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0. - Introduction

In the last few years R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss have developed a method of interpolation for families of Banach spaces that generalizes the complex method of A.P. Calderon (see [CCRSW 1] and [CCRSW 2]).

As a further development we presented in [T-V 2] a method of interpolation for families of quasi-Banach spaces. More precisely, we considered quasi-Banach spaces $B(\varphi)$ associated with the points $e^{i\varphi}$ of the boundary $\Gamma$ of the open unit disk $D$ in the complex plane $\mathbb{C}$. Intermediate spaces $B(z)$, for each $z \in D$, were constructed in such a way that interpolation theorems for linear operators hold.

The aim of this note is to find some results that easily yield the identification of families of quasi-Banach spaces commonly used in analysis.

A large number of interesting spaces of functions in analysis has a norm defined by a sublinear operator, generally a maximal operator. Consider, for example, the case of $H^p$-spaces. If $S$ denotes the space of test functions in $\mathbb{R}^n$ and $\varphi \in S$ with $\varphi(0) \neq 0$, we consider the operator $\mathcal{M}$ defined by

$$\mathcal{M}(x, f) = \sup_{|x-y| \leq a t} |f \ast \varphi_t(y)|,$$

for every tempered distribution $f$. We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\mathcal{M}(\cdot, f) \in L^p(\mathbb{R}^n)$ and $\|f\|_{H^p} = \|\mathcal{M}(\cdot, f)\|_{L^p}$, $0 < p \leq \infty$.

Observe that $\mathcal{M}$ can be described as the composition of two operators: the first one is linear and it associates to $f$ the convolution $f \ast \varphi_t$, the second one is a typical maximal operator. The aim of section 2 is to prove an interpolation theorem for operators of this kind.

In section 4 we study the relations between our interpolation method and the real interpolation method developed by J. Peetre and J.L. Lions. The theorem
that we obtain yields the identification of complex-intermediate spaces from the identification of real-intermediate ones.

As an example, in section 5, from the Fefferman-Riviere-Sagher result

\[(H^{p_0}, L^\infty)_{\theta,p} = H^p, \quad \frac{1}{p} = \frac{1 - \theta}{p_0},\]

we are able to characterize the intermediate spaces when on the boundary we have \(H^p\) spaces. As we shall see, these intermediate spaces are again \(H^p\) spaces.

Finally, I would like to thank my advisors, Richard Rochberg and Guido Weiss, for introducing me to the subject and for their valuable help during my graduate studies.

1. - The complex interpolation method

We now briefly describe the complex interpolation method for families of quasi-Banach spaces, as given in [T-V 2].

For each \(\vartheta \in \Gamma\) we assign a quasi-Banach space \((B(\vartheta), \| \cdot \|_\vartheta)\), and denote by \(c(\vartheta)\) the constants in the quasi-triangle inequalities.

We say that the family \(\{B(\vartheta)\}\) is an interpolation family (of quasi-Banach spaces) if each \(B(\vartheta)\) is continuously embedded in a Hausdorff topological vector space \(U\), the function \(\vartheta \to \|b\|_\vartheta\) is measurable for each \(b \in \bigcap_\vartheta B(\vartheta)\), and \(\log c(\vartheta) \in L^1(\Gamma)\).

The subset \(B\) of \(\bigcap_\vartheta B(\vartheta)\) of those elements \(b\) such that \(\log^+ \|b\|_\vartheta \in L^1(\Gamma)\) is called the log-intersection of the family \(\{B(\vartheta)\}\).

By \(G = G(B(\cdot), \Gamma)\) we denote the set of functions defined by

\[G = \{g(z) = \sum_{n=1}^{N} \varphi_j(z)a_j : \varphi_j \in N^+(D), \ a_j \in B, \ j = 1, \ldots, N \text{ and } \|\|g\||_\infty < \infty\}\]

where \(N^+(D)\) denotes the positive Nevanlinna class (see [Dur], ch. 2), and

\[\|\|g\||_\infty = \text{Ess sup}_{\vartheta} |g(\vartheta)|_{\vartheta}.

For every \(a \in B\) and \(z \in D\) we define

\[|a|_z = \text{Inf}\{\|\|g\||_\infty : g \in G, \ g(z) = a\}.

If \(N_z\) denotes the set of elements of \(B\) such that \(|a|_z = 0\), the completion \(B(z)\) of \((B/N_z, |\cdot|_z)\) will be called the interpolation space at \(z\) of the family \(\{B(\vartheta)\}\).
It can be proven that \((B(z), |\cdot|_z)\) is a quasi-Banach space with quasi-triangle inequality constant \(c(z) = \exp \int \log c(\theta) P_z(\theta) d\theta\), where \(P_z(\theta)\) is the Poisson kernel “centered” at \(z\).

2. - An interpolation theorem for log-subharmonic operators

Let \(M\) be the set of measurable complex-valued functions on some measure space \((Y, \nu)\). An operator \(M\) mapping \(M\) into the class \(N\) of non-negative-valued measurable functions on some other measure space \((X, \mu)\) is said to be of maximal type provided it satisfies:

\[(a) \quad M(\lambda a) = |\lambda|M a \quad \forall \lambda \in \mathbb{C}, \forall a \in M;\]
\[(b) \quad M(a) = M(|a|) \quad \forall a \in M;\]
\[(c) \quad M(a)(x) \leq M(b)(x) \quad \text{if } |a(y)| \leq |b(y)|, \ a, b \in M;\]
\[(d) \quad M \left[ \int_{\Gamma} f(\cdot, \theta) d\theta \right](x) \leq \int_{\Gamma} M(f(\cdot, \theta))(x) d\theta.\]

If \\{B(\theta)\}\ is an interpolation family, with containing space \(U\), we say that an operator \(\tilde{M} : U \to N\) is a log-subharmonic operator associated to the family \\{B(\theta)\}\ if it can be expressed as the composition \(M \cdot L\) of a linear operator \(L\) mapping \(U\) into \(M\) and of a maximal-type operator \(M\).

The reason for such a name is clear if we note that

**Proposition 2.1.** Let \(\tilde{M}\) be a log-subharmonic operator associated to the family \(\{B(\theta)\}\). If \(f(z) = \sum_{j=1}^{n} \varphi_j(z) a_j \in \mathcal{H}(B(\cdot), \Gamma)\), then \(\log \tilde{M}(f(z))(x)\) is a subharmonic function of \(z\) in the disk, for a.e. fixed \(x\).

Before proving the proposition we recall the following result, due to Radö (see [Aup]):

**Lemma 2.2.** Let \(\phi\) be a positive function defined on a domain \(\Omega\); then, \(\log \phi\) is subharmonic in \(\Omega\) if and only if the function

\[z \to |e^{\alpha z}|\phi(z)\]

is subharmonic in \(\Omega\), for every complex number \(\alpha\).

We shall refer to such functions \(\phi\) as log-subharmonic.

**Proof of 2.1.** By Radö’s criterion (lemma 2.2), it is enough to show that the functions \(z \to |e^{\alpha z}|\tilde{M}(f(z))(x)\) are subharmonic for every complex number
α. Fix $z_0 \in D$ and let $\rho > 0$ be such that the closure of the ball of radius $\rho$ centered at $z_0$ is contained in $D$. Since $L(f(z))$ is analytic, $\log|L(f(z))|$ is subharmonic in the disk and, thus, we have

$$|e^{\alpha z_0}| \tilde{M}(f(z_0))(x) = M(|e^{\alpha z_0}| |L(f(z_0))|)(x)$$

$$\leq M \left[ \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha (z_0 + \rho e^{i\theta})}| |L(f(z_0 + \rho e^{i\theta})|d\theta \right](x)$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} M \left[ |e^{\alpha (z_0 + \rho e^{i\theta})}| |L(f(z_0 + \rho e^{i\theta})|\right](x)d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha (z_0 + \rho e^{i\theta})}| \tilde{M}(f(z_0 + \rho e^{i\theta}))(x)d\theta.$$ 

The $H^p$-spaces introduced above are just one of several cases where the norm of an element $f$ in a given function space is defined by evaluating the Lebesgue norm of $\tilde{M}f$, $\tilde{M}$ a log-subharmonic operator.

For all these spaces we can apply the following interpolation theorem:

**Theorem 2.3.** Let $\tilde{M}$ be a log-subharmonic operator associated to an interpolation family of quasi-Banach spaces $\{B(\theta)\}$. Suppose that

$$\|	ilde{M}a\|_{L^{p_\theta}(X)} \leq \eta(\theta) |a|_\theta, \quad \forall a \in B,$$

where $0 < p(\theta) \leq \infty$ and $\frac{1}{p} \in L^1(\Gamma)$.

If $\log \eta \in L^1(\Gamma)$, then for all $a \in B$

$$\|	ilde{M}a\|_{L^{p_\theta}(X)} \leq \eta(z) |a|_z,$$

where $\frac{1}{p(z)} = \int_\Gamma \frac{1}{p(\theta)} P_z(\theta)d\theta$ and $\eta(z) = \exp \int_\Gamma \log \eta(\theta) P_z(\theta)d\theta$.

**Proof.** For $a \in B$, there exists $f(z) = \sum_{j=1}^n \varphi_j(z)a_j \in \mathcal{G}(B(\cdot), \Gamma)$ such that $f(z_0) = a$ and $\|f\|_{L^\infty} \leq |a|_{z_0} + \epsilon$, for $z_0 \in \overline{D}$ and $\epsilon > 0$ fixed. To prove the theorem it is enough to show that the function $z \to \log \|	ilde{M}(f(z))(\cdot)|_{L^{p_\theta}(X)}$ is subharmonic in the disk. Indeed

$$\log \|	ilde{M}(f(0))(\cdot)|_{L^{p_\theta}(X)} \leq \frac{1}{2\pi} \int_0^{2\pi} \log \|	ilde{M}(f(\theta))(\cdot)|_{L^{p_\theta}(X)}d\theta$$
is equivalent to

\[ \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(x_0)}(X)} \leq \exp \int_0^{2\pi} \log \|\tilde{M}(f(\vartheta))(\cdot)\|_{L^{p(\vartheta)}(X)} P_{z_0}(\vartheta) d\vartheta \]

via the Möbius transformation

\[ w = \frac{z - z_0}{1 + \bar{z}_0 z}, \quad d(\arg z) = P_{z_0}(\vartheta) d\vartheta. \]

Therefore,

\[ \|\tilde{M}a\|_{L^{p(x_0)}(X)} = \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \]

\[ \leq \exp \int_0^{2\pi} \eta(\vartheta) |f(\vartheta)| P_{z_0}(\vartheta) d\vartheta \]

\[ \leq \eta(z_0) \|f\|_{L^{\infty}} \]

\[ \leq \eta(z_0) (|a|_{z_0} + \varepsilon). \]

Thus, letting \( \varepsilon \to 0 \), we get

\[ \|\tilde{M}a\|_{L^{p(x_0)}(X)} \leq \eta(z_0) |a|_{z_0}. \]

To show our claim, we note that since the function \( p(z) \) is strictly positive on \( D \), for any \( \rho > 0 \) we can find \( r > 0 \) such that \( 0 < r < p(z) \) if \( z \in B_\rho(z_0) = \{ z : |z - z_0| \leq \rho \} \subset D \).

Moreover, since subharmonicity is a local property, it suffices to show

\[ \log \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(x_0)}(X)} \]

\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(x_0 + \rho e^{i\vartheta})}(X)} d\vartheta \]

for any such \( \rho > 0 \).

Define

\[ \ell(z) = 1 - \frac{r}{p(z)} \]

and let \( g \) be a simple and positive function on \( X \) of the form \( g(x) = \sum_{j=1}^N \alpha_j \chi_{E_j} \),

with \( \alpha_j > 0 \) and \( E_j \) pairwise disjoint sets of finite measure. Then \( g(x)\ell(x) \) is a log-subharmonic function in the disk for every fixed \( x \). Moreover, since \( \tilde{M}(f(z))(x) \) is a log-subharmonic function, also \( \tilde{M}(f(z))^\ell(x) \) is log-subharmonic in \( D \) for
every fixed $x$. We also have that $\tilde{M}(f(z))(\cdot)$ is in $L^r_{\text{loc}}(X)$ (i.e. $(\tilde{M}(f(z)))^r(\cdot)$ is integrable on sets of finite measure) for every $z \in D$, since $\tilde{M}(f(z)) \in \cap L^{p(\theta)}(X)$.

So

$$I(z) \equiv \int_X g(x) \ell(x)(\tilde{M}(f(z)))^r(x)dx$$

$$= \sum_{j=1}^N \alpha_j \int_{E_j} (\tilde{M}(f(z)))^r(x)dx$$

$$= \sum_{j=1}^N \beta_j(z)$$

is well defined. We claim that $I(z)$ is a log-subharmonic function in the disk.

We need only prove that every $\beta_j$ is log-subharmonic since, by Radó's criterion (Lemma 2.2), a finite sum of log-subharmonic functions is also log-subharmonic.

But

$$\log \beta_j(z) = \ell(z) \log \alpha_j + \log \int_{E_j} (\tilde{M}(f(z)))^r(x)dx.$$

Thus, it remains only to show that

$$\delta_j(z) \equiv \log \int_{E_j} (\tilde{M}(f(z)))^r(x)dx$$

is subharmonic.

We know that $(\tilde{M}(f(z)))^r(x)$ is log-subharmonic; therefore, for any complex number $\alpha$ we have:

$$\int_{E_j} |e^{\alpha z_0}|(\tilde{M}(f(z_0)))^r(x)dx$$

$$\leq \int_{E_j} \left[ \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha (z_0 + \rho e^{i\theta})}|(\tilde{M}(f(z_0 + \rho e^{i\theta})))^r(x)d\theta \right] dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha (z_0 + \rho e^{i\theta})}| \left[ \int_{E_j} (\tilde{M}(f(z_0 + \rho e^{i\theta})))^r(x)dx \right] d\theta.$$
Thus, again by Lemma 2.2, \( \delta_j \) is log-subharmonic in \( D \). So we have

\[
\log \int_X g(x)^{\ell(z_0)}(\bar{M}(f(z_0)))^\tau(x)dx = \log I(z_0)
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \log I(z_0 + \rho e^{i\theta})d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \log \left[ \int_X g(x)^{\ell(z_0+\rho e^{i\theta})}(\bar{M}(f(z_0 + \rho e^{i\theta})))^\tau(x)dx \right] d\theta.
\]

We can assume \( \|g\|_{L^1(X)} = 1 \) and use Hölder's inequality on the right hand side with indices \( q = \frac{p(z_0 + \rho e^{i\theta})}{\tau} \) and \( q' = \frac{1}{\ell(z_0 + \rho e^{i\theta})} \).

Thus, we have

\[
\log I(z_0)
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \log \|g\|_{L^1(X)}^{\ell(z_0+\rho e^{i\theta})} \|\bar{M}(f(z_0 + \rho e^{i\theta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}(X)}^{\tau} d\theta
\]

\[
= \frac{\tau}{2\pi} \int_0^{2\pi} \log \|\bar{M}(f(z_0 + \rho e^{i\theta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}(X)} d\theta.
\]

Taking the Supremum over all such \( g \)'s, we obtain

\[
\log \|\bar{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)}^{\tau}
\]

\[
= \log \sup_{g \text{ simple} \atop \|g\|_{L^1(X)} = 1} \int_X g(x)^{\ell(z_0)}(\bar{M}(f(z_0)))^\tau(x)dx
\]

\[
\leq \frac{\tau}{2\pi} \int_0^{2\pi} \log \|\bar{M}(f(z_0 + \rho e^{i\theta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}(X)} d\theta.
\]

Hence,

\[
\log \|\bar{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)}
\]

\[
\leq \int_0^{2\pi} \log \|\bar{M}(f(z_0 + \rho e^{i\theta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}(X)} d\theta.
\]
Theorem 2.3 generalizes a result by M. Cwickel, M. Milman and Y. Sagher (see [CMS]) for couples of quasi-Banach spaces.

3. - The real method of interpolation: the $K$, $\tilde{K}$, $J$ and $\tilde{J}$ functionals

We start by summarizing the $K$-method and the $J$-method of interpolation. We take the definitions and properties from [BL], where these methods are treated in a systematic way.

Let $(A_j, \| \cdot \|_j)$ be a quasi-Banach space with quasi-triangle inequality constant $c_j$, $j = 0, 1$. We suppose $A = (A_0, A_1)$ is a compatible pair; i.e. $A_0$, $A_1$ are continuously embedded in a common Hausdorff topological vector space. For $a \in \Sigma(A) = A_0 + A_1$, we define the $K$-functional by letting

$$K(t, a) = \inf_{a_0 + a_1 \in A_j} (\|a_0\|_0 + \|a_1\|_1), \quad t > 0.$$  

**Proposition 3.2.** For any $a \in \Sigma(A)$, $K(t, a)$ is a positive, increasing and concave function of $t$. Moreover,

$$K(t, a) \leq \max \left( 1, \frac{t}{s} \right) K(s, a)$$  

$$K(t, a + b) \leq c_0[K(c_1 t/c_0, a) + K(c_1 t/c_0, b)].$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$) we let $(A_0, A_1)_{\vartheta,q,K} = A_{\vartheta,q,K}$ denote the space of all $a \in \Sigma(A)$ such that

$$\|a\|_{\vartheta,q,K} = \left\{ \int_0^\infty [t^{-\vartheta} K(t, a)]^q \frac{dt}{t} \right\}^{1/q} < \infty.$$  

**Proposition 3.6.** For all $a \in A_{\vartheta,q,K}$ we have

$$K(t, a) \leq \gamma_{\vartheta,q} t^\vartheta \|a\|_{\vartheta,q,K}$$

where

$$\gamma_{\vartheta,q} = [q \vartheta(1 - \vartheta)]^{1/q}.$$  

A variant of the $K_{\vartheta,q}$-functor is the discrete $K_{\vartheta,q}$-method. We shall replace the continuous variable $t$ by a discrete variable $n$; the relation between them being $t = 2^n$. If $\lambda_{\vartheta,q}$ denotes the space of all sequences $\{\alpha_n\}_{n=0}^\infty$ such that

$$\|\{\alpha_n\}\|_{\lambda_{\vartheta,q}} = \left\{ \sum_n [2^{-n\vartheta} |\alpha_n|]^q \right\}^{1/q} < \infty,$$
we have

\textbf{THEOREM 3.7.} For $a \in \Sigma(A)$ we put $\alpha_v = K(2^v, a)$. Then $a \in A_{\theta,q,K}$ if and only if $\{\alpha_v\}_{v=0}^{\infty}$ belongs to $\lambda^{\theta,q}$.

Moreover,

\begin{equation}
2^{-\theta} \log 2 \|\{\alpha_v\}\|_{\lambda^{\theta,q}} \leq \|a\|_{\theta,q,K} \leq 2 \log 2 \|\{\alpha_v\}\|_{\lambda^{\theta,q}}.
\end{equation}

For every $a \in \Delta(A) = A_0 \cap A_1$, let the $J$-functional be defined by

\begin{equation}
J(t, a) = \max (\|a\|_0, t\|a\|_1), \quad t > 0.
\end{equation}

For $0 < \theta < 1$, $0 < q \leq \infty$ (and $0 \leq \theta \leq 1$, $q = \infty$), we define the space $(A_0, A_1)_{\theta,q,J} = A_{\theta,q,J}$ as the set of all those $a$ in $\Sigma(A)$ that can be represented as $a = \Sigma_v a_v$ (convergence in $\Sigma(A)$) where $a_v \in \Delta(A)$ and

\[ \|a\|_{\theta,q,J} = \inf_{\{a_v\}} \|\{J(2^v, a_v)\}\|_{\lambda^{\theta,q}} < \infty. \]

There is a continuous representation of the space $A_{\theta,q,J}$, which is analogous to the continuous representation of the space $A_{\theta,q,K}$, but we shall not need it.

It is possible to show that the spaces $A_{\alpha,q,K}$ and $A_{\theta,q,J}$ so defined are again quasi-Banach spaces with quasi-triangle inequality constants $c_0^{1-\theta} c_1^\theta \max(1, 2^{1/q-1})$. Moreover, the $K$ and $J$ methods give rise to the same space with equivalent quasi-norms, as long as $0 < \theta < 1$, $0 < q \leq \infty$. More precisely, we have:

\textbf{THEOREM 3.10.} Let $A$ be a compatible couple of quasi-Banach spaces and assume that $0 < \theta < 1$, $0 < q \leq \infty$. Then $A_{\theta,q,J} = A_{\theta,q,K} = A_{\theta,q}$ and

\begin{equation}
\frac{1}{16c} \|a\|_{\theta,q,J} \leq \|a\|_{\theta,q,K} \leq c_{\theta,q} \|a\|_{\theta,q,J},
\end{equation}

where

\begin{equation}
c_{\theta,q} = 2^{1+1/p} \log 2 \left\{ \frac{1 - 2^{-\rho}}{(1 - 2^{\rho(\theta-1)})(1 - 2^{-\theta})} \right\}^{1/p}
\end{equation}

and $c, \rho$ are such that $c \geq \max(c_0, c_1)$, $(2c)^\rho = 2$ and $q/\rho \geq 1$.

\textbf{PROPOSITION 3.13.} Let $A$ be a compatible couple of quasi-Banach spaces. Then, if $q < \infty$, $\Delta(A)$ is dense in $A_{\theta,q}$.

We shall introduce two new functionals, the $\tilde{K}$ and $\tilde{J}$ functionals. They are defined in a way similar to the $K$ and $J$ functionals, using the so called Gagliardo completion norms. For $a \in \Delta(A)$, let

\begin{equation}
|a|_0 = \lim_{t \to \infty} K(t, a)
\end{equation}

\begin{equation}
|a|_1 = \lim_{t \to 0} \frac{1}{t} K(t, a).
\end{equation}
Clearly \( |a_j|_j \leq \|a\|_j \), \( j = 0, 1 \).

In certain particular situations also the converse inequality is true. Consider, for example, the \( L^p \)-spaces case. Let \( A_0 = L^1(M) \), \( A_1 = L^\infty(M) \) where \((M, dx)\) is any measure space. We know the explicit expression of the \( K \)-functional,

\[
K(t, a) = \int_0^t f^*(s)ds,
\]

where \( f^* \) is the non-increasing rearrangement of the function \( f \).

Therefore,

\[
|f|_0 = \lim_{t \to \infty} K(t, a) = \int_0^\infty f^*(s)ds = \|f\|_{L^1(M)}
\]

\[
|f|_1 = \lim_{t \to 0} \frac{1}{t} K(t, a) = \lim_{t \to 0} \frac{1}{t} \int_0^t f^*(s)ds = f^*(0) = \|f\|_{L^\infty(M)}.
\]

In general we do not have equality; however, the new norms \( |\cdot|_j \), \( j = 0, 1 \), play a rôle similar to the old ones, \( \|\cdot\|_j \), \( j = 0, 1 \).

Back to our situation, for \( a \in \Delta(A) \) we define the \( \bar{K} \)-functional as

\[
(3.16) \quad \bar{K}(t, a) \equiv \inf_{\substack{a = a_0 + a_1 \in A_0 + A_1 \\text{a}_j \in A_j}} (|a_0|_0 + t|a_1|_1), \quad t > 0.
\]

**Proposition 3.17.** Let \( a \in \Delta(A) \); then

\[
\bar{K}(t, a) \leq K(t, a) \leq \max(c_0, c_1) \bar{K}(t, a).
\]

**Proof.** Clearly \( \bar{K}(t, a) \leq K(t, a) \). To prove the second inequality, note that \( K(t, a) \leq |a_0|_0 \), since \( K(t, a) \) is an increasing function of \( t \). Moreover \( K(t, a) \leq t|a_1|_1 \), since \( K(t, a)/t \) is a decreasing function of \( t \), by (3.3).

So, if \( a = a_0 + a_1 \), \( a_j \in \Delta(A) \), \( j = 0, 1 \), using (3.3) and (3.4) we obtain

\[
K(t, a) \leq c_0[K(c_1t/c_0, a_0) + K(c_1t/c_0, a_1)]
\]

\[
\leq \max(c_0, c_1) [K(t, a_0) + K(t, a_1)]
\]

\[
\leq \max(c_0, c_1) [\|a_0\|_0 + t|a_1|_1].
\]

Taking the infimum over all possible decompositions, we get

\[
K(t, a) \leq \max(c_0, c_1) \bar{K}(t, a).
\]

For \( a \in \Delta(A) \), let the \( \bar{J} \)-functional be defined by

\[
(3.18) \quad \bar{J}(t, a) \equiv \max (|a_0|_0, t|a_1|_1), \quad t > 0.
\]
Clearly \( \bar{J}(t, a) \leq J(t, a) \), but unlike the case of the \( K \) and \( \bar{K} \)-functionals one does not have that \( J \) and \( \bar{J} \) are equivalent.

It is possible, however, to prove the following result:

**Proposition 3.19.** Let \( \varepsilon > 0, \ a \in \Delta(A) \). There exists a representation 
\[
a = \sum_{|v| \leq N+1} a_v, \text{ with } a_v \in \Delta(A), \text{ such that } 
\]
\[
\bar{J}(2^v, a_v) \leq 3 \max (c_0, c_1)(1 + \varepsilon) \ K(2^v, a). 
\]

**Proof.** Take \( a \in \Delta(A) \) and \( \varepsilon > 0 \); then there exists \( N \) such that
\[
|a|_0 \leq (1 + \varepsilon) \ K(2^v, a) \quad \text{if } v \geq N \\
2^v|a|_1 \leq (1 + \varepsilon) \ K(2^v, a) \quad \text{if } -v \leq N.
\]

We can, therefore, find two sequences \( \{a_{j,v}\} \), \( j = 0, 1 \), such that
\[
a = a_{0,v} + a_{1,v} \\
|a_{0,v}|_0 + 2^v|a_{1,v}|_1 \leq (1 + \varepsilon) \ K(2^v, a) \\
a_{0,v} = 0 \quad \text{if } -v \leq -N \\
a_{1,v} = 0 \quad \text{if } v \geq N.
\]

Let \( a_v = a_{0,v} - a_{0,v-1} = a_{1,v-1} - a_{1,v} \). Then \( a_v = 0 \) for \( |v| \geq N + 1 \). Thus
\[
\sum_{|v| \leq N+1} a_v = a_{0,N} - a_{0,-N-1} = a - a_{1,N} - a_{0,-N-1} = a. \text{ Moreover,}
\]
\[
\bar{J}(2^v, a_v) = \max(|a_v|_0, 2^v|a_v|_1) \\
\leq \max(c_0, c_1) \max(|a_{0,v}|_0 + |a_{0,v-1}|_0, 2^v(|a_{1,v-1}|_1 + |a_{1,v}|_1)) \\
\leq \max(c_0, c_1) (1 + \varepsilon) \ [K(2^v, a) + K(2^{v-1}, a)] \\
\leq 3 \max(c_0, c_1) (1 + \varepsilon) \ K(2^v, a).
\]

For \( 0 < \vartheta < 1, \ 0 < q \leq \infty \) (and \( 0 \leq \vartheta \leq 1, \ q = \infty \)) we let
\[
(A_0, A_1)_{\vartheta,q;\bar{K}} = A_{\vartheta,q;\bar{K}} \text{ denote the completion of } \Delta(A) \text{ with respect to the quasi-norm} 
\]
\[
\|a\|_{\vartheta,q;\bar{K}} = \left\{ \int_0^\infty \left( t^{-\vartheta} \bar{K}(t, a) \right)^q \, dt/t \right\}^{1/q}.
\]

We define the space \( (A_0, A_1)_{\vartheta,q;\bar{J}} = A_{\vartheta,q;\bar{J}} \text{ as the completion of } \Delta(A) \text{ with respect to the quasi-norm} 
\]
\[
\|a\|_{\vartheta,q;\bar{J}} = \inf \{ \| \bar{J}(2^v, a_v) \| \}_{v \in \vartheta, q},
\]

where the infimum is taken over all the representations of \( a \) of the form \( a = \Sigma_v a_v \) with \( a_v \in \Delta(A) \).
THEOREM 3.20. Let $A$ be a compatible couple of quasi-Banach spaces. Then, for every $a \in \Delta(A)$ we have

$$
\|a\|_{\theta,q,K} \leq \max(c_0, c_1) \|a\|_{\theta,q,R} \leq \max(c_0, c_1) \|a\|_{\theta,q,K}
$$

and

$$
\|a\|_{\theta,q,J} \leq 12 \max(c_0, c_1) \|a\|_{\theta,q,K} \leq 12 \max(c_0, c_1) c_{\theta,q} \|a\|_{\theta,q,J}
$$

where $c_{\theta,q}$ is as in (3.12).

Before proving the theorem we need a couple of lemmas.

**Lemma 3.23** (see [BL]). Let $(B, \| \cdot \|)$ be a quasi-normed vector space with quasi-triangle inequality constant $c$. Let $\rho$ be defined by the equation $(2c)^\rho = 2$. If $a = \sum_{j=0}^{\infty} a_j$ converges in $B$, then

$$
|a| \leq 2^{1/\rho} \left( \sum_{j=0}^{\infty} |a_j|^\rho \right)^{1/\rho}.
$$

**Lemma 3.24.** Let $a \in \Delta(A)$; then

$$
K(t, a) \leq \min \left( 1, \frac{t}{s} \right) \bar{J}(s, a).
$$

**Proof.**

$$
K(t, a) \leq \min \left( |a|_0, \frac{t}{s} |a|_1 \right)
\leq \min(\bar{J}(s, a), \frac{t}{s} \bar{J}(s, a)) = \min \left( 1, \frac{t}{s} \right) \bar{J}(s, a).
$$

**Proof of Theorem 3.20.** By Proposition 3.17, we immediately obtain 3.21. Now take $a \in \Delta(A)$; by Proposition 3.19 there exists a representation $a = \sum_{|v| \leq N} a_v$ with $a_v \in \Delta(A)$ such that

$$
\bar{J}(2^v, a_v) \leq 4 \max(c_0, c_1) K(2^v, a).
$$

Thus,

$$
\|\{\bar{J}(2^v, a_v)\}\|_{\lambda^*} \leq 4 \max(c_0, c_1) \|\{K(2^v, a)\}\|_{\lambda^*}.
$$

Therefore, using (3.8), we obtain

$$
\|a\|_{\theta,q,J} \leq 4 \max(c, c_1) \frac{2^\theta}{\log 2} \|a\|_{\theta,q,K}
\leq 12 \max(c_0, c_1) \|a\|_{\theta,q,K}.
$$
To prove the second part of the inequality, take $a \in \Delta(A)$ and assume $a = \Sigma v a_v$, with $a_v \in \Delta(A)$. We know that $K(t, a)$ is a $c$-norm ($c \geq \max(c_0, c_1)$). Choosing $c$ large and $p$ so that $(2c)^p = 2$, we have $p = q/r \geq 1$. Then, using Lemmas 3.23 and 3.24 we obtain
\[
K(t, a) \leq 2^{1/p} \left\{ \Sigma v [K(t, a_v)]^r \right\}^{1/r}
\leq 2^{1/p} \left\{ \Sigma v [\min(1, t2^{-v}) \tilde{J}(2^v, a_v)]^r \right\}^{1/r}.
\]
So
\[
K(2^\mu, a) \leq 2^{1/p} \left\{ \Sigma v [\min(1, 2^{\mu-v}) \tilde{J}(2^v, a_v)]^r \right\}^{1/r}
= 2^{1/p} \left\{ \Sigma v [\min(1, 2^v) \tilde{J}(2^{\mu-v}, a_{\mu-v})]^r \right\}^{1/r}.
\]
Thus, using (3.8) and Minkowski’s inequality for series, we obtain
\[
\|a\|_{\theta,q,K} \leq 2 \log 2 \left\{ \Sigma \mu [2^{-\mu\theta} K(2^\mu, a)]^q \right\}^{1/q}
= 2 \log 2 \left\{ \left\{ \Sigma \mu [2^{-\mu\theta} K(2^\mu, a)]^p \right\}^{1/p} \right\}^{1/q}
\leq 2 \log 2 \left\{ \left\{ \Sigma \mu [2^{-\mu\theta}\mu 2(\Sigma v [\min(1, 2^v) J(2^{\mu-v}, a_{\mu-v})]^q)]^p \right\}^{1/p} \right\}^{1/q}
\leq 2^{1+1/p} \log 2 \left\{ \Sigma v \min(1, 2^{\mu v}) [\Sigma \mu (2^{-\mu\theta} J(2^{\mu-v}, a_{\mu-v})]^q] \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \Sigma v \min(1, 2^{\mu v}) [\Sigma \mu (2^{-\mu(\mu+v)} J(2^\mu, a_{\mu})]^q] \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \Sigma v \min(1, 2^{\mu v}) [\Sigma \mu (2^{-\mu(\mu+v)} J(2^\mu, a_{\mu})]^q] \right\}^{1/q}.
\]
Therefore, $\|a\|_{\theta,q,K} \leq c_{\theta,q} \|a\|_{\theta,q,F}$, with
\[
c_{\theta,q} = 2^{1+1/p} \log 2 \left\{ \Sigma v \min(1, 2^{\mu v}) \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \sum_{v \leq 0} 2^{\mu v - 2\theta v} + \sum_{v \geq 0} 2^{-\theta v} \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \sum_{v > 0} 2^{\mu v(\theta-1)} + (1 - 2^{-\theta p})^{-1} \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \frac{1}{1 - 2^{\mu(\theta-1)}} - 1 + \frac{1}{1 - 2^{-\theta p}} \right\}^{1/p}
= 2^{1+1/p} \log 2 \left\{ \frac{1 - 2^{-\theta p}}{(1 - 2^{\mu(\theta-1)})(1 - 2^{-\theta p})} \right\}^{1/p}.
\]

4. - Complex interpolation of real interpolation spaces

Let $0 < \alpha(\theta) < 1$, $0 < q(\theta) \leq \infty$ be two measurable functions defined on $\Gamma$ and let $B(\theta) = (B_0, B_1)_{\alpha(\theta), q(\theta)}$, where $(B_0, B_1)$ is a compatible pair of quasi-Banach spaces and $(B_0, B_1)_{\alpha, q}$ denotes the space obtained by the real method of interpolation.
PROPOSITION 4.1. If $\frac{1}{q} \in L^1(\Gamma)$, then $\{B(\vartheta)\}$ is an interpolation family of quasi-Banach spaces.

PROOF. The containing space $\mathcal{U}$ can be taken to be $B_0 + B_1$. The measurability of $\vartheta \to |b|_{\vartheta}$, $b \in \bigcap_{\vartheta} B(\vartheta)$, is a consequence of the measurability of the functions $\alpha$ and $q$ and the definition of

$$
|b|_{\vartheta} = \|b\|_{\alpha(\vartheta),q(\vartheta)} = \left\{ \begin{array}{ll}
\left\{ \int_0^\infty \left[ t^{-\alpha(\vartheta)} K(t,b) \right]^{q(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)} & \text{if } q(\vartheta) < \infty \\
\sup_t t^{-\alpha(\vartheta)} K(t,b) & \text{if } q(\vartheta) = \infty.
\end{array} \right.
$$

Moreover $c(\vartheta) = c_0^{1-\alpha(\vartheta)} c_{\vartheta}^{\alpha(\vartheta)} \max(1,2^{1/q(\vartheta)-1})$; thus, $\log c(\vartheta)$ is integrable, since $\frac{1}{q} \in L^1(\Gamma)$.

PROPOSITION 4.2. Suppose

$$
(4.3) \quad \frac{1}{q} \log q, \quad \frac{1}{q} \log \alpha, \quad \frac{1}{q} \log(1-\alpha) \in L^1(\Gamma);
$$

then $B_0 \cap B_1 \subset B$, where $B$ denotes the log-intersection of the family $\{B(\vartheta)\}$.

PROOF. Take $b \in B_0 \cap B_1$; since $\|b\|_{\Delta(B)} = \max(\|b\|_0, \|b\|_1)$, we have

$$
K(t,b) = \inf_{b_0+b_1} (\|b_0\|_0 + t\|b_1\|_1) \leq \min(\|b\|_0, \|b\|_1) \leq \min(1,t)\|b\|_{\Delta(B)}.
$$

Thus,

$$
\|b\|_{\alpha(\vartheta),q(\vartheta)} = \left\{ \int_0^\infty \left[ t^{-\alpha(\vartheta)} K(t,b) \right]^{q(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)}
\leq \left\{ \int_0^\infty \left[ t^{-\alpha(\vartheta)} \min(1,t)\|b\|_{\Delta(B)} \right]^{q(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)}
= \|b\|_{\Delta(B)} \left\{ \frac{1}{t_{(1-\alpha(\vartheta))}^{q(\vartheta)}} \int_0^1 t^{-\alpha(\vartheta)} \frac{dt}{t} + \int_1^\infty t^{-\alpha(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)}
= \|b\|_{\Delta(B)} \left\{ \frac{1}{q(\vartheta)\alpha(\vartheta)(1-\alpha(\vartheta))} \right\}^{1/q(\vartheta)}
$$

if $q(\vartheta) < \infty$. 

or

\[ \|b\|_{\alpha(\vartheta), q(z)} = \sup_t t^{-\alpha(\vartheta)} K(t, b) \leq \sup_t t^{-\alpha(\vartheta)} \min(1, t) \|b\|_{\Delta(B)} \]

\[ = \max \left[ \sup_{0 < t \leq 1} t^{-\alpha(\vartheta)+1}, \sup_{1 < t < \infty} t^{-\alpha(\vartheta)} \right] \|b\|_{\Delta(B)} = \|b\|_{\Delta(B)} \text{ if } q(\vartheta) = \infty. \]

Therefore, by (4.3), \( \log^+ \|b\|_{\alpha(\vartheta), q(\vartheta)} \in L^1(\Gamma) \).

**Proposition 4.4.** If (4.3) holds, then \( B \subset (B_0, B_1)_{\alpha(z), q(z)} \), where

\[ (4.5) \quad \alpha(z) = \int_{\Gamma} \alpha(\vartheta) P_z(\vartheta) d\vartheta \]

\[ (4.6) \quad \frac{1}{q(z)} = \int_{\Gamma} \frac{1}{q(\vartheta)} P_z(\vartheta) d\vartheta. \]

**Proof.** Fix \( z \in D; \) then there exist \( \vartheta, \varphi \in \Gamma \) such that \( \alpha(\vartheta) \leq \alpha(z) \leq \alpha(\varphi) \). Thus, \( A = (B_0, B_1)_{\alpha(z), q(z)} \cap (B_0, B_1)_{\alpha(\varphi), q(\varphi)} \subset (B_0, B_1)_{\alpha(z), q(z)} \). Indeed, if \( a \in A \), we have

\[ \phi_{\alpha(z), q(z)}(K(\cdot, a)) \]

\[ \leq \max(1, 2^{1/q(z)-1}) \left\{ \left[ \int_0^1 \left[ t^{-\alpha(z)} K(t, a) \right]^{q(z)} dt/t \right]^{1/q(z)} \right. \]

\[ + \left. \int_1^\infty \left[ t^{-\alpha(z)} K(t, a) \right]^{q(z)} dt/t \right]^{1/q(z)} \}

\[ = \max(1, 2^{1/q(z)-1}) \left\{ \left[ \int_0^1 \left[ t^{-\alpha(\varphi)} K(t, a) \right]^{q(z)} t^{(\alpha(\varphi)-\alpha(z))q(z)} dt/t \right]^{1/q(z)} \right. \]

\[ + \left. \int_1^\infty \left[ t^{-\alpha(\vartheta)} K(t, a) \right]^{q(z)} t^{(\alpha(\vartheta)-\alpha(z))q(z)} dt/t \right]^{1/q(z)} \}

Using Proposition 3.6 we get

\[ \phi_{\alpha(z), q(z)}(K(\cdot, a)) \]

\[ \leq \max(1, 2^{1/q(z)-1}). \]
THEOREM 4.7. Suppose (4.3) holds; then

for every \( b \in B_0 \cap B_1 \), where \( \alpha(z) \) and \( q(z) \) are defined as in (4.5) and (4.6).

\[
\max(1, 2^{1/q(z)-1}).
\]

\[
< \infty.
\]

Thus \( B \subset \cap_\vartheta \mathcal{B}(\vartheta) \subset A \subset (B_0, B_1)_{\alpha(z), q(z)} \).

THEOREM 4.7. Suppose (4.3) holds; then

\[ |b|_z \leq d(z) \| b \|_{\alpha(z), q(z)}; K \]

for every \( b \in B_0 \cap B_1 \), where \( \alpha(z) \) and \( q(z) \) are defined as in (4.5) and (4.6).

PROOF. Take \( 0 \neq b \in B_0 \cap B_1 \). By Proposition 2.6, there exists a representation of \( b \) of the form \( b = \sum b_v \), with \( b_v \in \Delta(B) \), such that

\[
J(2^u, b_v) \leq 4 \max \left( c_0, c_1 \right) K(2^u, b).
\]

(4.8)

Fix \( \xi \in D \), and let \( w(z) \) and \( 1/s(z) \) be the two unique analytic functions in \( D \) whose real parts are \( \alpha(z) \) and \( 1/q(z) \) respectively, and such that \( w(\xi) \) and \( 1/s(\xi) \) are real. Let \( \rho(\vartheta) = \min(\rho, q(\vartheta)) \), where \( \rho \) is defined by the equation \((2c)^\rho = 2\) and \( c = \max(c_0, c_1) \). Note that \( q(\vartheta)/\rho(\vartheta) \geq 1 \) and so we can apply Theorem 3.20 to every \( \| \cdot \|_{\alpha(\vartheta), q(\vartheta)} \).

Let \( f_\nu(z) = b_v B_\nu(z) \), where

\[
B_\nu(z) = 2^{\nu(w(z) - w(\xi))} \left\{ \frac{J(2^u, b_v) 2^{-\nu c(\xi)}}{\| b \|_{\alpha(\vartheta), q(\vartheta)}; K} \right\} \frac{2\nu}{\nu\xi}^{-1} \frac{k(\xi)}{k(z)} L(\xi) L(z),
\]

\[
k(z) = \exp \int_0^{2\pi} \log k(\vartheta) H_\nu(\vartheta) d\vartheta \quad \text{with} \quad k(\vartheta) = c_{\alpha(\vartheta), q(\vartheta)},
\]

\[
L(z) = \exp \int_0^{2\pi} \log L(\vartheta) H_\nu(\vartheta) d\vartheta \quad \text{with} \quad L(\vartheta) = \left\{ \frac{2\alpha(\xi)}{\log 2} \right\} \frac{4 \max(c_0, c_1)}{\nu\xi}.\]
Note that $k(z)$ and $L(z)$ are well defined, since (4.3) holds and $\log(1 - 2^{-t}) \approx \log(t \log 2)$ as $t \to 0$. Moreover, $B_v \in N^+(D)$ and $b_v \in B$ by Proposition 4.2. Finally, define

$$g(z) = \sum_{|u| \leq N} f_u(z).$$

Observe that $g(\xi) = b$ and, using (3.22), (4.8), (3.8),

$$\|g(\vartheta)\|_{\alpha(\vartheta), q(\vartheta); K} \leq k(\vartheta) \|g(\vartheta)\|_{\alpha(\vartheta), q(\vartheta); J}^{1/q(\vartheta)}$$

$$= k(\vartheta) \left\{ \sum_{|u| \leq N} \left[ 2^{-v_{0}(\vartheta)} J(2^{u}, B_v b_v) \right]^{q(\vartheta)} \right\}^{1/q(\vartheta)}$$

$$= k(\vartheta) \frac{L(\xi)}{L(\vartheta)} \|b\|_{\alpha(\xi), q(\xi); K}^{1 - \frac{\alpha(\theta)}{\alpha(\xi)}} \left\{ \sum_{|u| \leq N} \left[ 2^{-v_{0}(\xi)} J(2^{u}, b) \right]^{q(\xi)} \right\}^{1/q(\xi)}$$

$$\leq k(\xi) \frac{L(\xi)}{L(\vartheta)} \left[ 4 \max(c_0, c_1) \sum_{|u| \leq N} \left[ 2^{-v_{0}(\xi)} K(2^{u}, b) \right]^{q(\xi)} \right]^{1/q(\xi)}$$

$$= k(\xi) L(\xi) \|b\|_{\alpha(\xi), q(\xi); K} < \infty.$$

Therefore $g \in \mathcal{G}(B(\cdot), \Gamma)$ and, thus,

$$|b|_{\vartheta} = |g(\vartheta)|_{\vartheta} \leq \|g\| \|\vartheta\| \leq d(\xi) \|b\|_{\alpha(\xi), q(\xi); K}.$$

A similar result has been proven by E. Hernandez ([Her]) in the Banach space case, and by M. Cwikel, M. Milman and Y. Sagher ([CMS]) for couples of quasi-Banach spaces.

5. - Example: $H^p$ spaces

We shall use the results obtained in sections 2 and 4 to identify the intermediate spaces when on the boundary we have $H^p$ spaces. We shall follow the notations of Calderon and Torchinski and we refer to [CT 1] and [CT 2] for the most important properties of these spaces. We recall that if $0 < p < 1$, $H^p(\mathbb{R}^n)$ is a quasi-Banach space and the constant in the quasi-triangle inequality is $c_p = 2^{1/p - 1}$, while, if $p \geq 1$, $H^p(\mathbb{R}^n)$ is a Banach space, and coincides with the ordinary $L^p(\mathbb{R}^n)$, with equivalence of norms, if $p > 1$. 
Suppose $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$; we shall see that the interpolation space at $z$ is $B(z) = H^{p(z)}(\mathbb{R}^n)$ where:

\[(5.1) \quad \frac{1}{p(z)} = \int \frac{1}{p(\vartheta)} P_z(\vartheta) d\vartheta.\]

We shall prove this when $0 < p_0 < p(\vartheta) \leq \infty$. If we want to remove the restriction $0 < p_0 < p(\vartheta)$, we can not apply Theorem 4.7 and we have to use the definition of interpolation space directly. This was done in [T-V 1].

We first recall the following result due to C. Fefferman, N. Riviere and Y. Sagher (see [FRS]).

**Theorem 5.2.** $(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\vartheta,p} = H^p$, where $\frac{1}{p} = \frac{1 - \vartheta}{p_0}$, $0 < \vartheta < 1$.

**Theorem 5.3.** Suppose $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$, $0 < p_0 + \varepsilon < p(\vartheta) \leq \infty$. Then $B(z) = H^{p(z)}(\mathbb{R}^n)$, with equivalence of quasi-norms, where $p(z)$ is defined in (5.1).

**Proof.** Fix $z \in D$. We can assume $p(z) < \infty$, since otherwise we have $p(\vartheta) = \infty$ a.e. and thus $B(w) = H^\infty(\mathbb{R}^n)$ for every $w \in D$.

By Theorem 5.2, $H^{p(\vartheta)}(\mathbb{R}^n) = (H^{p_0}, L^\infty)_{1 - p_0/p(\vartheta), p(\vartheta)}$. Applying Theorem 4.7, we get $|f|_z \leq d(z) \|f\|_{1 - p_0/p(z), p(z)}$, or $\|f\|_z \leq k(z) \|f\|_{H^{p(z)}}$ for every $f \in H^p \cap L^\infty$.

Moreover, if $\varphi \in S$ and $\varphi(0) \neq 0$, we can define

$$\tilde{M}(x, f) = \sup_{|x - y| \leq a} |f \ast \varphi(y)|, \quad a > 0$$

for every tempered distribution $f$. We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\tilde{M}(\cdot, f)$ is in $L^p(\mathbb{R}^n)$, and $\|f\|_{H^p} = \|\tilde{M}(\cdot, f)\|_{L^p}$.

It is easy to see that $\tilde{M}$ is a log-subharmonic operator associated with the family $\{B(\vartheta)\}$. Therefore, using Theorem 2.3, we get

$$\|\tilde{M}(\cdot, f)\|_{L^p} = \|f\|_{H^{p(z)}} \leq |f|_z$$

for every $f \in B$. By Proposition 4.2, this is true in particular for $f \in H^{p_0} \cap L^\infty$.

Finally we observe that the completions of $H^{p_0} \cap L^\infty$ with respect to both norms $\|\cdot\|_{H^{p(z)}}$ and $|\cdot|_z$ give us $H^{p(z)}(\mathbb{R}^n)$ by Proposition 3.13. Consequently, since the log-intersection $B$ is contained in $H^{p(z)}(\mathbb{R}^n)$ by Proposition 4.4, the spaces $H^{p(z)}(\mathbb{R}^n)$ and $B(z)$ coincide with equivalence of norms.
REFERENCES


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