p_i\})$. As to the case of $f$ being a constant map, the same reasoning as above shows that for some subsequence $j'$, $u_j$, converges to $u \equiv 0$ in $C^\infty(\overline{D}\setminus\{p_i\})^m$. Now the proof of Theorem 1, part B in [1] yields the conclusion that convergence takes place in $C^0(\overline{D}\setminus\{p_i\})$. 

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3. A Division Theorem with Estimates

Assuming that \( h \in A(D) \) is bounded and that \( u h^m \in A^\infty(D) \) for all \( m \geq 0 \), we wish to show that \( h \) is in \( A^\infty(D) \) and give estimates for \( h \). Certainly, this cannot hold if \( u \) vanishes to infinite order at some point in \( D \). One is reduced to studying the question in the neighbourhood of a point \( p \) in \( \partial D \) at which \( u \) vanishes of finite order \( k \). We restrict the considered functions \( u, h \) to a complex line \( L_p \) at \( p \) with this property. The division of \( u h^m \) by \( u \) will be carried out on such lines \( L_x \) for \( x \in \partial D \cap U \), \( U \) a neighbourhood of \( p \), and we will prove that for every \( \ell \), the function \( h_{\mid D \cap L_x \cap U} \) is in \( C^\ell(D \cap L_x \cap U) \) with uniform estimates in \( x \). The point is to keep track of the \( C^\ell \)-sup norm estimate of \( h \) during the division process which proceeds by dividing the zeroes of \( u \) out of \( u h^m \) one at a time. To facilitate this procedure we introduce a normalizing transformation of \( D \cap U \) which preserves analyticity on the complex lines \( D \cap L_x \cap U \). We may choose holomorphic coordinates \( (z_1, \ldots, z_n) \) such that \( L_p \) is the \( z_1 \)-axis and \( p \) is the origin. Let \( x = (z_2, \ldots, z_n) \in \mathbb{R}^{2n-2} \) and \( G \) be a smooth domain in \( D \) with \( \partial D \cap U = \partial G \cap U \) for a neighbourhood \( U \) of the origin such that for some \( r > 0 \) and \( |x| < r \) the slices \( G_x = \{ z \in \mathbb{C} : (z, x) \in G \} \subset \mathbb{C} \) are simply connected. Let \( a \in \mathbb{C} \) be a fixed point which lies in all \( G_x \) and let \( \Phi_x \) be the Riemann mapping function from \( G_x \) onto the unit disc \( \Delta \), with \( \Phi_x(a) = 0 \) and \( \phi_x'(a) > 0 \). Note that \( 0 \in \partial G_0 \). Next let \( \Psi \) denote a conformal map of the unit disc onto \( \Delta_\ast = \Delta \cap \{ \text{Im} z < 0 \} \) which takes \( \Phi_0(0) \in \partial A \) to \( 0 \in \partial \Delta_\ast \). The coordinate change given by \( (z, x) \to (\Psi \circ \Psi_\ast(z), x/r) \) transforms \( \bigcup \ G_x \subset D \) to \( \Delta_\ast \times V \subset \mathbb{C} \times \mathbb{R}^{2n-2} \), where \( V = \{|z| < 1\} \). One knows from the classical theory of conformal mappings that this change is \( C^\infty \)-smooth up to \( \partial G \cap U_1 \) and maps this set onto \( \{(-1, +1) \times V \} \) (by normalization) for some neighbourhood \( U_1 \) of the origin in \( \mathbb{C}^n \). The function \( u \in A(G) \) is transformed to a smooth function \( u(z, x) \) on \( \Delta_\ast \times V \) which is holomorphic in \( z \) for fixed \( x \). For smooth functions \( u \) on \( \Delta_\ast \times V \) we define the norm

\[
|u(\cdot, x)|_{\ell, \Delta_\ast} := \sup_{z \in \Delta_\ast, i+j \leq \ell} \left| \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} u(z, x) \right|.
\]

Next we define the class of functions we will work in.

**DEFINITION:**

a) \( \Gamma^{-}(\ell, c) \) is the set of complex valued functions \( u \) on \( \Delta_\ast \times V \) with \( u(\cdot, x) \in A(\Delta_\ast) \cap C^\ell(\Delta_\ast) \), and \( |u(\cdot, x)|_{\ell, \Delta_\ast} \leq c \) for each \( x \in V \).

b) \( \Gamma(\ell, c) \) are the functions on \( \Delta \times V \) with \( u(\cdot, x) \in A(\Delta_\ast) \cap C^\ell(\Delta_\ast) \) and \( |u(\cdot, x)|_{\ell, \Delta} \leq c \) for each \( x \in V \).

Let \( u^{(i)}(z, x) = \frac{1}{i!} \frac{\partial^i}{\partial z^i} u(z, x) \) and \( u^{(i,j)} = \frac{1}{i! j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} u \). The objective of this section is to prove the following.

**THEOREM 5.** Let \( h(\cdot, x) \in A(\Delta_\ast) \) and \( |h(\cdot, x)| \leq c_1 \) for \( x \in V \), and assume that for all \( m \geq 0 \) there exists \( c_1(m) \) with \( u \cdot h^m \in \Gamma^{-}(\ell_1, c_1(m)) \). Assume
furthermore that \( u \) vanishes of order \( k \) at \((0,0)\) and that for some \( c_2 > 0 \):

i) \[ |u^{(k,0)}(0,0)| \geq c_2^{-1} \]

ii) \[ u(z, \cdot) \in C^1(V) \text{ and } \left| \frac{d}{dx} u(z, \cdot) \right| \leq c_2 \text{ on } \Delta \times V. \]

Then \( h \in \Gamma^-(\ell, c) \), where \( \ell = \left( \frac{1}{2} \right)^k \ell_1 - 2k - 2 \), and \( c \) depends only on \( \ell_1, k, \) finitely many \( c_1, \) and \( c_2. \)

Theorem 5 implies Theorem 2: Note that the assumption i) of Theorem 5 is verified uniformly for all \( u_i \) since they converge in \( C^\infty(D) \) to \( u \) which we assume to vanish of at most finite order in \( \overline{D} \). Assumption ii) follows from \( |u_i \cdot h_i^0|_{1,D} = |u_i|_{1,D} \leq c_1(0,1) \). Now the conclusion of Theorem 9 gives for all \( \ell \) uniform \( C^\ell \)-estimates for \( h_i|_{L \cap \partial D \cup U} \) for complex lines \( L \) transversal to the boundary of \( D \) and some neighbourhood \( U \) of any boundary point of \( D \).

Note that by Cauchy estimates, the uniform boundedness of \( h_i \) in \( D \) gives uniform boundedness of \( h_i|_K \) in \( C^\infty(K) \) for compact subsets \( K \) of \( D \). Pick any point \( p \) in \( D\setminus K \). Any \( \ell \)-th order derivative of \( u \) at \( p \) can be expressed as a finite linear combination of derivatives of \( u \) in the direction of complex lines \( L \) transversal to the boundary. Since we can choose the \( L \) from an open cone of directions at each boundary point, we conclude that the sequence \( h_i \) is bounded in \( C^\infty(D) \).

The proof of Theorem 5 proceeds by four propositions. We closely follow Diederich-Fornaess [7]. Here is a well-known fact on bounded extension [8, p. 277].

**PROPOSITION 6.** Let \( u \in \Gamma^-(\ell, c). \) Then there exists a \( v \in \Gamma(\ell, c) \) with \( v(\cdot, x)|_{\Delta} = u(\cdot, x) \).

**LEMMA 7.** Let \( u \in \Gamma(\ell, c_1) \) and \( \zeta, V \to \Delta \) be any map. Then there exists \( \tilde{u} \in \Gamma(\ell, c), c \) depending only on \( \ell \) and \( c_1, \) with

a) \( u(\cdot, x) = \tilde{u}(\cdot, x) \) in \( \Delta \)

b) \( \tilde{u}(z, x) = \sum_{i=0}^{\ell-1} u^{(i)}(\zeta_1(x), x) \cdot (z - \zeta_1(x))^i + \sigma_\ell(x, z), \) where \( \sigma_\ell \) vanishes of order \( \ell \) at \( z = \zeta_1(x) \) for all \( x \in V. \)

**PROOF.** Conclusion b) says that the anti-holomorphic derivatives of \( \tilde{u} \) up to order \( \ell - 1 \) vanish at \( \zeta_1. \)

The Taylor expansion for \( u \) at \( \zeta_1 \) is given by

\[
\begin{align*}
u(z, x) &= \sum_{i+j=0}^{\ell-1} u^{(i,j)}(\zeta_1, x)(z - \zeta_1)^i(\bar{z} - \bar{\zeta}_1)^j \\
&\quad + \frac{1}{(\ell - 1)!} (z - \zeta_1)^{-\ell} \sum_{i+j=\ell}^{z} \int (z - w)^{\ell-1} u^{(i,j)}(w, x) dw \cdot (z - \zeta_1)^i(\bar{z} - \bar{\zeta}_1)^j
\end{align*}
\]
Clearly b) holds for \( \tilde{u} \equiv u \) if \( \zeta_1(x) \in \overline{\Delta}_- \), and if \( \zeta_1(x) \not\in \overline{\Delta}_- \), we set

\[
\tilde{u} = u - \sum_{\substack{j \geq 1 \atop i+j \leq \ell-1}} \varphi \left( \frac{z - \zeta_1}{\text{Im} \, \zeta_1} \right) \cdot u^{(i,j)}(\zeta_1, x) \cdot (z - \zeta_1)^i \overline{(z - \zeta_1)}^j.
\]

Here, \( \varphi \in C^\infty_0 \left( \frac{1}{2} \Delta \right) \), \( \varphi \equiv 1 \) for \( |z| < \frac{1}{4} \). We see that a) and b) hold. Note that since \( u \in A(\Delta_-) \cap C^\ell(\overline{\Delta}_-) \), \( u^{(0,1)} \) vanishes of order \( \ell - 1 \) on \( \text{Im} \, z = 0 \). We may estimate

\[
|u^{(i,j)}(z, x)| \leq c_2 \cdot |u|_\ell \cdot |\text{Im} \, z|^{\ell-i-j}, \quad i+j \leq \ell, \; j > 1.
\]

Here \( c_2 \) depends only on \( \ell, \; c_1 \). Denote by \( A_{ij} \) the entries of the above sum. For \( z \in \text{supp} \varphi \left( \frac{z - \zeta_1}{\text{Im} \, \zeta_1} \right) \), we have \( |z - \zeta_1| \leq |\text{Im} \, z| \), and for \( z \not\in \text{supp} \), \( A_{ij} \) and all its derivatives vanish. Therefore

\[
|A_{ij}(z, x)| \leq c_3 |u|_\ell \cdot |\text{Im} \, z|^{\ell}.
\]

Now every derivative up to order \( \ell \) of \( A_{ij} \) with respect to \( z \) or \( \overline{z} \) will take away one power of \( |\text{Im} \, z| \) in this estimate and change the constant \( c_3 \), making it dependent on the first \( \ell \) derivatives of \( \varphi \).

We conclude that \( |\tilde{u}|_\ell \leq c_4 \cdot |u|_\ell \leq c_4 c_1 \). \( \square \)

**Lemma 8.** Let \( u \in \Gamma(\ell, c_1), \; \zeta_1 : V \to \Delta \) satisfy the conclusion b) of Lemma 7 and \( u(\zeta_1, x) = 0 \). Then there exists \( u_1 \in \Gamma(\ell-1, c), \) \( c \) depending only on \( \ell \) and \( c_1 \), with

\[
u = (z - \zeta_1) \cdot u_1 \quad \text{on} \; \Delta \times V.
\]

**Proof.** Let \( \sigma_\ell \) denote the \( \ell \)-th order Taylor remainder term in the development of \( u(z, x) \) around \( (\zeta_1(x), x) \). Define

\[
u_1 = \ell - 1 \sum_{i=1}^{\ell-1} u^{(i)}(\zeta_1, x) \cdot (z - \zeta_1)^{i-1} + \frac{\sigma_\ell(z, x)}{z - \zeta_1}.
\]

Then

\[
\left| \sum_{i=1}^{\ell-1} u^{(i)} \cdot (z - \zeta_1)^{i-1} \right|_{\ell-1} \leq c_2 \cdot |u|_\ell \; \text{in} \; \Delta \times V.
\]

The expression \( \left( \frac{\sigma_\ell}{z - \zeta_1} \right)^{(i,j)} \) for \( i+j \leq \ell-1 \) is a sum of terms of the form \( \sigma_\ell^{(p,q)}(z - \zeta_1)^{-r} \) with \( p+q+r \leq \ell, \; r \geq 1 \). From the integral formula for \( \sigma_\ell \) it follows that \( |\sigma_\ell^{(p,q)}(z, x)| \leq c_3 |u|_\ell \cdot |z - \zeta_1|^{\ell-p-q} \). This implies \( |u_1|_{\ell-1} \leq c_4 \cdot |u_\ell| \leq c_4 c_1 \). \( \square \)
PROPOSITION 9. Let $u(\cdot, x) \in \Gamma(\ell, c_1)$ vanish of order $k \leq \ell - 1$ at 0, and

i) $|u^{(k,0)}(0,0)| \geq c_1^{-1}$

ii) $u(z, \cdot) \in C^1(V)$ and $\left| \frac{d}{dx} u(z, \cdot) \right| \leq c_1$ on $\Delta \times V$.

Then, after shrinking $\Delta, V$ to $\Delta_\epsilon, V_\epsilon$, where $\epsilon, \rho$ depend only on $k, \ell, c_1$, there exist $u_k \in \Gamma(\ell - k, c_1), c$ depending only on $k, \ell, c_1$ and maps $\zeta_j : V \to \Delta, j = 1, \ldots, k$ with

a) $u = u_k \prod_{j=1}^{k} (z - \zeta_j)$ on $\Delta_\epsilon \times V$

b) $|u_k(z, x)| \geq 2^{k-3}c_1^{-1}$ on $\Delta \times V$.

PROOF. By the $k$-th order vanishing of $u$ at 0,

$$u(z, 0) = \sum_{i+j=k} u^{(i,j)}(0,0)z^j\bar{z}^j + \sigma_{k+1}(z, 0).$$

Since $u \in A(\Delta_-) \cap C^\ell(\Delta_-)$, we have $u^{(i,j)}(0,0) = 0$ for $j \geq 1$. Therefore

$$u(z, 0) = u^{(k,0)}(0)z^k + \sigma_{k+1}(z) = z^k \cdot \left( u^{(k,0)}(0) + \frac{\sigma_{k+1}}{z^k} \right) = z^k u(z).$$

Since $|\sigma_{k+1}(z, 0)| \leq c_2|u|_{k+1} \cdot |z|^k c_1^{k+1} \leq c_2 c_1 |z|^k$, we see that $|v(z)| \geq \frac{1}{2}c_1^{-1}$ for $|z| < \frac{1}{2c_1^2c_2} =: \epsilon_0$. Therefore $|u(z, 0)| \geq \frac{c_1^k}{2} \cdot c_1^{-1}$ for $\epsilon \leq |z| < \epsilon_0$. By assumption ii), there exists $r_0(\epsilon, c_1)$ with

(2) $|u(z, x)| \geq \frac{\epsilon_0 c_1^k}{4} \epsilon^{-1}$ for $\epsilon \leq |z| < \epsilon_0, |x| < r_0$.

We now see that

(3) $\log u(\cdot, x)$ increases its value by $2\pi ik$ around $|z| = \epsilon_0$ for $|x| \leq r_0$.

Therefore, there exists a map $\zeta_1 : V_\epsilon \to \Delta_{\epsilon_0/2}$ with $u(\zeta_1(x), x) = 0$. Applying Lemma 7 to $u, \zeta_1$ gives a $\tilde{u} \in \Gamma(\ell, c_3)$ with properties a) and b). Next apply Lemma 8 to $\tilde{u}, \zeta_1$ and get $u_1 \in \Gamma(\ell - 1, c_4)$ with

(4) $\tilde{u} = u_1 \cdot (z - \zeta_1)$ on $\Delta_{\epsilon_0} \times V_\epsilon$.

Since $\tilde{u} = u$ on $|z| = \epsilon_0$, we conclude from (4) that $u_1 \neq 0$ on $|z| = \epsilon_0$ and (3) holds for $u_1$ and $k - 1$. 
We may repeat this argument \( k \) times and conclude that

\[
\tilde{u}_j = u_{j+1} \cdot (z - \zeta_{j+1}) \text{ on } \Delta_{e_0} \times V_r,
\]

with \( u_{j+1} \in \Gamma(\ell - k, c_5), \ j = 1, \ldots, k - 1 \). Therefore

\[
\tilde{u} = u_k \cdot \prod_{1}^{k} (z - \zeta_1) \text{ on } \Delta_{e_0} \times V_r.
\]

Since \( u = \tilde{u} = \tilde{u} \) on \( \Delta_{e_0} \times V_r \), this proves a). To prove b), we make the following claim:

Given \( \epsilon > 0 \), \( \exists \epsilon_0 (\epsilon, c_1, k) \) such that \( |\zeta_j(x)| < \epsilon \) for \( |x| < r_j \), \( j = 1, \ldots, k \).

PROOF. Let \( \epsilon < \epsilon_0 \) be given. By (4), \( \tilde{u}(\zeta_1(x), x) = 0 \) for \( |x| < r_1 \).

Since \( \tilde{u}(\zeta_1, x) = 0 = u(\zeta_1, x) \), (2) implies that \( |\zeta_1(x)| \leq \epsilon \). Continuing inductively, assume that \( \zeta_1, \ldots, \zeta_j \) have modulus smaller that \( \epsilon \). By (5), we have

\[
\tilde{u}_j(\zeta_{j+1}, x) = 0 \text{ for } |x| < r_1. \text{ Now } \tilde{u} = u_j \cdot \prod_{1}^{j} (z - \zeta_1). \text{ Outside } \Delta_-, \text{ this } \tilde{u}
\]

never has to coincide with the \( \tilde{u} \) above. Since in Lemma 7 we have \( \tilde{u}_j(\zeta_{j+1}) = u_j(\zeta_{j+1}), \) we conclude that \( \tilde{u}(\zeta_{j+1}) = 0. \) If now also \( u(\zeta_{j+1}) = 0, \) then (2) implies that \( |\zeta_{j+1}(x)| < \epsilon. \) Otherwise one has \( \tilde{u}(\zeta_{j+1}) \neq u(\zeta_{j+1}), \) and since by construction \( \tilde{u} \) differs from \( u \) only in \( \epsilon \)-neighbours of \( \zeta_1, \ldots, \zeta_j, \) we have for some \( q \)

\[
|\zeta_{j+1}(x)| \leq |\zeta_{j+1} - \zeta_q| + |\zeta_q| < 2\epsilon.
\]

This proves the claim.

To conclude the proof, note that given \( \epsilon > 0 \), we have by (2) for \( |z| = \epsilon \)

\[
|u_k(z, x)| = \frac{|u(z, x)|}{\prod_{1}^{k} |z - \zeta_j|} \geq \frac{\epsilon^k}{(2\epsilon)^k} = 2^{-k} c_1^{-1}
\]

for \( |z| < r \), \( r \) chosen as in the claim. Since \( u_k \in \Gamma(\ell - k, c_5) \), we may choose \( \epsilon \) small enough, depending on \( k, c_1, c_5 \) such that this implies \( |u_k(z, x)| \geq 2^{-k} c_1^{-1} \) for \( |z| < \epsilon. \)

Proof of Theorem 5: First we apply Proposition 9 to \( u \), which gives \( u = u_k \prod_{1}^{k} (z - \zeta_j) \) on \( \Delta_- \times V. \)

We will successively divide the \((z - \zeta_j)\) out of \( uh. \) To retain estimates on the way, we need to take into account those for \( uh^m \in \Gamma^-(\ell_1, c_1(m)) \) which by Proposition 6 we may assume to lie in \( \Gamma(\ell_1, c_1(m)). \)

a) Note that \( \frac{(uh)^2}{z - \zeta_1} = (uh^2) \cdot u_k \prod_{1}^{k} (z - \zeta_j) \) on \( \Delta_- \times V. \) By the assumption concerning \( uh^2, \) and Proposition 9 concerning \( u_k, \) we may conclude that

\[
\frac{(uh)^2}{z - \zeta_1} \in \Gamma^-(\ell_1 - k, c_2).
\]
We wish to show \( e^{-\frac{uh}{z - \zeta_1}} \in \Gamma^-(\ell, c) \) for some \( \ell, c \). If \( \zeta_1 \in \Delta_\infty \), then (6) implies that \((hu)(\zeta_1, x) = 0\), and we are done by Lemma 8 with \( \ell = \ell_1 - 1 \).

If \( \zeta_1 \not\in \Delta_\infty \), we proceed as follows. Since \( uh \in \Gamma(\ell_1, c_1) \), we may apply Lemma 7 to \( uh, \zeta_1 \):

\[
(\tilde{u}h) = \sum_{i=0}^{\ell-1} (uh)^{(i)}(\zeta_1, x)(z - \zeta_1)^i + \sigma_{\ell}(z, x).
\]

Now

\[
\frac{(\tilde{u}h)^2}{z - \zeta_1} = \frac{(uh^{(0)})^2}{z - \zeta_1} + 2uh^{(0)} \left( \sum_{i=1}^{\ell-1} uh^{(i)}(z - \zeta_1)^{i-1} + \frac{\sigma_{\ell}}{z - \zeta_1} \right)
+ \frac{1}{z - \zeta_1} \left( \sum_{i=1}^{\ell-1} uh^{(i)}(z - \zeta_1)^i + \sigma_{\ell} \right)^2.
\]

Since \( \tilde{u}h = uh \) on \( \Delta_\infty \times V \), the left hand side is in \( \Gamma^-(\ell_1 - k, c_2) \). As in the proof of Lemma 8 we may estimate the second and third terms on the right hand side to see that they are in \( \Gamma(\ell_1 - 1, c_3) \). Therefore \( \frac{(uh^{(0)})^2}{z - \zeta_1} \in \Gamma^-(\ell_1 - k, c_4) \). By differentiating, this implies \( \frac{(uh^{(0)})^2}{(z - \zeta_1)^{i-1-k+1}} \in \Gamma^-(0, c_5) \), and \( \frac{uh^{(0)}}{(z - \zeta_1)^p} \in \Gamma^-(0, c_6) \).

Here, \( p = \left[ \frac{\ell_1 - k + 1}{2} \right] \). Assume \( p \geq 2 \), and consider the integral

\[
\int_{\overline{z}}^z \frac{uh^{(0)}}{(w - \zeta_1)^p} dw = \frac{1}{1 - p} \left( \frac{uh^{(0)}}{(z - \zeta_1)^{p-1}} - \frac{uh^{(0)}}{-\frac{i}{2} - \zeta_1} \right),(p-1),
\]

where we integrate along a straight line. The left hand side is in \( \Gamma^-(1, c_7) \), and since \( \zeta_1 \not\in \Delta_\infty \), the second term on the right hand side, independent of \( z \), is bounded by \( \frac{1}{p-1}2^{p-1}c_7 \). We conclude that \( \frac{uh^{(0)}}{(z - \zeta_1)^{p-1}} \in \Gamma^-(1, c_8) \). Repeating this gives \( \frac{uh^{(0)}}{z - \zeta_1} \in \Gamma^-(p - 1, c_9) \). Now we have on \( \Delta_\infty \times V \):

\[
\frac{uh}{z - \zeta_1} = \frac{uh^{(0)}}{z - \zeta_1} + \sum_{i=1}^{\ell-1} uh^{(i)} \cdot (z - \zeta_1)^{i-1} + \frac{\sigma_{\ell}}{z - \zeta_1} \in \Gamma^- (p - 1, c_{10}).
\]

b) We prove by induction the following statement:

\[
(7i) \quad u_k \prod_i^k (z - \zeta_j) h^m \in \Gamma^-(\ell, c) \quad \forall m \geq 0.
\]

\((i = 1)\) is the assumption of the Theorem. \((i = 2)\) was proved in part a) for \( m = 1 \). We will show that \((7i)\) implies \((7i+1)\). Let \( g_m = u_k \prod_i^k (z - \zeta_j) h^m \in \Gamma^- (\ell, c) \), and note that
\[
\frac{g_m^2}{z - \zeta_i} = g_{2m} \cdot u_k \cdot \prod_{i+1}^k (z - \zeta_i) \text{ on } \Delta_\times V.
\]

The right hand side is in \( \Gamma^-(\ell - k, c_1) \). We may now conclude as in a) that this implies

\[
\frac{g_m}{z - \zeta_1} \in \Gamma^-(\ell_1, c_2) \text{ where } \ell_1 = \left\lfloor \frac{\ell - k + 1}{2} \right\rfloor - 1.
\]

c) Letting \( m = 1 \) in \( i = k \), we get \( u_k \cdot h \in \Gamma^-(\ell, c) \). Now the conclusions of Proposition 9 allow us to infer that \( h = \frac{(u_k \cdot h)}{u_k} \in \Gamma^-(\ell - k, c_1) \). The expression given for \( \ell_1 \) in Theorem 5 now follows from the one in part b) above. \( \square \)

REFERENCES