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Some techniques for the characterization of intermediate spaces

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0. - Introduction

In the last few years R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss have developed a method of interpolation for families of Banach spaces that generalizes the complex method of A.P. Calderon (see [CCRSW 1] and [CCRSW 2]).

As a further development we presented in [T-V 2] a method of interpolation for families of quasi-Banach spaces. More precisely, we considered quasi-Banach spaces $B(\vartheta)$ associated with the points $e^{i\vartheta}$ of the boundary Γ of the open unit disk D in the complex plane $\mathbb{C}$. Intermediate spaces $B(z)$, for each $z \in D$, were constructed in such a way that interpolation theorems for linear operators hold.

The aim of this note is to find some results that easily yield the identification of families of quasi-Banach spaces commonly used in analysis.

A large number of interesting spaces of functions in analysis has a norm defined by a sublinear operator, generally a maximal operator. Consider, for example, the case of H^p-spaces. If S denotes the space of test functions in $\mathbb{R}^n$ and $\varphi \in S$ with $\hat{\varphi}(0) \neq 0$, we consider the operator $\tilde{M}$ defined by

$$\tilde{M}(x, f) = \operatorname*{Sup}_{|x-y| \leq at} |f * \varphi_t(y)|, \qquad a > 0$$

for every tempered distribution f. We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\tilde{M}(\cdot, f) \in L^p(\mathbb{R}^n)$ and $\|f\|_{H^p} = \|\tilde{M}(\cdot, f)\|_{L^p}$, $0 < p \leq \infty$.

Observe that $\tilde{M}$ can be described as the composition of two operators: the first one is linear and it associates to f the convolution $f * \varphi_t$, the second one is a typical maximal operator. The aim of section 2 is to prove an interpolation theorem for operators of this kind.

In section 4 we study the relations between our interpolation method and the real interpolation method developed by J. Peetre and J.L. Lions. The theorem

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that we obtain yields the identification of complex-intermediate spaces from the identification of real-intermediate ones.

As an example, in section 5, from the Fefferman-Riviere-Sagher result

$$(H^{p_0}, L^\infty)_{\vartheta,p} = H^p, \qquad \frac{1}{p} = \frac{1-\vartheta}{p_0},$$

we are able to characterize the intermediate spaces when on the boundary we have H^p spaces. As we shall see, these intermediate spaces are again H^p spaces.

Finally, I would like to thank my advisors, Richard Rochberg and Guido Weiss, for introducing me to the subject and for their valuable help during my graduate studies.

1. - The complex interpolation method

We now briefly describe the complex interpolation method for families of quasi-Banach spaces, as given in [T-V 2].

For each $e^{i\vartheta} \in \Gamma$ we assign a quasi-Banach space $(B(\vartheta), |\cdot|_\vartheta)$, and denote by $c(\vartheta)$ the constants in the quasi-triangle inequalities.

We say that the family $\{B(\vartheta)\}$ is an *interpolation family* (of quasi-Banach spaces) if each $B(\vartheta)$ is continuously embedded in a Hausdorff topological vector space $\mathcal{U}$, the function $\vartheta \to |b|_\vartheta$ is measurable for each $b \in \bigcap_\vartheta B(\vartheta)$, and $\log\, c(\vartheta) \in L^1(\Gamma)$.

The subset $\mathcal{B}$ of $\bigcap_\vartheta B(\vartheta)$ of those elements b such that $\log^+ |b|_\vartheta \in L^1(\Gamma)$ is called the *log-intersection* of the family $\{B(\vartheta)\}$.

By $\mathcal{G} = \mathcal{G}(B(\cdot), \Gamma)$ we denote the set of functions defined by

$$\mathcal{G} = \{g(z) = \sum_{n=1}^{N} \varphi_j(z) a_j : \varphi_j \in N^+(D),\ \ a_j \in \mathcal{B},$$
$$j = 1, \ldots, N \text{ and } \|\, \|g\|\, \|_\infty < \infty\}$$

where $N^+(D)$ denotes the positive Nevanlinna class (see [Dur], ch. 2), and

$$\|\, \|g\|\, \|_\infty = \operatorname{Ess\,sup}_\vartheta\ |g(\vartheta)|_\vartheta.$$

For every $a \in \mathcal{B}$ and $z \in D$ we define

$$|a|_z = \operatorname{Inf}\{\|\, \|g\|\, \|_\infty : g \in \mathcal{G},\ \ g(z) = a\}.$$

If N_z denotes the set of elements of $\mathcal{B}$ such that $|a|_z = 0$, the completion $B(z)$ of $(\mathcal{B}/N_z, |\cdot|_z)$ will be called the interpolation space at z of the family $\{B(\vartheta)\}$.

It can be proven that $(B(z), |\cdot|_z)$ is a quasi-Banach space with quasi-triangle inequality constant $c(z) = \exp \int_\Gamma \log c(\vartheta)\, P_z(\vartheta) d\vartheta$, where $P_z(\vartheta)$ is the Poisson kernel "centered" at z.

2. - An interpolation theorem for log-subharmonic operators

Let $\mathcal{M}$ be the set of measurable complex-valued functions on some measure space (Y, v). An operator M mapping $\mathcal{M}$ into the class $\mathcal{N}$ of non-negative-valued measurable functions on some other measure space (X, μ) is said to be of *maximal type* provided it satisfies:

(a) $M(\lambda a) = |\lambda| Ma \qquad \forall \lambda \in \mathbb{C},\ \forall a \in \mathcal{M};$

(b) $M(a) = M(|a|) \qquad \forall a \in \mathcal{M};$

(c) $M(a)(x) \leq M(b)(x) \qquad$ if $|a(y)| \leq |b(y)|,\ a, b \in \mathcal{M};$

(d) $$M\left[\int_\Gamma f(\cdot, \vartheta) d\vartheta\right](x) \leq \int_\Gamma M(f(\cdot, \vartheta))(x) d\vartheta.$$

If $\{B(\vartheta)\}$ is an interpolation family, with containing space $\mathcal{U}$, we say that an operator $\tilde{M} : \mathcal{U} \to \mathcal{N}$ is a *log-subharmonic operator* associated to the family $\{B(\vartheta)\}$ if it can be expressed as the composition $M \cdot L$ of a linear operator L mapping $\mathcal{U}$ into $\mathcal{M}$ and of a maximal-type operator M.

The reason for such a name is clear if we note that

PROPOSITION 2.1. *Let* $\tilde{M}$ *be a* log-*subharmonic operator associated to the family* $\{B(\vartheta)\}$. *If* $f(z) = \sum_{j=1}^{n} \varphi_j(z) a_j \in \mathcal{G}(B(\cdot), \Gamma)$, *then* $\log \tilde{M}(f(z))(x)$ *is a subharmonic function of* z *in the disk, for a.e. fixed* x.

Before proving the proposition we recall the following result, due to Radö (see [Aup]):

LEMMA 2.2. *Let* ϕ *be a positive function defined on a domain* Ω; *then,* $\log \phi$ *is subharmonic in* Ω *if and only if the function*

$$z \to |e^{\alpha z}| \phi(z)$$

is subharmonic in Ω, *for every complex number* α.

We shall refer to such functions ϕ as log-*subharmonic*.

PROOF OF 2.1. By Radö's criterion (lemma 2.2), it is enough to show that the functions $z \to |e^{\alpha z}| \tilde{M}(f(z))(x)$ are subharmonic for every complex number

α. Fix $z_0 \in D$ and let $\rho > 0$ be such that the closure of the ball of radius ρ centered at z_0 is contained in D. Since $L(f(z))$ is analytic, $\log|L(f(z))|$ is subharmonic in the disk and, thus, we have

$$\begin{aligned}
|e^{\alpha z_0}|\tilde{M}(f(z_0))(x) &= M(|e^{\alpha z_0}|\ |L(f(z_0))|)(x) \\
&\le M\left[\frac{1}{2\pi}\int_0^{2\pi} |e^{\alpha(z_0+\rho e^{i\vartheta})}|\ |L(f(z_0+\rho e^{i\vartheta}))|d\vartheta\right](x) \\
&\le \frac{1}{2\pi}\int_0^{2\pi} M\left[|e^{\alpha(z_0+\rho e^{i\vartheta})}|\ |L(f(z_0+\rho e^{i\vartheta}))|\right](x)d\vartheta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |e^{\alpha(z_0+\rho e^{i\vartheta})}|\ \tilde{M}(f(z_0+\rho e^{i\vartheta}))(x)d\vartheta.
\end{aligned}$$

The H^p-spaces introduced above are just one of several cases where the norm of an element f in a given function space is defined by evaluating the Lebesgue norm of $\tilde{M}f$, $\tilde{M}$ a log-subharmonic operator.

For all these spaces we can apply the following interpolation theorem:

THEOREM 2.3. *Let $\tilde{M}$ be a* log-*subharmonic operator associated to an interpolation family of quasi-Banach spaces $\{B(\vartheta)\}$. Suppose that*

$$||\tilde{M}a||_{L^{p(\vartheta)}(X)} \le \eta(\vartheta)\ |a|_\vartheta, \qquad \forall a \in \mathcal{B},$$

where $0 < p(\vartheta) \le \infty$ and $\frac{1}{p} \in L^1(\Gamma)$.

If log $\eta \in L^1(\Gamma)$, *then for all $a \in \mathcal{B}$*

$$||\tilde{M}a||_{L^{p(z)}(X)} \le \eta(z)\,|a|_z,$$

where $\frac{1}{p(z)} = \int_\Gamma \frac{1}{p(\vartheta)}\ P_z(\vartheta)d\vartheta$ and $\eta(z) = \exp\int_\Gamma \log\ \eta(\vartheta)\ P_z(\vartheta)d\vartheta$.

PROOF. For $a \in \mathcal{B}$, there exists $f(z) = \sum_{j=1}^{n} \varphi_j(z)a_j \in \mathcal{G}(B(\cdot),\Gamma)$ such that $f(z_0) = a$ and $||\ ||f||\ ||_\infty \le |a|_{z_0} + \varepsilon$, for $z_0 \in D$ and $\varepsilon > 0$ fixed. To prove the theorem it is enough to show that the function $z \to \log||\tilde{M}(f(z))(\cdot)||_{L^{p(z)}(X)}$ is subharmonic in the disk. Indeed

$$\log||\tilde{M}(f(0))(\cdot)||_{L^{p(0)}(X)} \le \frac{1}{2\pi}\int_0^{2\pi} \log||\tilde{M}(f(\vartheta))(\cdot)||_{L^{p(\vartheta)}(X)}d\vartheta$$

is equivalent to

$$\|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \le \exp \int_0^{2\pi} \log \ \|\tilde{M}(f(\vartheta))(\cdot)\|_{L^{p(\vartheta)}(X)} P_{z_0}(\vartheta)d\vartheta$$

via the Möbius transformation

$$w = \frac{z - z_0}{1 + \overline{z}_0 z}, \qquad d(\arg \ z) = P_{z_0}(\vartheta)d\vartheta.$$

Therefore,

$$\begin{aligned} \|\tilde{M}a\|_{L^{p(z_0)}(X)} &= \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \\ &\le \exp \int_0^{2\pi} \log \ \eta(\vartheta)|f(\vartheta)|_\vartheta \, P_{z_0}(\vartheta)d\vartheta \\ &\le \eta(z_0) \ \| \, \|f\| \, \|_\infty \\ &\le \eta(z_0) \ (|a|_{z_0} + \varepsilon). \end{aligned}$$

Thus, letting $\varepsilon \to 0$, we get

$$\|\tilde{M}a\|_{L^{p(z_0)}(X)} \le \eta(z_0) \ |a|_{z_0}.$$

To show our claim, we note that since the function $p(z)$ is strictly positive on D, for any $\rho > 0$ we can find $r > 0$ such that $0 < r < p(z)$ if $z \in \overline{B_\rho(z_0)} = \{z : |z - z_0| \le \rho\} \subset D$.

Moreover, since subharmonicity is a local property, it suffices to show

$$\log \ \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \le \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\vartheta})}(X)} d\vartheta$$

for any such $\rho > 0$.

Define

$$\ell(z) = 1 - \frac{r}{p(z)}$$

and let g be a simple and positive function on X of the form $g(x) = \sum_{j=1}^{N} \alpha_j \chi_{E_j}$, with $\alpha_j > 0$ and E_j pairwise disjoint sets of finite measure. Then $g(x)^{\ell(z)}$ is a log-subharmonic function in the disk for every fixed x. Moreover, since $\tilde{M}(f(z))(x)$ is a log-subharmonic function, also $(\tilde{M}(f(z)))^r(x)$ is log-subharmonic in D for every fixed x. We also have that $\tilde{M}(f(z))(\cdot)$ is in $L^r_{\text{loc}}(X)$ (i.e. $(\tilde{M}(f(z)))^r(\cdot)$ is integrable on sets of finite measure) for every $z \in D$, since $\tilde{M}(f(z)) \in \bigcap_{\vartheta} L^{p(\vartheta)}(X)$.

So

$$\begin{aligned} I(z) &\equiv \int_X g(x)^{\ell(z)} (\tilde{M}(f(z)))^r \ (x) dx \\ &= \sum_{j=1}^{N} \alpha_j^{\ell(z)} \int_{E_j} (\tilde{M}(f(z)))^r (x) dx \\ &= \sum_{j=1}^{N} \beta_j(z) \end{aligned}$$

is well defined. We claim that $I(z)$ is a log-subharmonic function in the disk. We need only prove that every β_j is log-subharmonic since, by Radö's criterion (Lemma 2.2), a finite sum of log-subharmonic functions is also log-subharmonic.

But

$$\log \ \beta_j(z) = \ell(z) \ \log \ \alpha_j + \log \int_{E_j} (\tilde{M}(f(z)))^r (x) dx.$$

Thus, it remains only to show that

$$\delta_j(z) \equiv \log \int_{E_j} (\tilde{M}(f(z)))^r (x) dx$$

is subharmonic.

We know that $(\tilde{M}(f(z)))^r(x)$ is log-subharmonic; therefore, for any complex number α we have:

$$\begin{aligned} &\int_{E_j} |e^{\alpha z_0}| (\tilde{M}(f(z_0)))^r (x) dx \\ &\leq \int_{E_j} \left[\frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| (\tilde{M}(f(z_0 + \rho e^{i\vartheta})))^r (x) d\vartheta \right] dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| \left[\int_{E_j} (\tilde{M}(f(z_0 + \rho e^{i\vartheta})))^r (x) dx \right] d\vartheta. \end{aligned}$$

Thus, again by Lemma 2.2, δ_j is log-subharmonic in D. So we have

$$\log \int_X g(x)^{\ell(z_0)}(\tilde{M}(f(z_0)))^r(x)dx = \log\ I(z_0)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log\ I(z_0+\rho e^{i\vartheta})d\vartheta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log\left[\int_X g(x)^{\ell(z_0+\rho e^{i\vartheta})}(\tilde{M}(f(z_0+\rho e^{i\vartheta})))^r(x)dx\right]d\vartheta.$$

We can assume $\|g\|_{L^1(X)} = 1$ and use Hölder's inequality on the right hand side with indices $q = \frac{p(z_0+\rho e^{i\vartheta})}{r}$ and $q' = \frac{1}{\ell(z_0+\rho e^{i\vartheta})}$.

Thus, we have

$$\log\ I(z_0)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log \|g\|_{L^1(X)}^{\ell(z_0+\rho e^{i\vartheta})}\|\tilde{M}(f(z_0+\rho e^{i\vartheta}))(\cdot)\|^r_{L^{p(z_0+\rho e^{i\vartheta})}(X)}d\vartheta$$

$$= \frac{r}{2\pi}\int_0^{2\pi} \log \|\tilde{M}(f(z_0+\rho e^{i\vartheta})(\cdot)\|_{L^{p(z_0+\rho e^{i\vartheta})}(X)}d\vartheta.$$

Taking the Supremum over all such g's, we obtain

$$\log\ \|\tilde{M}(f(z_0))(\cdot)\|^r_{L^{p(z_0)}(X)}$$

$$= \log \sup_{\substack{g \text{ simple} \\ \|g\|_{L^1(X)}=1}} \int_X g(x)^{\ell(z_0)}(\tilde{M}(f(z_0)))^r(x)dx$$

$$\leq \frac{r}{2\pi}\int_0^{2\pi} \log\ \|\tilde{M}(f(z_0+\rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\vartheta})}(X)}d\vartheta.$$

Hence,

$$\log \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)}$$

$$\leq \int_0^{2\pi} \log \|\tilde{M}(f(z_0+\rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0+\rho e^{i\vartheta})}(X)}d\vartheta.$$

Theorem 2.3 generalizes a result by M. Cwickel, M. Milman and Y. Sagher (see [CMS]) for couples of quasi-Banach spaces.

3. - The real method of interpolation: the K, $\tilde{K}$, J and $\tilde{J}$ functionals

We start by summarizing the K-method and the J-method of interpolation. We take the definitions and properties from [BL], where these methods are treated in a systematic way.

Let $(A_j, \|\cdot\|_j)$ be a quasi-Banach space with quasi-triangle inequality constant c_j, $j = 0, 1$. We suppose $A = (A_0, A_1)$ is a compatible pair; i.e. A_0, A_1 are continuously embedded in a common Hausdorff topological vector space. For $a \in \Sigma(A) = A_0 + A_1$, we define the K-functional by letting

$$(3.1) \qquad K(t,a) = \operatorname*{Inf}_{\substack{a=a_0+a_1 \\ a_j \in A_j}} (\|a_0\|_0 + \|a_1\|_1), \qquad t > 0.$$

PROPOSITION 3.2. *For any $a \in \Sigma(A)$, $K(t,a)$ is a positive, increasing and concave function of t. Moreover,*

$$(3.3) \qquad K(t,a) \leq \max\left(1, \frac{t}{s}\right) K(s,a)$$

$$(3.4) \qquad K(t, a+b) \leq c_0[K(c_1t/c_0, a) + K(c_1t/c_0, b)].$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$) we let $(A_0, A_1)_{\vartheta,q;K} = A_{\vartheta,q;K}$ denote the space of all $a \in \Sigma(A)$ such that

$$(3.5) \qquad \|a\|_{\vartheta,q;K} = \left\{ \int_0^\infty [t^{-\vartheta} K(t,a)]^q \, dt/t \right\}^{1/q} < \infty.$$

PROPOSITION 3.6. *For all $a \in A_{\vartheta,q;K}$ we have*

$$K(t,a) \leq \gamma_{\vartheta,q} t^{\vartheta} \|a\|_{\vartheta,q;K}$$

where

$$\gamma_{\vartheta,q} = [q\vartheta(1-\vartheta)]^{1/q}.$$

A variant of the $K_{\vartheta,q}$-functor is the discrete $K_{\vartheta,q}$-method. We shall replace the continuous variable t by a discrete variable n; the relation between them being $t = 2^n$. If $\lambda^{\vartheta,q}$ denotes the space of all sequences $\{\alpha_v\}_{-\infty}^{\infty}$ such that

$$\|\{\alpha_v\}\|_{\lambda^{\vartheta,q}} \equiv \left\{\Sigma_v [2^{-v\vartheta}|\alpha_v|]^q\right\}^{1/q} < \infty,$$

we have

THEOREM 3.7. *For $a \in \Sigma(A)$ we put $\alpha_v = K(2^v, a)$. Then $a \in A_{\vartheta,q;K}$ if and only if $\{\alpha_v\}_{-\infty}^{\infty}$ belongs to $\lambda^{\vartheta,q}$.*
Moreover,

$$2^{-\vartheta} \log 2 \ \|\{\alpha_v\}\|_{\lambda^{\vartheta,q}} \leq \|a\|_{\vartheta,q;K} \leq 2 \log 2 \ \|\{\alpha_v\}\|_{\lambda^{\vartheta,q}}. \tag{3.8}$$

For every $a \in \Delta(A) = A_0 \cap A_1$, let the J-functional be defined by

$$J(t,a) = \max(\|a\|_0, t\|a\|_1), \qquad t > 0. \tag{3.9}$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$), we define the space $(A_0, A_1)_{\vartheta,q;J} = A_{\vartheta,q;J}$ as the set of all those a in $\Sigma(A)$ that can be represented as $a = \Sigma_v a_v$ (convergence in $\Sigma(A)$) where $a_v \in \Delta(A)$ and

$$\|a\|_{\vartheta,q;J} = \operatorname*{Inf}_{(a_v)} \|\{J(2^v, a_v)\}\|_{\lambda^{\vartheta,q}} < \infty.$$

There is a continuous representation of the space $A_{\vartheta,q;J}$, which is analogous to the continuous representation of the space $A_{\vartheta,q;K}$, but we shall not need it.

It is possible to show that the spaces $A_{\alpha,q;K}$ and $A_{\vartheta,q;J}$ so defined are again quasi-Banach spaces with quasi-triangle inequality constants $c_0^{1-\vartheta} c_1^{\vartheta} \max(1, 2^{1/q-1})$. Moreover, the K and J methods give rise to the same space with equivalent quasi-norms, as long as $0 < \vartheta < 1$, $0 < q \leq \infty$. More precisely, we have:

THEOREM 3.10. *Let A be a compatible couple of quasi-Banach spaces and assume that $0 < \vartheta < 1$, $0 < q \leq \infty$. Then $A_{\vartheta,q;J} = A_{\vartheta,q;K} = A_{\vartheta,q}$ and*

$$\frac{1}{16c} \ \|a\|_{\vartheta,q;J} \leq \|a\|_{\vartheta,q;K} \leq c_{\vartheta,q} \ \|a\|_{\vartheta,q;J}, \tag{3.11}$$

where

$$c_{\vartheta,q} = 2^{1+1/p} \ \log 2 \ \left\{ \frac{1 - 2^{-\rho}}{(1 - 2^{\rho(\vartheta-1)})(1 - 2^{-\vartheta\rho})} \right\}^{1/\rho} \tag{3.12}$$

and c, ρ are such that $c \geq \max(c_0, c_1)$, $(2c)^\rho = 2$ and $q/\rho \geq 1$.

PROPOSITION 3.13. *Let A be a compatible couple of quasi-Banach spaces. Then, if $q < \infty$, $\Delta(A)$ is dense in $A_{\vartheta,q}$.*

We shall introduce two new functionals, the $\tilde{K}$ and $\tilde{J}$ functionals. They are defined in a way similar to the K and J functionals, using the so called Gagliardo completion norms. For $a \in \Delta(A)$, let

$$|a|_0 = \lim_{t \to \infty} \ K(t,a) \tag{3.14}$$

$$|a|_1 = \lim_{t \to 0} \ \frac{1}{t} \ K(t,a). \tag{3.15}$$

Clearly $|a|_j \leq \|a\|_j, \ j = 0, 1$.

In certain particular situations also the converse inequality is true. Consider, for example, the L^p-spaces case. Let $A_0 = L^1(M), \ A_1 = L^\infty(M)$ where (M, dx) is any measure space. We know the explicit expression of the K-functional, $K(t, a) = \int_0^t f^*(s)ds$, where f^* is the non-increasing rearrangement of the function f.

Therefore,

$$|f|_0 = \lim_{t\to\infty} K(t, a) = \int_0^\infty f^*(s)ds = \|f\|_{L^1(M)}$$

$$|f|_1 = \lim_{t\to 0} \frac{1}{t} \ K(t, a) = \lim_{t\to 0} \ \frac{1}{t} \int_0^t f^*(s)ds = f^*(0) = \|f\|_{L^\infty(M)}.$$

In general we do not have equality; however, the new norms $|\cdot|_j, \ j = 0, 1$, play a rôle similar to the old ones, $\|\cdot\|_j, \ j = 0, 1$.

Back to our situation, for $a \in \Delta(A)$ we define the $\tilde{K}$-functional as

$$\tilde{K}(t, a) \equiv \operatorname*{Inf}_{\substack{a=a_0+a_1 \\ a_j\in A_j}} (|a_0|_0 + t|a_1|_1), \qquad t > 0. \tag{3.16}$$

PROPOSITION 3.17. *Let $a \in \Delta(A)$; then*

$$\tilde{K}(t, a) \leq K(t, a) \leq \max(c_0, c_1) \ \tilde{K}(t, a).$$

PROOF. Clearly $\tilde{K}(t, a) \leq K(t, a)$. To prove the second inequality, note that $K(t, a) \leq |a|_0$, since $K(t, a)$ is an increasing function of t. Moreover $K(t, a) \leq t|a|_1$, since $K(t, a)/t$ is a decreasing function of t, by (3.3).

So, if $a = a_0 + a_1, \ a_j \in \Delta(A), \ j = 0, 1$, using (3.3) and (3.4) we obtain

$$\begin{aligned} K(t, a) &\leq c_0[K(c_1 t/c_0, a_0) + K(c_1 t/c_0, a_1)] \\ &\leq \max(c_0, c_1) \ [K(t, a_0) + K(t, a_1)] \\ &\leq \max(c_0, c_1) \ [|a_0|_0 + t|a_1|_1]. \end{aligned}$$

Taking the infimum over all possible decompositions, we get

$$K(t, a) \leq \max(c_0, c_1) \ \tilde{K}(t, a).$$

For $a \in \Delta(A)$, let the $\tilde{J}$-functional be defined by

$$\tilde{J}(t, a) \equiv \max \ (|a|_0, \ t|a|_1), \qquad t > 0. \tag{3.18}$$

Clearly $\tilde{J}(t,a) \leq J(t,a)$, but unlike the case of the K and $\tilde{K}$-functionals one does not have that J and $\tilde{J}$ are equivalent.

It is possible, however, to prove the following result:

PROPOSITION 3.19. *Let $\varepsilon > 0$, $a \in \Delta(A)$. There exists a representation $a = \sum_{|v| \leq N+1} a_v$, with $a_v \in \Delta(A)$, such that*

$$\tilde{J}(2^v, a_v) \leq 3 \max\,(c_0, c_1)(1+\varepsilon)\; K(2^\nu, a).$$

PROOF. Take $a \in \Delta(A)$ and $\varepsilon > 0$; then there exists N such that

$$|a|_0 \leq (1+\varepsilon)\; K(2^v, a) \qquad \text{if } v \geq N$$
$$2^v |a|_1 \leq (1+\varepsilon)\; K(2^v, a) \qquad \text{if } -\nu \leq N.$$

We can, therefore, find two sequences $\{a_{j,v}\}$, $j = 0, 1$, such that

$$a = a_{0,v} + a_{1,v}$$
$$|a_{0,v}|_0 + 2^v |a_{1,v}|_1 \leq (1+\varepsilon)\; K(2^v, a)$$
$$a_{0,v} = 0 \quad \text{if } -v \leq -N$$
$$a_{1,v} = 0 \quad \text{if } v \geq N.$$

Let $a_v = a_{0,v} - a_{0,v-1} = a_{1,v-1} - a_{1,v}$. Then $a_v = 0$ for $|v| \geq N+1$. Thus $\sum_{|v| \leq N+1} a_v = a_{0,N} - a_{0,-N-1} = a - a_{1,N} - a_{0,-N-1} = a$. Moreover,

$$\begin{aligned}\tilde{J}(2^v, a_v) &= \max(|a_v|_0, 2^v |a_v|_1) \\ &\leq \max(c_0, c_1)\; \max(|a_{0,v}|_0 + |a_{0,v-1}|_0, 2^v[|a_{1,v-1}|_1 + |a_{1,v}|_1]) \\ &\leq \max(c_0, c_1)\; (1+\varepsilon)\; [K(2^v, a) + K(2^{v-1}, a)] \\ &\leq 3\; \max(c_0, c_1)\; (1+\varepsilon)\; K(2^v, a).\end{aligned}$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$) we let $(A_0, A_1)_{\vartheta,q;\tilde{K}} = A_{\vartheta,q;\tilde{K}}$ denote the completion of $\Delta(A)$ with respect to the quasi-norm $\|a\|_{\vartheta,q;\tilde{K}} = \left\{ \int_0^\infty [t^{-\vartheta}\; \tilde{K}(t,a)]^q\; dt/t \right\}^{1/q}$.

We define the space $(A_0, A_1)_{\vartheta,q;\tilde{J}} = A_{\vartheta,q,\tilde{J}}$ as the completion of $\Delta(A)$ with respect to the quasi-norm

$$\|a\|_{\vartheta,q;\tilde{J}} = \operatorname*{Inf}_{\{a_v\}}\; \|\{\tilde{J}(2^v, a_v)\}\|_{\lambda^{\vartheta,q}},$$

where the infimum is taken over all the representations of a of the form $a = \Sigma_v a_v$ with $a_v \in \Delta(A)$.

THEOREM 3.20. *Let A be a compatible couple of quasi-Banach spaces. Then, for every $a \in \Delta(A)$ we have*

$$(3.21) \qquad \|a\|_{\vartheta,q;K} \leq \max(c_0, c_1) \; \|a\|_{\vartheta,q;\tilde{K}} \leq \max(c_0, c_1) \; \|a\|_{\vartheta,q;K}$$

and

$$(3.22) \qquad \|a\|_{\vartheta,q;\tilde{J}} \leq 12 \max(c_0, c_1)\|a\|_{\vartheta,q,K} \leq 12 \max(c_0, c_1) c_{\vartheta,q}\|a\|_{\vartheta,q,\tilde{J}}$$

where $c_{\vartheta,q}$ is as in (3.12).

Before proving the theorem we need a couple of lemmas.

LEMMA 3.23 (see [BL]). *Let $(B, |\cdot|)$ be a quasi-normed vector space with quasi-triangle inequality constant c. Let ρ be defined by the equation $(2c)^\rho = 2$. If $a = \sum_{j=0}^{\infty} a_j$ converges in B, then*

$$|a| \leq 2^{1/\rho} \; (\sum_{j=0}^{\infty} |a_j|^\rho)^{1/\rho}.$$

LEMMA 3.24. *Let $a \in \Delta(A)$; then*

$$K(t, a) \leq \min\left(1, \frac{t}{s}\right) \; \tilde{J}(s, a).$$

PROOF. $K(t, a) \leq \min\left(|a|_0, \frac{t}{s} s|a|_1\right)$

$$\leq \min(\tilde{J}(s, a), \frac{t}{s} \; \tilde{J}(s, a)) = \min\left(1, \frac{t}{s}\right) \; \tilde{J}(s, a).$$

PROOF OF THEOREM 3.20. By Proposition 3.17, we immediately obtain 3.21. Now take $a \in \Delta(A)$; by Proposition 3.19 there exists a representation $a = \sum_{|v| \leq N} a_v$ with $a_v \in \Delta(A)$ such that

$$\tilde{J}(2^v, a_v) \leq 4 \; \max(c_0, c_1) \; K(2^v, a).$$

Thus,

$$\|\{\tilde{J}(2^v, a_v)\}\|_{\lambda^{\vartheta,q}} \leq 4 \; \max \; (c_0, c_1) \; \|\{K(2^v, a)\}\|_{\lambda^{\vartheta,q}}.$$

Therefore, using (3.8), we obtain

$$\|a\|_{\vartheta,q;\tilde{J}} \leq 4 \; \max(c, c_1) \; \frac{2^\vartheta}{\log 2} \; \|a\|_{\vartheta,q;K}$$

$$\leq 12 \; \max(c_0, c_1) \; \|a\|_{\vartheta,q;K}.$$

To prove the second part of the inequality, take $a \in \Delta(A)$ and assume $a = \Sigma_v a_v$, with $a_v \in \Delta(A)$. We know that $K(t,a)$ is a c-norm ($c \geq \max(c_0, c_1)$). Choosing c large and ρ so that $(2c)^\rho = 2$, we have $p = q/\rho \geq 1$. Then, using Lemmas 3.23 and 3.24 we obtain

$$\begin{aligned} K(t,a) &\leq 2^{1/\rho}\{\Sigma_v[K(t,a_v)]^\rho\}^{1/\rho} \\ &\leq 2^{1/\rho}\{\Sigma_v[\min(1, t2^{-v})\ \tilde{J}(2^v, a_v)]^\rho\}^{1/\rho}. \end{aligned}$$

So

$$\begin{aligned} K(2^\mu, a) &\leq 2^{1/\rho}\{\Sigma_v[\min(1, 2^{\mu-v})\ \tilde{J}(2^v, a_v)]^\rho\}^{1/\rho} \\ &= 2^{1/\rho}\{\Sigma_v[\min(1, 2^v)\ \tilde{J}(2^{\mu-v}, a_{\mu-v})]^\rho\}^{1/\rho}. \end{aligned}$$

Thus, using (3.8) and Minkowski's inequality for series, we obtain

$$\begin{aligned} &\|a\|_{\vartheta,q;K} \leq 2\log 2\{\Sigma_\mu[2^{-\mu\vartheta}K(2^\mu,a)]^q\}^{1/q} \\ &= 2\log 2\ \{[\Sigma_\mu[2^{-\mu\vartheta}K(2^\mu,a)]^{\rho p}]^{1/p}\}^{1/\rho} \\ &\leq 2\log 2\ \{[\Sigma_\mu[2^{-\mu\vartheta\rho}2(\Sigma_v[\min(1,2^v)J(2^{\mu-v},a_{\mu-v})]^\rho)]^p]^{1/p}\}^{1/\rho} \\ &\leq 2^{1+1/\rho}\log 2\ \{\Sigma_v \min(1,2^{v\rho})[\Sigma_\mu(2^{-\mu\vartheta}J(2^{\mu-v},a_{\mu-v}))^q]^{1/p}\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \{\Sigma_v \min(1,2^{v\rho})[\Sigma_\mu(2^{-(\mu+v)\vartheta}J(2^\mu,a_\mu))^q]^{1/p}\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \{\Sigma_v \min(1,2^{v\rho})2^{-v\rho\vartheta}\}^{1/\rho}\{\Sigma_\mu(2^{-\mu\vartheta}J(2^\mu,a_\mu))^q\}^{1/q}. \end{aligned}$$

Therefore, $\|a\|_{\vartheta,q;K} \leq c_{\vartheta,q}\ \|a\|_{\vartheta,q;\tilde{J}}$, with

$$\begin{aligned} c_{\vartheta,q} &= 2^{1+1/\rho}\log 2\ \{\Sigma_v \min(1,2^{v\rho})2^{-v\rho\vartheta}\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \{\sum_{v<0} 2^{v\rho - v\vartheta\rho} + \sum_{v\geq 0} 2^{-\vartheta v\rho}\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \{\sum_{v>0} 2^{v\rho(\vartheta-1)} + (1-2^{-\vartheta\rho})^{-1}\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \left\{\frac{1}{1-2^{\rho(\vartheta-1)}} - 1 + \frac{1}{1-2^{-\vartheta\rho}}\right\}^{1/\rho} \\ &= 2^{1+1/\rho}\log 2\ \left\{\frac{1-2^{-\rho}}{(1-2^{\rho(\vartheta-1)})(1-2^{-\vartheta\rho})}\right\}^{1/\rho}. \end{aligned}$$

4. - Complex interpolation of real interpolation spaces

Let $0 < \alpha(\vartheta) < 1$, $0 < q(\vartheta) \leq \infty$ be two measurable functions defined on Γ and let $B(\vartheta) = (B_0, B_1)_{\alpha(\vartheta),q(\vartheta)}$, where (B_0, B_1) is a compatible pair of quasi-Banach spaces and $(B_0, B_1)_{\alpha,q}$ denotes the space obtained by the real method of interpolation.

PROPOSITION 4.1. *If* $\frac{1}{q} \in L^1(\Gamma)$, *then* $\{B(\vartheta)\}$ *is an interpolation family of quasi-Banach spaces.*

PROOF. The containing space $\mathcal{U}$ can be taken to be $B_0 + B_1$. The measurability of $\vartheta \to |b|_\vartheta$, $b \in \bigcap_\vartheta B(\vartheta)$, is a consequence of the measurability of the functions α and q and the definition of

$$|b|_\vartheta = \|b\|_{\alpha(\vartheta),q(\vartheta)} = \begin{cases} \left\{ \int_0^\infty [t^{-\alpha(\vartheta)} K(t,b)]^{q(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)} & \text{if } q(\vartheta) < \infty \\ \operatorname{Sup}_t \ t^{-\alpha(\vartheta)} K(t,b) & \text{if } q(\vartheta) = \infty. \end{cases}$$

Moreover $c(\vartheta) = c_0^{1-\alpha(\vartheta)} c_1^{\alpha(\vartheta)} \max(1, 2^{1/q(\vartheta)-1})$; thus, log $c(\vartheta)$ is integrable, since $\frac{1}{q} \in L^1(\Gamma)$.

PROPOSITION 4.2. *Suppose*

$$\frac{1}{q} \log q, \ \frac{1}{q} \log \alpha, \ \frac{1}{q} \log(1-\alpha) \in L^1(\Gamma); \tag{4.3}$$

then $B_0 \cap B_1 \subset \mathcal{B}$, *where* $\mathcal{B}$ *denotes the* log-*intersection of the family* $\{B(\vartheta)\}$.

PROOF. Take $b \in B_0 \cap B_1$; since $\|b\|_{\Delta(B)} = \max(\|b\|_0, \|b\|_1)$, we have

$$K(t,b) = \operatorname*{Inf}_{b=b_0+b_1} (\|b_0\|_0 + t\|b_1\|_1) \le \min(\|b\|_0, t\|b\|_1) \le \min(1,t)\|b\|_{\Delta(B)}.$$

Thus,

$$\|b\|_{\alpha(\vartheta),q(\vartheta)} = \left\{ \int_0^\infty [t^{-\alpha(\vartheta)} K(t,b)]^{q(\vartheta)} dt/t \right\}^{1/q(\vartheta)}$$

$$\le \left\{ \int_0^\infty [t^{-\alpha(\vartheta)} \min(1,t)\|b\|_{\Delta(B)}]^{q(\vartheta)} dt/t \right\}^{1/q(\vartheta)}$$

$$= \|b\|_{\Delta(B)} \left\{ \int_0^1 t^{(1-\alpha(\vartheta))q(\vartheta)} dt/t + \int_1^\infty t^{-\alpha(\vartheta)q(\vartheta)} dt/t \right\}^{1/q(\vartheta)}$$

$$= \|b\|_{\Delta(B)} \left\{ \frac{1}{q(\vartheta)\alpha(\vartheta)(1-\alpha(\vartheta))} \right\}^{1/q(\vartheta)} \qquad \text{if } q(\vartheta) < \infty,$$

or

$$\|b\|_{\alpha(\vartheta),q(\alpha)} = \operatorname*{Sup}_t \; t^{-\alpha(\vartheta)}K(t,b) \leq \operatorname*{Sup}_t \; t^{-\alpha(\vartheta)} \min(1,t)\|b\|_{\Delta(B)}$$

$$= \max\left[\operatorname*{Sup}_{0<t\leq 1} t^{-\alpha(\vartheta)+1}, \operatorname*{Sup}_{1<t<\infty} t^{-\alpha(\vartheta)}\right] \|b\|_{\Delta(B)} = \|b\|_{\Delta(B)} \text{ if } q(\vartheta) = \infty.$$

Therefore, by (4.3), $\log^+ \|b\|_{\alpha(\vartheta),q(\vartheta)} \in L^1(\Gamma)$.

PROPOSITION 4.4. *If* (4.3) *holds, then* $\mathcal{B} \subset (B_0, B_1)_{\alpha(z),q(z)}$, *where*

$$\alpha(z) = \int_\Gamma \alpha(\vartheta) \; P_z(\vartheta)d\vartheta \tag{4.5}$$

$$\frac{1}{q(z)} = \int_\Gamma \frac{1}{q(\vartheta)} \; P_z(\vartheta)d\vartheta. \tag{4.6}$$

PROOF. Fix $z \in D$; then there exist $\vartheta, \varphi \in \Gamma$ such that $\alpha(\vartheta) \leq \alpha(z) \leq \alpha(\varphi)$. Thus, $A \equiv (B_0, B_1)_{\alpha(\vartheta),q(\vartheta)} \cap (B_0, B_1)_{\alpha(\varphi),q(\varphi)} \subset (B_0, B_1)_{\alpha(z),q(z)}$. Indeed, if $a \in A$, we have

$$\phi_{\alpha(z),q(z)}(K(\cdot, a))$$

$$\leq \max(1, 2^{1/q(z)-1})\left\{\left[\int_0^1 [t^{-\alpha(z)}K(t,a)]^{q(z)}dt/t\right]^{1/q(z)} + \left[\int_1^\infty [t^{-\alpha(z)}K(t,a)]^{q(z)}dt/t\right]^{1/q(z)}\right\}$$

$$= \max(1, 2^{\frac{1}{q(z)}-1})\left\{\left[\int_0^1 [t^{-\alpha(\varphi)}K(t,a)]^{q(z)}t^{(\alpha(\varphi)-\alpha(z))q(z)} \; \frac{dt}{t}\right]^{\frac{1}{q(z)}} + \left[\int_1^\infty [t^{-\alpha(\vartheta)}K(t,a)]^{q(z)} \; t^{(\alpha(\vartheta)-\alpha(z))q(z)} \; \frac{dt}{t}\right]^{\frac{1}{q(z)}}\right\}.$$

Using Proposition 3.6 we get

$$\phi_{\alpha(z),q(z)}(K(\cdot, a))$$

$$\leq \max(1, 2^{1/q(z)-1}).$$

$$\cdot \left\{ \gamma_{\alpha(\varphi),q(\varphi)} \|a\|_{\alpha(\varphi),q(\varphi)} \left[\int_0^1 t^{(\alpha(\varphi)-\alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right.$$

$$\left. + \gamma_{\alpha(\vartheta),q(\vartheta)} \|a\|_{\alpha(\vartheta),q(\vartheta)} \left[\int_1^\infty t^{(\alpha(\vartheta)-\alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right\}$$

$$= \max(1, 2^{1/q(z)-1}) \cdot$$

$$\cdot \left\{ \frac{\gamma_{\alpha(\varphi),q(\varphi)} \|a\|_{\alpha(\varphi),q(\varphi)}}{[q(z)(\alpha(\varphi) - \alpha(z))]^{1/q(z)}} + \frac{\gamma_{\alpha(\vartheta),q(\vartheta)} \|a\|_{\alpha(\vartheta),q(\vartheta)}}{[q(z)(\alpha(z) - \alpha(\vartheta))]^{1/q(z)}} \right\} < \infty.$$

Thus $B \subset \bigcap_\vartheta B(\vartheta) \subset A \subset (B_0, B_1)_{\alpha(z),q(z)}$.

THEOREM 4.7. *Suppose* (4.3) *holds; then*

$$|b|_z \leq d(z) \ \|b\|_{\alpha(z),q(z);K}$$

for every $b \in B_0 \cap B_1$, *where* $\alpha(z)$ *and* $q(z)$ *are defined as in* (4.5) *and* (4.6).

PROOF. Take $0 \neq b \in B_0 \cap B_1$. By Proposition 2.6, there exists a representation of b of the form $b = \sum_{|v| \leq N} b_v$, with $b_v \in \Delta(B)$, such that

$$\tilde{J}(2^v, b_v) \leq 4 \max\ (c_0, c_1)\ K(2^v, b). \tag{4.8}$$

Fix $\xi \in D$, and let $w(z)$ and $1/s(z)$ be the two unique analytic functions in D whose real parts are $\alpha(z)$ and $1/q(z)$ respectively, and such that $w(\xi)$ and $1/s(\xi)$ are real. Let $\rho(\vartheta) = \min(\rho, q(\vartheta))$, where ρ is defined by the equation $(2c)^\rho = 2$ and $c = \max(c_0, c_1)$. Note that $q(\vartheta)/\rho(\vartheta) \geq 1$ and so we can apply Theorem 3.20 to every $\| \cdot \|_{\alpha(\vartheta),q(\vartheta)}$.

Let $f_v(z) = b_v B_v(z)$, where

$$B_v(z) = 2^{v(w(z)-w(\xi))} \left\{ \frac{\tilde{J}(2^v, b_v) 2^{-v\alpha(\xi)}}{\|b\|_{\alpha(\xi),q(\xi);K}} \right\}^{\frac{q(\xi)}{s(z)}-1} \frac{k(\xi)}{k(z)} \frac{L(\xi)}{L(z)},$$

$$k(z) = \exp \int_0^{2\pi} \log\ k(\vartheta)\ H_z(\vartheta) d\vartheta \quad \text{with } k(\vartheta) = c_{\alpha(\vartheta),q(\vartheta)},$$

$$L(z) = \exp \int_0^{2\pi} \log\ L(\vartheta)\ H_z(\vartheta) d\vartheta \quad \text{with } L(\vartheta) = \left\{ \frac{2^{\alpha(\xi)}}{\log 2}\ 4 \max(c_0, c_1) \right\}^{\frac{q(\xi)}{q(\vartheta)}}.$$

Note that $k(z)$ and $L(z)$ are well defined, since (4.3) holds and $\log(1-2^{-t}) \approx \log(t \log 2)$ as $t \to 0$. Moreover, $B_v \in N^+(D)$ and $b_v \in \mathcal{B}$ by Proposition 4.2. Finally, define

$$g(z) = \sum_{|v| \leq N} f_v(z).$$

Observe that $g(\xi) = b$ and, using (3.22), (4.8), (3.8),

$$\begin{aligned}
\|g(\vartheta)\|_{\alpha(\vartheta),q(\vartheta);K} &\leq k(\vartheta)\ \|g(\vartheta)\|_{\alpha(\vartheta),q(\vartheta);\bar{J}} \\
&\leq k(\vartheta) \left\{ \sum_{|v|\leq N} [2^{-v\alpha(\vartheta)} \bar{J}(2^v, B_v b_v)]^{q(\vartheta)} \right\}^{1/q(\vartheta)} \\
&= k(\xi) \frac{L(\xi)}{L(\vartheta)}\ \|b\|^{1-\frac{q(\xi)}{q(\vartheta)}}_{\alpha(\xi),q(\xi);K} \left\{ \sum_{|v|\leq N} [2^{-v\alpha(\xi)} \bar{J}(2^v, b_v)]^{q(\xi)} \right\}^{1/q(\vartheta)} \\
&\leq k(\xi) \frac{L(\xi)}{L(\vartheta)}\ \|b\|^{1-\frac{q(\xi)}{q(\vartheta)}}_{\alpha(\xi),q(\xi);K}\ [4 \max(c_0, c_1)]^{\frac{q(\xi)}{q(\vartheta)}} \cdot \\
&\qquad \cdot \left\{ \sum_{|v|\leq N} [2^{-v\alpha(\xi)} K(2^v, b)]^{q(\xi)} \right\}^{1/q(\vartheta)} \\
&\leq k(\xi) \frac{L(\xi)}{L(\vartheta)}\ [4 \max(c_0, c_1) \frac{2^{\alpha(\xi)}}{\log 2}]^{\frac{q(\xi)}{q(\vartheta)}}\ \|b\|_{\alpha(\xi),q(\xi);K} \\
&= k(\xi) L(\xi)\ \|b\|_{\alpha(\xi),q(\xi);K} < \infty.
\end{aligned}$$

Therefore $g \in \mathcal{G}(B(\cdot), \Gamma)$ and, thus,

$$|b|_\xi = |g(\xi)|_\xi \leq \|\ \|g\|\ \|_\infty \leq d(\xi)\ \|b\|_{\alpha(\xi),q(\xi);K}.$$

A similar result has been proven by E. Hernandez ([Her]) in the Banach space case, and by M. Cwikel, M. Milman and Y. Sagher ([CMS]) for couples of quasi-Banach spaces.

5. - Example: H^p spaces

We shall use the results obtained in sections 2 and 4 to identify the intermediate spaces when on the boundary we have H^p spaces. We shall follow the notations of Calderon and Torchinski and we refer to [CT 1] and [CT 2] for the most important properties of these spaces. We recall that if $0 < p < 1$, $H^p(\mathbb{R}^n)$ is a quasi-Banach space and the constant in the quasi-triangle inequality is $c_p = 2^{1/p-1}$, while, if $p \geq 1$, $H^p(\mathbb{R}^n)$ is a Banach space, and coincides with the ordinary $L^p(\mathbb{R}^n)$, with equivalence of norms, if $p > 1$.

Suppose $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$; we shall see that the interpolation space at z is $B(z) = H^{p(z)}(\mathbb{R}^n)$ where:

$$\frac{1}{p(z)} = \int_\Gamma \frac{1}{p(\vartheta)} P_z(\vartheta) d\vartheta. \tag{5.1}$$

We shall prove this when $0 < p_0 < p(\vartheta) \leq \infty$. If we want to remove the restriction $0 < p_0 < p(\vartheta)$, we can not apply Theorem 4.7 and we have to use the definition of interpolation space directly. This was done in [T-V 1].

We first recall the following result due to C. Fefferman, N. Riviere and Y. Sagher (see [FRS]).

THEOREM 5.2. $(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\vartheta,p} = H^p$, *where* $\frac{1}{p} = \frac{1-\vartheta}{p_0}$, $0 < \vartheta < 1$. #

THEOREM 5.3. *Suppose* $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$, $0 < p_0 + \varepsilon < p(\vartheta) \leq \infty$. *Then* $B(z) = H^{p(z)}(\mathbb{R}^n)$, *with equivalence of quasi-norms, where* $p(z)$ *is defined in* (5.1).

PROOF. Fix $z \in D$. We can assume $p(z) < \infty$, since otherwise we have $p(\vartheta) = \infty$ a.e. and thus $B(w) = H^\infty(\mathbb{R}^n)$ for every $w \in D$.

By Theorem 5.2, $H^{P(\vartheta)}(\mathbb{R}^n) = (H^{p_0}, L^\infty)_{1-p_0/p(\vartheta),p(\vartheta)}$. Applying Theorem 4.7, we get $|f|_z \leq d(z)\ \|f\|_{1-p_0/p(z),p(z)}$, or $\|f\|_z \leq k(z)\ \|f\|_{H^{p(z)}}$ for every $f \in H^p{}_\circ \cap L^\infty$.

Moreover, if $\varphi \in S$ and $\hat{\varphi}(0) \neq 0$, we can define

$$\tilde{M}(x, f) = \operatorname*{Sup}_{|x-y|\leq at} |f * \varphi_t(y)|, \quad a > 0$$

for every tempered distribution f. We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\tilde{M}(\cdot, f)$ is in $L^p(\mathbb{R}^n)$, and $\|f\|_{H^p} = \|\tilde{M}(\cdot, f)\|_{L^p}$.

It is easy to see that $\tilde{M}$ is a log-subharmonic operator associated with the family $\{B(\vartheta)\}$. Therefore, using Theorem 2.3, we get

$$\|\tilde{M}(\cdot, f)\|_{L^{p(z)}} = \|f\|_{H^{p(z)}} \leq |f|_z$$

for every $f \in \mathcal{B}$. By Proposition 4.2, this is true in particular for $f \in H^{p_0} \cap L^\infty$.

Finally we observe that the completions of $H^{p_0} \cap L^\infty$ with respect to both norms $\|\cdot\|_{H^{p(z)}}$ and $|\cdot|_z$ give us $H^{p(z)}(\mathbb{R}^n)$ by Proposition 3.13. Consequently, since the log-intersection $\mathcal{B}$ is contained in $H^{p(z)}(\mathbb{R}^n)$ by Proposition 4.4, the spaces $H^{p(z)}(\mathbb{R}^n)$ and $B(z)$ coincide with equivalence of norms.

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