ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 16, nº 4 (1989), p. 603-636
<http://www.numdam.org/item?id=ASNSP_1989_4_16_4_603_0>

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# Local Regularity for Minimizers of Non Convex Integrals (*)

E. ACERBI - N. FUSCO

## 1. - Introduction

In this paper we study the regularity of minimizers of the functional

$$I(u) = \int_\Omega f(x, u(x), Du(x))dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \to \mathbb{R}^N$, and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a continuous function satisfying

$$|f(x, s, \xi)| \leq L(1 + |\xi|^p)$$

with $p \geq 2$. This problem has been studied under various ellipticity assumptions on $f$; for the case when $f$ is uniformly strictly convex in $\xi$, i.e.,

$$f_{\xi\xi}(x_0, s_0, \xi_0)\eta\eta \geq \gamma|\eta|^2 \tag{1.2}$$

for all $(x_0, s_0, \xi_0)$, see e.g. [10], and a comprehensive account in [8].

If convexity is replaced by uniform strict quasiconvexity, i.e., there is some $\gamma > 0$ such that

$$\begin{aligned} &\int_{\mathrm{spt}\varphi} f(x_0, s_0, \xi_0 + D\varphi(y))dy \\ &\qquad \geq \int_{\mathrm{spt}\varphi} [f(x_0, s_0, \xi_0) + \gamma(|D\varphi(y)|^2 + |D\varphi(y)|^p)]\ dy \end{aligned} \tag{1.3}$$

for all $\varphi \in C_0^1$ and all $(x_0, s_0, \xi_0)$, partial regularity of minimizers has been studied in [5], [6] in the case independent of $(x, u)$, in [7], [11] in the case

Pervenuto alla Redazione il 21 Dicembre 1988 e in forma definitiva il 10 settembre 1989.
(*) This work has been supported by the Italian Ministry of Education.

with $(x, u)$, but with second derivatives with respect to $\xi$ bounded by $|\xi|^{p-2}$, and in the general case in [2], see also [9].

These papers are motivated by the fact that in the vector-valued case $(N > 1)$ quasiconvexity, i.e., condition (1.3) with $\gamma = 0$, is essentially equivalent to the semicontinuity of (1.1): see e.g. [15], [14], [1].

Of course the uniform ellipticity conditions (1.2) or (1.3) are not necessary in order for the functional (1.1) to have a minimizer (this happens for example if $f(\xi) = |\xi|^p$ with $p \neq 2$; however, this particular functional may be treated in a special way as far as regularity is concerned, see e.g. [16]).

A new kind of result has been recently proved in [3] which is useful for studying regularity in cases of degenerate ellipticity, by showing that $Du$ is Hölder continuous near points where it is "close" to a value $\xi_0$ where $f$ is uniformly strictly convex. Precisely, in the case independent of $(x, u)$, if $f$ is convex and with growth $p \geq 1$, and $u$ is a minimizer of $I$, then if

$$\lim_{r \to 0} \fint_{B_r(x_0)} |Du - \xi_0|^p dx = 0$$

for some $x_0$ such that (1.2) holds, and $f$ is of class $C^2$ in a neighbourhood of $x_0$, then $Du$ is Hölder continuous of any exponent $\alpha < 1$ in a neighbourhood of $x_0$. A similar result is given when $f$ depends also on $(x, u)$.

In the same spirit, we prove the following result (Therem 2.1):
*Let $p \geq 2$, and let $f : \mathbb{R}^{nN} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying*

$$|f(\xi)| \leq L(1 + |\xi|^p), \qquad |Df(\xi)| \leq L(1 + |\xi|^{p-1}).$$

*Fix $\xi_0 \in \mathbb{R}^{nN}$ such that $f \in C^2$ in a neighbourhood of $\xi_0$ and*

$$\int_{\mathrm{spt}\varphi} f(\xi_0 + D\varphi(y))dy$$
$$\geq \int_{\mathrm{spt}\varphi} [f(\xi_0) + \gamma(|D\varphi(y)|^2 + |D\varphi(y)|^p)]dy \text{ for all } \varphi \in C_0^1.$$

*Then if $u$ is a minimizer of $\int f(Dv)dx$ and*

$$\lim_{r \to 0} \fint_{B_r(x_0)} |Du - \xi_0|^p dx = 0,$$

*there is a neighbourhood of $x_0$ in which the function $u$ is of class $C^{1,\alpha}$ for all $\alpha < 1$.*

An extension to the case with $(x, u)$ is also provided (Theorem 3.1).

We remark that in the above theorem we do not require a global quasiconvexity assumption. On the other hand the theorem covers only the case $p \geq 2$; however, it is not clear whether a function which is genuinely quasiconvex at some point $\xi_0$ and has growth $p < 2$ may exist.

These result allow us to generalize the former partial regularity results of [5], [6], [2]: the strict quasiconvexity need no longer be uniform (Corollaries 4.1 and 4.2).

The last part of the paper is devoted to the study of the set of regular points in the scalar case $N = 1$; as an example, an application to an energy functional of interest in nonlinear elasticity is also provided.

## 2. - The case independent of $(x, u)$

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, fix $p \geq 2$ and let $f : \mathbb{R}^{nN} \to \mathbb{R}$ satisfy:

$$f \text{ is locally Lipschitz continuous} \tag{2.1}$$

$$|f(\xi)| \leq L(1 + |\xi|^p) \tag{2.2}$$

$$|Df(\xi)| \leq L(1 + |\xi|^{p-1}). \tag{2.3}$$

We say that $\xi_0 \in \mathbb{R}^{nN}$ is a regular point for $f$ if there exist $\sigma > 0, \gamma > 0$ such that $f \in C^2(B_\sigma(\xi_0))$ and

$$\int_{\mathrm{spt}\varphi} f(\xi_0 + D\varphi(x))dx \geq \int_{\mathrm{spt}\varphi} [f(\xi_0) + \gamma(|D\varphi(x)|^2 + |D\varphi(x)|^p)]dx$$

for every $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$.

Set for every $u \in W^{1,p}(\Omega; \mathbb{R}^N)$

$$I(u) = \int_\Omega f(Du(x))dx;$$

we say that $u$ is a minimizer of $I$ if

$$I(u) \leq I(u + \varphi) \text{ for every } \varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N).$$

Then we have:

THEOREM 2.1. *Let $f$ satisfy* (2.1), (2.2), (2.3), *and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a minimizer of $I$. If for some $x_0 \in \Omega$ and some regular point $\xi_0$*

$$\lim_{r \to 0} \fint_{B_r(x_0)} |Du(x) - \xi_0|^p dx = 0$$

*then in a neighbourhood of $x_0$ the function $u$ is of class $C^{1,\alpha}$ for all $\alpha < 1$.*

In the sequel we denote by the same letter $c$ any positive constant, which may vary from to line; if $\varphi$ is any vector-valued function, we denote by $(\varphi)_{x_0,r}$ or simply by $(\varphi)_r$ the mean value of $\varphi$ on $B_r(x_0)$. Finally, we set

$$g_p(t) = |t|^2 + |t|^p.$$

We shall use the following lemmas:

LEMMA 2.2. *Let $f$ satisfy* (2.1), (2.2), (2.3) *and let $\xi_0$ be a regular point for $f$, i.e.,*

$$\int_{\mathrm{spt}\varphi} f(\xi_0 + D\varphi(y))dy \geq \int_{\mathrm{spt}\varphi} [f(\xi_0) + \gamma g_p(D\varphi(y))]dy \text{ for all } \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N)$$

*and $f \in C^2(B_{2\sigma}(\xi_0))$. There exists $\varrho > 0$ such that for every $\xi \in B_\varrho(\xi_0)$*

$$\int_{\mathrm{spt}\varphi} f(\xi + D\varphi(y))dy \geq \int_{\mathrm{spt}\varphi} [f(\xi) + \frac{\gamma}{2} g_p(D\varphi(y))]dy \text{ for all } \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N). \tag{2.4}$$

PROOF. Set $\omega_\varrho = \sup\{|D^2 f(\xi) - D^2 f(\eta)| : \xi, \eta \in B_\sigma(\xi_0), |\xi - \eta| < \varrho\}$, and fix $\xi$ such that $|\xi - \xi_0| < \varrho < \sigma/2$. Then

$$\begin{aligned}\int_\Omega [f(\xi + D\varphi) - f(\xi)]dy &= \int_\Omega [f(\xi_0 + D\varphi) - f(\xi_0)]dy \\ &\quad + \int_\Omega [G(D\varphi) - G(0) - DG(0)D\varphi]dy,\end{aligned} \tag{2.5}$$

where we set

$$G(\eta) = f(\xi + \eta) - f(\xi_0 + \eta).$$

The first integral which appears at the right hand side of (2.5) is greater than $\int_\Omega \gamma g_p(D\varphi)dy$. As for the second, we can set

$$S_\sigma = \{y \in \Omega : |D\varphi(y)| \leq \sigma/2\}, \qquad L_\sigma = \{y \in \Omega : |D\varphi(y)| > \sigma/2\};$$

remarking that $G(D\varphi) - G(0) - DG(0)D\varphi = \frac{1}{2} D^2 G(\vartheta D\varphi) D\varphi D\varphi$ with $0 < \vartheta < 1$,

and that $\xi + D\varphi(y) \in B_\sigma(\xi_0)$ for $y \in S_\sigma$, we have

$$\begin{aligned}\int_\Omega [f(\xi + D\varphi) - f(\xi)]dy \\ \geq \int_\Omega \gamma g_p(D\varphi)dy - \frac{\omega_\varrho}{2}\int_{S_\sigma} g_p(D\varphi)dy \\ + \int_{L_\sigma} [G(D\varphi) - G(0) - DG(0)D\varphi]dy,\end{aligned} \tag{2.6}$$

Set

$$H(\eta, y) = f(\eta + D\varphi(y)) - f(\eta) - Df(\eta)D\varphi(y),$$

and

$$M_\sigma = \sup\{|D^2 f(\eta)| : \eta \in B_\sigma(\xi_0)\};$$

we remark that the last integral may be written $\int_{L_\sigma}[H(\xi, y) - H(\xi_0, y)]\ dy$, therefore its absolute value is bounded by

$$\begin{aligned}\int_{L_\sigma} |\xi - \xi_0||D_\eta H(\tau, y)|dy \\ \leq |\xi - \xi_0|\int_{L_\sigma} [|Df(\tau + D\varphi) - Df(\tau)| + M_\sigma|D\varphi|]dy \\ \leq c(\sigma, |\xi_0|, L)|\xi - \xi_0|\int_{L_\sigma} (1 + |D\varphi| + |D\varphi|^{p-1})dy \\ \leq \tilde{c}\varrho\int_{L_\sigma} g_p(D\varphi)dy.\end{aligned}$$

If we choose $\varrho$ such that $\frac{1}{2}\omega_\varrho + \tilde{c}\varrho < \frac{1}{2}\gamma$, the result follows from (2.6). ∎

LEMMA 2.3. *Let $f$ satisfy* (2.1), (2.2), (2.3) *and assume $f \in C^2(B_\sigma(\xi_0))$. Set for all $\lambda > 0$ and $\xi, \eta \in \mathbb{R}^{nN}$*

$$f_{\xi,\lambda}(\eta) = \lambda^{-2}[f(\xi + \lambda\eta) - f(\xi) - \lambda Df(\xi)\eta].$$

*There exists $c > 0$ such that for every $\xi \in B_{\sigma/3}(\xi_0)$*

$$|f_{\xi,\lambda}(\eta)| \leq c(|\eta|^2 + \lambda^{p-2}|\eta|^p), \qquad |Df_{\xi,\lambda}(\eta)| \leq c(|\eta| + \lambda^{p-2}|\eta|^{p-1}).$$

PROOF. Set $K = \max\{|D^2 f(\eta)| : |\eta - \xi_0| < 2\sigma/3\}$; then

$$|\lambda\eta| \leq \sigma/3 \Rightarrow |f_{\xi,\lambda}(\eta)| = \frac{1}{2}|D^2 f(\xi + \vartheta\lambda\eta)\eta\eta| \leq \frac{1}{2}K|\eta|^2;$$

$$|\lambda\eta| > \sigma/3 \Rightarrow |f_{\xi,\lambda}(\eta)| \leq \lambda^{-2}c(\sigma)(1 + |\lambda\eta| + |\lambda\eta|^p) \leq c(\sigma)\lambda^{p-2}|\eta|^p,$$

and the first inequality is proven. The second is analogous. ∎

The following result may be easily derived from [8], p. 161.

LEMMA 2.4. *$f : [\frac{r}{2}, r] \to [0, +\infty)$ be a bounded function satisfying*

$$f(t) \le \vartheta f(s) + \frac{A}{(s-t)^2} + \frac{B}{(s-t)^p} + D$$

*for some $0 < \vartheta < 1$ and all $\frac{r}{2} \le t < s \le r$. Then there exists a constant $c(\vartheta, p)$ such that*

$$f\left(\frac{r}{2}\right) \le c\left(\frac{A}{r^2} + \frac{B}{r^p} + D\right).$$

The following lemma may be found in [2], Lemma II.4; since we will later refer to the proof, we include it for the readers' convenience.

LEMMA 2.5. *Let $g : \mathbb{R}^{nN} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying*

$$|g(\xi)| \le c_1(|\xi|^2 + \lambda^{p-2}|\xi|^p), \qquad |Dg(\xi)| \le c_1(|\xi| + \lambda^{p-2}|\xi|^{p-1}),$$
$$\int g(D\varphi)dx \ge \gamma \int [|D\varphi|^2 + \lambda^{p-2}|D\varphi|^p]dx \text{ for all } \varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$$

*for suitable constants $c_1, \lambda$ and $\gamma$. Fix $\nu \ge 0$ and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ satisfy*

$$\int_\Omega g(Du)dx \le \int_\Omega [g(Du + D\varphi) + \nu|D\varphi|]dx \text{ for all } \varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N).$$

*Then there exists $c_2 > 0$, depending only on $c_1, \gamma$, such that for every $B_r \subset \Omega$*

$$\fint_{B_{r/2}} (|Du|^2 + \lambda^{p-2}|Du|^p)dx$$
$$\le c_2 \fint_{B_r} \left(\nu^2 + \frac{|u - (u)_r|^2}{r^2} + \lambda^{p-2}\frac{|u - (u)_r|^p}{r^p}\right) dx.$$

PROOF. Fix $B_r \subset \Omega$, let $\frac{r}{2} < t < s < r$ and take a cut-off function $\varsigma \in C_0^1(B_s)$ such that $0 \le \varsigma \le 1, \varsigma = 1$ on $B_t$ and $|D\varsigma| \le \frac{2}{s-t}$. If we set

$$\varphi_1 = [u - (u)_r]\varsigma, \qquad \varphi_2 = [u - (u)_r](1 - \varsigma),$$

then $D\varphi_1 + D\varphi_2 = Du$, and

$$\gamma \int_{B_s} [|D\varphi_1|^2 + \lambda^{p-2}|D\varphi_1|^p]dx$$
$$\le \int_{B_s} g(D\varphi_1)dx = \int_{B_s} g(Du - D\varphi_2)dx. \tag{2.7}$$

In addition, by the minimality of $u$,

$$\begin{aligned}\int_{B_s} g(Du)dx &\\ &\le \int_{B_s} g(Du - D\varphi_1)dx + \nu \int_{B_s} |D\varphi_1|dx\\ &\le \int_{B_s\setminus B_t} g(D\varphi_2)dx + \frac{\gamma}{2}\int_{B_s} |D\varphi_1|^2 dx + \frac{\nu^2}{2\gamma}\ \text{meas}\ (B_r).\end{aligned}$$

Then

$$\begin{aligned}\int_{B_s} g(Du - D\varphi_2)dx &\\ &= \int_{B_s} g(Du)dx + \int_{B_s} [g(Du - D\varphi_2) - g(Du)]dx\\ &\le \int_{B_s\setminus B_t} g(D\varphi_2)dx + \frac{\gamma}{2}\int_{B_s} |D\varphi_1|^2 dx + \frac{\nu^2}{2\gamma}\ \text{meas}\ (B_r)\\ &\quad + c\int_{B_s\setminus B_t} [|Du| + |D\varphi_2| + \lambda^{p-2}(|Du|^{p-1} + |D\varphi_2|^{p-1})]|D\varphi_2|dx.\end{aligned} \tag{2.8}$$

By (2.7), (2.8) and the assumptions on $g$ it then follows

$$\begin{aligned}\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]dx &\\ &\le \int_{B_s} [|D\varphi_1|^2 + \lambda^{p-2}|D\varphi_1|^p]dx\\ &\le c(\gamma, c_1)\Big[\nu^2 r^n + \int_{B_s\setminus B_t} [|Du|^2 + |D\varphi_2|^2\\ &\qquad + \lambda^{p-2}(|Du|^p + |D\varphi_2|^p)]dx\Big]\\ &\le \tilde{c}\Big[\nu^2 r^n + \int_{B_s\setminus B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]dx\\ &\qquad + \int_{B_s\setminus B_t} \left(\frac{|u - (u)_r|^2}{(s-t)^2} + \lambda^{p-2}\frac{|u - (u)_r|^p}{(s-t)^p}\right) dx\Big].\end{aligned}$$

We fill the hole by adding to both sides the term

$$\tilde{c}\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]dx;$$

then we divide by $\tilde{c}+1$, thus obtaining

$$\begin{aligned}\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]dx \\ &\le \vartheta \int_{B_s} [|Du|^2 + \lambda^{p-2}|Du|^p]dx \\ &\quad + c\int_{B_r} \left[\nu^2 + \frac{|u-(u)_r|^2}{(s-t)^2} + \lambda^{p-2}\frac{|u-(u)_r|^p}{(s-t)^p}\right] dx,\end{aligned}$$

with $\vartheta < 1$, and the result follows by Lemma 2.4. ∎

In the sequel we assume that $f$ satisfies (2.1), (2.2), (2.3), and that $\xi_0$ is a regular point for $f$, so that (2.4) holds in $B_\varrho(\xi_0)$ and we may assume that $f \in C^2(B_{4\varrho})$. If $u$ is a minimizer of $I(u) = \int_\Omega f(Du)dx$, for every $B_r(x_0) \subset \Omega$ we define

$$U(x_0, r) = \fint_{B_r(x_0)} g_p(Du - (Du)_r)dx.$$

The main ingredient to prove Theorem 2.1 is the following decay estimate:

PROPOSITION 2.6. *There is a constant $C$, depending only on $\xi_0$, such that for every $\tau < 1/4$ there exists $\varepsilon(\tau)$ such that if $u$ is a minimizer of $I(u)$ and*

$$|(Du)_{x_0,r} - \xi_0| < \varrho, \qquad |(Du)_{x_0,\tau r} - \xi_0| < \varrho, \qquad U(x_0, r) < \varepsilon$$

*then*

$$U(x_0, \tau r) < C\tau^2 U(x_0, r).$$

PROOF. Fix $\tau$; we shall determine $C$ later. Reasoning by contradiction, we assume that there is a sequence of balls $B_{r_h}(x_h) \subset \Omega$ satisfying

$$|(Du)_{x_h,r_h} - \xi_0| < \varrho, \qquad |(Du)_{x_h,\tau r_h} - \xi_0| < \varrho, \qquad U(x_h, r_h) = \lambda_h^2 \to 0,$$

and

$$U(x_h, \tau r_h) > C\tau^2\lambda_h^2. \tag{2.9}$$

Set $A_h = (Du)_{x_h,r_h}$, $a_h = (u)_{x_h,r_h}$, $\tilde{A}_h = (Du)_{x_h,\tau r_h}$, and

$$v_h(z) = \frac{u(x_h + r_h z) - a_h - r_h A_h z}{\lambda_h r_h},$$

so that

$$(v_h)_{0,1} = 0, \qquad \fint_{B_1} \lambda_h^{-2} g_p(\lambda_h Dv_h)dz = \fint_{B_1} (|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p)dz = 1.$$

Then we may assume

$$v_h \rightharpoonup v \text{ weakly in } W^{1,2}(B_1;\mathbb{R}^N),$$
$$\lambda_h^{1-2/p} v_h \rightharpoonup 0 \text{ weakly in } W^{1,p}(B_1;\mathbb{R}^N)$$

and also $A_h \to A$. Set $L_h = \{z \in B_1 : \lambda_h |Dv_h(z)| \geq \varrho\}, S_h = B_1 \backslash L_h$; then

$$\text{meas } L_h \leq \lambda_h^2 \varrho^{-2}, \qquad \int_{L_h} |Dv_h|^{p-1} dz \leq \lambda_h^{3-p} \varrho^{-2/p}. \tag{2.10}$$

Now fix $\varphi \in C_0^1(B_1;\mathbb{R}^N)$: by the minimality of $u$

$$\int_{B_1} [f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)] dz \geq 0.$$

Dividing by $t > 0$ we have

$$\begin{aligned}
&\frac{1}{t}\int_{S_h} [f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)] dz \\
&\qquad \geq -\int_{L_h} \frac{f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)}{t} dz \\
&\qquad \geq -c_\varphi \int_{L_h} (1 + \lambda_h^{p-1} |Dv_h|^{p-1}) dz \\
&\qquad \geq -c_{\varphi,\varrho} \lambda_h^2
\end{aligned}$$

by (2.10). If $t$ is smaller than $\varrho/\|\varphi\|_{C_1}$, in the first integral above the argument of $f$ is always in $B_{3\varrho}(\xi_0)$, and the integrand is bounded, therefore as $t \to 0$ we have

$$\int_{S_h} Df(\lambda_h Dv_h + A_h) D\varphi dz \geq -c\lambda_h^2.$$

Again by (2.10)

$$\begin{aligned}
&\frac{1}{\lambda_h}\int_{S_h} [Df(\lambda_h Dv_h + A_h) - Df(A_h)] D\varphi dz \\
&\qquad \geq -c\lambda_h + \frac{1}{\lambda_h}\int_{L_h} Df(A_h) D\varphi dz \geq -c\lambda_h.
\end{aligned}$$

This may be written also as

$$\int_{B_1}\int_0^1 D^2 f(A_h + s\lambda_h Dv_h) \mathbf{1}_{S_h} Dv_h D\varphi ds \; dz \geq -c\lambda_h, \tag{2.11}$$

and remarking that $\lambda_h Dv_h \to 0$ a.e. we have as $h \to \infty$

$$\int_{B_1} D^2 f(A) Dv D\varphi dz \geq 0,$$

which yields $\int_{B_1} D^2 f(A) Dv D\varphi dz = 0$ for every $\varphi \in C_0^1(B_1; \mathbb{R}^N)$. Then $v$ solves a linear system with constant coefficients; remarking that (2.4) implies $D^2 f(A)\lambda_i\lambda_j\eta_\alpha\eta_\beta \geq \frac{\gamma}{2}|\lambda|^2|\eta|^2$, by the standard regularity theory we have for every $\tau < 1/4$

$$\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \leq c\tau^2. \tag{2.12}$$

Set

$$w_h = v_h - \frac{\tilde{A}_h - A_h}{\lambda_h} z$$

and remark that

$$\frac{\tilde{A}_h - A_h}{\lambda_h} = (Dv_h)_\tau, \qquad (w_h)_s = (v_h)_s \text{ for all } s < 1,$$

and $w_h$ minimizes $\int_{B_1} f_{\tilde{A}_h;\lambda_h}(D\psi)dz$; therefore Lemma 2.5 holds with $\nu = 0$, and

$$\begin{aligned}
U(x_h, \tau r_h) &= \fint_{B_{\tau r_h}(x_h)} g_p(Du)dx \\
&= \fint_{B_\tau} g_p[\lambda_h(Dv_h - (Dv_h)_\tau)]dz \\
&= \fint_{B_\tau} g_p(\lambda_h Dw_h)dz \\
&\leq \tilde{c}\fint_{B_{2\tau}} g_p\left(\lambda_h \frac{w_h - (w_h)_{2\tau}}{\tau}\right)dz \\
&= \tilde{c}\fint_{B_{2\tau}} g_p\left(\lambda_h \frac{v_h - (v_h)_{2\tau} - (Dv_h)_\tau z}{\tau}\right)dz \\
&= \tilde{c}\lambda_h^2\left[\fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau z|^2}{\tau^2}dz \right. \\
&\qquad \left. + \lambda_h^{p-2}\fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Du_h)_\tau z|^p}{\tau^p}dz\right] \\
&= \tilde{c}\lambda_h^2(I_h^1 + I_h^2).
\end{aligned} \tag{2.13}$$

By the Sobolev-Poincaré inequality and (2.12)

$$
\begin{aligned}
\lim_{h\to\infty} I_h^1 &= \tau^{-2} \fint_{B_{2\tau}} |v - (v)_{2\tau} - (Dv)_\tau z|^2 dz \\
&\le c \fint_{B_{2\tau}} |Dv - (Dv)_\tau|^2 dz \\
&\le c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 dz + c|(Dv)_\tau - (Dv)_{2\tau}|^2 \\
&\le c\tau^2 + c \fint_{B_\tau} |Dv - (Dv)_{2\tau}|^2 dz \\
&\le c\tau^2 + c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 dz \\
&\le \hat{c}\tau^2,
\end{aligned}
\tag{2.14}
$$

whereas if $\frac{\vartheta}{2} + \frac{1-\vartheta}{p^*} = \frac{1}{p}$ we have

$$
\begin{aligned}
I_h^2 &\le \lambda_h^{p-2} \left( \fint_{B_{2\tau}} \tau^{-2} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau z|^2 dz \right)^{p\vartheta/2} \cdot \\
&\qquad \cdot \left( \fint_{B_{2\tau}} \tau^{-p^*} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau z|^{p^*} dz \right)^{p(1-\vartheta)/p^*} \\
&\le c\lambda_h^{p-2} (I_h^1)^{p\vartheta/2} \left( \fint_{B_{2\tau}} |Dv_h - (Dv_h)_\tau|^p dz \right)^{1-\vartheta} \\
&\le c\tau^{p\vartheta} \lambda_h^{(p-2)\vartheta} \left[ \fint_{B_{2\tau}} \lambda_h^{p-2} |Dv_h|^p dz + \fint_{B_\tau} \lambda_h^{p-2} |Dv_h|^p dz \right] \\
&\le c_\tau \lambda_h^{(p-2)\vartheta},
\end{aligned}
$$

so that

$$
\lim_{h\to\infty} I_h^2 = 0. \tag{2.15}
$$

By (2.13), (2.14), (2.15) we get

$$
\limsup_{h\to\infty} \lambda_h^{-2} U(x_h, \tau r_h) \le \tilde{c}\hat{c}\tau^2,
$$

which contradicts (2.9) if we chose $C > \tilde{c}\hat{c}$.

The fact (which we do not need in the sequel) that $C$ does not depend on the particular minimizer $u$, could have been proven by taking a different minimizer $u_h$ in each $B_{r_h}(x_h)$. ∎

PROPOSITION 2.7. *Let $\xi_0$ be a regular point for $f$, and take $\alpha < 1$; if $C$ is as in Proposition 2.6, fix $\tau < 1/4$ such that $C\tau^2 < \tau^{2\alpha}$. Let $u$ be a minimizer of $I$ and assume that for some $B_r(x_0) \subset \Omega$*

$$|(Du)_{x_0;r} - \xi_0| < \varrho/2, \qquad |(Du)_{x_0;\tau r} - \xi_0| < \varrho/2, \qquad U(x_0, r) < \delta(\tau),$$

*where $\delta(\tau) < \min\{\varepsilon(\tau), \varrho^2\tau^{2n}(1-\tau^\alpha)^2/4\}$. Then for every $k$*

$$|(Du)_{x_0;\tau^{k+1}r} - \xi_0| < \varrho, \qquad U(x_0, \tau^k r) < \tau^{2k\alpha}U(x_0, r). \tag{2.16}$$

PROOF. The result is true for $k = 0$; we proceed by induction, assuming (2.16) holds for $0 \le k \le m-1$. Then $U(x_0, \tau^{m-1}r) < \varepsilon(\tau)$, and by Proposition 2.6 we have

$$U(x_0, \tau^m r) \le C\tau^2 U(x_0, \tau^{m-1}r) \le \tau^{2\alpha}\tau^{2(m-1)\alpha}U(x_0, r) = \tau^{2m\alpha}U(x_0, r).$$

Now,

$$\begin{aligned}
|(Du)_{\tau^{m+1}r} - \xi_0| &\le |(Du)_r - \xi_0| + \sum_{k=0}^{m} |(Du)_{\tau^{k+1}r} - (Du)_{\tau^k r}| \\
&\le \frac{\varrho}{2} + \sum_{k=0}^{m} \fint_{B_{\tau^{k+1}r}} |Du - (Du)_{\tau^k r}|dx \\
&\le \frac{\varrho}{2} + \tau^{-n}\sum_{k=0}^{m}\left[\fint_{B_{\tau^k r}} |Du - (Du)_{\tau^k r}|^2 dx\right]^{1/2} \\
&\le \frac{\varrho}{2} + \tau^{-n}\sum_{k=0}^{m}[U(x_0, \tau^k r)]^{1/2} \\
&\le \frac{\varrho}{2} + \tau^{-n}\sum_{k=0}^{m}\tau^{k\alpha}[U(x_0, r)]^{1/2} \\
&\le \frac{\varrho}{2} + \frac{1}{\tau^n(1-\tau^\alpha)}[\delta(\tau)]^{1/2} < \varrho,
\end{aligned}$$

thus concluding the proof. ∎

PROOF OF THEOREM 2.1. Suppose $\xi_0$ is a regular point and

$$\lim_{r\to 0} \fint_{B_r(x_0)} |Du - \xi_0|^p dx = 0;$$

fix a particular $\hat{\alpha} < 1$: for a suitable $r > 0$ the assumptions of Proposition 2.7 are verified uniformly in a neighbourhood of $x_0$, i.e.,

$$|(Du)_{x;r} - \xi_0| < \varrho, \qquad |(Du)_{x;\tau r} - \xi_0| < \varrho, \qquad U(x, r) < \varepsilon$$

for all $x \in B_s(x_0)$. Then (2.16) implies

$$U(x,\varrho) \le c\left(\frac{\varrho}{r}\right)^{2\hat{\alpha}} U(x,r) \text{ for all } x \in B_s(x_0),$$

and $u \in C^{1,\hat{\alpha}}(B_s(x_0))$ by a standard argument – see e.g. [8], Chapter 3.

Since $Du$ is now continuous, by Lemma 2.2. we may suppose that $s$ is so small that $Du(x)$ is a regular point for $f$ for all $x \in B_s(x_0)$; moreover, clearly

$$\lim_{r\to 0} \fint_{B_r(x)} |Du(y) - Du(x)|^p dy = 0 \text{ for all } x \in B_s(x_0).$$

Now fix any $\alpha < 1$: the same argument employed above shows that $u \in C^{1,\alpha}$ in a neighbourhood of $x$ for all $x \in B_s(x_0)$, therefore $u \in C^{1,\alpha}(B_s(x_0))$ for all $\alpha < 1$. ∎

## 3. - The case with $(x, u)$

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, fix $p \ge 2$ and assume that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfies:

(3.1) $f(x,s,\xi)$ is locally Lipschitz continuous with respect to $\xi$;

$$|f(x,s,\xi)| \le L(1+|\xi|^p); \tag{3.2}$$

$$|f_\xi(x,s,\xi)| \le L(1+|\xi|^{p-1}); \tag{3.3}$$

$$|f(x,s,\xi) - f(y,t,\xi)| \le L(1+|\xi|^p)\ \omega(|x-y|^p + |s-t|^p), \tag{3.4}$$

where $\omega(t) \le t^\delta, 0 < \delta < 1/p$, and $\omega$ is bounded, concave and increasing;

$$f(x,s,\xi) \ge \psi(\xi) \tag{3.5}$$

for a suitable continuous function $\psi$ satisfying

$$\int_\Omega \psi(D\varphi(y))dy \ge \int_\Omega [\psi(0) + \mu|D\varphi(y)|^p]dy \text{ for all } \varphi \in C_0^1(\Omega;\mathbb{R}^N),$$

with $\mu > 0$; finally, we assume that

(3.6) either $f \ge 0$ or $f$ is quasiconvex.

We remark that if $f$ is quasiconvex then (3.3) is implied by (3.2); assumption (3.5) was introduced in [12].

We say that $(x_0, s_0, \xi_0)$ is a regular point for $f$ if there exist $\sigma > 0$, $\gamma > 0$ such that for every $x \in B_\sigma(x_0)$ and $s \in B_\sigma(s_0)$ the function $f(x, s, \cdot)$ is of class $C^2$ in $B_\sigma(\xi_0)$, and

$$\int_{\mathrm{spt}\varphi} f(x, s, \xi + D\varphi(y))dy$$
$$\geq \int_{\mathrm{spt}\varphi} [f(x, s, \xi) + \gamma(|D\varphi(y)|^2 + |D\varphi(y)|^p)]dy$$

for every $x \in B_\sigma(x_0)$, $s \in B_\sigma(s_0)$, $\xi \in B_\sigma(\xi_0)$ and $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$.

Set for every $u \in W^{1,p}(\Omega; \mathbb{R}^N)$

$$I(u) = \int_\Omega f(x, u(x), Du(x))dx;$$

then we have:

THEOREM 3.1. *Let $f$ satisfy* (3.1),...,(3.6) *and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a minimizer of $I$. Then there exists $\alpha \in (0, 1)$ such that if for some regular point $(x_0, s_0, \xi_0)$ of $f$ we have*

$$\lim_{r \to 0} \fint_{B_r(x_0)} [|u(x) - s_0|^p + |Du(x) - \xi_0|^p]dx = 0$$

*then $u$ is of class $C^{1,\alpha}$ in a neighbourhood of $x_0$.*

In the proof we shall use the following results:

LEMMA 3.2. *Let $(X, d)$ be a metric space, and $J : X \to [0, +\infty]$ a lower semicontinuous functional not identically $+\infty$. if*

$$J(u) \leq \alpha + \inf J,$$

*there is a $v \in X$ such that*

$$d(u, v) \leq 1$$

*and*

$$J(v) \leq J(w) + \alpha d(v, w) \text{ for all } w \in X.$$

The result above may be found in [4], the next one in [8], p. 122.

LEMMA 3.3. *Let $Q$ be a cube in $\mathbb{R}^n$, and suppose that for every ball $B_r(x_0) \subset Q$ such that $2r < \min\{r_0, \mathrm{dist}(x_0, \partial Q)\}$*

$$\fint_{B_{r/2}} g^q dx \leq a \left(\fint_{B_r} g dx\right)^q + \fint_{B_r} f^q dx$$

*with* $f \in L^k(Q)$, $k > q$. *Then* $g \in L^{q+\epsilon}_{loc}(Q)$ *for some positive* $\epsilon(a, q, k)$ *and*

$$\left(\fint_{B_{r/2}} g^{q+\epsilon} dx\right)^{1/(q+\epsilon)} \leq c\left[\left(\fint_{B_r} g^q dx\right)^{1/q} + \left(\fint_{B_r} f^{q+\epsilon} dx\right)^{1/(q+\epsilon)}\right].$$

LEMMA 3.4. *Let* $f$ *satisfy* (3.1), (3.2), (3.3), (3.5); *there are* $q_0 > p$ *and* $c_0 > 0$, *depending only on* $\mu, L, p$, *such that if* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *is a minimizer of* $I$, *then* $u \in W^{1,q_0}_{\text{loc}}(\Omega; \mathbb{R}^N)$ *and for every* $B_r \subset \Omega$

$$\left(\fint_{B_{r/2}} |Du|^{q_0} dx\right)^{1/q_0} \leq c_0 \left(\fint_{B_r} (1 + |Du|^p) dx\right)^{1/p}.$$

PROOF. The argument is similar to Lemma 2.5; fix $B_r \subset \Omega$, let $\frac{r}{2} < t < s < r$, take the cut-off function $\varsigma$ of 2.5, and again set

$$\varphi_1 = [u - (u)_r]\varsigma, \qquad \varphi_2 = [u - (u)_r](1 - \varsigma);$$

then $\varphi_1 + \varphi_2 = u - (u)_r$ and $D\varphi_1 + D\varphi_2 = Du$. Now, by (3.5)

$$\begin{aligned} &\int_{B_s} [\mu|D\varphi_1|^p + \psi(0)] dx \\ &\qquad \leq \int_{B_s} \psi(D\varphi_1) dx \\ &\qquad \leq \int_{B_s} f(x, u, D\varphi_1) dx \\ &\qquad = \int_{B_s} f(x, u, Du - D\varphi_2) dx. \end{aligned} \tag{3.7}$$

By the minimality of $u$ we have

$$\begin{aligned} &\int_{B_s} f(x, u, Du) dx \\ &\qquad \leq \int_{B_s} f(x, u - \varphi_1, Du - D\varphi_1) dx \\ &\qquad = \int_{B_s} f(x, \varphi_2 + (u)_r, D\varphi_2) dx \\ &\qquad = \int_{B_s \setminus B_t} f(x, \varphi_2 + (u)_r, D\varphi_2) dx + \int_{B_t} f(x, (u)_r, 0) dx, \end{aligned}$$

so that by (3.2)

$$\int_{B_s} f(x, u, Du) dx \leq L \int_{B_s \setminus B_t} |D\varphi_2|^p dx + cr^n,$$

and by (3.3)

$$\begin{aligned}
&\int_{B_s} f(x,u,Du-D\varphi_2)dx \\
&\qquad = \int_{B_s} f(x,u,Du)dx \\
&\qquad\quad + \int_{B_s} [f(x,u,Du-D\varphi_2) - f(x,u,Du)]dx \\
&\qquad \le \int_{B_s} f(x,u,Du)dx \\
&\qquad\quad + c\int_{B_s\setminus B_t} (1+|Du|^{p-1}+|D\varphi_2|^{p-1})|D\varphi_2|dx \\
&\qquad \le cr^n + c\int_{B_s\setminus B_t} (|D\varphi_2|^p + |Du|^p)dx \\
&\qquad \le cr^n + c\int_{B_s\setminus B_t} \left(|Du|^p + \frac{|u-(u)_r|^p}{(s-t)^p}\right)dx.
\end{aligned}$$

Then by (3.7) we obtain

$$\mu\int_{B_s} |D\varphi_1|^p dx \le c\int_{B_s\setminus B_t} |Du|^p dx + c\int_{B_r}\left(1+\frac{|u-(u)_r|^p}{(s-t)^p}\right)dx, \tag{3.8}$$

therefore

$$\int_{B_t} |Du|^p dx \le c\int_{B_s\setminus B_t} |Du|^p dx + c\int_{B_r}\left(1+\frac{|u-(u)_r|^p}{(s-t)^p}\right)dx;$$

we fill the hole, and by Lemma 2.4 we obtain

$$\fint_{B_{r/2}} |Du|^p dx \le c\fint_{B_r}\left(1+\frac{|u-(u)_r|^p}{r^p}\right)dx \le c\left(\fint_{B_r}(1+|Du|^{p_*})dx\right)^{p/p_*},$$

where $p_* = np/(n+p)$. The conclusion then follows by Lemma 3.3. ∎

LEMMA 3.5. *Let $f$ satisfy* (3.1), (3.2), (3.3), (3.5) *and fix any $(\hat{x},\hat{s})$. Let $B\subset\Omega$ be a ball, and let $u\in W^{1,q}(B;\mathbb{R}^N)$ with $q>p$. There exist $q_0\in(p,q)$ and $c_0>0$, depending only on $\mu, L, p, q$, such that if $0\le\eta<2^{-p}\mu$ and $v\in u+W_0^{1,p}(B;\mathbb{R}^N)$ satisfies*

$$\begin{aligned}
&\int_B f(\hat{x},\hat{s},Dv)dx \\
&\qquad\qquad \le \int_B [f(\hat{x},\hat{s},Dv+D\varphi)+\eta|D\varphi|]dx \text{ for all } \varphi\in W_0^{1,p}(B;\mathbb{R}^N)
\end{aligned}$$

*then* $v \in W^{1,q_0}(B;\mathbb{R}^N)$ *and*

$$\left(\fint_B |Dv|^{q_0}dx\right)^{1/q_0} \le c_0\left[\left(\fint_B (1+|Dv|^p)dx\right)^{1/p} + \left(\fint_B (1+|Du|^q)dx\right)^{1/q}\right].$$

PROOF. We begin with the interior estimate; fix any ball $B_r(x_0) \subset B$, and let $t, s$ and $\varsigma$ be as in Lemma 3.4; define

$$\varphi_1 = [v-(v)_r]\varsigma, \qquad \varphi_2 = [v-(v)_r](1-\varsigma).$$

Following the proof of Lemma 3.4, an additional term $\eta\int_{B_s}|D\varphi_1|dx$ appears, and instead of (3.8) we are led to

$$\mu\int_{B_s}|D\varphi_1|^p dx \le c\int_{B_s\setminus B_t}|Dv|^p dx + c\int_{B_r}\left(1+\frac{|v-(v)_r|^p}{(s-t)^p}\right)dx$$
$$+\eta\int_{B_s}|D\varphi_1|dx;$$

by the bounds on $\eta$ we have

$$\eta\int_{B_s}|D\varphi_1|dx \le \frac{\mu}{2}\int_{B_s}(1+|D\varphi_1|^p)dx,$$

and we may conclude, as in Lemma 3.4, that if $B_r(x_0) \subset B$ then

$$\fint_{B_{r/2}}|Dv|^p dx \le c\left(\fint_{B_r}(1+|Dv|^{p_*})dx\right)^{p/p_*}. \tag{3.9}$$

Now we estimate $v$ near the boundary: assume $B_r(x_0)\cap B \ne \emptyset, B_r(x_0)\cap(\mathbb{R}^n\setminus B)\ne\emptyset$, and fix $t, s, \varsigma$ as before; define

$$\varphi_1 = (v-u)\varsigma, \qquad \varphi_2 = (v-u)(1-\varsigma),$$

so that $\varphi_1 \in W_0^{1,p}(B_s\cap B)$. Following again the proof of Lemma 3.4 we find

$$\mu\int_{B_s\cap B}|D\varphi_1|^p dx$$
$$\le \int_{B_s\cap B}\{-\psi(0)+f(\hat{x},\hat{s},Du-D\varphi_2)$$
$$+[f(\hat{x},\hat{s},Dv-(Du+D\varphi_2))-f(\hat{x},\hat{s},Dv)]\}dx$$
$$+\eta\int_{B_s\cap B}|D\varphi_1|dx.$$

The last integral is dealt with as above, and using (3.2), (3.3) we have

$$\begin{aligned}
&\frac{\mu}{2}\int_{B_t\cap B}|Dv-Du|^p dx \\
&\qquad \le c\int_{B_s\cap B}[1+|Du|^p+|D\varphi_2|^p]dx \\
&\qquad\quad + c\int_{B_s\cap B}(|Dv-Du|^{p-1}+|D\varphi_2|^{p-1}+|Dv|^{p-1})\cdot \\
&\qquad\qquad \cdot|Du+D\varphi_2|dx \\
&\qquad \le c_\mu\int_{B_s\cap B}(1+|Du|^p+|D\varphi_2|^p)dx \\
&\qquad\quad + \frac{\mu}{4}\int_{B_s\cap B}|Dv-Du|^p dx,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{B_t\cap B}|Dv-Du|^p dx \\
&\qquad \le c\int_{(B_s\setminus B_t)\cap B}|Dv-Du|^p dx \\
&\qquad\quad + c\int_{B_r\cap B}\left(1+|Du|^p+\frac{|u-v|^p}{(s-t)^p}\right)dx.
\end{aligned}$$

The usual hole-filling argument and Lemma 2.4 yield

$$\int_{B_{r/2}\cap B}|Dv-Du|^p dx \le c\int_{B_r\cap B}(1+|Du|^p)dx + cr^{-p}\int_{B_r\cap B}|v-u|^p dx.$$

Since $v-u$ can be extended as zero outside $B$, and since the measure of $B_{2r}\setminus B$ is greater than $c_n r^n$, we may apply a modification of Sobolev-Poincaré inequality, and we have

$$\int_{B_r\cap B}|v-u|^p dx \le c\left(\int_{B_{2r}\cap B}|Dv-Du|^{p_*}dx\right)^{p/p_*},$$

so that

$$\begin{aligned}
&\int_{B_{r/2}\cap B}|Dv|^p dx \\
&\qquad \le c\int_{B_{2r}\cap B}(1+|Du|^p)dx + cr^{-p}\left(\int_{B_{2r}\cap B}|Dv|^{p_*}dx\right)^{p/p_*}.
\end{aligned} \tag{3.10}$$

Then if we set

$$V(x) = \begin{cases} |Dv|^{p_*} & \text{in } B \\ 0 & \text{outside } B \end{cases} \qquad U(x) = \begin{cases} 1 + |Du|^p & \text{in } B \\ 0 & \text{outside } B \end{cases}$$

by (3.9), (3.10) we have for any ball $B_r$ in $\mathbb{R}^n$

$$\fint_{B_{r/2}} V^{p/p_*} dx \leq c \left( \fint_{B_{2r}} V dx \right)^{p/p_*} + c \fint_{B_{2r}} U dx,$$

with $U \in L^{q/p}(\mathbb{R}^n)$. Aplying Lemma 3.3 the result follows. ∎

LEMMA 3.6. *Let $f$ satisfy* (3.1), (3.2), (3.5) *and fix any* $(\hat{x}, \hat{s})$. *If $B$ is any ball in $\mathbb{R}^n$, and $u \in W^{1,p}(B; \mathbb{R}^N)$, then the functional $\int_B f(\hat{x}, \hat{s}, Dw(x))dx$ satisfies*

$$\int_B f(\hat{x}, \hat{s}, Dw(x))dx \geq \mu \int_B |Dw|^p dx - c \int_B (1 + |Du|^p) dx$$

*for every $w \in u + W_0^{1,p}(B; \mathbb{R}^N)$; moreover, if $f$ is also quasiconvex with respect to $\xi$, then the functional $\int_B f(\hat{x}, \hat{s}, Dw(x))dx$ is sequentially weakly semicontinuous on $u + W_0^{1,p}(B; \mathbb{R}^N)$.*

PROOF. The semicontinuity on the Dirichlet classes follows from [14], Theorem 5.

Let $B'$ be the ball with same center as $B$, and twice the radius, and let $\tilde{u} \in (u)_B + W_0^{1,p}(B'; \mathbb{R}^N)$ be an extension of $u$ such that $\int_{B'} |D\tilde{u}|^p dx \leq c \int_B |Du|^p dx$; if we set for every $w \in u + W_0^{1,p}(B; \mathbb{R}^N)$

$$\tilde{w} = \begin{cases} w & \text{in } B \\ \tilde{u} & \text{in } B' \backslash B, \end{cases}$$

then by (3.5)

$$\begin{aligned} \int_{B'} [\mu |D\tilde{w}|^p + \psi(0)] dx & \\ & \leq \int_{B'} f(\hat{x}, \hat{s}, D\tilde{w}) dx \\ & \leq \int_B f(\hat{x}, \hat{s}, Dw) dx + c \int_{B' \backslash B} (1 + |D\tilde{u}|^p) dx, \end{aligned}$$

and the result follows. ∎

LEMMA 3.7. *Let $f$ satisfy* (3.1),...,(3.6). *There exist two constants, $0 < \beta_1 < \beta_2 < 1$, a radius $r_0 < 1$, and for every $K > 0$ a constant $c_K$, such that if*

*$u$ is a minimizer of $I, r < r_0, B_{2r}(x_0) \subset \Omega$ and $(|Du|^p)_{x_0,2r} \leq K$ then there is a $v \in u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N)$ such that*

$$\left(\fint_{B_r} |Dv - Du|^p dx\right)^{1/p} \leq c_K r^{\beta_1}$$

*and*

$$\int_{B_r} f(x_0, (u)_{x_0,r}, Dv(x))dx$$
$$\leq \int_{B_r} f(x_0, (u)_{x_0,r}, Dv(x) + D\varphi(x))dx + r^{\beta_2} \int_{B_r} |D\varphi(x)|dx$$

*for every $\varphi \in C_0^1(B_r(x_0);\mathbb{R}^N)$.*

PROOF. By Lemma 3.4 and the minimality of $u$ follows the existence of $q_0 > p$ and $c_0 > 0$ such that $u \in W^{1,q_0}_{\text{loc}}(\Omega;\mathbb{R}^N)$ and

$$\left(\fint_{B_r} |Du|^{q_0} dx\right)^{1/q_0} \leq c_0 \left(\fint_{B_{2r}} (1 + |Du|^p)dx\right)^{1/p} \tag{3.11}$$

for every $B_{2r} \subset \Omega$. Set

$$I_r^0(w) = \fint_{B_r(x_0)} f(x_0, (u)_{x_0,r}, Dw(x))dx$$

and

$$J(w) = \begin{cases} I_r^0(w) & \text{if } w \in u + W_0^{1,p}(B_r;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that

$$I_r^0(u) - \inf J \leq \tilde{c}(K) r^{\beta}, \tag{3.12}$$

where $\beta < 1$ depends only on $\delta, L, p, \mu$.

FIRST CASE. We assume that in (3.6) the condition $f \geq 0$ holds.

Denote by $(u_h)$ a sequence in $u + W_0^{1,p}(B_r;\mathbb{R}^N)$ such that

$$I_r^0(u_h) - \inf J \leq 1/h,$$

and consider the functional $J$ on the space $u + W_0^{1,1}(B_r;\mathbb{R}^N)$ endowed with the metric

$$d_1(v, w) = \fint_{B_r} |Dv - Dw|dx.$$

Since $f \geq 0$, by Fatou's lemma $J$ is semicontinuous in this space; we may then apply Lemma 3.2, so that there exists a sequence $(v_h)$ in $u + W_0^{1,1}(B_r;\mathbb{R}^N)$ such that

$$\fint_{B_r} |Dv_h - Du_h| dx \leq 1$$

and

$$J(v_h) \leq J(w) + \frac{1}{h}\fint_{B_r} |Dv_h - Dw| dx \text{ for all } w \in W_0^{1,p}(B_r;\mathbb{R}^N). \tag{3.13}$$

In particular $J(v_h) \leq J(u_h) + 1/h$, hence

$$\lim_{h\to\infty} J(v_h) = \inf J \tag{3.14}$$

and $J(v_h)$ is finite. Therefore $v_h \in u + W_0^{1,p}(B_r;\mathbb{R}^N)$ and by Lemma 3.6

$$\fint_{B_r} |Dv_h|^p dx \leq c\fint_{B_r} (1 + |Du|^p) dx. \tag{3.15}$$

Moreover by (3.13) we may apply Lemma 3.5 for $h$ large enough, and there exist $c_1$ and $q_1 \in (p, q_0)$ such that $v_h \in W^{1,q_1}(B_r;\mathbb{R}^N)$ and

$$\begin{aligned}
&\left(\fint_{B_r} |Dv_h|^{q_1} dx\right)^{1/q_1} \\
&\quad \leq c_1 \left(\fint_{B_r} (1 + |Dv_h|^p) dx\right)^{1/p} + \left(\fint_{B_r} (1 + |Du|^{q_0}) dx\right)^{1/q_0} \\
&\quad \leq c \left(\fint_{B_{2r}} (1 + |Du|^p) dx\right)^{1/p},
\end{aligned} \tag{3.16}$$

where we used (3.15) and (3.11). Now

$$\begin{aligned}
I_r^0(u) - I_r^0(v_h) &\leq \fint_{B_r} [f(x_0, (u)_r, Du) - f(x, u, Du)] dx \\
&\quad + \fint_{B_r} [f(x, u, Du) - f(x, v_h, Dv_h)] dx \\
&\quad + \fint_{B_r} [f(x, v_h, Dv_h) - f(x_0, (u)_r, Dv_h)] dx \\
&= a_h + b_h + c_h.
\end{aligned}$$

Then by (3.4)

$$
\begin{aligned}
|a_h| &\le \fint_{B_r} L(1+|Du|^p)\ \omega(r^p+|u-(u)_r|^p)dx \\
&\le L\left(\fint_{B_r}(1+|Du|^{q_0})dx\right)^{p/q_0}\cdot \\
&\qquad\qquad \cdot\left(\fint_{B_r}\omega^{q_0/(q_0-p)}(r^p+|u-(u)_r|^p)dx\right)^{1-p/q_0}.
\end{aligned}
$$

Since $\omega$ is bounded and concave,

$$
\begin{aligned}
|a_h| &\le c\left(\fint_{B_r}(1+|Du|^{q_0})dx\right)^{p/q_0}\omega^{1-p/q_0}\left(cr^p\fint_{B_r}(1+|Du|^p)dx\right) \\
&\le c\left(\fint_{B_{2r}}(1+|Du|^p)dx\right)^{1+\delta(q_0-p)/q_0} r^{\delta p(q_0-p)/q_0} \\
&\le c_1(K)r^{\delta p(q_0-p)/q_0}
\end{aligned}
$$

by (3.11). Analogously

$$
\begin{aligned}
|c_h| &\le c\left(\fint_{B_r}(1+|Dv_h|^{q_1})dx\right)^{p/q_1}\cdot \\
&\qquad\qquad \cdot\omega^{1-p/q_1}\left(cr^p\fint_{B_r}(1+|Du|^p+|D(v_h-u)|^p)dx\right) \\
&\le c_2(K)r^{\delta p(q_1-p)/q_1}
\end{aligned}
$$

by (3.15), (3.16). Since $b_h \le 0$ by the minimality of $u$, we deduce from the estimates above

$$
I_r^0(u)-I_r^0(v_h)\le \tilde{c}(K)r^\beta,
$$

with $\beta=\delta p(q_1-p)/q_1$, which together with (3.14) proves (3.12) in the first case; the idea of passing to the sequence $(v_h)$ was first used in [13].

SECOND CASE. We assume that in (3.6) the quasiconvexity condition holds.

In this case, by Lemma 3.6 the functional $J$ is semicontinuous, and has a minimum point $\overline{u}\in u+W_0^{1,p}(B_r;\mathbb{R}^N)$ which satisfies

$$
\int_{B_r}|D\overline{u}|^p dx \le c\int_{B_r}(1+|Du|^p)dx;
$$

then, by Lemma 3.5 applied with $\eta=0$, there exist $c_1$ and $q_1\in(p,q_0)$ such

that $\overline{u} \in W^{1,q_1}(B_r;\mathbb{R}^N)$ and

$$\begin{aligned}\left(\fint_{B_r} |D\overline{u}|^{q_1}\,dx\right)^{1/q_1} &\\ &\le c_1\left[\left(\fint_{B_r}(1+|D\overline{u}|^p)dx\right)^{1/p} + \left(\fint_{B_r}(1+|Du|^{q_0})dx\right)^{1/q_0}\right]\\ &\le c\left(\fint_{B_{2r}}(1+|Du|^p)dx\right)^{1/p}.\end{aligned}$$

The inequality

$$I_r^0(u) - \inf J = I_r^0(u) - I_r^0(\overline{u}) \le \tilde{c}(K)r^\beta$$

may be proved as above, and also the second case is concluded.

We now consider on the space $u + W_0^{1,1}(B_r;\mathbb{R}^N)$ the metric

$$d_2(v,w) = (\tilde{c}(K)r^{\beta/2})^{-1}\fint_{B_r}|Dv - Dw|dx.$$

By (3.12), applying again Lemma 3.2 we find $v \in u + W_0^{1,p}(B_r;\mathbb{R}^N)$ such that

$$\fint_{B_r}|Dv - Du|dx \le \tilde{c}(K)r^{\beta/2} \tag{3.17}$$

and

$$J(v) \le J(v+\varphi) + r^{\beta/2}\fint_{B_r}|D\varphi|dx \text{ for all } \varphi \in W_0^{1,1}(B_r;\mathbb{R}^N).$$

This proves the last assertion of the lemma, with $\beta_2 = \beta/2$. In addition, by Lemma 3.6

$$\begin{aligned}\mu\fint_{B_r}|Dv|^p dx &\le I_r^0(u) + r^{\beta/2}\fint_{B_r}|Dv - Du|dx\\ &\qquad + c\fint_{B_r}(1+|Du|^p)dx\\ &\le c\fint_{B_r}(1+|Du|^p)dx\end{aligned} \tag{3.18}$$

since $r < 1$. We now select $r_0 = (2^{-p-1}\mu)^{2/\beta}$, so that we may apply Lemma 3.5 to the functional $w \mapsto I_r^0(w) + r^{\beta/2}\fint_{B_r}|Dv - Dw|dx$. Then there exist $c$ and $q_1 \in (p, q_0)$ such that

$$\left(\fint_{B_r}|Dv|^{q_1}dx\right)^{1/q_1} \le c\left(\fint_{B_{2r}}(1+|Du|^p)dx\right)^{1/p}, \tag{3.19}$$

where we used also (3.18) and (3.11). Now if $\vartheta = \dfrac{q_1 - p}{p(q_1 - 1)}$ we have

$$\begin{aligned}\left(\fint_{B_r} |Dv - Du|^p dx\right)^{1/p} &\\ &\le \left(\fint_{B_r} |Dv - Du| dx\right)^{\vartheta} \cdot \left(\fint_{B_r} |Dv - Du|^{q_1} dx\right)^{(1-\vartheta)/q_1} \\ &\le c(K) r^{\beta\vartheta/2}\end{aligned}$$

by (3.17), (3.11) and (3.19), and the result is proved with $\beta_1 = \beta\vartheta/2$. ∎

The next result is analogous to Proposition 2.6, and after that only the iteration remains to be made. Fix $d < \beta_1$, and set

$$U(x_0, r) = r^d + \fint_{B_r(x_0)} g_q(Du - (Du)_{x_0,r}) dx.$$

Then we have:

PROPOSITION 3.8. *Let $f$ satisfy* (3.1),...,(3.6) *and let $(x_0, s_0, \xi_0)$ be a regular point for $f$. There exists a constant $C$ such that for every $\tau < 1/4$ there exists $\varepsilon(\tau)$ such that if $u$ is a minimizer of $I$ satisfying*

$$\begin{gathered}|\hat{x} - x_0| < \frac{\sigma}{3}, \quad |(u)_{\hat{x},r} - s_0| < \frac{\sigma}{3}, \quad |(Du)_{\hat{x},r} - \xi_0| < \frac{\sigma}{3}, \\ |(Du)_{\hat{x},\tau r} - \xi_0| < \frac{\sigma}{3}, \quad (|Du|^p)_{\hat{x},2r} \le |\xi_0|^p + 1\end{gathered} \tag{3.20}$$

*and*

$$U(\hat{x}, r) < \varepsilon$$

*then*

$$U(\hat{x}, \tau r) < C\tau^d U(\hat{x}, r).$$

PROOF. Fix $\tau$; we shall determine $C$ later. Reasoning as in Proposition 2.6, assume that (3.20) holds in $B_{r_h}(x_h)$, and that

$$U(x_h, r_h) = \lambda_h^2 \to 0, \tag{3.21}$$

but

$$U(x_h, \tau r_h) > C\tau^d \lambda_h^2. \tag{3.22}$$

Applying Lemma 3.7 in each $B_{r_h}(x_h)$ we find a sequence $(u_h)$ in $u + W_0^{1,p}(B_{r_h}(x_h); \mathbb{R}^N)$ satisfying

$$\left(\fint_{B_{r_h}(x_h)} |Du_h - Du|^p dx\right)^{1/p} \le c r_h^{\beta_1} \tag{3.23}$$

and

$$\int_{B_{r_h}(x_h)} f(x_h, (u)_{x_h,r_h}, Du_h)dx$$

$$\leq \int_{B_{r_h}(x_h)} f(x_h, (u)_{x_h,r_h}, Du_h + D\varphi)dx \tag{3.24}$$

$$+ r_h^{\beta_2} \int_{B_{r_h}(x_h)} |D\varphi|dx$$

for every $\varphi \in C_0^1$. Set

$$A_h = (Du_h)_{x_h,r_h}, \qquad \tilde{A}_h = (Du_h)_{x_h,\tau r_h}, \qquad a_h = (u_h)_{x_h,r_h}.$$

From (3.21) we deduce that

$$r_h^d \leq \lambda_h^2, \tag{3.25}$$

therefore in particular $r_h \to 0$; from (3.23) we then get, if $h$ is sufficiently large,

$$|A_h - \xi_0| < \frac{\sigma}{3}, \qquad |\tilde{A}_h - \xi_0| < \frac{\sigma}{3}, \qquad |a_h - s_0| < \frac{\sigma}{3}.$$

Now $g_p(x+y+z) \leq (1+\epsilon)g_p(x) + c_\epsilon[g_p(y) + g_p(z)]$, and by the convexity of $g_p$

$$\frac{1}{\lambda_h^2} \fint_{B_{r_h}} g_p(Du_h - (Du_h)_{r_h})dx$$

$$\leq \frac{(1+\epsilon)}{\lambda_h^2} \fint_{B_{r_h}} g_p(Du - (Du)_{r_h})dx$$

$$+ \frac{c_\epsilon}{\lambda_h^2} \fint_{B_{r_h}} [g_p(Du_h - Du) + g_p((Du_h)_{r_h} - (Du)_{r_h})]dx$$

$$\leq \frac{(1+\epsilon)}{\lambda_h^2} \fint_{B_{r_h}} g_p(Du - (Du)_{r_h})dx$$

$$+ \frac{2c_\epsilon}{\lambda_h^2} \fint_{B_{r_h}} g_p(Du_h - Du)dx$$

$$\leq \frac{(1+\epsilon)}{\lambda_h^2} \fint_{B_{r_h}} g_p(Du - (Du)_{r_h})dx + c(\epsilon)\frac{r_h^{2\beta_1}}{\lambda_h^2}$$

by (3.23). Since this holds also with $u_h$ and $u$ interchanged, if we set

$$U_h(x_h, r) = r^d + \fint_{B_r(x_h)} g_p(Du_h - (Du_h)_{x_h,r})dx$$

using (3.25) we deduce easily

$$\lim_{h\to\infty} \frac{U(x_h, r_h) - U_h(x_h, r_h)}{\lambda_h^2} = 0; \tag{3.26}$$

similarly one has

$$\lim_{h\to\infty} \frac{U(x_h, \tau r_h) - U_h(x_h, \tau r_h)}{\lambda_h^2} = 0. \tag{3.27}$$

We now define

$$v_h(z) = \frac{u_h(x_h + r_h z) - a_h - r_h A_h z}{\lambda_h r_h},$$

and remark that $(v_h)_{0,1} = 0$, and that by (3.26)

$$\limsup_{h\to\infty} \int_{B_1} [|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p]dz \le 1,$$

so that we may suppose

$$v_h \rightharpoonup v \text{ weakly in } W^{1,2}(B_1;\mathbb{R}^N),$$
$$\lambda_h^{1-2/p} v_h \rightharpoonup 0 \text{ weakly in } W^{1,p}(B_1;\mathbb{R}^N),$$

and also

$$A_h \to A, \qquad (u)_{x_h,r_h} \to a, \qquad x_h \to \overline{x}.$$

Remark that $(\overline{x}, a, A)$ is a regular point for $f$. Now define

$$L_h = \{z \in B_1 : \lambda_h |Dv_h(z)| \ge \sigma/3\}, \qquad S_h = B_1 \backslash L_h,$$

and use (3.24) as in Proposition 2.6 to obtain, instead of (2.11),

$$\int_{B_1}\int_0^1 f_{\xi\xi}(x_h, a_h, A_h + s\lambda_h Dv_h)\mathbf{1}_{S_h} Dv_h D\varphi ds\, dz$$
$$\ge -\frac{r_h^{\beta_2}}{\lambda_h}\int_{B_1} |D\varphi| dz - c\lambda_h,$$

whence again

$$\int_{B_1} f_{\xi\xi}(\overline{x}, a, A) Dv D\varphi dz = 0 \text{ for all } \varphi \in C_0^1(B_1;\mathbb{R}^N),$$

and for all $\tau < 1/4$

$$\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 dz \le c\tau^2.$$

Define

$$w_h = v_h - \frac{\tilde{A}_h - A_h}{\lambda_h} z$$

and remark that

$$\int_{B_1} f_{\tilde{A}_h;\lambda_h}(x_h, (u)_{x_h;r_h}, Dw_h)dz$$
$$\leq \int_{B_1} f_{\tilde{A}_h;\lambda_h}(x_h, (u)_{x_h;r_h}, Dw_h + D\psi)dz + r_h^{\beta_2} \int_{B_1} |D\psi|dz.$$

Then we may apply Lemma 2.5, this time with $\nu = r_h^{\beta_2}$; repeating the argument of Proposition 2.6, and recalling (3.25), we get from (3.27)

$$\limsup_{h\to\infty} \frac{U(x_h, \tau r_h)}{\lambda_h^2} \leq c\tau^2 + \limsup_{h\to\infty} \tau^d r_h^d/\lambda_h^2 \leq c\tau^d,$$

which gives the required contradiction with (3.22). ∎

PROPOSITION 3.9. *Take a regular point $(x_0, s_0, \xi_0)$ and $\alpha < d$; if $C$ is as in Proposition 3.8, fix $\tau < 1/4$ such that $C\tau^d < \tau^\alpha$. Let $u$ be a minimizer of $I$ and assume that for some $B_r(x_0)$*

$$|(u)_{x_0;r} - s_0| < \frac{\sigma}{6}, \quad |(Du)_{x_0;r} - \xi_0| < \frac{\sigma}{6}, \quad |(Du)_{x_0;\tau r} - \xi_0| < \frac{\sigma}{6},$$
$$(|Du|^p)_{x_0;r} \leq |\xi_0|^p + \frac{1}{2}, \quad (|Du|^p)_{x_0;2r} \leq |\xi_0|^p + \frac{1}{2}$$

*and $U(x_0, r) < \eta$, with $\eta > 0$ sufficiently small. Then for all $k$*

$$|(u)_{x_0;\tau^k r} - s_0| < \frac{\sigma}{3}, \quad |(Du)_{x_0;\tau^k r} - \xi_0| < \frac{\sigma}{3},$$
$$|(Du)_{x_0;\tau^{k+1} r} - \xi_0| < \frac{\sigma}{3}, \quad (|Du|^p)_{x_0;2\tau^k r} \leq |\xi_0|^p + 1$$

*and*

$$U(x_0, \tau^k r) \leq \tau^{k\alpha} U(x_0, r).$$

PROOF. Reasoning as in Proposition 2.7 we have

$$|(Du)_{\tau^{m+1} r} - \xi_0| \leq \frac{\sigma}{6} + \tau^{-n} \sum_{k=0}^{m} [U(x_0, \tau^k r)]^{1/2} \tag{3.28}$$

$$|(u)_{\tau^m r} - s_0| \leq \frac{\sigma}{6} + c\tau^{-n} \sum_{k=0}^{m-1} \tau^k r (|Du|^p)_{x_0;2\tau^k r}^{1/p}; \tag{3.29}$$

now, since $\tau < 1/4$, for $m \geq 2$ we have

$$\begin{aligned}
&\left(\fint_{B_{2\tau^m r}} |Du|^p dx\right)^{1/p} \\
&\qquad \leq |(Du)_{2\tau^{m-1}r}| + \left(\fint_{B_{2\tau^m r}} |Du - (Du)_{2\tau^{m-1}r}|^p dx\right)^{1/p} \\
&\qquad \leq |(Du)_{2\tau^{m-1}r}| + \tau^{-n/p}\left(\fint_{B_{2\tau^{m-1}r}} |Du - (Du)_{2\tau^{m-1}r}|^p dx\right)^{1/p} \\
&\qquad \leq (|Du|^p)^{1/p}_{2\tau^{m-1}r} + (2\tau^2)^{-n/p}[U(x_0, \tau^{m-2}r)]^{1/p},
\end{aligned}$$

whereas for $m = 1$

$$\begin{aligned}
&\left(\fint_{B_{2\tau r}} |Du|^p dx\right)^{1/p} \\
&\qquad \leq |(Du)_r| + \left(\fint_{B_{2\tau r}} |Du - (Du)_r|^p dx\right)^{1/p} \\
&\qquad \leq (|Du|^p)^{1/p}_r + (2\tau)^{-n/p}[U(x_0, r)]^{1/p}.
\end{aligned}$$

Thus, combining these two estimates,

$$\begin{aligned}
(|Du|^p)^{1/p}_{2\tau^m r} \leq &\left(|\xi_0|^p + \frac{1}{2}\right)^{1/p} + (2\tau)^{-n/p}[U(x_0, r)]^{1/p} \\
&+ (2\tau^2)^{-n/p}\sum_{k=0}^{m-2}[U(x_0, \tau^k r)]^{1/p}.
\end{aligned}$$

From (3.28), (3.29) and this inequality, an induction argument proves the result if $\eta$ was chosen sufficiently small. ∎

PROOF OF THEOREM 3.1. One may follow the lines of the proof of Theorem 2.1, except that $\alpha$ must be less than $d$. ∎

## 4. - Additional remarks

In this section we state two corollaries which follow from our results, then we apply Theorem 2.1 in a case which is relevant in nonlinear elasticity, and finally we study the scalar-valued case $N = 1$, identifying exactly the set of regular points.

Theorems 2.1 and 3.1 yield two new (global) partial regularity results: precisely, we have

COROLLARY 4.1. *Let* $f : \mathbb{R}^{nN} \to \mathbb{R}$ *be a function of class* $C^2$ *satisfying for some* $p \geq 2$

$$|f(\xi)| \leq c(1 + |\xi|^p);$$

*assume that for every* $\xi$ *there exists a positive number* $\gamma(\xi)$ *such that*

$$\int_{\mathrm{spt}\varphi} f(\xi + D\varphi(y))dy \geq \int_{\mathrm{spt}\varphi} [f(\xi) + \gamma(\xi)(|D\varphi(y)|^2 + |D\varphi(y)|^p)]dy$$

*for every* $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$. *Then if* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *is a local minimizer of* $\int f(Dv(x))dx$ *there exists an open subset* $\Omega_0$ *of* $\Omega$ *with* meas $(\Omega \backslash \Omega_0) = 0$ *such that* $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ *for all* $\alpha < 1$.

COROLLARY 4.2. *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ *satisfy for some* $p \geq 2$

$$f \text{ is twice differentiable with respect to } \xi;$$

$$f_{\xi\xi}(x, s, \xi) \text{ is continuous;}$$

$$|f(x, s, \xi)| \leq c(1 + |\xi|^p);$$

$$|f(x, s, \xi) - f(y, t, \xi)| \leq L(1 + |\xi|^p)\ \omega(|x - y|^p + |s - t|^p),$$

*where* $\omega(t) \leq t^\delta, 0 < \delta < 1/p$, *and* $\omega$ *is bounded, concave and increasing;*

$$f(x, s, \xi) \geq \psi(\xi)$$

*for a suitable continuous function* $\psi$ *satisfying*

$$\int_\Omega \psi(D\varphi(y))dy \geq \int_\Omega [\psi(0) + \mu|D\varphi(y)|^p]dy \text{ for all } \varphi \in C_0^1(\Omega; \mathbb{R}^N),$$

*with* $\mu > 0$; *finally, we assume that there exists a positive lower semicontinuous function* $\gamma(x, s, \xi)$ *such that for every* $(x, s, \xi)$

$$\int_{\mathrm{spt}\varphi} f(x, s, \xi + D\varphi(y))dy$$
$$\geq \int_{\mathrm{spt}\varphi} [f(x, s, \xi) + \gamma(x, s, \xi)(|D\varphi(y)|^2 + |D\varphi(y)|^p)]dy$$

*for every* $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$. *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be a local minimizer of* $\int_\Omega f(x, v(x), Dv(x))dx$. *Then there exists an open subset* $\Omega_0$ *of* $\Omega$ *with* meas $(\Omega \backslash \Omega_0) = 0$ *such that* $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ *for some* $\alpha < 1$.

To prove this second Corollary, it is enough to remark that (see Propositions 3.8 and 3.9) the Hölder exponent $\alpha$ must satisfy only $\alpha < d$, and the number $d$ is independent of $\gamma$.

These results improve the former general regularity theorems of [5], [6]: not only, as already in [2], the boundedness of the second derivatives of $f$ is dropped, but also the strict quasiconvexity need no longer be uniform.

EXAMPLE 4.3. Let $n = N$, and define $f$ by

$$f(\xi) = |{}^t\xi\xi - I|^2,$$

where $I$ is the $n \times n$ identity matrix; if $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is the deformation of an $n$-dimensional body $\Omega$, the functional $\int_\Omega f(Du(x))dx$ is an important model of nonlinear elastic energy associated with $u$. The "expansion points" of $u$ are the points $x$ at which the $n$ eigenvalues of the matrix $\sqrt{{}^tDu\ Du}$ are greater than 1. A not too hard computation shows that if the eigenvalues of $\sqrt{{}^t\xi\xi}$ are all greater than 1, then $\xi$ is a regular point for $f$; therefore, a deformation $u$ is of class $C^{1,\alpha}$ around each of its expansion points.

We shall henceforth confine ourselves to the scalar-valued case $N = 1$; in this case, it is well known that a function being quasiconvex everywhere is equivalent to its being convex (everywhere). This is not true for quasiconvexity and convexity at a single point, as is shown by the following proposition (for any function $f$ we denote by

$$f^{**} = \max\{g \le f : g \text{ is convex}\}$$

the convex hull of $f$).

PROPOSITION 4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, and assume there exists some $\xi_0$ such that*

$$\int_{\mathrm{spt}\varphi} f(\xi_0 + D\varphi(x))dx \ge \int_{\mathrm{spt}\varphi} [f(\xi_0) + \gamma(|D\varphi(x)|^2 + |D\varphi(x)|^p)]dx$$

*for all $\varphi \in C_0^{0,1}(\mathbb{R}^n)$, where $p \ge 2$ and $\gamma \ge 0$ is a constant. Then*

$$f(\xi_0) = f^{**}(\xi_0),$$

*and, if $\gamma > 0$, we have for some positive constants $c$, $c'$ depending only on $(\xi_0, \gamma, p)$*

$$f(\xi) \ge c|\xi|^p - c' \quad \text{for all } \xi.$$

*If in addition $f$ is twice differentiable at $\xi_0$, then*

$$D_{ij}f(\xi_0)\xi_i\xi_j \ge 2\gamma|\xi|^2.$$

PROOF. The idea is not new; take any $\tilde{\xi}, \tilde{\eta}$ and $\lambda \in (0,1)$ such that

$$\lambda\tilde{\xi} + (1-\lambda)\tilde{\eta} = \xi_0,$$

and set $\xi = \tilde{\xi} - \xi_0, \eta = \tilde{\eta} - \xi_0$, so that $\lambda\xi + (1-\lambda)\eta = 0$. Let $Q$ be a unit cube with an edge parallel to $\xi$, and fix a face $F$ of $Q$ which is orthogonal to $\xi$; for every positive integer $m$ slice $Q$ into $m$ stripes orthogonal to $\xi$, and call $F_m$ the union of their faces parallel to $F$, then divide again each stripe in two, a stripe with thickness $\lambda/m$, the other with thickness $(1-\lambda)/m$, and call $Q_m^\lambda$, $Q_m^{1-\lambda}$ the union of the $\lambda$-stripes and the union of the $(1-\lambda)$-stripes respectively (they are thus intertwined). Then we may define a Lipschitz continuous function $v_m$ on $Q$ by setting

$$v_m = 0 \text{ on } F_m$$

$$Dv_m = \begin{cases} \xi & \text{in } Q_m^\lambda \\ \eta & \text{in } Q_m^{1-\lambda}. \end{cases}$$

In addition, $\max|v_m| = \lambda|\xi|/m$ : therefore, if $Q_\delta$ is the cube concentric with $Q$ and whose side is $1+\delta$, we may extend $v_m$ to a Lipschitz continuous function $\varphi_m$ vanishing outside $Q_\delta$ and such that

$$\sup_{Q_\delta \setminus Q} |D\varphi_m| \le c\left(|\xi| + |\eta| + \frac{\lambda|\xi|}{m\delta}\right).$$

Set $\omega(t) = \sup\{f(\varsigma) : |\varsigma| \le t\}$; then the quasiconvexity inequality yields

$$\begin{aligned} \gamma \int_Q (|D\varphi_m|^2 + |D\varphi_m|^p)dx &+ (1+\delta)^n f(\xi_0) \\ &\le \int_Q f(\xi_0 + D\varphi_m)dx \\ &\quad + \int_{Q_\delta/Q} \omega\left[c\left(1 + |\xi_0|^p + |\xi|^p + |\eta|^p + \frac{|\xi|^p}{m^p\delta^p}\right)\right]dx \\ &\le \int_Q f(\xi_0 + D\varphi_m)dx + c\delta\omega[c(1 + |\xi_0|^p + |\xi|^p + |\eta|^p)] \end{aligned}$$

for $m$ large enough, but $D\varphi_m = Dv_m$ in $Q$, hence

$$\xi_0 + D\varphi_m = \begin{cases} \tilde{\xi} & \text{in } Q_m^\lambda \\ \tilde{\eta} & \text{in } Q_m^{1-\lambda}, \end{cases}$$

thus

$$\begin{aligned} &\gamma[\lambda(|\xi|^2 + |\xi|^p) + (1-\lambda)(|\eta|^2 + |\eta|^p)] + (1+\delta)^n f(\xi_0) \\ &\qquad \le \lambda f(\tilde{\xi}) + (1-\lambda)f(\tilde{\eta}) + c\delta\omega[c(1 + |\xi_0|^p + |\xi|^p + |\eta|^p)] \end{aligned}$$

for $m$ large enough; by letting $m \to \infty$, then $\delta \to 0$, we get

$$\lambda f(\tilde{\xi}) + (1-\lambda)f(\tilde{\eta}) \ge f(\xi_0) + \gamma[\lambda(|\xi|^2 + |\xi|^p) + (1-\lambda)(|\eta|^2 + |\eta|^p)]. \tag{4.1}$$

From this we deduce in particular

$$\lambda f(\tilde{\xi}) + (1-\lambda) f(\tilde{\eta}) \geq f(\xi_0),$$

and by taking the infimum for $\lambda\tilde{\xi} + (1-\lambda)\tilde{\eta} = \xi_0$

$$f^{**}(\xi_0) \geq f(\xi_0),$$

thus proving the first assertion since the opposite inequality is obvious.

Set $M(\xi_0) = \max\{f(\xi_0+\eta) : |\eta| \leq 1\}$, and take $|\eta| = 1$ in (4.1), so that $\lambda = 1/(1+|\xi|)$; dropping $|\xi|^2$ and some other terms on the right-hand side, we have

$$\begin{aligned} f(\tilde{\xi}) &\geq -2M(\xi_0)(1+|\xi|) + \gamma|\xi|^p \\ &\geq -2M(\xi_0)(1+|\xi_0|+|\tilde{\xi}|) - \gamma|\xi_0|^p + 2^{1-p}\gamma|\tilde{\xi}|^p, \end{aligned}$$

and the second assertion follows easily.

Finally, this time dropping $|\xi|^p$ and $|\eta|^p$, and taking $\eta = -\xi$, so that $\lambda = 1/2$, we have again from (4.1)

$$\frac{1}{2}[f(\xi_0+\xi) + f(\xi_0-\xi) - 2f(\xi_0)] \geq \gamma|\xi|^2,$$

and the last assertion follows by taking $\xi = t\varsigma$, dividing by $t^2$ and letting $t \to 0$. ∎

As an example, note that the function $f(x) = x^2 - 3x^4 + \frac{5}{3}x^6$ is convex at 0, but it is not quasiconvex at 0, since $f^{**}(0) = -\frac{1}{3} < f(0)$.

We also remark that in the proof we did not fully use the continuity of $f$, but almost only the fact that it is bounded on bounded sets.

PROPOSITION 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying for some $p \geq 2$*

$$|f(\xi)| \leq c(1+|\xi|^p), \qquad |Df(\xi)| \leq c(1+|\xi|^{p-1})$$

*and such that the set of its regular points is not empty. Then this set is*

$$\{\xi_0 : f^{**} \in C^2(B_\sigma(\xi_0)) \text{ for some } \sigma > 0, \quad D^2 f^{**}(\xi_0)\eta\eta > 0 \text{ for all } \eta \neq 0\}.$$

PROOF. If there is at least a regular point, then by Proposition 4.4 we have

$$f(\xi) \geq c|\xi|^p - c'; \tag{4.2}$$

take a regular point $\xi_0$ : by Lemma 2.2 a whole ball $B_\sigma(\xi_0)$ is made of regular points, so by Proposition 4.4 $f(\xi) = f^{**}(\xi)$ in $B_\sigma(\xi_0)$ : thus $f^{**} \in C^2(B_\sigma(\xi_0))$ and (again by Proposition 4.4) $D^2 f^{**}(\xi_0)$ is positive definite.

To prove the converse, assume $f^{**}$ is of class $C^2$ around $\xi_0$, and $D^2 f^{**}(\xi) \geq \gamma I$ for $|\xi - \xi_0| < \sigma$; then necessarily $f(\xi) = f^{**}(\xi)$ in $B_\sigma(\xi_0)$, so $f \in C^2(B_\sigma(\xi_0))$ too. Now for $|\xi - \xi_0| < \sigma/2$

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{\gamma}{2}|\xi - \xi_0|^2, \tag{4.3}$$

and for $|\xi - \xi_0| \geq \sigma/2$

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{\gamma\sigma^2}{8}. \tag{4.4}$$

By (4.2) we have for a suitably large $R$ that if $|\xi - \xi_0| \geq R$ then

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{c}{2}|\xi - \xi_0|^p. \tag{4.5}$$

Now if $\lambda \leq c/2$ satisfies

$$\lambda(t^2 + t^p) \leq \frac{\gamma\sigma^2}{8} \text{ for all } t \leq R,$$

we immediately deduce from (4.3), (4.4), (4.5)

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \lambda(|\xi - \xi_0|^2 + |\xi - \xi_0|^p),$$

and the strict quasiconvexity at $\xi_0$ follows immediately. ∎

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