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About a theorem of Paolo Codecà’s and omega estimates for arithmetical convolutions. Second part

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About a Theorem of Paolo Codecà's
and Omega Estimates for Arithmetical Convolutions

Second Part

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1. - Introduction

Consider the real valued functions $h$ defined on $[1, \infty)$

$$h(x) = \sum_{n \leq z} \alpha(n)n^a f \left( \frac{x}{n} \right),$$

where $\alpha(n)$ is a sequence of real numbers satisfying

$$\sum_{n \leq z} |\alpha(n)| = O(x),$$

$-1 \leq a < 0$, $f$ is a periodic function of period 1, of bounded variation on $[0, 1]$ and such that

$$\int_{0}^{1} f(u)du = 0,$$

and $z = z(x) \leq x$ is a positive, strictly increasing, continuous and unbounded function ($z$ will always be assumed to satisfy these properties in the sequel).

We say that $h$ is $C_z(a, \alpha, f)$.

In the first part of this work [8], inspired by an article of Codecà's [1], I considered functions $g$ nearly $C_z(-1, \alpha, f)$ (i.e. short of a $o(1)$), where $\alpha$ possesses an asymptotic mean $K$, and such that

$$g - K \int_{1}^{\infty} u^{-1} f(u)du$$

is nearly \( C_z(-1, \alpha, f) \) for some \( z = z(x) = o(x) \), as \( x \to \infty \).

In the case where \( z \) may be taken small enough, I obtained a general expression for the mean of \( g \) on an arithmetical progression \( An + B, \ n \leq x \), (Theorem 1), from which I could then, for particular functions \( g \), prove omega estimates by suitably choosing the parameters \( A, B, \) and \( x \).

Among the functions for which I found this process to be successful are the classical error terms \( H \) and \( E \) related to the Euler function \( \phi \) and to the sum-of-divisors function \( \sigma \), an error term of Landau (see e.g. [14]) related to the function \( n/\phi(n) \), and the "Chowla-Walum functions"

\[
G_{a,k}(x) := \sum_{n \leq x} n^a \psi_k \left( \frac{x}{n} \right)
\]

(where \( \psi_k(y) = B_k(\{y\}) \) denotes the \( k \)-th Bernoulli polynomial of argument the fractional part \( \{y\} \) of \( y \)) when \( a \leq -1 \). This brings us to the main purpose of this sequel.

The \( G_{a,k} \) are related to various divisor problems (see e.g. [5], [6], [9], [10], [11]). Conjectures were proposed as to the "best" \( O \) and \( \Omega \) estimates satisfied by the functions \( G_{a,k} \), originating with the Piltz-Hardy-Landau conjecture on the famous Dirichlet divisor problem, generalized in 1963 by Chowla and Walum [2]. If we gather the conjectures that appear reasonable so far in view of the various investigations made by a number of authors, we can state them in the compact form described below.

Let \( \alpha_k(a) \) and \( \beta^*_k(a) \), where \( * \) is allowed to denote \( +, - , \pm, \) or nothing at all (i.e. not even a blank), be the smallest \( \alpha \), respectively the largest \( \beta \), for which \( G_{a,k}(x) = O(x^{a+\varepsilon}) \), resp. \( G_{a,k}(x) = \Omega_*(x^{\beta-\varepsilon}) \), for every \( \varepsilon > 0 \). Set

\[
g(a) = \begin{cases} \frac{a}{2} + \frac{1}{4} & \text{if } a \geq -\frac{1}{2}, \\ 0 & \text{if } a \leq -\frac{1}{2}. \end{cases}
\]

**CONJECTURE.** For every real number \( a \), every positive integer \( k \), and for \( * \) denoting \( + \) and \( - \), we have

\((O.k.a.)\) \quad \alpha_k(a) \leq g(a) \quad \text{and} \quad \beta^*_k(a) \geq g(a).\)

**REMARKS.**

(i) The Piltz-Hardy-Landau conjecture is in this notation \((O.1.0)\); the Chowla-Walum conjecture is: \((O.k.a)\) for all \( a \geq 0 \) and all positive integers \( k \).

(ii) The assertions "\((O.k.a)\) and \((\Omega.k.a)\)" and "\( \alpha_k(a) = g(a) \)" are equivalent.
(iii) For a brief review of the results known to date towards these conjectures see [11].

In the first part of this paper [8] (see the Addendum), the truth of $(\Omega_{\pm}k, a), \ k = 1, 2, \ldots$, is proved for $a \leq -1$. Here, through an extension of the main result of [8] to $C_\alpha(a, \alpha, f), \ -1 < a < 0$, for suitable $z$ and $\alpha$ (Theorem 1 in Section 2 below), we obtain $\Omega$-estimates for the $G_{a,k}(x)$ (Theorem 2 just below), and as a corollary the truth of $\Omega_{\pm}k, a), \ k = 1, 2, \ldots$, for $a \leq -\frac{1}{2}$.

**THEOREM 2.** For $-1 < a < 0$ and every positive integer $k$, we have

\[G_{a,k}(x) = \Omega_\pm \left( \exp \left\{ (1 + o(1)) \xi_{a,k} \left( \frac{(-a/2)^{1+a}}{1 + a} \frac{(\log x)^{1+a}}{\log \log x} \right) \right\} \right),\]

where

\[\xi_{a,k} = \begin{cases} 
1 & \text{if } k = 1 \text{ or } k \text{ is even}, \\
2 & \text{if } k > 1 \text{ is odd}.
\end{cases}\]

As another corollary of Theorem 2, we obtain in Section 3:

**THEOREM 3.** Let $E_a(x)$ be the error term

\[E_a(x) := \sum_{n \leq x} \sigma_a(n) \left( \frac{\zeta(1 + a)}{1 + a} x^{1+a} + \frac{\zeta(1 + a)x - \zeta(-a)}{2} \right), \quad a \neq -1, 0.
\]

Then, for $\frac{25}{38} < |a| < 1$,

\[E_a(x) = \Omega_\pm \left( x^{\frac{\text{sgn}(a)}{2}} \exp \left\{ (1 + o(1)) \frac{|a|/2 - |a|}{1 - |a|} \frac{(\log x)^{1-|a|}}{\log \log x} \right\} \right).
\]

It appears that no nontrivial $\Omega$-estimate for $E_a(x)$, with $-1 < a < -\frac{1}{2}$, was known so far; and, when $a$ is positive, (1.7) improves in the indicated range both results

\[E_a(x) = \Omega_\pm \left( x \log x^{\frac{3}{2} + \frac{1}{2}} \right)\]

and

\[E_a(x) = \Omega(x^a)\]

of Hafner’s [3], and should be compared, on the one hand with MacLeod’s [7]

\[\lim_{x \to \infty} \frac{E_a(x)}{x^a} = \pm \frac{\zeta(a)}{2}, \quad a > 1,
\]
and our \([12]\)

\[(1.12)\]
\[
\lim_{x \to \infty} \frac{E_1(x)}{x \log \log x} \leq \pm \frac{e^\gamma}{2},
\]

and, on the other hand, with the following consequence of conjecture \((O.1.a)\)

\[(1.H)\]
\[
E_a(x) = O(x^{a+\varepsilon}), \quad a \geq \frac{1}{2},
\]

(see (3.10)).

Finally in Section 4 we give another application of Theorem 1. We define the functions

\[(1.13)\]
\[
P_a(x) = \sum_{n \leq \sqrt{x}} n^a \cos \left( \frac{x}{n} \right)
\]

and

\[(1.14)\]
\[
Q_a(x) = \sum_{n \leq \sqrt{x}} n^a \sin \left( \frac{x}{n} \right).
\]

In [8] we prove (see [4])

\[(1.15)\]
\[
P_{-1}(x) = \Omega_\pm (\log \log x)
\]

and

\[(1.16)\]
\[
Q_{-1}(x) = \Omega_\pm \left( (\log \log x)^{\frac{1}{2}} \right).
\]

Here we obtain

**Theorem 4.** For \(-1 < a < 0\), we have

\[(1.17)\]
\[
P_a(x) = \Omega_\pm \left( \exp \left\{ (1 + o(1)) \frac{(-a/2)^{1+a}}{1+a} \frac{\log x}{{\log \log x}} \right\} \right)
\]

and

\[(1.18)\]
\[
Q_a(x) = \Omega_\pm \left( \exp \left\{ (1 + o(1)) 2^a \frac{(-a/2)^{1+a}}{1+a} \frac{\log x}{{\log \log x}} \right\} \right).
\]

2. - The main result

Let the notation be that of Section 1 and consider a function \(h\) being some \(C_\ast(a, \alpha, f)\), where \(-1 < a < 0\) and, in addition to (1.2), the arithmetical function \(\alpha\) satisfies the submultiplicative property

\[(2.1)\]
\[
|\alpha(nm)| = O(|\alpha(n)\alpha(m)|)
\]
for all positive integers \( n \) and \( m \). Let \( A = A(x) > 0 \) be an integer valued function, and \( B = B(x) \geq 0 \) (we do not require that \( B \) be an integer: see [8, Addendum]). Then we have

**THEOREM 1.** Set

\[
\begin{align*}
    u &= u(x) := x(Ax + B), \\
    v_i &= \left( \frac{Ax}{2^i} + B \right), \quad i \geq 0, \\
    \alpha_b(A) &= \sum_{d|A} |\alpha(d)|d^b,
\end{align*}
\]

and suppose that there exists a function \( \eta = \eta(x) \) decreasing to 0 as \( x \to \infty \) and such that

\[
(2.2) \quad A \leq v_N,
\]

where \( N := \left\lfloor \frac{\log \eta}{\log 2} \right\rfloor \). Then

\[
(2.3) \quad \frac{1}{x} \sum_{n \leq x} h(An + B) = \sum_{k \leq u} \alpha(k)k^a \left( \frac{1}{k^*} \sum_{n \leq k^*} f \left( \frac{n}{k} + \frac{B}{k} \right) \right) + O \left( \frac{u^{a+2}}{x} + \nu_N^a \alpha_0(A) + \eta \alpha_a(A) \right),
\]

where \( k^* \) denotes \( \frac{k}{(A,k)} \).

**PROOF.** The proof goes along the same line as that of Theorem 1 in [8]. We let \( w(k) \) be the inverse function of \( v(y) := x(Ay + B) \), \( 1 \leq y \leq x \) if \( k \geq u(1) \), and \( w(k) = 1 \) otherwise, and we obtain as in [8]

\[
(2.4) \quad \frac{1}{x} \sum_{n \leq x} h(An + B) = \beta + \delta + \varepsilon,
\]

where

\[
(2.5) \quad \beta := \sum_{k \leq u} k^a \alpha(k) \left( \frac{1}{k^*} \sum_{n \leq k^*} f \left( \frac{n}{k} + \frac{B}{k} \right) \right),
\]
where by (1.2),
\[
\varepsilon = O \left( \frac{1}{x} \sum_{k \leq u} k^a k^* |\alpha(k)| \right) = O \left( \frac{u^{a+2}}{x} \right),
\]
and where
\[
\delta = O \left( \frac{1}{x} \sum_{k \leq u} k^{a-1} (A, k) |\alpha(k)| w(k) \right).
\]
In order to estimate \( \delta \) we define, as in [8],
\[
R_i := \begin{cases} 
\{ k \in \mathbb{N} | \max(v(1), v_i) < k \leq v_{i-1} \} & \text{if } i = 1, 2, \ldots, M := \left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1, \\
\{ k \in \mathbb{N} | k \leq v := v(1) \} & \text{if } i = M + 1,
\end{cases}
\]
and may thus rewrite the sum in (2.7) as
\[
\sum_{i=1}^{M+1} \sum_{k \in R_i} (A, k) k^{a-1} |\alpha(k)| w(k) \leq \sum_{1} + \sum_{2} + \sum_{3},
\]
say, where \( \sum_{3} := \sum_{k \leq v} k^a |\alpha(k)| \), and
\[
\sum_{1,2} := \sum_{i} \frac{x}{2i-1} \sum_{k \in R_i} (A, k) k^{a-1} |\alpha(k)|,
\]
the ranges of summation in (2.9) being respectively
\[
1 \leq i \leq N \text{ and } N + 1 \leq i \leq M
\]
(\( \sum_{3} \) corresponding to \( i = M + 1 \)). By (1.2)
\[
\sum_{3} = O(v^{a+1}) = O(u^{a+2}).
\]
By (2.1) the inside sum on the right side of (2.9) is a \( O \) of
\[
\sum_{d | A} d^a |\alpha(d)| \sum_{d^* \geq \max(1, \frac{x}{d})} d^{a-1} |\alpha(d^*)|,
\]
which in turn is, by (1.2), a \( O \) of
\[
\sum_{d | A} d^a |\alpha(d)| = \alpha_a(A),
\]
and, in the case of $\sum_1$, is by (2.2) a $O$ of

$$\sum_{d|A} |\alpha(d)|v_i^d \leq v_N^s \alpha_0(A).$$

Thus

$$\sum_2 = O(x\eta \alpha_a(A))$$

and

$$\sum_1 = O(xv_N^s \alpha_0(A)).$$

The theorem now follows from (2.6), (2.7), (2.8), (2.10), (2.14) and (2.15).

3. - Proofs of Theorems 2 and 3

In this section we let $h$ be a $G_{a,\ell}$ as in (1.4). Thus $z = \sqrt{x}$, $f = \psi_\ell$ and $\alpha(n) = 1$ for all $n$. A well known identity for Bernoulli polynomials [13, (6.1)] implies that

$$\frac{1}{k^*} \sum_{n \leq k^*} \psi_\ell \left( \frac{n}{k^*} + \frac{B}{k} \right) = \frac{1}{k^*} \psi_\ell \left( \frac{B}{(A, k)} \right),$$

whence, if $B = O(A)$ and $A = o(x)$, an application of Theorem 1 yields

$$\frac{1}{x} \sum_{n \leq x} G_{a,\ell}(An + B)$$

$$= \sum_{k \leq u} (A, k)^\ell k^{a-\ell} \psi_\ell \left( \frac{B}{(A, k)} \right) + O \left[ A(Ax)^{\frac{2}{3}} + A^a \alpha_0(A) \right] + o(\alpha_0(A))$$

and, with a special choice of the parameters $A$, $B$ and $x$, the

**Lemma 1.** If $\ell = -1 < a < 0$ and

$$A = \prod_{p \leq y} p = x^{-\frac{\log y}{\ell}},$$

where $P_\ell = \{p \equiv 1(2), p \text{ prime}\}$ if $\ell$ is either 1 or even, and $P_\ell = \{p \equiv 1(3), p \text{ prime}\}$ if $\ell > 1$ is odd, then there are non negative numbers $B_i \prec A$, $i =$
1, 2, such that as $y \to \infty$,

$$
\frac{1}{x} \sum_{n \leq x} G_{\alpha, \ell}(An + B) = \begin{cases} 
\Omega_+(\sigma_{\alpha}(A)), & \text{if } B = B_1 \\
\Omega_-(\sigma_{\alpha}(A)), & \text{if } B = B_2
\end{cases}
$$

PROOF. We shall make use of the following elementary properties of the Bernoulli polynomials (see [13], Chapter I).

\((i)\) With the choice of $A$ and with $B = 0$ the right side of (3.2) becomes, for a certain set $\mathcal{D}$ of integers containing 1,

$$
\psi_\ell(0) \psi_\ell \left( \frac{1}{2} \right) < 0 \quad \text{when } \ell \text{ is even}, \\
\psi_\ell \left( \frac{1}{3} \right) \psi_\ell \left( \frac{2}{3} \right) < 0 \quad \text{when } \ell \text{ is odd}.
$$

\((ii)\) Let $B = A - 1$ and $\ell = 1$. The right side of (3.2) becomes

$$
\psi_\ell(0) \sum_{d \mid A} d^a \sum_{d \leq \frac{\delta}{\delta^x} \in \mathcal{D}} d^\alpha - \ell + o(\sigma_{\alpha}(A)),
$$

and we thus obtain the $\Omega_-$-estimate for $\ell = 1$, and one of the $\Omega$-estimates for $\ell$ even.

\((iii)\) With $B = A/2$ we obtain, by virtue of (3.4), the other $\Omega$-estimate for $\ell$ even.

\((iv)\) Finally, when $\ell > 1$ is odd, each one of the choices $B = A/3, B = 2A/3$ yields one of the $\Omega$-estimates (again we use (3.4)).

Now we need an estimate for $\sigma_{\alpha}(A)$.

**Lemma 2.** For $-1 < a < 0$ and

$$
A = \prod_{p \leq y} p,
$$

where $(k, n) = 1$, we have

$$
\sigma_{\alpha}(A) = \exp \left( (1 + o(1)) \left\{ \frac{\phi(n)^a}{a + 1} \left( \frac{\log A^a}{\log \log A} \right) \right\} \right).
$$
PROOF. For this choice of $A$ we have

$$\log \sigma_a(A) = \sum_{p \leq y \atop p \equiv k(n)} \log(1 + p^a)$$

(3.8)

$$= \sum_{p \leq y \atop p \equiv k(n)} p^a + \begin{cases} O(1), & -1 < a < -\frac{1}{2}, \\ O(\log \log y), & a = -\frac{1}{2}, \\ O(y^{2a+1}), & -\frac{1}{2} < a < 0. \end{cases}$$

Now the Euler summation formula and the prime number theorem for arithmetical progressions yield

(3.9) $$\sum_{p \leq y} p^a = (1 + o(1)) \frac{\phi(n)^a}{a+1} \frac{y^{a+1}}{\log y},$$

and the lemma follows, after another application of the prime number theorem.

Theorem 2 is now a direct consequence of Lemmata 1 and 2.

As for theorem 3, it easily follows from Theorem 2, from \[9, (1.3)\] and from \[10, (5.4)\]

(3.10) $$E_a(x) = -x^a G_{a,1}(x) - G_{a,1}(x) + O(x^{\delta}), \quad |a| < 2,$$

and from \[10, (5.4)\]

(3.11) $$\alpha_1(b) \leq \frac{b}{2} + \frac{25}{76}, \quad b > 0.$$

4. - Proof of Theorem 4

We shall need

LEMMA 3. With the notation of Theorem 1, we have

(4.1) $$\sum_{n \leq k^*} \sin \left(2\pi \left(\frac{n}{k^*} + \frac{B}{k}\right)\right) = \begin{cases} \sin \left(\frac{2\pi B}{k}\right) & \text{if } k^* = 1, \\ 0 & \text{otherwise}. \end{cases}$$

The proof of which is quite straightforward.

To prove (1.17) we set

(4.2) $$A = \prod_{2 < p \leq y} p = x^{-\epsilon_4}, \quad B = 0, \quad z = \sqrt{x},$$
and we obtain, with Theorem 1 and Lemma 3,

\[
\frac{1}{x} \sum_{n \leq x} P_a(2\pi A \nu + B) = \sum_{k | A} k^a + o(\sigma_a(A)) = \sigma_a(A)(1 + o(1))
\]

which, with Lemma 2, implies the \( \Omega_+ \)-estimate; as for the \( \Omega_- \)-estimate, it is obtained similarly with \( B = A/2 \) instead of \( B = 0 \) in (4.2).

To prove the \( \Omega_- \)-estimate in (1.18), we set

\[
A = 4B = x^{-\alpha_2}, \quad B = \prod_{\substack{p \leq y \quad \text{p \equiv 1(4)}}} p, \quad z = \sqrt{x}.
\]

Theorem 1 and Lemma 3 yield this time

\[
\frac{1}{x} \sum_{n \leq x} Q_a(2\pi B(4n + 1)) = 4^a \sum_{k | B} k^a + o(\sigma_a(A)) = 4^a \sigma_a(B)(1 + o(1)),
\]

and we conclude again with Lemma 2. The \( \Omega_- \)-estimate is similarly obtained with

\[
A = \frac{4B}{3} = x^{-\alpha_2} = 4D, \quad D = \prod_{\substack{p \leq y \quad \text{p \equiv 1(4)}}} p, \quad z = \sqrt{x}.
\]

\[\Box\]

**REFERENCES**


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