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# Algebraic and Topological Selections of Multi-Valued Linear Relations

SUNG J. LEE - M. ZUHAIR NASHED<sup>1</sup>

## 1. - Introduction

In this paper we develop a comprehensive theory of algebraic and topological single-valued linear selections of a multi-valued linear relation. Let  $X, Y$  be (real or complex) vector spaces. A vector subspace (or simply a subspace, or a linear manifold)  $M$  of  $X \times Y$  is called a *linear relation*. We identify  $M$  also as the “graph” of a multi-valued linear mapping defined by  $M(x) := \{y : (x, y) \in M\}$  where  $\text{Dom } M$ , the domain of  $M$ , consists of those  $x \in X$  such that  $M(x)$  is not empty. The null space and the range of  $M$  are denoted by  $\text{Null } M$  and  $\text{Range } M$ , respectively. The graph inverse of  $M$  is denoted by  $M^{-1}$ . If  $Z$  is another vector space and  $S$  is a linear relation in  $Y \times Z$ , then the composition  $SM$  is defined by

$$SM := \{(x, z) \mid (x, y) \in M, (y, z) \in S \text{ for some } y\}.$$

Let  $M$  be an arbitrary, but fixed linear relation in  $X \times Y$ . An *algebraic operator part* (or *algebraic selection*)  $R$  of  $M$  is (the graph of) a single-valued linear operator with  $\text{Dom } R = \text{Dom } M$  such that  $R \subset M$ . Equivalently,  $R = \text{Null } \mathbb{P}$  where  $\mathbb{P}$  is an algebraic projector (i.e., a linear idempotent operator) of  $M$  onto  $M_\infty := \{0\} \times M(0)$ . If  $\mathbb{P}$  satisfies the extra condition  $\mathbb{P}(x, 0) = (0, 0)$  for all  $x \in \text{Null } M$ , then  $R$  is called a *principal algebraic operator part* of  $M$ . Equivalently, a principal algebraic operator part is an algebraic selection  $R$  such that  $\text{Null } R = \text{Null } M$ . If  $X$  and  $Y$  are normed linear spaces, then an algebraic selection (resp., a principal algebraic selection)  $R$  of  $M$  is called a *topological operator part* or *topological selection* (resp., a *principal topological operator part* or *principal topological selection*) of  $M$  if the associated projector  $\mathbb{P}$  is continuous.

We now describe briefly the contents of this paper. In §2 we develop the theory of algebraic linear selections of linear relations. In §3 we study

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topological and  $w^*$ -topological operator parts and some invariant properties. Unlike algebraic operator parts, topological operator parts do not always exist. Moreover, the existence of a topological operator part of  $M^{-1}$  does not imply the existence of a principal topological operator part of  $M^{-1}$ . We will show that the latter situation never occurs when  $M$  is single-valued.

The theory of linear relations in functional analysis was initiated by von Neumann [11], [12]. In the case when  $X, Y$  are Hilbert spaces and  $M$  is closed, Arens [1] was the first to note that  $M \cap (\{0\} \times M(0))^\perp$  is a single-valued selection of  $M$ . Coddington and Dijkstra [3] generalized this concept to a Banach space case: If  $M$  is *closed*, then a vector subspace  $R$  of  $M$  is an "operator part" of  $M$  if  $R$  is a single-valued, *closed* and  $M$  is the direct sum of  $R$  and  $M_\infty$ . We have extended this concept to the case when  $M$  is *not necessarily closed*. Our definition requires only the use of *continuous projectors*.

The study of multi-valued operator equations (inclusions) has gained considerable importance in recent years due to the increasing occurrence of multi-valued linear or nonlinear mappings in various areas of analysis, differential equations, continuum mechanics, control theory, and mathematical economics. Various results on (single-valued) selections of multi-valued *nonlinear* relations have been obtained by many authors, such as Michael's continuous selection theorem, Carathéodory-type selection theorems, measurable selections, minimal selections, etc. (See, e.g., Aubin and Cellina [2] and related references cited therein). The selection theorems for *nonlinear* relations do not provide much information about algebraic and topological linear selections for multi-valued *linear* relations. The results of this paper complement and do not overlap with existing selection theorems in the literature.

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## 2. - Algebraic Operator Parts of Multi-Valued Linear Mappings

Throughout this section,  $X$  and  $Y$  are (real or complex) vector spaces and  $M$  is a linear relation contained in the product space  $X \times Y$ . We denote  $\{0\} \times M(0)$  by  $M_\infty$ . If  $M_1$  and  $M_2$  are subspaces of  $X \times Y$ ,  $M_1 + M_2 := \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$ .

**DEFINITION 2.1.** A subspace (a vector subspace)  $R$  of  $M$  is called a *linear selection* or an *algebraic operator part* of  $M$  if  $\text{Dom } R = \text{Dom } M$  and  $R$  is the graph of a single-valued linear operator such that  $R \subset M$ .

**PROPOSITION 2.1.** *The following statements are equivalent: (i)  $R$  is a linear selection of  $M$ . (ii)  $R$  is a subspace and  $M$  is the direct algebraic sum of  $R$  and  $M_\infty$ . (iii)  $R = \text{Null } \mathbb{P}$  for some algebraic projector  $\mathbb{P}$  of  $M$  onto  $M_\infty$ .*

**PROOF.** Assume (i). It is clear that  $R + M_\infty$  is an algebraic direct sum and

that  $M \supset R + M_\infty$ . Let  $(x, y) \in M$ . Then  $(x, y) - (x, R(x)) = (0, y - R(x)) \in M$ . Thus  $(x, y) = (0, y - R(x)) + (x, R(x)) \in M_\infty + R$ . Hence (i) implies (ii). Assume (ii). Then there exists an algebraic projector  $\mathbb{P}$  of  $M$  onto  $M_\infty$  such that  $R = \text{Null } \mathbb{P}$ . Thus (ii) implies (iii). Now assume (iii). Then it is clear that  $R$  is a subspace of  $M$  and  $\text{Dom } M = \text{Dom } R$ . Suppose  $(0, y) \in R$ . Then  $(0, y) \in M$  and so  $(0, y) \in M_\infty$ . Therefore,  $\mathbb{P}(0, y) = (0, y)$ . But  $(0, y) \in \text{Null } \mathbb{P}$ . Hence  $\mathbb{P}(0, y) = (0, 0)$  and so  $y = 0$ . Thus  $R$  is single-valued. Hence (iii) implies (i). ■

By the above proposition, an algebraic selection  $R$  determines an algebraic projector  $\mathbb{P}$ , and conversely, an algebraic projector  $\mathbb{P}$  of  $M$  onto  $M_\infty$  determines an algebraic selection of  $M$ . For this reason, we will say that  $R$  is generated by  $\mathbb{P}$ , and denote it by  $R_{\mathbb{P}}$  to indicate its dependence on  $\mathbb{P}$ . The existence of a selection is a consequence of the Axiom of Choice. A linear selection of  $M$  always exists since  $M_\infty$  is always algebraically complemented in  $M$ , or equivalently there exists an algebraic projector  $\mathbb{P}$  of  $M$  onto  $M_\infty$ .

By considering an algebraic operator part of  $M^{-1}$ , we can easily characterize all solutions of a linear inclusion. The following proposition is immediate from the definition of an algebraic operator part.

**PROPOSITION 2.2.** *Let  $R$  be an algebraic selection of  $M^{-1}$ . Then  $\text{Dom } R = \text{Range } M$ ,  $\text{Range } R \subset \text{Dom } M$ . Moreover,  $(x, y) \in M$  if and only if  $y \in \text{Range } M$  and  $x = R(y) + u$  for some  $u \in \text{Null } M$ .*

In what follows we state results mainly for operator parts of  $M^{-1}$ , rather than  $M$ , since we are interested in solvability of linear inclusions and selection theorems for inverse mappings. Analogous results for operator parts of  $M$  follow immediately from those for  $M^{-1}$  by replacing  $M^{-1}$  by  $M$ , and  $\text{Null } M$  by  $M(0)$ . A linear relation  $M$  may have “nonlinear selections”. For example if  $M = X \times Y$ , then every single-valued mapping with domain  $X$  and range in  $Y$  is a “selection” of  $M$ . In this paper we shall only be interested in *linear* selections of multi-valued linear mappings.

**PROPOSITION 2.3.** *Let  $R$  be an arbitrary algebraic operator part of  $M^{-1}$ . Then  $M = MRM$ , and  $RMR$  is the direct sum of  $R$  and  $\{0\} \times RM(0)$ . In particular,  $R$  is an algebraic operator part of  $RMR$ .*

**PROOF.** Take  $(x, y) \in M$ . Then  $x = R(y) + u$  for some  $u \in \text{Null } M$ . In particular,  $(y, x - u) \in R$ , and  $(x - u, y) \in M$ . Thus  $(x, y) \in MRM$ . Suppose now  $(x, y) \in MRM$ . Then  $(x, z) \in M$ ,  $(z, h) \in R$  and  $(h, y) \in M$  for some  $z, h$ . In particular,  $(h, z) \in M$ . Thus  $(0, y - z) \in M$ , and so  $(x, y) = (0, y - z) + (x, z) \in M$ . This shows that  $M = MRM$ . Take any  $(y, x)$  in  $RMR$ . Then  $(y, h) \in R$ ,  $(h, z) \in M$  and  $(z, x) \in R$  for some  $h, z$ . In particular  $(y, h)$ ,  $-(z, h)$ ,  $(z, x)$  are in  $M^{-1}$ . Thus  $(y, x) \in M^{-1}$ . This shows that  $RMR \subset M^{-1}$ . Take any  $(z, x) \in RMR$ . Then  $(z, x_1) \in R$ ,  $(x_1, y) \in M$ ,  $(y, x) \in R$  for some  $x_1$  and  $y$ . Then  $(0, y - z) = (x_1, y) - (x_1, z) \in M$  and  $(R(z), y) \in M$ ,  $x = R(y)$ . Thus  $R(z) = R(y) + k$  for some  $k \in \text{Null } M$ . Hence  $x = R(z) - k = R(z) + R(y - z)$ . Since  $y - z \in M(0)$  and  $x = R(z) + R(y - z)$ , we see that  $(z, x) = (z, R(z) + R(y - z)) \in (\{0\} \times RM(0)) + R$ .

Thus  $RM R = (\{0\} \times RM(0)) + R$ . Clearly this algebraic sum is direct. ■

REMARK 2.1. It is not always true that  $R = RM R$  for any algebraic operator part  $R$  of  $M^{-1}$  (see Example 2.1 below). But the equality holds if  $M$  is single-valued or, more generally, if  $R$  is a principal algebraic operator part of  $M^{-1}$  as defined below (see Theorem 2.6).

Suppose now that  $P$  is an algebraic projector of Range  $M$  onto  $M(0)$  and let  $\mathbb{P} := 0 \times P : M \rightarrow \{0\} \times M(0)$ , be defined by  $\mathbb{P}(x, y) = (0, P(y))$  for all  $(x, y) \in M$ . Then  $\mathbb{P}$  is an algebraic projector and is surjective. Thus  $\text{Null } \mathbb{P} = \{(x, y) \in M : P(y) = 0\}$  is an algebraic operator part of  $M$ . This motivates the following definition.

DEFINITION 2.2.  $R \subset X \times Y$  is called a *principal algebraic operator part* of  $M$  if  $R := \{(x, y) \in M : P(y) = 0\}$  for some algebraic projector  $P$  of Range  $M$  onto  $M(0)$ . To indicate the dependence on  $P$ , we also write  $R = R_P$ .

REMARK 2.2. It is easy to show that  $R$  is a principal algebraic part of  $M$  if and only if  $R = \text{Null } \mathbb{P}$  for some algebraic projector  $\mathbb{P}$  of  $M$  onto  $M_\infty$  such that  $\mathbb{P}(x, 0) = (0, 0)$  for all  $x \in \text{Null } M$ . Equivalently,  $R$  is an algebraic selection such that  $\text{Null } R = \text{Null } M$ . It follows from the construction preceding Definition 2.2 that a principal algebraic operator part always exists. We will see soon that not every algebraic operator part is principal.

THEOREM 2.4. Let  $R := \text{Null } \mathbb{P}$  be an algebraic operator part of  $M^{-1}$ , where  $\mathbb{P}$  is an algebraic projector of  $M^{-1}$  onto  $\{0\} \times \text{Null } M$ . Define  $\mathbb{P}_2$  by  $\mathbb{P}(y, x) = (0, \mathbb{P}_2(y, x))$ , for all  $(x, y) \in M$ . Then we have the following:

- (i)  $\text{Null } R = \{y \in M(0) : \mathbb{P}_2(y, 0) = 0\} \subset M(0)$ .
- (ii)  $\text{Null } R = M(0)$  if and only if the map  $Q$  defined by  $Q(x) = \mathbb{P}_2(y, x)$ ,  $(x, y) \in M$ , is single-valued. Equivalently,  $R$  is a principal algebraic operator part of  $M^{-1}$ .
- (iii) If  $M$  is a single-valued, then every algebraic operator part of  $M^{-1}$  is a principal operator part of  $M^{-1}$ .

PROOF. (i) is easy to check. (ii) Assume that  $\text{Null } R = M(0)$ . Suppose that  $(x, y)$ ,  $(x, z)$  are in  $M$ . Then  $(0, y - z) \in M$ , and so  $\mathbb{P}_2(y, x) - \mathbb{P}_2(z, x) = \mathbb{P}_2(y - z, 0) = 0$ . Thus  $Q$  is single-valued. Conversely, suppose that  $Q$  is single-valued. Let  $y \in M(0)$ , so  $\{y, 0\} \in M^{-1}$ . Since  $(0, 0) \in M^{-1}$ ,  $0 = Q(0) = \mathbb{P}_2(y, 0)$ . Thus  $y \in \text{Null } R$ , and so  $\text{Null } R = M(0)$ . This proves the first part of (ii). The second part of (ii) then follows immediately from Remark 2.2. Finally, (iii) is an immediate consequence of (i) and (ii). ■

Note that the converse to part (iii) of Theorem 2.4 is false: Let  $M = \{0\} \times S$ , where  $S$  is any subspace. Then  $M^{-1}$  is the zero operator with domain  $S$ . Thus (trivially) every algebraic selection of  $M^{-1}$  is principal, but  $M$  is not single-

valued.

EXAMPLE 2.1. In this example, there exists a unique principal algebraic selection and all other selections are nonprincipal. Let  $M = \{(x, x+a) : x, a \in X\}$ . Then  $\text{Null } M = \text{Dom } M = \text{Range } M = X$ . Thus any algebraic projector of  $\text{Dom } M$  onto  $\text{Null } M$  is the identity operator. Consequently, the only principal operator part of  $M^{-1}$  is the zero operator on  $X$ . Now let  $\mathbb{P} : M^{-1} \rightarrow \{0\} \times \text{Null } M$  be defined by  $\mathbb{P}(x, x+a) = (0, a)$  for  $x, a \in X$ . Then  $\mathbb{P}^2 = \mathbb{P}$  and  $\mathbb{P}$  is onto; thus  $R_{\mathbb{P}} := \text{Null } \mathbb{P} = \{(x, x) : x \in X\}$  is an algebraic operator part of  $M^{-1}$ . Moreover, the inclusion  $R_{\mathbb{P}} \subset R_{\mathbb{P}} M R_{\mathbb{P}}$  is proper and  $R_{\mathbb{P}} M$  is not single-valued. Notice that  $R_{\mathbb{P}}$  is one-to-one even though  $M$  is not single-valued. This phenomenon contrasts with the fact that a principal algebraic selection of  $M^{-1}$  is one-to-one if and only if  $M$  is single-valued (see below).

REMARK 2.3. Let  $R$  be a principal algebraic operator part of  $M^{-1}$ . Then  $\text{Null } R = M(0)$ . Consequently,  $R$  is one-to-one if and only if  $M$  is single-valued.

We now investigate the structure, form, and distinctive properties of principal algebraic operator parts (Theorems 2.6, 2.5, 2.7 and 2.10; Proposition 2.11; and Corollary 2.12). The graph of an operator  $A$  will be denoted by  $\mathcal{G}rA$ , and the restriction of a relation  $M$  to a set  $S$  will be denoted by  $M|_S$ .

THEOREM 2.5. Let  $R := \{(y, x) : (y, x) \in M^{-1}, P(x) = 0\}$ , where  $P$  is an algebraic projector of  $\text{Dom } M$  onto  $\text{Null } M$ . Then  $\text{Range } R = \text{Dom } M \cap \text{Null } P$ , and  $R = [\mathcal{G}r(I - P)]M^{-1}$ . Moreover, if  $M$  is single-valued, then  $R = (M|_{\text{Null } P})^{-1}$ .

PROOF. Clearly  $\text{Range } R = \text{Dom } M \cap \text{Null } P$ . Take any  $(y, x) \in R$ . Then  $(x, y) \in M$  and  $P(x) = 0$ . Now  $P(x) = 0$  if and only if  $x = (I - P)(z)$  for some  $z$  as the range of  $I - P$  is  $\text{Null } P$ . Thus  $(y, x) = (y, z - P(z))$ , and, since  $P(z) \in \text{Null } M$ ,  $(y, z) \in M^{-1}$ . But then  $(z - P(z), y) \in M$  and  $P(z - P(z)) = 0$ . This shows that  $R = [\mathcal{G}r(I - P)]M^{-1}$ . The last part of the theorem is obvious.

THEOREM 2.6. Let  $R$  be an algebraic selection of  $M^{-1}$ . Then the following are equivalent: (i)  $R$  is a principal algebraic selection of  $M^{-1}$ . (ii)  $RM$  is single-valued. (iii)  $R = RMR$ .

PROOF. By Proposition 2.3,  $RMR = R + (\{0\} \times RM(0))$ . In particular,  $RMR(0) = RM(0)$ . This immediately yields that (ii) and (iii) are equivalent. Assume (i). Since  $R$  is principal, by Remark 2.3,  $\text{Null } R = M(0)$ . But then  $RM(0) = \{R(z) : z \in M(0)\} = \{R(z) : z \in \text{Null } R\} = \{0\}$ . Hence  $R = RMR$ . Assume (iii). Then (ii) also holds. Thus  $RM$  is single-valued. Define  $P$  by  $P(x) = x - RM(x)$  for  $x \in \text{Dom } M$ . Then by (iii),  $P$  is an algebraic projector of  $\text{Dom } M$ . We will show that  $[\mathcal{G}r(I - P)]M^{-1} = R$ . Clearly  $(\mathcal{G}r(I - P))M^{-1} = RMM^{-1} \supset R$ . Take any  $(y, x) \in RMM^{-1}$ . Then  $(y, z) \in M^{-1}$ ,  $(z, h) \in M$  and  $(h, x) \in R$  for some  $z, h$ . In particular,  $y - h \in M(0)$ . Thus  $R(y - h) = 0$ , and so  $x = R(h) = R(y)$ . Hence  $(y, x) \in R$ . Therefore  $R = RMR$ . This shows that  $R$  is principal and so (iii) implies (ii). ■

REMARK 2.4. In the above theorem  $RM(0) = \{R(y) : y \in M(0)\}$ . Hence

if  $M$  is single-valued, then any algebraic selection of  $M^{-1}$  is a principal selection. Thus there is a subtle difference, which heretofore seems to have been undiscovered, between selections of a multi-valued linear mapping  $M$  whose inverse is single-valued and those of a multi-valued linear mapping  $M$  whose inverse is also multi-valued.

**THEOREM 2.7.** *Let  $R$  be a principal algebraic selection of  $M^{-1}$  given by  $[\mathcal{G}r(I - P)]M^{-1}$ . Then  $MR = \{(x, x + s) : x \in \text{Range } M, s \in M(0)\}$ , and  $RM = \{(x, (I - P)x) : x \in \text{Dom } M\}$ .*

**PROOF.** Take  $(x, y) \in MR$ . Then  $(x, h) \in R$ ,  $(h, y) \in M$ . Thus  $x, y \in \text{Range } M$ ,  $h = R(x) = R(y) + k$  for some  $k \in \text{Null } M$ . Hence  $R(x - y) = k$  and so  $k = 0$  as  $R$  is principal. Thus  $(0, x - y) \in M(0)$  and so  $y = x + s$  for some  $s \in M(0)$ . Conversely, suppose that  $x \in \text{Range } M$ ,  $s \in M(0)$  and  $y = x + s$ . Then  $(0, x - y) \in M$  and so  $0 = R(x) - R(y) + k$  for some  $k \in \text{Null } M$ . But  $k = 0$  as  $k \in \text{Null } P \cap \text{Range } P$ . Thus  $R(x) = R(y)$ . Let  $h = R(x)$ . Then  $(x, h) \in R$  and  $(h, y) \in M$  and so  $(x, y) \in MR$ . This proves the statement about  $MR$ . Take any  $(x, y) \in RM$ . Then  $(x, z) \in M$  and  $(z, y) \in R$ . Thus  $x \in \text{Dom } M$ ,  $y \in \text{Range } R$ ,  $y = R(z)$ ,  $x = R(z) + u$  for some  $u \in \text{Null } M$ . Then  $x = y + u$ . Hence  $P(x) = P(y) + u = u$  as  $y \in \text{Range } R \cap \text{Null } M$ . This shows that  $y = x - P(x)$  and  $(x, x - P(x)) \in RM$ . But by Theorem 2.6,  $RM$  is single-valued. Therefore  $(RM)(x) = x - P(x)$  for all  $x \in \text{Dom } M$ . ■

We remark here that the conclusions of Theorem 2.7 can be stated equivalently as  $MR \subset MM^{-1}$  and  $RM \subset \mathcal{G}r(I - P)$ . This is in sharp contrast with inner and outer inverses of single-valued linear operators (see [8]).

In some applications, it is not enough to know one operator part of  $M^{-1}$ . We may need to choose a particular selection which satisfies a certain preferential property, or to determine among all operator parts a selection that would minimize a prescribed operator norm or that would have a minimal property with respect to an induced partial ordering. Examples of the first type include linearized analysis in bifurcation theory and alternative methods, where a particular inner inverse is to be selected. Examples of the second type arise in control theory and constrained minimization problems (see, e.g., [4], [6], [7]). Often we are also interested in determining properties (e.g., continuity, compactness, etc.) that are invariant with respect to various selections of operator parts. These considerations require that (i) we obtain tractable descriptions and characterizations of *all* operator parts in terms of a *fixed* operator part, and (ii) we determine relationships between two operator parts corresponding to two distinct choices of projectors. Suppose that  $R_1$  is a given algebraic operator part of  $M$  and  $R_2$  is the graph a single-valued linear operator defined on  $\text{Dom } M$ . Clearly if  $R_2$  is also an algebraic selection of  $M$ , then  $g(x) := R_1(x) - R_2(x) \in M(0)$  for all  $x \in \text{Dom } M$ . The converse is false. In order that  $R_2$  be also an algebraic selection,  $g(x)$  must satisfy more subtle conditions. We next address this problem and the related questions (i) and (ii) mentioned above.

In order to describe all possible algebraic selections of  $M^{-1}$  we need the

following fundamental lemma which follows easily from Sobczyk's lemma [10]; see also [8].

LEMMA 2.8. let  $P_0$  be an arbitrary but fixed algebraic projector of a vector space  $Y$  onto a subspace  $X$  of  $Y$ . Then  $P$  is an algebraic projector of  $Y$  onto  $X$  if and only if  $P = P_0 + A$  for some linear operator  $A : Y \rightarrow X$  such that  $Ay = 0$  for all  $y \in X$ . The above result remains valid if  $X, Y$  are normed linear spaces and  $P_0, A$  are required further to be continuous.

We establish a fundamental relationship between any two algebraic operator parts of  $M^{-1}$ .

THEOREM 2.9. Let  $R_0 = [\mathcal{G}r(I - P_0)]M^{-1}$  be a principal algebraic operator part of  $M^{-1}$  where  $P_0 : \text{Dom } M \rightarrow \text{Null } M$  is an algebraic projector onto. Then

- (i)  $R$  is an algebraic operator part of  $M^{-1}$  if and only if  $R(y) = R_0(y) - A_2(y, R_0(y))$  for all  $y \in \text{Range } M$ , for some linear operator  $A_2 : M^{-1} \rightarrow \text{Null } M$  such that  $A_2(0, x) = 0$  for all  $x \in \text{Null } M$ .
- (ii) For any algebraic operator part  $R$  of  $M^{-1}$ ,  $(I - P_0)R(y) = R_0(y)$  for all  $y \in \text{Range } M$ .
- (iii) If  $R := [\mathcal{G}r(I - P)]M^{-1}$  is any principal algebraic operator part of  $M^{-1}$  (where  $P : \text{Dom } M \rightarrow \text{Null } M$  is an algebraic projector onto), then

$$R_0(y) = (I - P_0)R(y), \quad R(y) = (I - P)R_0(y) \text{ for all } y \in \text{Range } M.$$

PROOF. We show first that  $R$  is an algebraic operator part of  $M^{-1}$  if and only if

$$(2.1) \quad R := \{(y, x) \in M^{-1} \mid P_0(x) = -A_2(y, x)\}$$

for some linear operator  $A_2 : M^{-1} \rightarrow \text{Null } M$  such that  $A_2(0, x) = 0$  for all  $x \in \text{Null } M$ . Define  $\mathbb{P}$  by  $\mathbb{P}(y, x) := (0, P_0(x))$ ,  $(y, x) \in M^{-1}$ . Then  $\mathbb{P}$  is an algebraic projector from  $M^{-1}$  onto  $\{0\} \times \text{Null } M$ . It follows from Lemma 2.8 that  $Q$  is an algebraic projector from  $M^{-1}$  onto  $\{0\} \times \text{Null } M$  if and only if

$$(2.2) \quad Q(y, x) := \mathbb{P}(y, x) + A(y, x)$$

for some linear operator  $A$  from  $M^{-1}$  into  $\{0\} \times \text{Null } M$  such that  $A(0, x) = (0, 0)$  for all  $x \in \text{Null } M$ . We can write  $A$  in the form  $A(y, x) = (0, A_2(y, x))$ ,  $(y, x) \in M^{-1}$ . Then  $A_2$  is a linear map into  $\text{Null } M$ , and  $A_2(0, x) = 0$  for all  $x \in \text{Null } M$ . Moreover,  $Q(y, x) = (0, P_0(x) + A_2(y, x))$ .

Now by definition of an algebraic operator part,  $R$  is an algebraic operator part of  $M^{-1}$  if and only if  $R = M^{-1} \cap \text{Null } Q$  for some  $Q$  as in (2.2). But then  $R = \{(x, y) \in M \mid A_2(y, x) + P_0(x) = 0\}$  for some such  $A_2$ . This proves assertion

(2.1). Take  $(y, x) \in R$ . Then  $x = R_0(y) + u$  for some  $u \in \text{Null } M$ , and

$$\begin{aligned} 0 &= A_2(y, x) + P_0(x) = A_2(y, R_0(y) + u) + P_0(R_0(y) + u) \\ &= A_2(y, R_0(y)) + u, \end{aligned}$$

and so

$$y = R_0(y) + u = R_0(y) - A_2(y, R_0(y)).$$

Thus (2.1) can be rewritten as

$$(2.3) \quad R(y) = R_0(y) - A_2(y, R_0(y)), \quad \text{all } y \in \text{Range } M.$$

We now prove (ii). Since  $\text{Range } A \subset \text{Null } M$ , we have for all  $y \in \text{Range } M$ ,  $(I - P_0)R(y) = (I - P_0)R_0(y) - (I - P_0)A(y, R_0(y)) = R_0(y)$ .

Suppose now that  $R$  is as in (iii). Then using (i), for all  $x \in \text{Range } M$ ,

$$R(y) = (I - P)R(y) = (I - P)R_0(y) - (I - P)A_2(y, R_0(y)) = (I - P)R_0(y)$$

as  $\text{Range } A \subset \text{Null } A$ . ■

REMARK 2.5. It follows from the above theorem that for any two algebraic operator parts  $R_1, R_2$  of  $M^{-1}$ ,  $(I - P_0)R_1(y) = (I - P_0)R_2(y)$ , for all  $y \in \text{Range } M$ , where  $P_0$  is any algebraic projector of  $\text{Dom } M$  onto  $\text{Null } M$ .

We now provide an important characterization (and describe the general form) of a principal algebraic operator of  $M^{-1}$ .

THEOREM 2.10. *Let  $R$  be the graph of a single-valued linear operator on  $\text{Range } M$  into  $\text{Dom } M$ . Then  $R$  is a principal algebraic operator part of  $M^{-1}$  if and only if the following two conditions are satisfied:*

- (i)  $RM(0) = \{0\}$ ; equivalently  $RM$  is single-valued.
- (ii) The map  $P$  on  $\text{Dom } M$  defined by  $P(x) := x - RM(x)$  is an algebraic projector of  $\text{Dom } M$  onto  $\text{Null } M$ .

Moreover, when (i) and (ii) hold,  $R = [\mathcal{G}r(I - P)]M^{-1}$ .

PROOF. The “only if” part follows from Theorems 2.6 and 2.7. Assume that (i), (ii) hold. Since  $\mathcal{G}r(I - P) = RM$ , we see that  $[\mathcal{G}r(I - P)]M^{-1} = RMM^{-1}$ . But  $R \subset RMM^{-1}$ . Thus

$$(2.4) \quad R \subset [\mathcal{G}r(I - P)]M^{-1} =: R_1.$$

By definition of a principal algebraic operator part of  $M^{-1}$ ,  $R_1$  is a principal algebraic operator of  $M^{-1}$ . Since  $R$  and  $R_1$  are single-valued and  $\text{Dom } R = \text{Dom } R_1$ , it follows from (2.4) that  $R = R_1$ . ■

The following proposition provides several algebraic operational characterizations of a principal algebraic operator part of  $M^{-1}$ .

**PROPOSITION 2.11.** *Let  $R$  be the graph of a single-valued linear operator on Range  $M$  into Dom  $M$ . Then the following are equivalent:*

- (i)  $R$  is a principal algebraic operator part of  $M^{-1}$ .
- (ii)  $RM(0) = \{0\}$ ,  $M = MRM$  and  $\text{Null } RM = \text{Null } M$ .
- (iii)  $R = RMR$ ,  $M = MRM$ ,  $\text{Null } RM = \text{Null } M$ .
- (iv)  $R = RMR$ ,  $\text{Null } RM = \text{Null } M$ .

Moreover, if any of (ii), (iii), (iv) holds, then  $R = [\mathcal{G}_r(I - P)]M^{-1}$  where  $P(x) := x - RM(x)$  for all  $x \in \text{Dom } M$ .

**PROOF.** Assume (i). Then by Proposition 2.10,  $RM(0) = \{0\}$ ,  $\text{Null } RM = \text{Null } M$ . By Proposition 2.3,  $M = MRM$ . Thus (i) implies (ii). Assume (ii). Then  $I - RM$  defines an algebraic projector of Dom  $M$  onto Null  $M$ . Thus (ii) implies (i), and moreover,  $R = [\mathcal{G}_r(I - P)]M^{-1}$ . Thus (i) and (ii) are equivalent. Clearly (iii) implies (ii), and (iii) implies (iv). Assume (ii). Then since (ii) and (i) are equivalent, by Proposition 2.3,  $R = RMR$ . Hence (ii) implies (iii). Assume (iv). Then  $RM(0) = \{0\}$ , and  $I - RM$  defines an algebraic projector of Dom  $M$  onto Null  $M$ . Thus by Proposition 2.10,  $R$  is a principal algebraic operator part of  $M^{-1}$ , and so (iv) implies (i). ■

**COROLLARY 2.12.** Assume that  $M$  is single-valued. Let  $R$  be the graph of a linear operator on Range  $M$  into Dom  $M$ . Then the following are equivalent: (i)  $R$  is an algebraic operator part of  $M^{-1}$ . (ii)  $R$  is a principal algebraic operator part of  $M^{-1}$ . (iii)  $R = RMR$ ,  $\text{Null } RM = \text{Null } M$ . (iv)  $M = MRM$ .

**PROOF.** The equivalence of (i) and (ii) follows from the definition of an algebraic operator part and part (iii) of Theorem 2.4. Since  $R, M$  are single-valued,  $RM(0) = \{0\}$ . When  $M = MRM$ ,  $\text{Null } M = \text{Null } RM$ . Thus the remaining assertions follow from Proposition 2.11. ■

### 3. - Topological Selections of Multi-Valued Linear Mappings

In the previous section we have defined and studied algebraic operator parts. In this section we will study certain algebraic operator parts (of a multi-valued linear mapping) which have also some topological properties, in particular those which are induced by continuous projectors. Throughout this section  $X$  and  $Y$  are real or complex Banach spaces, and  $X^*$  and  $Y^*$  their (topological) duals. Throughout this section,  $M$  is a linear manifold in  $X \times Y$ , and  $M^+$  is a linear manifold in  $Y^* \times X^*$ . We do not assume that  $M$  is closed or  $M^+$  is  $w^*$ -closed. All the results of this section are also valid in topological vector

spaces.

DEFINITION 3.1. We say that  $R$  is a *topological operator part* (TOP for short) or a *topological selection of  $M$*  if there exists a *continuous* projector  $\mathbb{P}$  of the normed space  $M$  (whose norm is inherited from  $X \times Y$ ) onto  $M_\infty$  such that  $R = \text{Null } \mathbb{P}$ . This  $R$  will be denoted by  $R_{\mathbb{P}}$  when the dependence on  $\mathbb{P}$  is to be emphasized.

For a subspace of the product of two dual spaces, one could define a topological operator part as in Definition 3.1. However, this would not lead to “symmetry properties” that are desirable. There is a more important definition that uses continuous projectors in the  $w^*$ -topology.

DEFINITION 3.2. Let  $M^+$  be a linear relation in  $Y^* \times X^*$ . Then  $R^+$  is called a  *$w^*$ -topological operator part ( $w^*$ -TOP)* of  $M^+$  if  $R^+ = \text{Null } \mathbb{P}^+$  for some algebraic projector  $\mathbb{P}^+$  of  $M^+$  onto  $M_\infty^+ := \{0\} \times M^+(0)$  such that  $\mathbb{P}^+$  is  *$w^*$ -continuous* on the normed space  $M^+$ . This operator part will be denoted by  $R_{\mathbb{P}^+}^+$ .

By the above definitions, a topological part and a  $w^*$ -operator part are algebraic selections.

REMARK 3.1. Suppose that  $R_{\mathbb{P}^+}^+$  is a topological operator part of  $M^+$  where  $\mathbb{P}^+$  is a continuous projector of  $M^+$  onto  $M_\infty^+$ . If  $\mathbb{P}^+$  can be extended to a continuous linear operator defined everywhere on  $Y^* \times X^*$ , then  $R_{\mathbb{P}^+}^+$  is a  $w^*$ -topological operator part of  $(M^+)^{-1}$ . The converse is not true.

REMARK 3.2. For a linear relation  $M$ , Coddington and Dijksma used the term “operator part” of  $M$  to mean a *closed algebraic operator part* (see, e.g., [3]). For a *closed* linear relation  $M$ , the notion of topological operator part in our sense coincides with the notion of “operator part” in the sense of Coddington. For suppose  $R$  is an operator part in the sense of Coddington. Then  $M$  is the direct algebraic sum of  $R$  and  $M_\infty$ , and  $R$  is closed. Let  $\mathbb{P}$  be the projector of  $M$  onto  $M_\infty$  such that  $R = \text{Null } \mathbb{P}$ . Since  $M$ ,  $M_\infty$  and  $R$  are closed, and  $M$  is the direct sum of  $R$  and  $M_\infty$ ,  $\mathbb{P}$  must be continuous (see, e.g., Rudin [9]). Hence  $R$  is a topological operator part of  $M$ . Conversely, let  $M$  be closed and suppose that  $S$  is a topological operator part of  $M$ . Then  $S = \text{Null } \mathbb{P}$  for some continuous projector of  $M$  onto  $M_\infty$ . Thus  $S$  is closed since  $M$  is closed. Moreover,  $S$  is an algebraic selection of  $M$ . Hence  $S$  is an operator part in the sense of Coddington. Our notion of topological operator part does not assume  $M$  to be closed, nor  $M(0)$  to be topologically complemented. Moreover, topological selections in our sense will not be closed in general.

We now consider several examples which highlight different aspects of topological operator parts.

EXAMPLE 3.1. A topological selection of  $M^{-1}$  always exists and is closed if  $M$  is closed and  $\text{Null } M$  is topologically complemented in  $X$ . To prove this, let  $P$  be a continuous projector of  $X$  onto  $\text{Null } M$ , and define  $\mathbb{P} : M^{-1} \rightarrow \{0\} \times$

Null  $M$  by  $\mathbb{P}(y, x) = (0, P(x))$ . Then  $\mathbb{P}$  is continuous and surjective, and so Null  $\mathbb{P}$  is a topological selection. Since  $\mathbb{P}$  is continuous and  $M^{-1}$  is closed, Null  $\mathbb{P}$  is closed.

EXAMPLE 3.2. If  $\text{Dom } M$  is topologically decomposable with respect to a continuous projector  $P : X \rightarrow X$ , that is, if  $P(\text{Dom } M) = \text{Null } M$ , then  $M^{-1}$  has a topological operator part. For, let  $\mathbb{P}(y, x) := (0, P(x))$  for  $(y, x) \in M^{-1}$ . Then  $\mathbb{P}$  is continuous and is onto  $\{0\} \times \text{Null } M$ . In this case  $R := \text{Null } \mathbb{P} = \{(y, x) : P(x) = 0\}$  is called a *principal* topological operator part of  $M^{-1}$ . In view of Theorem 2.5, we may recast definition of a principal topological operator part of  $M$  in the following equivalent, but more explicit, form:

DEFINITION 3.3. A *principal topological operator part* of  $R$  of  $M$  is a subspace of  $M$  of the form  $R = [\mathcal{G}r(I - P)]M$ , where  $P$  is a *continuous* projector of  $\text{Range } M$  onto  $M(0)$ . Similarly, a  $w^*$ -topological operator part  $R^+$  of  $M^+$  is called a *principal  $w^*$ -topological operator part* if  $R^+ = [\mathcal{G}r(I - P^+)]M^+$  for some  $w^*$ -continuous projector  $P^+$  of  $\text{Range } M^+$  onto  $M^+(0)$ .

Thus by Theorem 2.5 any principal topological operator part and any principal  $w^*$ -topological operator part are principal algebraic operator parts.

Suppose now that  $X, Y$  are Hilbert spaces and  $M(0)$  is *closed* ( $M$  need not be closed). Then  $R := M \cap (\{0\} \times M(0))^\perp = [\mathcal{G}r(I - P)]M$ , where  $P$  is the *orthogonal* projector of  $Y$  onto  $M(0)$ . Thus, in particular,  $R$  is an algebraic operator of  $M$ . Moreover, if  $\mathbb{P}(x, y) := \{0, P(y)\}$  for all  $(x, y) \in M$ , then  $\mathbb{P}$  is a continuous projector of  $M$  onto  $\{0\} \times M(0)$ , and  $R = R_{\mathbb{P}}$ . Hence  $R$  is a topological operator of  $M$ . The assumption that  $M(0)$  is closed may be replaced by the weaker assumption that the domain of  $M^{-1}$  is decomposable with respect to an orthogonal projector, i.e.,  $M(0) \subset \text{Range } P$  and

$$\text{Range } M = M(0) + (\text{Range } M \cap \text{Null } P),$$

where  $P$  is an orthogonal projector on  $Y$ . This is equivalent to the condition  $P(\text{Dom } M^{-1}) = M(0)$ , with  $P$  being an orthogonal projector. Clearly if  $M(0)$  is closed, then  $\text{Dom } M^{-1}$  is orthogonally decomposable, but the converse is false. This motivates the following definition.

DEFINITION 3.4. Suppose  $X, Y$  are Hilbert spaces and  $\text{Dom } M^{-1}$  is orthogonally decomposable with respect to an orthogonal projector  $P$ . Then  $R := [\mathcal{G}r(I - P)]M$  is called the *orthogonal operator part* of  $M$ .

Orthogonal operator parts and orthogonal generalized inverses possess certain extremal properties. These have been studied by the authors [6], [7] and applied to linear inclusions and minimization problems.

EXAMPLE 3.3. A topological operator part may not exist. Let  $T$  be a *unbounded linear functional* on a Banach space  $X$ . Thus  $\text{Null } T$  is dense in  $X$  and so  $\text{Null } T$  is not closed. Let  $M = (\mathcal{G}rT)^{-1}$ . Then  $R$  is an algebraic operator part of  $M$  if and only if there exists  $x_0 \in X$  with  $T(x_0) = 1$  such that  $R(t) = tx_0$

for all  $t \in \mathbb{R}$ . For such  $R$ ,  $RT(x_0) = x_0$  and  $RT = 0$  on a dense subset of  $X$ . Therefore  $RT$  is discontinuous. Suppose there exists a topological operator part  $R$  of  $M$ . Then, since  $T$  is single-valued, by Theorem 2.4,  $R$  must be a principal topological operator part of  $M$  generated by a continuous projector  $P$  of Range  $M = \text{Dom } T$  onto  $M(0) = \text{Null } T$  such that  $R = [\mathcal{G}_r(I - P)]M$ . But then by Theorem 2.7,  $RM^{-1}(x) = RT(x) = x - P(x)$  for all  $x \in \text{Dom } T$ . This, in particular, implies that  $RT$  is continuous, a contradiction. Thus  $(\mathcal{G}_r T)^{-1}$  has no topological operator part.

EXAMPLE 3.4. A topological operator part of  $M$  always exists and is closed if  $M$  is closed and if  $\{0\} \times M(0)$  is topologically complemented in  $M$ . This follows from Example 3.1. However, neither assumption is necessary for the existence of a topological operator part.

EXAMPLE 3.5. A topological operator part may not be closed. For example, let  $M = A \times \{0\}$  where  $A$  is a *nonclosed* vector subspace of  $X$ . Then the topological operator part of  $M$  is  $M$  itself, and so it is *not* closed.

EXAMPLE 3.6. A topological operator part of  $M$  can exist even if  $M(0)$  is *not* complemented in  $Y$ . For example, take  $M = \{0\} \times A$  where  $A$  is a *closed* subspace of  $Y$  which is not complemented in  $Y$ . Then  $\{(0, 0)\}$  is the topological operator part of  $M$ .

We now give a characterization of a principal TOP of  $M^{-1}$  without using projectors.

THEOREM 3.1. *Let  $R$  be the graph of a single-valued linear operator on Range  $M$  into  $\text{Dom } M$ . Then  $R$  is a principal topological operator part of  $M^{-1}$  if and only if  $RM$  is single-valued and continuous,  $\text{Null } RM = \text{Null } M$ , and  $M = MRM$ . In particular if, in addition,  $M$  is single-valued, then  $R$  is a topological operator part of  $M^{-1}$  if and only if  $M = MRM$  and  $RM$  is continuous.*

PROOF. This follows from Proposition 2.11, Corollary 2.12 and Proposition 2.10 by considering the projector  $P$  defined by  $P(x) = x - RM(x)$ ,  $x \in \text{Dom } M$ . ■

The existence of a TOP of  $M$  does not always imply the existence of a principal TOP of  $M$  (see Remark 3.4 below). But we still can relate any two TOP of  $M$  via the following theorem.

THEOREM 3.2. *Suppose that  $M$  has a topological operator part, say  $R_0 := \{(x, y) \in M : \mathbb{P}_2(x, y) = 0\}$  where  $\mathbb{P}_2 : M \rightarrow M(0)$  is a linear operator such that the map  $(x, y) \rightarrow (0, \mathbb{P}_2(x, y))$  defines a continuous projector of  $M$  onto  $M_\infty$ . Then  $R$  is a topological operator part of  $M$  if and only if*

$$(3.1) \quad R(x) = R_0(x) - \mathbb{P}_2(x, R_0(x)) - A_2(x, R_0(x))$$

for all  $x \in \text{Dom } M$ , for some continuous linear operator  $A_2 : M \rightarrow M(0)$  such

that  $A_2(0, y) = 0$  for all  $y \in M(0)$ .

PROOF. Assume that  $R$  is a topological operator part of  $M$ . Then  $R = \text{Null } \tilde{\mathbb{P}}$  for some continuous projector  $\tilde{\mathbb{P}}$  of  $M$  onto  $M_\infty$ . Thus by Lemma 2.8, there exists a continuous linear operator  $A : M \rightarrow M_\infty$  such that  $A(0, y) = (0, 0)$  for all  $y \in M(0)$ , and for all  $(x, y) \in M$  we have

$$(3.2) \quad \tilde{\mathbb{P}}(x, y) = (0, \mathbb{P}_2(x, y)) + A(x, y)$$

Let  $\tilde{\mathbb{P}} := (0, \tilde{\mathbb{P}}_2)$ ,  $A := (0, A_2)$ . Then  $\tilde{\mathbb{P}}_2(x, y) = \mathbb{P}_2(x, y) + A_2(x, y)$ , and  $A_2(0, y) = 0$  for all  $y \in M(0)$ . Now  $R = \{(x, y) \in M : \mathbb{P}_2(x, y) + A_2(x, y) = 0\}$ . Take any  $(x, y) \in M$ . Then  $y = R_0(x) + u$  for some  $u \in M(0)$ . Set  $z = R_0(x)$ . Then  $0 = \mathbb{P}_2(x, z+u) + A_2(x, z+u) = \mathbb{P}_2(x, z) + A_2(x, z) + u$ . Thus  $y = z - \mathbb{P}_2(x, z) - A_2(x, z)$ . Conversely, suppose that  $R$  is defined as in the theorem for some  $A_2$ . Take any  $x \in \text{Dom } M$  and let  $y = z - \mathbb{P}_2(x, z) - A_2(x, z)$ ,  $z = R_0(x)$ . Then  $(x, y) = (x, z) - (0, \mathbb{P}_2(x, z)) - (0, A_2(x, z))$ . Thus  $(x, y) \in M$  and  $\mathbb{P}_2(x, y) + A_2(x, y) = \mathbb{P}_2(x, z) - \mathbb{P}_2(x, z) - A_2(x, z) + A_2(x, z) = 0$ . Define  $\mathbb{P} : M \rightarrow M_\infty$  by  $\mathbb{P}(x, y) := (0, \mathbb{P}_2(x, y) + A_2(x, y))$ . Then  $\mathbb{P}$  is a continuous projector of  $M$  onto  $M_\infty$  and  $R = \text{Null } \mathbb{P}$ . Thus  $R$  is a topological operator part of  $M$ . ■

REMARK 3.3. Theorem 3.1 remains valid algebraically, i.e., when all topological statements are deleted.

REMARK 3.4. The existence of a topological operator part does *not* always imply the existence of a principal topological operator part. This is in contrast with the algebraic setting, where a principal algebraic operator part always exists. For example, take a Banach space  $X$  and a closed subspace  $A$  of  $X$  which is not topologically complemented in  $X$ . Let  $M = \{(x, x+a) : x \in X, a \in A\}$ . Then  $\text{Dom } M = X$ ,  $\text{Null } M = A$ . Moreover,  $M$  is closed. Thus there exists no *continuous* projector of  $\text{Dom } M$  onto  $\text{Null } M^{-1}$ , and so there is no principal topological operator part of  $M^{-1}$ . Define  $\mathbb{P} : M^{-1} \rightarrow \{0\} \times \text{Null } M$  by  $\mathbb{P}(x+a, x) = (0, -a)$ ,  $x \in X, a \in A$ . Then  $\mathbb{P}$  is an algebraic projector of  $M^{-1}$  onto  $\{0\} \times \text{Null } M$ . For, take any  $x \in X, a \in A$ . Then  $\mathbb{P}^2(x+a, x) = \mathbb{P}(0, -a) = \mathbb{P}(-a+a, -a) = (0, -a) = \mathbb{P}(x+a, x)$ . Take any  $a \in A$ . Then  $(0, a) = (a-a, a) = \mathbb{P}(a-a, a)$ . Thus  $\mathbb{P}$  is surjective. Also,  $\mathbb{P}$  is continuous. For take any  $(x+a, x) \in M^{-1} (x \in X, a \in A)$ . Suppose  $(x_n + a_n, x_n) \in M^{-1}$  and  $(x_n + a_n, x_n) \rightarrow (x+a, x)$ . Then  $x_n \rightarrow x$  and  $x_n + a_n \rightarrow x+a$ . Thus  $a_n \rightarrow a$ . Hence  $\mathbb{P}(x_n + a_n, x_n) = (0, -a_n) \rightarrow (0, -a) = \mathbb{P}(x+a, x)$ . This shows that  $\mathbb{P}$  is continuous and so  $\text{Null } \mathbb{P} = \{(x+a, x) : x \in X, \mathbb{P}(x+a, x) = (0, -a) = (0, 0)\} = \{(x, x) : x \in X\}$  is a topological operator part of  $M^{-1}$ .

The above example shows that there exist marked differences between principal algebraic operator parts and principal topological operator parts. Theorem 3.2 can be simplified if a principal TOP if  $M^{-1}$  exists; the following corollary addresses this simplification.

COROLLARY 3.3. Assume that  $\text{Dom } M^{-1}$  is topologically decomposable with respect to a continuous projector  $P$  on  $Y$ , i.e.,  $P(\text{Dom } M^{-1}) = M(0)$ . Let

$R_0 := [\mathcal{G}r(I - P)]M$  (so  $R_0$  is a *principal* topological operator part of  $M$ ). Then  $R$  is a topological operator part of  $M$  if and only if  $R(x) = R_0(x) - A_2(x, R_0(x))$ , for all  $x \in \text{Dom } M$ , for some *continuous* linear operator  $A_2 : M \rightarrow M(0)$  such that  $A_2(0, y) = 0$  for all  $y \in M(0)$ .

PROOF. We can write  $R_0 = \{(x, y) \in M : \mathbb{P}_2(x, y) = 0\}$ , where  $\mathbb{P}_2(x, y) = P(y)$ ,  $(x, y) \in M$ . By Theorem 3.2, any TOP  $R$  of  $M$  has the form:

$$R(x) = R_0(x) - \mathbb{P}_2(x, R_0(x)) - A_2(x, R_0(x)), \quad x \in \text{Dom } M,$$

for some continuous linear operator  $A_2$  as in the theorem. But  $R_0(x) - \mathbb{P}_2(x, R_0(x)) = R_0(x) - P(R_0(x)) = (I - P)R_0(x) = (I - P)(\mathcal{G}r(I - P)M)(x) = [(\mathcal{G}r(I - P))M](x) = R_0(x)$ . ■

REMARK. Under the assumptions on  $M$  and  $M(0)$  as in Corollary 3.3, the statements (ii) and (iii) of Theorem 2.9 are valid for topological selections when  $X, Y$  are Banach spaces and the linear operators  $P_0$  and  $A_2$  are assumed to be continuous.

The following is a dual of Theorem 3.2.

THEOREM 3.2 (Dual). *Assume that  $M^+$  is  $w^*$ -closed and  $\{0\} \times M^+(0)$  is  $w^*$ -complemented in  $M^+$ . Let  $R_0^+ := \{(x, y) \in M^+ : \mathbb{P}_2^+(x, y) = 0\}$  where  $\mathbb{P}^+(x, y) := \{0, \mathbb{P}_2^+(x, y)\}$  is a  $w^*$ -continuous projector of  $M^+$  onto  $\{0\} \times M^+(0)$ . Then  $R$  is a  $w^*$ -topological operator part of  $M^+$  if and only if there exists a  $w^*$ -continuous linear operator  $A_2^+ : M^+ \rightarrow M^+(0)$  such that  $A_2^+(0, y) = 0$  for all  $y \in M^+(0)$ ,*

$$R(x) = R_0^+(x) - \mathbb{P}_2^+(x, R_0^+(x)) - A_2^+(x, R_0^+(x)); \quad x \in \text{Dom } M.$$

COROLLARY 3.2 (Dual). *Assume that  $M^+$  is  $w^*$ -closed and  $M^+(0)$  is  $w^*$ -complemented in  $Y^*$ . Let  $R_0^+$  be a principal  $w^*$ -topological operator part of  $M^+$ . Then  $R^+$  is a  $w^*$ -topological operator part of  $M^+$  if and only if there exists a  $w^*$ -continuous linear operator  $A_2^+$  of  $M^+$  into  $M^+(0)$  such that  $R^+(x) = R_0^+(x) - A_2^+(x, R_0^+(x))$ ,  $x \in \text{Dom } M^+$ ;  $A^+(0, y) = 0$ , for all  $y \in M^+(0)$ .*

PROPOSITION 3.4. *Assume that  $M$  is closed and  $M_\infty$  is topologically complemented in  $M$ . Then the following statements are equivalent: (i) Some topological operator part of  $M$  is continuous. (ii) All topological operator parts of  $M$  are continuous. (iii)  $\text{Dom } M$  is closed. If, in addition,  $M(0)$  is complemented in  $Y$ , then each of (i), (ii), (iii) is also equivalent to: (iv) Some principal operator part of  $M$  is continuous.*

PROOF. The equivalence of (i) and (iii) follows from Theorem 3.2. Assume (i). Let  $R$  be a topological operator part of  $M$ . Since  $M$  is closed, so is  $R$ . Since  $R$  is continuous, the extension  $\hat{R}$  of  $R$  by continuity coincides with the extension  $\tilde{R}$  of  $R$  by closure. In particular,  $(\text{Dom } M)^c = \text{Dom } \hat{R} = \text{Dom } \tilde{R} = \text{Dom } R = \text{Dom } M$ , where the superscript  $c$  denotes closure. Hence  $\text{Dom } M$  is

closed. Assume (iii). Take any topological operator part  $R$  of  $M$ . Since  $R$  is closed and  $\text{Dom } R = \text{Dom } M$  is closed, by the closed graph theorem,  $R$  is continuous. Thus (iii) implies (ii). Assume now that  $M(0)$  is complemented in  $Y$ . Then clearly (ii) and (iv) are equivalent. ■

**PROPOSITION 3.5.** *Assume that  $M$  is closed,  $M(0)$  is topologically complemented in  $Y$  and  $M^*(0)$  is  $w^*$ -complemented in  $X^*$ . Then each of (i)-(iv) of Proposition 3.4 is equivalent to any of the following:*

- (i)  $\text{Dom } M^*$  is  $w^*$ -closed.
- (ii) Some  $w^*$ -topological operator part of  $M^*$  is  $w^*$ -continuous.
- (iii) Every  $w^*$ -topological operator parts of  $M^*$  is  $w^*$ -continuous.
- (iv) Some principal  $w^*$ -topological operator part of  $M^*$  is continuous.

**PROOF.** For a closed linear relation  $M$ , it is well-known that  $\text{Dom } M$  is closed if and only if  $\text{Dom } M^*$  is  $w^*$ -closed (see [3]). Proposition 3.5 can then be proved in a manner similar to the proof of Proposition 3.4. ■

**REMARK 3.6.** Even though we have used the closed graph theorem to prove Proposition 3.4, we can regard this proposition as a generalized closed graph theorem. For, let  $M = X \times Y$ . Then clearly  $M_\infty$  is topologically complemented in  $M$ . Moreover, the graph of the zero operator on  $X$  into  $Y$  is a topological selection which is continuous. Hence by the above proposition any closed linear operator on  $X$  into  $Y$  is continuous.

**REMARK 3.7.** When  $M$  is closed and  $M(0)$  is topologically complemented in  $Y$ , it is not always true that  $M^*(0)$  is  $w^*$ -complemented in  $X^*$ . For example, let  $S^+$  be a  $w^*$ -closed vector subspace of  $X^*$  which is not  $w^*$ -complemented in  $X^*$ . Let  $M = {}^\perp(S^+) \times \{0\}$ . Then  $M(0)$  is complemented in  $Y$ , but  $M^*(0) = S^+$  is not  $w^*$ -complemented in  $X^*$ .

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