Annali della
Scuola Normale Superiore di Pisa
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 17, nº 2 (1990), p. 283-296

<http://www.numdam.org/item?id=ASNSP_1990_4_17_2_283_0>

# Algebraic Methods in the Theory of Theta Functions

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The functions of theta type were introduced for the first time in 1968 by I. Barsotti [1] as a generalization of the classical theta functions. This generalization consists in considering formal power series over an algebraically closed field $k$: a non-zero element $\vartheta(t) \in k[[t]]$ is called a theta type if

$$F(t_1, t_2, t_3) = \frac{\vartheta(t_1 + t_2 + t_3)\vartheta(t_1)\vartheta(t_2)\vartheta(t_3)}{\vartheta(t_1 + t_2)\vartheta(t_1 + t_3)\vartheta(t_2 + t_3)}$$

belongs to the quotient field of the tensor product over $k$, $k[[t_1]] \otimes k[[t_2]] \otimes k[[t_3]]$ (for a more detailed description see Section 1).

The first construction of theta types was strongly geometric and could not be generalized to characteristic $p > 0$. Only several years later (cfr. [2] and [7]) the true cohomological nature of $F$ was discovered, and this allowed the direct construction of $\vartheta$ from the function $F$. The new technique, which is called the "$F$ method", applies in quite different situations, and in particular in the case of positive characteristic.

More recently (cfr. [3]), the introduction of another function, called g, was proposed. This is simply a specialization of the function $F$, by means of which a simpler and more useful definition of theta types can be given; but the proof of this fact is once more geometric.

In this paper we propose first of all to develop the "$g$ method" and to show that it is perfectly equivalent to the previous "$F$ method", and finally to give an algebraic proof of the following fundamental result: the so called "prosthaferesis formula"

$$\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$$

is sufficient to define theta types ([3], Theorem 3.7).

We begin, in Section 1, by recalling some basic definitions and results on the theory of theta types; then, in Section 2, we introduce the function $g$ and

Pervenuto alla Redazione il 6 Marzo 1989.

show that there exists a functional relation which is a necessary and sufficient condition for a power series $g(t_1, t_2)$ to split as

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}.$$

When $g$ splits, we give a completely algebraic way to construct $\vartheta$ starting from $g$.

Finally, in Section 3, we show that the definition of theta type can be given in terms of the function $g$, thus proving the complete equivalence of the two methods. The proof we give here is almost completely algebraic: more precisely, we will show in a purely algebraic way that $\vartheta^2$ is a theta type but, to conclude that also $\vartheta$ is a theta type, we must use a geometric argument, involving the group variety and the divisor of $\vartheta$.

The section ends with some remarks on the hyperfield $C$ of a theta type $\vartheta$: more precisely, we show that $C$ is finitely generated over $k$ by the coefficients of the Taylor expansion of $g$, together with their first order partial derivatives.

## 1. - Preliminaries

We recall some basic facts on functions of theta type, refering the reader to the fundamental works of I. Barsotti [1] and [3] for an introduction and a detailed treatment of the subject.

Let $k$ be an algebraically closed field of characteristic zero and $k[[t]]$, $t = (t^{(1)}, \dots, t^{(n)})$, the ring of formal power series in $n$ variables over $k$. If $I$ is an integral domain, we denote by $Q(I)$ its quotient field. A non-zero element $\vartheta(t) \in Q(k[[t]])$ is called a *function of theta type*, or simply a *theta type*, if the function

$$F(t_1, t_2, t_3) = \frac{\vartheta(t_1 + t_2 + t_3)\vartheta(t_1)\vartheta(t_2)\vartheta(t_3)}{\vartheta(t_1 + t_2)\vartheta(t_1 + t_3)\vartheta(t_2 + t_3)}$$

belongs to the quotient field of the tensor product over $k$, $k[[t_1]] \otimes k[[t_2]] \otimes k[[t_3]]$. Two theta types are *associate* if their ratio is a quadratic exponential, i.e. a factor of the form $c \exp q(t)$, where $c \in k$ and $q(t)$ is a polynomial of degree $\leq 2$ with vanishing constant term. To a theta type $\vartheta$, one can associate a hyperfield $C$ in the following way: $C$ is the smallest subfield of $Q(k[[t]])$, containing $k$, such that $F \in Q(C \otimes C \otimes C)$; the coproduct $\mathbf{P}$ of $C$ is induced by the coproduct of $k[[t]]$,

$$\mathbf{P} : k[[t]] \longrightarrow k[[t]] \hat{\otimes} k[[t]] \cong k[[t, t']]$$

$$t^{(i)} \longrightarrow t^{(i)} \hat{\otimes} 1 + 1 \hat{\otimes} t^{(i)}$$

(for the definition of hyperfield, see the brief exposition in [1] or the more detailed treatment in [8]).

We define the *transcendency* of $\vartheta$, in symbols transc $\vartheta$, as transc $(C/k)$ and the *dimension* of $\vartheta$, dim $\vartheta$, as the least positive integer $m$ such that there exists a theta type $\theta$, associate to $\vartheta$, and linear combinations $u^{(1)}, \dots, u^{(m)}$ of $t^{(1)}, \dots, t^{(n)}$, with coefficients in $k$, such that $\theta(t) \in Q(k[[u]])$. We always have $\dim \vartheta \leq n$, and $\vartheta$ is called *degenerate* if $\dim \vartheta < n$. Moreover it is $\dim \vartheta \leq$ transc $\vartheta$, and $\vartheta$ is a *theta function* if the equality holds.

A fundamental result, on the hyperfield $C$ of a theta type $\vartheta$, states that it is finitely generated over $k$ by the logarithmic derivatives of $\vartheta$ from the seconds on, hence it is the function field $C = k(A)$ of a commutative group variety $A$ over $k$, called the group variety of $\vartheta$. By definition $F \in Q(C \otimes C \otimes C) = k(A \times A \times A)$, so it defines a divisor on $A \times A \times A$. It can be shown that there exists a unique divisor $X$ on $A$ such that the divisor of $F$ on $A \times A \times A$ is

$$(p_1+p_2+p_3)^*X + p_1^*X + p_2^*X + p_3^*X(p_1+p_2)^*X - (p_1+p_3)^*X - (p_2+p_3)^*X,$$

where $p_i : A \times A \times A \to A$, denotes the $i$-th canonical projection, $i = 1, 2, 3$. This divisor $X$ on $A$, which is automatically on $A - A_d$, where $A_d$ denotes the degeneration locus of the group variety $A$, is the divisor of the theta type $\vartheta : X = \operatorname{div} \vartheta$. If $\vartheta$ and $\theta$ are associated theta types, they define the same hyperfield $C$, the same variety $A$ and the same divisor $X$. Moreover the following properties hold: if $X = \operatorname{div} \vartheta_X$ and $Y = \operatorname{div} \vartheta_Y$, then $\operatorname{div}(\vartheta_X \vartheta_Y) = X + Y; X = 0$ if and only if $\vartheta_X = 1$ and $X \sim 0$ if and only if $\vartheta_X \in k(A)$, where all equalities between theta types are considered modulo substitution of a theta type with an associate one. It can also be shown that, if $\vartheta$ is non-degenerate, its divisor $X$ has the property that $T_P^* X = X$ if and only if $P = 0$, the identity point of $A$, where $T_P : A \to A$ denotes translation by $P$, and a necessary and sufficient condition, for $X$ to be an effective divisor, is that $\vartheta$ satisfy the following relation, called *holomorphic prosthaferesis*:

$$\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in k[[t_1]] \otimes k[[t_2]],$$

in this case we say that $\vartheta$ is a holomorphic theta type (if $k = \mathbb{C}$, the complex field, a holomorphic theta type is precisely an entire function).

To conclude, we just mention a result which explains the relationships between theta types and theta functions; it asserts that a theta type is just a theta whose arguments are replaced by "generic" linear combinations of fewer arguments, precisely:

THEOREM 1.1. *If $\vartheta(u) \in Q(k[[u_1, \dots, u_n]])$ is a non-degenerate theta type, then there exists a non-degenerate theta $\theta(\nu) \in Q(k[[\nu_1, \dots, \nu_m]])$ and elements $c_{ij} \in k$ $(i = 1, \dots, m; j = 1, \dots, n)$ such that $m \geq n$, the matrix $(c_{ij})$ has rank $n$ and $\vartheta(u) = \theta(x_1, \dots, x_m)$, where $x_i = \sum_j c_{ij} u_j$. The homomorphism of $k[[\nu]]$ onto $k[[u]]$, which sends $\nu_i$ to $x_i$, induces an isomorphism $\sigma$ of $C_\theta$ into $C_\vartheta$, such that $\sigma^*(\operatorname{div} \vartheta) = \operatorname{div} \theta$.*

*Conversely if $\theta(\nu) \in Q(k[[\nu_1, \dots, \nu_m]])$ is a non-degenerate theta and if the homomorphism just described, with rank $(c_{ij}) = n$, induces an isomorphism* *of $C_\theta$ into $Q(k[[u]])$, then $\theta(x)$ is a non-degenerate theta type with hyperfield $C_\theta$.*

## 2. - The function $g$

For a given $\vartheta(t) \in Q(k[[t]]), t = (t^{(1)}, \dots, t^{(n)}), \vartheta(t) \neq 0$, let us denote by $g$ the following function

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}. \tag{2.1}$$

In the sequel we will always assume that $\vartheta(t) \in k[[t]]$ and $\vartheta(0) = 1$ (this is not restrictive if $\vartheta(0) \neq 0$, i.e. if $\vartheta$ is a unit in $k[[t]]$) and we will call such an element *normalized*. Under these hypotheses, we have $g(t_1, t_2) \in k[[t_1, t_2]], g(t_1, 0) = g(0, t_2) = 1$ and, in particular, we note that $g(t_1, t_2) = F(t_1, t_2, -t_2)^{-1}$.

By a simple calculation, we can check that $g$ satisfies the following functional relation:

$$\begin{aligned} &g(t_1 + t_2, t_3 + t_4)g(t_1 - t_2, t_3 - t_4)g(t_1, t_2)^2 g(t_3, t_4)g(-t_3, t_4) = \\ &= g(t_1 + t_3, t_2 + t_4)g(t_1 - t_3, t_2 - t_4)g(t_1, t_3)^2 g(t_2, t_4)g(-t_2, t_4), \end{aligned} \tag{2.2}$$

which states the invariance of the left hand side under the mutual exchange of $t_2$ and $t_3$.

There are other properties of $g$ which can be derived from (2.2): if we let $t_1 = t_2 = 0$, we get $g(-t_3, t_4) = g(-t_3, -t_4)$, which shows that $g$ is an even function of the second variable; if we let $t_1 = t_4 = 0$, we have

$$g(t_2, t_3)g(-t_2, t_3) = g(t_3, t_2)g(-t_3, t_2),$$

and finally, letting $t_3 = 0$ and using the two preceding relations, we find another functional relation already pointed out by I. Barsotti in the introduction of [3]:

$$g(t_1 + t_2, t_4)g(t_1 - t_2, t_4)g(t_1, t_2)^2 = g(t_1, t_2 + t_4)g(t_1, t_2 - t_4)g(t_4, t_2)g(-t_4, t_2).$$

Now we come to the most important result of this section, i.e. to the proof that the relation (2.2) is not only necessary but also sufficient in order that a power series $g(t_1, t_2)$ splits as in (2.1).

THEOREM 2.3. *Let $g(t_1, t_2) \in k[[t_1, t_2]]$ satisfy* (2.2), *and suppose also that $g(t_1, 0) = g(0, t_2) = 1$. Then there exists a power series $\vartheta(t) \in k[[t]]$, uniquely determined up to multiplication by a quadratic exponential, such that* (2.1) *holds.*

PROOF. First of all we must introduce some notations. If $\mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ are multiindices and $r$ is a positive integer, we let $\mu+\nu=(\mu_1+\nu_1,\dots,\mu_n+\nu_n), r\mu=(r\mu_1,\dots,r\mu_n)$, $|\mu|=\mu_1+\dots+\mu_n$ and $\mu! = \mu_1!\dots\mu_n!$; $\mu\le\nu$ means $\mu_i\le\nu_i$ for all $i$, and $\mu<\nu$ means $\mu_i\le\nu_i$ but $\mu_j<\nu_j$ for some $j$. In the sequel $\varepsilon_i$ will always denote the multiindex $(\delta_{1i},\dots,\delta_{ni})$, where $\delta_{ij}$ is Kronecker's symbol. If $t=(t^{(1)},\dots,t^{(n)})$, $t^\mu$ means $t^{(1)\mu_1}\cdot\ldots\cdot t^{(n)\mu_n}$, $\partial t^{(1)},\dots,\partial t^{(n)}$ are the differentials of $t^{(1)},\dots,t^{(n)}$ and d denotes derivation with respect to the variables $t$. When there are more than one set of variables, we use $\mathrm{d}_i$ to mean derivation with respect to the variables $t_i=(t_i^{(1)},\dots,t_i^{(n)})$; more precisely, we let

$$\mathrm{d}_i^\mu=\frac{\partial^{|\mu|}}{\partial t_i^{(1)\mu_1}\dots\partial t_i^{(n)\mu_n}}.$$

Let us start with $g(t_1,t_2)\in k[[t_1,t_2]]$ as in the statement of the theorem. The normalization of $g$ assures us of the existence of $\log g(t_1,t_2)$ and from (2.2) it follows that $g$, and also $\log g$, is an even function of the second variable; so we can expand log $g$ in a power series as follows:

$$\log g(t_1,t_2)=\sum_\mu A_\mu(t_1)t_2^\mu,\qquad A_\mu(t_1)\in k[[t_1]],\ A_\mu(0)=0,$$

where the sum is over all $\mu\in\mathbb{N}^n-\{0\}$ such that $|\mu|\equiv 0 \bmod 2$.

Let us consider the 1-forms

$$\omega_j=\frac{1}{2}A_{\varepsilon_1+\varepsilon_j}(t)\partial t^{(1)}+\dots+A_{\varepsilon_j+\varepsilon_j}(t)\partial t^{(j)}+\dots+\frac{1}{2}A_{\varepsilon_n+\varepsilon_j}(t)\partial t^{(n)},$$

for $j=1,\dots,n$: we shall prove that they are closed.

In order for $\omega_j$ to be closed, we must have

$$\text{(2.4)}\qquad \begin{aligned} &\mathrm{d}^{\varepsilon_r}\left(\tfrac{1}{2}A_{\varepsilon_s+\varepsilon_j}\right)=\mathrm{d}^{\varepsilon_s}\left(\tfrac{1}{2}A_{\varepsilon_r+\varepsilon_j}\right), &&\text{if } s\ne j\ne r,\\ &\mathrm{d}^{\varepsilon_r}\left(A_{\varepsilon_j+\varepsilon_j}\right)=\mathrm{d}^{\varepsilon_j}\left(\tfrac{1}{2}A_{\varepsilon_r+\varepsilon_j}\right), &&\text{if } j\ne r.\end{aligned}$$

To show this, we apply log to (2.2) and use the power series expansion of log $g$, getting

$$\text{(2.5)}\qquad \begin{aligned} &\sum_\mu A_\mu(t_1+t_2)(t_3+t_4)^\mu+\sum_\mu A_\mu(t_1-t_2)(t_3-t_4)^\mu+2\sum_\mu A_\mu(t_1)t_2^\mu\\ &+\sum_\mu A_\mu(t_3)t_4^\mu+\sum_\mu A_\mu(-t_3)t_4^\mu=\sum_\mu A_\mu(t_1+t_3)(t_2+t_4)^\mu\\ &+\sum_\mu A_\mu(t_1-t_3)(t_2-t_4)^\mu+2\sum_\mu A_\mu(t_1)t_3^\mu\\ &+\sum_\mu A_\mu(t_2)t_4^\mu+\sum_\mu A_\mu(-t_2)t_4^\mu.\end{aligned}$$

Now if we apply $d_2^{\varepsilon_r} d_3^{\varepsilon_s} d_4^{\varepsilon_j}$ to (2.5) and let $t_2 = t_3 = t_4 = 0$, we easily obtain (2.4).

This proves that $\omega_j$ is closed, hence it is exact (remember we are in a ring of formal power series over a field of characteristic zero) and we can consider its integral $\eta_j$, normalized by letting $\eta_j(0) = 0$. Let $\varsigma = \sum_j \eta_j(t)\partial t^{(j)}$, where $j$ ranges from 1 up to $n$: it follows immediately from the definition of $\eta_j$ that $\varsigma$ is closed, so we can take its integral $\gamma$, again normalized by letting $\gamma(0) = 0$. Now let $\vartheta = \exp\gamma$: we claim this is the function we are looking for. We have only to show that

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)},$$

or equivalently:

$$\log g(t_1, t_2) = \gamma(t_1 + t_2) + \gamma(t_1 - t_2) - 2\gamma(t_1) - \gamma(t_2) - \gamma(-t_2). \tag{2.6}$$

Expanding the right hand side of (2.6) in a power series in $t_2$, we find

$$2 \sum_{\substack{\mu\neq 0 \\ |\mu|\equiv 0 \bmod 2}} \frac{1}{\mu!}(d^\mu\gamma(t_1) - d^\mu\gamma(0))t_2^\mu,$$

while the left hand side is simply

$$\log g(t_1, t_2) = \sum_{\substack{\mu\neq 0 \\ |\mu|\equiv 0 \bmod 2}} A_\mu(t_1)t_2^\mu.$$

This shows that (2.6) is equivalent to

$$A_\mu(t_1) = 2(\mu!)^{-1}(d^\mu\gamma(t_1) - d^\mu\gamma(0)), \qquad \text{for all } \mu \text{ s.t. } |\mu| \equiv 0 \bmod 2. \tag{2.7}$$

Now let us apply $d_3^\nu d_4^\lambda$ to (2.5) and let $t_2 = t_3 = t_4 = 0$, we get:

$$\begin{aligned}(\nu + \lambda)!A_{\nu+\lambda}(t_1) + (\nu + \lambda)!(-1)^{|\lambda|}A_{\nu+\lambda}(t_1) + \lambda!\, d^\nu A_\lambda(0) + \lambda!(-1)^{|\nu|}\, d^\nu A_\lambda(0) \\ = \lambda!\, d^\nu A_\lambda(t_1) + \lambda!(-1)^{|\lambda+\nu|}\, d^\nu A_\lambda(t_1).\end{aligned}$$

From this, under the hypotheses $|\lambda| \equiv 0 \bmod 2, |\nu| \equiv 0 \bmod 2$ and $t = t_1$, we find

$$A_{\mu+\nu}(t) = \lambda!(\nu + \lambda)!^{-1}(d^\nu A_\lambda(t) - d^\nu A_\lambda(0)), \tag{2.8}$$

which becomes, by taking $\lambda = \varepsilon_i + \varepsilon_j, \nu = \mu - \varepsilon_i - \varepsilon_j$ with $i \neq j$,

$$A_\mu(t) = \frac{1}{\mu!}\left[d^{\mu-\varepsilon_i-\varepsilon_j}A_{\varepsilon_i+\varepsilon_j}(t) - d^{\mu-\varepsilon_i-\varepsilon_j}A_{\varepsilon_i+\varepsilon_j}(0)\right]. \tag{2.9}$$

This holds, however, only if $\mu_i$ and $\mu_j$ are both $\geq 1$, otherwise, if there is only one $\mu_i \geq 2$ (recall that $|\mu|$ must be even), we must take $i = j$ and get

$$A_\mu(t) = \frac{2}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(0)\right]. \tag{2.10}$$

These two last relations are really meaningful. They show that all $A_\mu$'s are completely determined by $A_{\varepsilon_i+\varepsilon_j}$'s and give explicit formulas by which to construct them. Now recall that

$$\sum_{i=1}^{n}(\mathrm{d}^{\varepsilon_i}\gamma)\partial t^{(i)} = \partial\gamma = \varsigma = \sum_{i=1}^{n}\eta_i\partial t^{(i)},$$

i.e. $\mathrm{d}^{\varepsilon_i}\gamma = \eta_i$, from which it follows immediately that

$$\mathrm{d}^{\varepsilon_i+\varepsilon_j}\gamma = \mathrm{d}^{\varepsilon_i}\eta_j = \tfrac{1}{2}A_{\varepsilon_i+\varepsilon_j}, \qquad \text{if } i \neq j,$$
$$\mathrm{d}^{\varepsilon_i+\varepsilon_i}\gamma = \mathrm{d}^{\varepsilon_i}\eta_i = A_{\varepsilon_i+\varepsilon_i}.$$

In order to prove (2.7), just substitute $A_{\varepsilon_i+\varepsilon_j}(t) = 2\,\mathrm{d}^{\varepsilon_i+\varepsilon_j}\gamma(t)$ in (2.9) or $A_{\varepsilon_i+\varepsilon_i}(t) = \mathrm{d}^{\varepsilon_i+\varepsilon_i}\gamma(t)$ in (2.10) according to whether there exist $i, j$ with $i \neq j, \mu_i \geq 1$ and $\mu_j \geq 1$, or there is only one $\mu_i \geq 2$.

It is now straightforward to verify that any other solution of

$$g(t_1, t_2) = \frac{\vartheta(t_1+t_2)\vartheta(t_1-t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}$$

is of the form $c\ \exp(q(t))\vartheta(t)$, where $q(t)$ is a polynomial of degree $\leq 2$ such that $q(0) = 0$ and $c$ is a non-zero constant; the normalization of $g$ then implies that $c = 1$ or $c = -1$. In the sequel, we shall always choose the normalization $\vartheta(0) = 1$. Q.E.D.

By now we have shown how to construct $\vartheta$ starting from $g$, then, using $\vartheta$, we can also construct $F$; but we can find a more direct relation between the functions $F$ and $g$.

Let us consider $\log F(t_1, t_2, t_3)$ and expand in power series, we find:

$$\log F(t_1, t_2, t_3) = \sum_{\mu,\nu} B_{\mu\nu}(t_1)t_2^\mu t_3^\nu, \qquad B_{\mu\nu}(t_1) \in k[[t_1]],\ B_{\mu\nu}(0) = 0,$$

the sum being performed over all multiindices $\mu, \nu \in \mathbb{N}^n\{0\}$.

We have already observed that $g(t_1, t_2) = F(t_1, t_2, -t_2)^{-1}$; from this, substituting the power series expansions of $\log F$ and $\log g$, by some simple calculations, we conclude that

$$A_\mu(t) = -\sum_{\substack{\alpha+\beta=\mu \\ \alpha,\beta\neq 0}}(-1)^{|\beta|}B_{\alpha\beta}(t), \tag{2.11}$$

which holds for $|\mu| \equiv 0 \bmod 2$.

We can also find an expression for the $B_{\mu\nu}$'s in terms of the $A_\mu$'s: from the proof of Theorem 2.3, we have

$$\mathrm{d}^{\varepsilon_i+\varepsilon_j} \log \vartheta(t) = \tfrac{1}{2} A_{\varepsilon_i+\varepsilon_j}(t), \quad \text{if } i \neq j,$$
$$\mathrm{d}^{\varepsilon_i+\varepsilon_i} \log \vartheta(t) = A_{\varepsilon_i+\varepsilon_i}(t),$$

and also

$$A_\mu(t) = \frac{1}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_j} A_{\varepsilon_i+\varepsilon_j}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_j} A_{\varepsilon_i+\varepsilon_j}(0)\right], \quad \text{if } i \neq j,$$
$$A_\mu(t) = \frac{2}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(0)\right].$$

With similar considerations, made on the function $F$, it can be shown that (cfr. [6], Theorem A.4):

$$\mathrm{d}^{\varepsilon_i+\varepsilon_j} \log \vartheta(t) = B_{\varepsilon_i\varepsilon_j}(t),$$

and

$$B_{\mu\nu}(t) = \frac{1}{\mu!\nu!}\left[\mathrm{d}^{\mu+\nu-\varepsilon_i-\varepsilon_j} B_{\varepsilon_i\varepsilon_j}(t) - \mathrm{d}^{\mu+\nu-\varepsilon_i-\varepsilon_j} B_{\varepsilon_i\varepsilon_j}(0)\right]. \tag{2.12}$$

From these relations it follows immediately that

$$B_{\mu\nu}(t) = \frac{(\mu+\nu)!}{2\mu!\nu!} A_{\mu+\nu}(t), \quad \text{if } |\mu+\nu| \equiv 0 \bmod 2. \tag{2.13}$$

Note that (2.13) holds under the restrictive condition $|\mu+\nu| \equiv 0 \bmod 2$; if we want to find an expression for $B_{\mu\nu}(t)$ in case $|\mu+\nu|$ is odd, we must use (2.12) (or other equivalent relations), and the derivatives of the $A_\mu$'s are also involved in such an expression.

## 3. - The prosthaferesis

For the sake of simplicity in this section we shall denote $(\mu!)^{-1}\,\mathrm{d}^\mu \log \varphi(t)$ by $\varphi_\mu(t)$, for every $\varphi(t) \in Q(k[[t]])$ and every multiindex $\mu > 0$. It can be shown that (cfr. [3], Section 3):

$$(\mu!)^{-1}\,\mathrm{d}^\mu \varphi(t) = \varphi(t) Q_\mu(\varphi), \tag{3.1}$$

where the $Q_\mu$'s are polynomial functions with positive rational coefficients in the $\varphi_\nu$'s, $0 < \nu \leq \mu$. More precisely, we have:

LEMMA 3.2. *If $\varphi(t) \in Q(k[[t]])$ and $\mu = (\mu_1, \ldots, \mu_n)$ is a multiindex $> 0$ and if $\nu_1, \ldots, \nu_h$ are all multiindices with $n$ components, such that $0 < \nu_i \leq \mu, i = 1, \ldots, h$, then*

$$Q_\mu(\varphi) = \sum_j (j!)^{-1} \varphi_{\nu_1}^{j_1} \ldots \varphi_{\nu_h}^{j_h},$$

*where the sum is over all $h$-tuples $j = (j_1, \ldots, j_h)$ of non-negative integers, satisfying the condition $j_1\nu_1 + \ldots + j_h\nu_h = \mu$.*

For the proof of this result see [3], Section 3.

We need one more lemma, which we cite without proof (cfr. [3], Lemma 3.3):

LEMMA 3.3. *Let $\varphi(t_1, t_2) \in k[[t_1, t_2]]$. If $\varphi(t_1, t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, the field generated over $k$ by the derivatives $d_2^\mu \varphi(t_1, 0)$ for all $\mu$, is a finitely generated subfield $C_1 \subset Q(k[[t_1]])$. Analogously the field $C_2$, generated over $k$ by the derivatives $d_1^\nu \varphi(0, t_2)$, is a finitely generated subfield of $Q(k[[t_2]])$. $C_1$ is the smallest subfield $C$ of $Q(k[[t_1]])$, containing $k$, such that $\varphi(t_1, t_2) \in Q(C[[t_2]])$, or equivalently such that $\varphi(t_1, t_2) \in Q(C \otimes Q(k[[t_2]]))$. Moreover we have $\varphi(t_1, t_2) \in Q(C_1 \otimes C_2)$.*

We can now prove the following

LEMMA 3.4. *Let $\vartheta(t) \in k[[t]]$ be a formal power series such that $\vartheta(0) = 1$. The following conditions are equivalent:*

i) $\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$;

ii) *$g(t_1, t_2) \in Q(C \otimes C)$, where $C$ is the subfield of $Q(k[[t]])$ generated over $k$ by the logarithmic derivatives $d^\mu \log \vartheta(t)$, for all $\mu$ such that $|\mu| \geq 2$.*

*Moreover, under these hypotheses, $C$ is a finitely generated hyperfield over $k$.*

PROOF. That ii) $\Rightarrow$ i) is obvious; the hard part is to show that i) $\Rightarrow$ ii).

Let $\varsigma_i(t) = d^{\varepsilon_i} \log \vartheta(t), i = 1, \ldots, n$. By applying $d_1^{\varepsilon_i}$ log to i), we obtain

$$\varsigma_i(t_1 + t_2) + \varsigma_i(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]]),$$

while, if we apply $d_2^{\varepsilon_i}$ log, we get

$$\varsigma_i(t_1 + t_2) - \varsigma_i(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]]);$$

from these relations it follows that $\varsigma_i(t_1 + t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, for $i = 1, \ldots, n$.

We are now under the hypotheses of Lemma 3.3, therefore there exists a subfield $C$ of $Q(k[[t]])$ such that $\varsigma_i(t_1 + t_2) \in Q(C \otimes C)$. $C$ is finitely generated over $k$ by the derivatives of $\varsigma_i(t)$, i.e. by the derivatives $d^\mu \log \vartheta(t)$ with $|\mu| \geq 2$, hence $\mathbf{P}(C)$ is generated by $d^\mu \log \vartheta(t_1 + t_2)$, actually by a finite number of them. This shows that $\mathbf{P}(C) \subset Q(C \otimes C)$.

Let $C'$ be the field generated over $k$ by $\mathrm{d}^\mu \log \vartheta(-t), |\mu| \geq 2$, considered as functions of $t$: the same reasoning proves that $\mathbf{P}(C') \subset Q(C' \otimes C')$. Now let $L$ be the smallest subfield of $Q(k[[t]])$ containing both $C$ and $C'$: we have $\mathbf{P}(L) \subset Q(L \otimes L)$ and also $\rho(L) \subset L$, where $\rho$ denotes the inversion of $k[[t]]$, moreover $L$ is the quotient field of $k[[t]] \cap L$, since $\mathrm{d}^\mu \log \vartheta(t)$ and $\mathrm{d}^\mu \log \vartheta(-t)$ are in $k[[t]]$. This sufficies to conclude that $L$ is a finitely generated hyperfield over $k$ (cfr. [1], Section 2). Now, from [1], Lemma 2.1, it follows that $C$ is also a finitely generated hyperfield over $k$. To complete the proof we need only check that $g(t_1, t_2) \in Q(C \otimes C)$.

Let $\varphi(t_1, t_2) = \vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$: from Lemma 3.3, it follows that $\varphi(t_1, t_2) \in Q(C_1 \otimes C_2)$, where $C_1$ and $C_2$ are the subfields of $Q(k[[t_1]])$ and $Q(k[[t_2]])$ generated over $k$ by $\mathrm{d}_2^\mu \varphi(t_1, 0)$ and $\mathrm{d}_1^\nu \varphi(0, t_2)$ respectively.

Lemma 3.2 states that

$$(\mu!)^{-1}\, \mathrm{d}_2^\mu \varphi(t_1, t_2) = \varphi(t_1, t_2) Q_\mu(\varphi),$$

where the $Q_\mu(\varphi)$'s are polynomials in $\mathrm{d}_2^\nu \log \varphi(t_1, t_2)$, with $0 < \nu \leq \mu$, and recalling the definition of $\varphi(t_1, t_2)$, we can immediately check that

$$(\mu!)^{-1}\, \mathrm{d}_2^\mu \varphi(t_1, 0) = \vartheta(t_1)^2 Q'_\mu(\varphi),$$

where the $Q'_\mu(\varphi)$'s are obtained from the $Q_\mu(\varphi)$'s by replacing $\mathrm{d}_2^\nu \log \varphi(t_1, t_2)$ with $2\, \mathrm{d}^\nu \log \vartheta(t_1)$, if $|\nu|$ is even and with 0 if $|\nu|$ is odd. This shows that all $Q'_\mu(\varphi)$'s are elements of $C$, hence $\mathrm{d}_2^\mu \varphi(t_1, 0)$ is written as a product of $\vartheta(t_1)^2$ by an element of $C$.

In a similar way we have:

$$(\mu!)^{-1}\, \mathrm{d}_1^\mu \varphi(t_1, t_2) = \varphi(t_1, t_2) Q_\mu(\varphi),$$

where now the $Q_\mu(\varphi)$'s are polynomials in $\mathrm{d}_1^\nu \log \varphi(t_1, t_2)$, with $0 < \nu \leq \mu$, and we can easily prove that

$$(\mu!)^{-1}\, \mathrm{d}_1^\mu \varphi(0, t_2) = \vartheta(t_2)\vartheta(-t_2) Q'_\mu(\varphi),$$

where the $Q'_\mu(\varphi)$'s are obtained from the $Q_\mu(\varphi)$'s by replacing $\mathrm{d}_1^\nu \log \varphi(t_1, t_2)$ with $\mathrm{d}^\nu \log \vartheta(t_2) + \mathrm{d}^\nu \log \vartheta(-t_2)$. As before, these are all elements of $C$, except at most those with $|\nu| = 1$, i.e. $\varsigma_i(t_2) + \varsigma_i(-t_2)$; but recall that $\varsigma_i(t_1 + t_2) \in Q(C \otimes C)$, hence $\varsigma_i(t_1 + t_2) - \varsigma_i(t_1) - \varsigma_i(t_2) \in Q(C \otimes C)$, and if we let $t_1 = -t_2$ in this last expression, we find that $\varsigma_i(t_2) + \varsigma_i(-t_2) \in C$. Thus we have shown that $\mathrm{d}_1^\mu \varphi(0, t_2)$ is the product of $\vartheta(t_2)\vartheta(-t_2)$ by an element of $C$, therefore we can conclude that

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)} \in Q(C \otimes C),$$

Q.E.D.

Now we come to the main result of this section:

THEOREM 3.5. *Let $\vartheta(t) \in k[[t]]$ be a normalized power series (i.e. $\vartheta(0) = 1$). $\vartheta(t)$ is a theta type if and only if it satisfies the prosthaferesis formula*

$$\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]]).$$

PROOF. The necessity of this condition is straightforward: just recall that $g(t_1, t_2) = F(t_1, t_2, -t_2)^{-1}$ and $\vartheta$ is a theta type if $F(t_1, t_2, t_3) \in Q(k[[t_1]] \otimes k[[t_2]] \otimes k[[t_3]])$.

In order to prove that it is also sufficient, we recall that the prosthaferesis formula is equivalent, by Lemma 3.4, to the fact that $g(t_1, t_2) \in Q(C \otimes C)$, where $C$ is a finitely generated hyperfield over $k$. From this, it follows immediately that

$$g(t_1 + t_2, t_3)g(t_1, t_3)^{-1}g(t_2, t_3)^{-1} \in Q(C \otimes C \otimes C).$$

Recalling the definition of $F$, we can easily check that

$$g(t_1 + t_2, t_3)g(t_1, t_3)^{-1}g(t_2, t_3)^{-1} = F(t_1, t_2, t_3)F(t_1, t_2, -t_3),$$

hence

$$F(t_1, t_2, t_3)F(t_1, t_2, -t_3) \in Q(C \otimes C \otimes C). \tag{3.6}$$

In the same way, using

$$g(t_1 + t_3, t_2)g(t_1, t_2)^{-1}g(t_3, t_2)^{-1} \in Q(C \otimes C \otimes C)$$

and

$$g(t_1, t_2 + t_3)g(t_1, t_2)^{-1}g(t_1, t_3)^{-1} \in Q(C \otimes C \otimes C),$$

we get respectively

$$F(t_1, t_2, t_3)F(t_1, -t_2, t_3) \in Q(C \otimes C \otimes C) \tag{3.7}$$

and

$$F(t_1, t_2, t_3)F(t_1, -t_2, -t_3) \in Q(C \otimes C \otimes C). \tag{3.8}$$

Now, if we divide (3.6) by (3.8), we find that

$$F(t_1, t_2, -t_3)F(t_1, -t_2, -t_3)^{-1} \in Q(C \otimes C \otimes C),$$

i.e.

$$F(t_1,t_2,t_3)F(t_1,-t_2,t_3)^{-1} \in Q(C \otimes C \otimes C),$$

and multiplying this last relation by (3.7), we finally get

$$F(t_1,t_2,t_3)^2 \in Q(C \otimes C \otimes C),$$

which proves that $\vartheta^2(t)$ is a theta type.

To show that $\vartheta(t)$ is also a theta type, we recall that $C$ is a finitely generated hyperfield over $k$, i.e. it is the function field of a group variety $A$ over $k$, hence $\vartheta^2(t)$, being a theta type, has a divisor $X$ on $A$.

But we have shown that $g(t_1,t_2) \in Q(C \otimes C)$, so it defines a divisor $Y$ on $A \times A$, and

$$g(t_1,t_2)^2 = \frac{\vartheta^2(t_1+t_2)\vartheta^2(t_1-t_2)}{\vartheta^4(t_1)\vartheta^2(t_2)\vartheta^2(-t_2)},$$

hence we must have:

$$2Y = (p_1+p_2)^*X + (p_1-p_2)^*X - 2p_1^*X - p_2^*X - (-p_2)^*X,$$

where $p_i : A \times A \to A$, denotes the $i$-th canonical projection, $i = 1,2$. This implies that $X = 2V$, for some divisor $V$ on $A$.

Let $\vartheta_V(u)$ be the non-degenerate theta function of the divisor $V$ (see [1]), $\vartheta_V(u) \in Q(k[[u]]) = Q(k[[u_1,\dots,u_m]])$, where $k[[u_1,\dots,u_m]]$ is the completion of the local ring of the identity point of $A$. We know that $C$ is embedded in $Q(k[[u]])$, but also $C \subset Q(k[[t]])$; this gives a homomorphism

$$\sigma : k[[u]] \to k[[t]],$$

which induces an isomorphism on the hyperfields, $C \cong C$.

From $X = 2V$, it follows that $\vartheta^2(t)$ is associated to $\vartheta_V(\sigma u)^2$, hence $\vartheta(t)$ is associated to $\vartheta_V(\sigma u)$. Now use Theorem 1.1 to conclude that $\vartheta(t)$ is a theta type. Q.E.D.

We end this section with a remark on the hyperfield $C$. Let us recall that the hyperfield $C$ of a theta type $\vartheta$ is the smallest subfield of $Q(k[[t]])$, containing $k$, such that $F \in Q(C \otimes C \otimes C)$. It can be shown that such a $C$ exists, and is generated over $k$ by $\mathrm{d}^{\mu} \log \vartheta(t)$, with $|\mu| \geq 2$. At this point, we may ask what are the relationships between the hyperfield $C$ and the function $g$. The answer is given by the following

PROPOSITION 3.9. *Let* $g(t_1,t_2) \in k[[t_1,t_2]]$ *and* $\vartheta(t) \in k[[t]]$ *be as in the statement of* Theorem 2.3. *Consider the power series expansion of* $g$*:*

$$g(t_1,t_2) = 1 + \sum_{\mu} D_{\mu}(t_1)t_2^{\mu}, \qquad D_{\mu}(t_1) \in k[[t_1]],\ D_{\mu}(0) = 0.$$

*Then the fields $C$, generated over $k$ by* $\mathrm{d}^{\mu}\log\vartheta(t)$, *with* $|\mu| \geq 2$, *and* $C'$, *generated over $k$ by* $D_{\mu}(t)$ *and* $\mathrm{d}^{\varepsilon_i}D_{\mu}(t)$, *for every* $\mu \neq 0$ *with* $|\mu| \equiv 0 \bmod 2$ *and* $i = 1,\ldots,n$, *coincide.*

*Moreover if $\vartheta$ is a theta type, i.e. if* $g(t_1,t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, *then* $C = C'$ *is a finitely generated hyperfield over $k$, with the coproduct* $\mathbf{P}$ *and the inversion* $\rho$ *induced by those of* $k[[t]]$.

PROOF. Let $\log\ g(t_1,t_2) = \sum_{\mu} A_{\mu}(t_1)t_2^{\mu}$, where the sum is over all $\mu \in \mathbb{N}^n - \{0\}$, with $|\mu| \equiv 0 \bmod 2$. From the proof of Theorem 2.3, we know that

$$A_{\mu}(t) = \frac{2}{\mu!}\left[\mathrm{d}^{\mu}\log\vartheta(t) - \mathrm{d}^{\mu}\log\vartheta(0)\right], \qquad |\mu| \equiv 0 \bmod 2.$$

Therefore it is clear that the fields $k(A_{\mu}(t),\ \mathrm{d}^{\varepsilon_i}A_{\mu}(t))$, where $\mu \in \mathbb{N}^n - \{0\}, |\mu| \equiv 0 \bmod 2$ and $i = 1,\ldots,n$, and $k(\mathrm{d}^{\nu}\log\vartheta(t))$ where $|\nu| \geq 2$, are equal. Thus we have only to show that $k(A_{\mu}(t),\ \mathrm{d}^{\varepsilon_i}A_{\mu}(t)) = k(D_{\mu}(t),\ \mathrm{d}^{\varepsilon_i}D_{\mu}(t))$.

Let $\varphi(t_1,t_2) = \sum_{\mu} D_{\mu}(t_1)t_2^{\mu}$, hence $g(t_1,t_2) = 1 + \varphi(t_1,t_2)$ and

$$\log g(t_1,t_2) = \varphi(t_1,t_2) - \frac{1}{2}\varphi(t_1,t_2)^2 + \frac{1}{3}\varphi(t_1,t_2)^3 - \ldots.$$

Now if we substitute the power series expansion of $\varphi(t_1,t_2)^n$ and compare with that of log $g(t_1,t_2)$, we can easily conclude that

$$A_{\mu}(t) = D_{\mu}(t) + (\text{poly. in } D_{\nu}(t), \text{ with } \nu < \mu).$$

In a similar way, letting $\Psi(t_1,t_2) = \sum_{\mu} A_{\mu}(t_1)t_2^{\mu} = \log g(t_1,t_2)$, we have

$$g(t_1,t_2) = \exp\Psi(t_1,t_2) = 1 + \Psi(t_1,t_2) + \frac{1}{2!}\Psi(t_1,t_2)^2 + \ldots$$

and finally

$$D_{\mu}(t) = A_{\mu}(t) + (\text{poly. in } A_{\nu}(t), \text{ with } \nu < \mu).$$

This proves what we wanted. The last statement of the proposition, being included in Lemma 3.4, is now obvious. Q.E.D.