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# A Construction of Unitary Representations in Parabolic Rank Two

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## 0. - Introduction

Some years ago, B. Spelh and the second author [7] investigated unitary representations of  $G = SU(N, 2)$  that arise as Langlands quotients of the nonunitary principal series. The nonunitary principal series consists of the induced representations

$$(0.1) \quad U(MAN, \sigma, \nu) = \text{ind}_{MAN}^G (\sigma \otimes e^\nu \otimes 1),$$

where  $MAN$  is a minimal parabolic subgroup. For this  $G$ ,  $M$  is a (compact) double cover of the unitary group  $U(N-2)$  and  $A$  is Euclidean of dimension 2. To visualize matters, we fix an irreducible representation  $\sigma$  of  $M$  and consider the two-dimensional picture of  $A$  parameters  $e^\nu$ , where  $\nu$  is a real-valued linear functional on the Lie algebra  $\mathcal{A}$  of  $A$ . In interesting cases, the typical picture of unitary points (i.e., those points for which the corresponding Langlands quotient can be made unitary) within the positive Weyl chamber, is as in Figure 1.

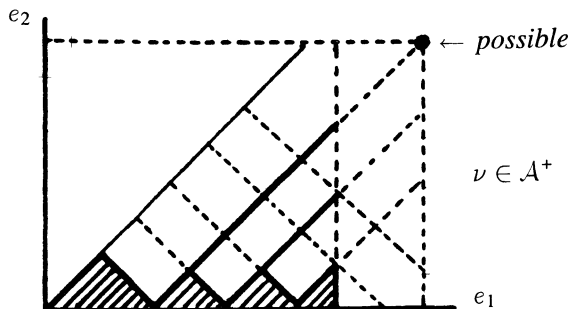


Figure 1: Typical picture of unitary points in  $SU(N, 2)$

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There are some two-dimensional regions of unitary points along the horizontal axis, there are some upward-sloping diagonal lines, and there is sometimes one isolated point.

Our concern here is with the lines of unitarity in the picture. The two-dimensional regions are a rank-one phenomenon, and we can regard them as already understood in an inductive argument. What about the lines? The proof in [7] that the lines give unitary points uses the fact that the lines correspond to irreducible degenerate series; the argument for irreducibility is rather complicated and shows little hope of generalization.

In the present paper, we shall show that the unitarity along these lines is indeed a rank-one phenomenon, provided we take into account some formal properties of intertwining operators. As we shall see, the argument generalizes to other real-rank-two groups and also to other parabolic-rank-two situations. In some cases, the argument produces isolated unitary points, as well as lines. The main results are Theorems 4.11 and 5.2.

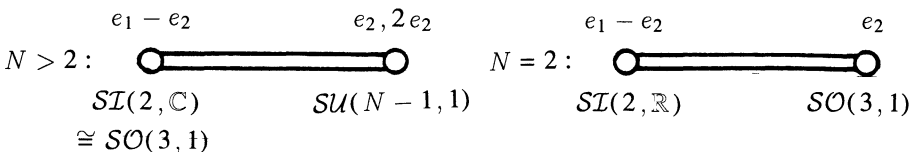
However, when we try to generalize the argument to parabolic rank  $n$ , we find that it does not seem to handle  $(n - 1)$ -dimensional sets in parabolic rank  $n$ , just one-dimensional sets and certain isolated points. This failure raises a question that we mention in Section 7.

We use the following notational conventions throughout. The group  $G$  is a connected linear semisimple Lie group with maximal compact subgroup  $K$ . Subgroups are denoted by upper-case Latin letters, and their Lie algebras are denoted by the corresponding lower-case German letters.

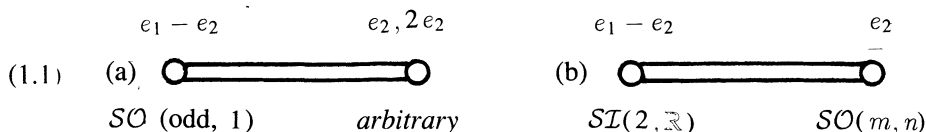
We are indebted to O.S. Rothaus for supplying a proof of Lemma 1.2 for us, to Chin-Han Sah for helping us to cope with some of the intricacies of  $SO(n, 1)$  in Section 4, and to H. Schlichtkrull for giving us independent confirmation of some of the unitarity we discovered.

## 1. - Oversimplified idea

In  $SU(N, 2)$  the configuration of roots of  $(\mathcal{G}, \mathcal{A})$  is of type  $(BC)_2$ , for  $N > 2$ , and may be taken to be of type  $B_2$  for  $N = 2$ . We give the precise diagrams below.



In the parabolic-rank-two generalization, we assume that  $G$  is linear, that  $MAN$  is a cuspidal parabolic subgroup of  $G$ , that all roots of  $(\mathcal{G}, \mathcal{A})$  are useful in the sense of [4], and that the corresponding configuration of roots is as in (1.1a) or (1.1b).



We work with induced representations (0.1) in which  $\sigma$  is a discrete series or (nonzero) limit of discrete series representation of  $M$ ,  $\mathcal{A}$  has dimension 2, and  $\nu$  is a real-valued linear functional on  $\mathcal{A}$  in the closed positive Weyl chamber.

Configuration (1.1a) arises with  $MAN$  minimal parabolic in  $G$  when  $G$  is locally  $SU(N, 2)$ ,  $Sp(N, 2)$ ,  $SO^*(10)$ , or the real form of  $E_6$  whose symmetric space is Hermitian. It arises with  $MAN$  general whenever  $\mathcal{A}$  has dimension 2 and  $G$  has restricted roots of type  $(BC)_n$  for some  $n$ ; it can occur in other situations as well. The prototype of configuration (1.1b) is a certain parabolic-rank-two situation with  $G = \widetilde{SO}(m+1, n+1)$ ; the parabolic subgroup is minimal when  $G = \widetilde{SO}(N, 2)$ . Our detailed discussion will concentrate on (1.1a); adjustments to the arguments that allow one to handle (1.1b) will be discussed in Section 5.

The Weyl group  $W(\mathcal{A})$  for the parabolic is the usual 8-element group for our root system, and we let  $s_0$  be its long element, which acts on  $\mathcal{A}$  by  $-1$ . We shall see in Lemma 4.1 for configuration (1.1a) that  $s_0$  fixes the class of  $\sigma$ .

Then unitarity of the Langlands quotient of (0.1) is detected by the standard intertwining operator of [9] corresponding to  $s_0$ . This operator maps  $U(MAN, \sigma, \nu)$  to  $U(MAN, \sigma, -\nu)$ . It is Hermitian on each  $K$  type, and the Langlands quotient is unitary if this operator is semidefinite. This operator factors as a product  $ABCD$  in a fashion that mirrors a decomposition of  $s_0$  as a product of four simple reflections. Here  $A$  and  $C$  are disguised intertwining operators for the subalgebra attached to  $e_1 - e_2$  [ $SO(\text{odd}, 1)$  and  $SI(2, \mathbb{R})$  in the two cases of (1.1)], and  $B$  and  $D$  are disguised intertwining operators attached to  $(e_2, 2e_2)$  or to  $e_2$ . We shall make use (on each  $K$  type) of the following fact from linear algebra.

**PROPOSITION 1.1.** *Suppose that  $A$ ,  $B$ ,  $C$ , and  $D$  are  $n$ -by- $n$  complex matrices with  $ABCD$  Hermitian. Suppose further that*

- (1)  *$B$  and  $D$  are Hermitian positive semidefinite*
- (2)  *$A = PC^*$  with  $P$  Hermitian positive semidefinite.*

*Then  $ABCD$  is positive semidefinite.*

In fact,  $ABCD = P(C^*BC)D$ . So the proposition reduces to the following lemma, which will be proved in Section 2.

LEMMA 1.2. *If  $R, S, T$  are positive semidefnite Hermitian  $n$ -by- $n$  matrices and  $RST$  is Hermitian, then  $RST$  is positive semidefnite.*

In our application  $B$  and  $D$  are intertwining operators for the parabolic-rank-one situation built from  $(e_2, 2e_2)$  or from  $e_2$ . Positivity for them corresponds to unitarity of certain Langlands quotients obtained from maximal parabolic subgroups, and this unitarity is understood completely [1]. Thus we know exactly when (1) holds in the proposition.

In considering condition (2), we must take into account slight differences between (1.1a) and (1.1b). For now, we consider (1.1a). Condition (2) for (1.1a) is a condition only on  $A$  and  $C$ , which are operators for  $\widetilde{SO}(\text{odd}, 1)$ , and these operators are well understood. (For formulas on all  $K$  types, see Klimyk-Gavrilik [3, p. 54].) The following properties of the operators and corresponding representations (nonunitary principal series) will be relevant for us in the situation (1.1a):

- (i) The  $K$  types have multiplicity one. This is well known and follows from Frobenius reciprocity and the branching theorem for  $\widetilde{SO}(n) \rightarrow \widetilde{SO}(n-1)$ . As a consequence, the intertwining operator is a scalar for each  $K$  type and each  $\nu$ . As  $\nu$  varies through real values, this scalar moves through real multiples of some complex number. (This last fact can be read off from Klimyk-Gavrilik [3] and presumably can be derived directly from Theorem 5.1 of [2]).
- (ii) The total operator (on all  $K$  types) has a kernel at  $\nu$  if and only if (0.1) is reducible at  $\nu$ . This is part of the Langlands theory; see Theorem 7.24 of [6].
- (iii) Reducibility of (0.1) can occur only at integral points of the infinitesimal character. See Proposition 6.1 of Speh-Vogan [12].
- (iv) For fixed  $M$  parameter and varying  $A$  parameter, Figure 2 gives the connection between reducibility of (0.1) and singularities of the infinitesimal character at points where the infinitesimal character is integral. This will be proved in Section 3.

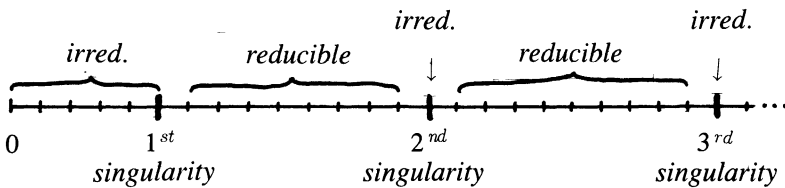


Figure 2: Reducibility points in  $\widetilde{SO}(\text{odd}, 1)$

- (v) In Figure 2 on any interval between two consecutive singularities of the infinitesimal character, the kernel of the intertwining operator at integral points decreases as one moves to the right. This can be read off from Klimyk-Gavrilik [3]. We do not have an abstract proof, but possibly one can be constructed with the aid of Zuckerman's translation functors [14].

Let us see how these properties are relevant for condition (2). Let  $A^{SO}$  be the standard intertwining operator for  $\widehat{SO}$  (odd, 1) attached to  $e_1 - e_2$ , and let  $\alpha$  be the positive restricted root for this group. If we write  $\nu = ae_1 + be_2$ , then the operators  $A$  and  $C$  are essentially

$$A \approx A^{SO} \left( s_\alpha, \sigma_0, \frac{1}{2}(a-b)\alpha \right),$$

$$C \approx A^{SO} \left( s_\alpha, s_\alpha \sigma_0, \frac{1}{2}(a+b)\alpha \right),$$

for a suitable  $\sigma_0$  constructed from  $\sigma$ . Since  $\nu$  is real,

$$C^* \approx A^{SO} \left( s_\alpha, \sigma_0, \frac{1}{2}(a+b)\alpha \right) = A^{SO} \left( s_\alpha, \sigma_0, \left[ \frac{1}{2}(a-b) + b \right] \alpha \right).$$

The upward-sloping lines in Figure 1 are the set of  $\nu$ 's where the factor  $A$  of  $ABCD$  has a kernel. Each line has  $\frac{1}{2}(a-b)$  constant and equal to the value at one of the reducibility points in Figure 2. Fix attention on one such line.

As  $b$  varies (increasing from 0), the argument of  $C^*$  increases from its starting value, which is the argument for  $A$ . In terms of Figure 2, the argument of  $C^*$  is moving to the right from the initial reducibility point. According to property (v) above,  $\ker A \supseteq \ker C^*$  until the argument of  $C^*$  gets to a singularity of the infinitesimal character.

Let us see the effect on the scalar function in (i). On a  $K$  type where  $A$  is 0, certainly  $A$  is a nonnegative multiple of  $C^*$ . On a  $K$  type where  $A$  is not 0, the operator  $A^{SO} \left( s_\alpha, \sigma_0, \left[ \frac{1}{2}(a-b) + t \right] \alpha \right)$  is not 0 for  $0 \leq t \leq b$ , because of the inclusion of kernels of (v). By (i) this operator is a real multiple of  $A$  on the  $K$  type, and continuity says that it is a positive multiple. We conclude that  $A = PC^*$  for an operator  $P$  that is a nonnegative function of  $\nu$  on each  $K$  type, this condition holding until the argument of  $C^*$  gets to a singularity of the infinitesimal character. Beyond that, the equality  $A = PC^*$  extends until the next integral point, by (iv) and continuity.

In other words, the two conditions in Proposition 1.1 are satisfied for the intertwining operator  $ABCD$  in configuration (1.1a) provided

- (1) the horizontal and vertical coordinates of  $\nu$  (namely  $ae_1$  and  $be_2$ ) correspond to unitary points in the appropriate parabolic-rank-one situations
- (2a)  $\nu$  is on an upwardly sloping line corresponding to reducibility coming from  $\ker A$  nontrivial

- (2b) the parameters of  $A$  and  $C^*$  for the group  $\widetilde{SO}$  (odd, 1) are not so far apart that they are separated by a singularity of the infinitesimal character, followed by a reducibility point.

Then Proposition 1.1 says that  $ABCD$  is semidefinite, and the corresponding Langlands quotient must be unitary.

For the special case of  $SU(N, 2)$ , the group  $\widetilde{SO}$  (odd, 1) is  $SL(2, \mathbb{C})$ . There is at most one singularity of the infinitesimal character in Figure 2. Thus there are only two intervals in the picture. Since the argument of  $A$  corresponds to a reducibility point, the arguments of  $A$  and  $C^*$  both correspond to the right-hand interval. Thus (2b) is automatic above. So when (1) and (2a) are satisfied, we have unitarity. This fact accounts for the lines of unitarity in Figure 1.

In the situation (1.1b), the above argument needs some minor modifications and additional hypotheses. We defer discussion of these points to Section 5.

## 2. - Proof of linear algebra lemma

In this section we prove Lemma 1.2. The argument is due to O.S. Rothaus.

Let  $K = \ker T$ , and write  $T = U^2$  with  $U$  Hermitian. Define  $U'$  to be  $U^{-1}$  on  $K^\perp$  and to be 0 on  $K$ . Then  $TU' = U$ , and  $U'RSU = U'RSTU'$ . Since  $RST(K) = 0 \subseteq K$ ,  $RST(K^\perp) \subseteq K^\perp$ . Therefore  $U'(RST)U'$  leaves stable both  $K$  and  $K^\perp$ . On  $K^\perp$  it has the same signature as  $RST$ , while on  $K$  it is 0 (and so is  $RST$ ). Thus  $U'RSTU'$  has the same signature as  $RST$ .

We claim that the nonzero eigenvalues of  $U'RSU (= U'RSTU')$  are a subset of the eigenvalues of  $RS$ . In fact,  $RST(K^\perp) \subseteq K^\perp$  and  $T$  carries  $K^\perp$  onto  $K^\perp$  (being invertible there). So  $RS(K^\perp) \subseteq K^\perp$ . Thus we can write  $U'(RS)U$  in block form as

$$\begin{aligned} & \begin{matrix} K^\perp & K \end{matrix} \\ & \begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 & * \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} = \\ & = \begin{pmatrix} u^{-1}m_1 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u^{-1}m_1u & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and its nonzero eigenvalues are those for  $m_1$ .

To complete the proof, it is enough to show that all eigenvalues of  $RS$  are  $\geq 0$ . This is well known. We simply write  $S = V^2$  with  $V$  Hermitian. Then  $RS = (RV)V$  has the same eigenvalues as  $VRV$ , which is positive semidefinite.

### 3. - Reducibility in $\widetilde{SO}$ (odd, 1)

In this section we prove property (iv) of Section 1 for the nonunitary principal series of  $\widetilde{SO}$  (odd, 1).

**PROPOSITION 3.1.** *For  $\widetilde{SO}$  (odd, 1), let  $\alpha$  be the positive restricted root, let  $\sigma$  be the  $M$  parameter, and let the integral points for the  $A$  parameter be  $\nu = (c + n)\alpha$ , with  $c$  fixed (equal to 0 or  $\frac{1}{2}$ ) and  $n$  varying in  $\mathbb{Z}$ . Let  $n_0$  be the least  $n \in \mathbb{Z}$  such that  $c + n \geq 0$  and such that the infinitesimal character of  $U(MAN, \sigma, (c + n)\alpha)$  is singular. For  $n \in \mathbb{Z}$  and  $c + n \geq 0$ ,  $U(MAN, \sigma, (c + n)\alpha)$  is irreducible if  $n \leq n_0$  or if the infinitesimal character is singular, and it is reducible otherwise.*

**PROOF.** For  $n < n_0$ , this result follows from Proposition 6.1 of Speh-Vogan [12]. Next suppose that  $(c + n_1)\alpha$  gives a singular point. Then the Plancherel factor  $p_\sigma(\nu)$  of Knapp-Stein [8] is 0 at  $\nu = (c + n_1)\alpha$  (see p. 543 of that paper). If  $w$  is a representative in  $K$  of the reflection  $s_\alpha$ , then Proposition 27 and Theorem 4 of [8] give

$$(3.1) \quad A(w^{-1}, w\sigma, -\nu)A(w, \sigma, \nu) = Cp_\sigma(\nu)^{-1}I$$

as an identity of meromorphic functions. It follows that  $A(w^{-1}, w\sigma, -\nu)$  has a pole at  $\nu = (c + n_1)\alpha$ . This pole can only be simple, according to Theorem 3 of [8]. Since the right side of (3.1) has a pole (on all functions  $\neq 0$ ) at  $\nu = (c + n_1)\alpha$ , we see that  $A(w, \sigma, (c + n_1)\alpha)$  has kernel 0. Then the Langlands theory says that  $U(MAN, \sigma, (c + n_1)\alpha)$  is irreducible.

Finally suppose that  $(c + n_1)\alpha$  is not a singular point and that  $n_1 > n_0$ . The previous paragraph shows that  $A(w^{-1}, w\sigma, -\nu)$  has a pole at  $\nu = (c + n_0)\alpha$ . We shall deduce from this fact that  $A(w^{-1}, w\sigma, -\nu)$  has a pole at  $\nu = (c + n_1)\alpha$ . Assuming this conclusion temporarily, we examine (3.1). The right side is regular at  $\nu = (c + n_1)\alpha$ , since  $(c + n_1)\alpha$  is not a singular point of the infinitesimal character. It follows that  $A(w, \sigma, (c + n_1)\alpha)$  has a nonzero kernel. Since  $A(w, \sigma, (c + n_1)\alpha)$  cannot be the 0 operator,  $U(MAN, \sigma, (c + n_1)\alpha)$  is reducible.

To get at the pole, it is enough to show that if  $A(w^{-1}, w\sigma, -\nu)$  has a pole at  $\nu = (c + n)\alpha$ , then it has a pole also at  $\nu = (c + n + 1)\alpha$ . We refer to pages 524 and 550 of [8]. In the notation of that paper, we expand

$$(3.2) \quad e^{(1+z)\rho H(y)} f(K(y)) = f(1) + g_1(z, y) + \cdots + g_{2n}(z, y) + R_n(z, y),$$

where  $g_j$  is entire in  $z$  and  $\alpha$ -homogeneous of degree  $j$  in  $y$ . Then a pole occurs at  $z = -\frac{j}{p+2q}$  (which corresponds to  $\nu = -\frac{1}{2}j\alpha$ ) if and only if  $\sigma^{-1}(yw)g_j(z, y)$  has mean value nonzero at  $z = -\frac{j}{p+2q}$ . When there is a pole, we can multiply  $f$  by a function on  $K$  such that the new expansion (3.2) is  $\|y\|^2$  times the old one, except for the remainder; here  $\|y\|^2$  is the Euclidean norm. Unwinding

the notation, we find that the new  $f$  gives us a pole at  $z = -\frac{j+2}{p+2q}$ , which corresponds to  $\nu = -\frac{1}{2}(j+2)\alpha$ .

#### 4. - Precise results

The treatment in Section 1 is an oversimplification in two ways. The first way is hidden by the vague description of what intertwining operator from [9] we are actually using. As we shall see, there are three slightly different kinds of operators coming into play. Moreover the multiplicative relations that the operators satisfy usually mirror multiplication only in the normalizer  $N_K(\mathcal{A})$ , not in its quotient  $W(\mathcal{A})$ .

The other way that oversimplification occurs is in the passage from operators for  $G$  to operators for a subgroup  $\widetilde{SO}$  (odd, 1). When intertwining operators for  $G$  are realized in the noncompact picture in the sense of [9], the operators for  $G$  corresponding to simple reflections are tensor products of operators for  $\widetilde{SO}$  (odd, 1) and identity operators. But it is necessary to use the compact picture in the present investigation, and the relevant formula ((7.10) of [9]) is not very helpful for deducing condition (2) in Proposition 1.1.

So let us be more precise. In this section we work exclusively with the configuration (1.1a). For any  $\mathcal{A}$  root  $\beta$ , we let  $\mathcal{G}_\beta$  be the corresponding root space, and we put

$$\mathcal{G}^{(\beta)} = \mathcal{M} \oplus \mathbb{R} H_\beta \oplus \sum_{t \neq 0} \mathcal{G}_{t\beta}.$$

Situation (1.1a) is to mean that  $G$  is connected linear, that the  $\mathcal{A}$  roots form a system of type  $(BC)_2$ , and that  $\mathcal{G}^{(e_1-e_2)}$  is the direct sum of ideals  $\mathcal{SO} \oplus \mathcal{SO}'$  with  $H_\beta \in \mathcal{SO}$  and  $\mathcal{SO} \cong \mathcal{SO}$  (odd, 1).

In this situation, Lemma 2.7 of [5] shows that  $2e_1$  and  $2e_2$  extend to real roots (relative to a Cartan subalgebra  $\mathcal{A} \oplus \mathcal{B}$  in which  $\mathcal{B}$  is a compact Cartan subalgebra of  $\mathcal{M}$ ). Then Lemma 7.8 of [5] says that the reflections  $s_{2e_1}$  and  $s_{2e_2}$  have representatives  $w_1$  and  $w_2$  in  $N_K(\mathcal{A})$  that commute with each other and centralize the identity component  $M_0$ . We fix  $w_1$  and  $w_2$  with these properties. Also we fix  $w_{12}$  in  $N_K(\mathcal{A})$  to be a representative of  $s_{e_1-e_2}$ .

**LEMMA 4.1.** *In the situation (1.1a), let  $\sigma$  be a discrete series or (nonzero) limit of discrete series representation of  $\mathcal{M}$ . Then  $w_1\sigma \cong \sigma$  and  $w_2\sigma \cong \sigma$ . Moreover  $\sigma$  extends to a representation (on the same Hilbert space) of the group generated by  $\mathcal{M}, w_1$ , and  $w_2$ .*

**PROOF.** To see that  $w_1\sigma \cong \sigma$ , we apply Lemma 10.3 of [10]. Since  $2e_1$  is a real root, Lemma 3.8b of [5] says that  $s_{2e_1}$  fixes the Harish-Chandra parameter of  $\sigma$ . The central character of  $\sigma$  has domain the group generated by the center  $Z_{M_0}$  and the elements  $\gamma_{2e_1}$  and  $\gamma_{2e_2}$  defined in (2.3) of [10], according to (1.6) and Lemma 2.1b of [10]. Since  $w_1$  centralizes these generators,  $w_1$  fixes the central character. Then Lemma 10.3 of [10] says  $w_1\sigma \cong \sigma$ . Similarly  $w_2\sigma \cong \sigma$ .

Since  $w_2\sigma \cong \sigma$ , Lemma 7.9 of [9] says that  $\sigma$  extends to a representation  $\sigma_2$  of the group  $M_2$  generated by  $M$  and  $w_2$ . Meanwhile the same lemma allows us to define  $\sigma(w_1)$  so as to extend  $\sigma$  to the group generated by  $M$  and  $w_1$ . We shall invoke the lemma a third time to extend  $\sigma_2$  to a representation of the group generated by  $M_2$  and  $w_1$ . To do so, we are to prove that  $w_1\sigma_2 \cong \sigma_2$ . Put  $E = \sigma(w_1)$ . For  $m$  in  $M$ , we have

$$(4.1a) \quad w_1\sigma_2(m) = \sigma_2(w_1^{-1}mw_1) = \sigma(w_1^{-1}mw_1) = E^{-1}\sigma(m)E = E^{-1}\sigma_2(m)E,$$

and for  $w_2$ , we have

$$\begin{aligned} (4.1b) \quad w_1\sigma_2(w_2) &= \sigma_2(w_1^{-1}w_2w_1) \\ &= \sigma_2(w_2) && \text{since } w_1w_2 = w_2w_1 \\ &= \sigma(w_2) \\ &= \sigma(w_1)^{-1}\sigma(w_2)\sigma(w_1) && \text{by an argument below} \\ &= E^{-1}\sigma_2(w_2)E. \end{aligned}$$

For (4.1b) we have used that  $\sigma(w_1)$  and  $\sigma(w_2)$  commute; this follows since  $\sigma(w_1)$  and  $\sigma(w_2)$  are in the commuting algebra of  $\sigma|_{M_0}$ , which is commutative by Lemma 4.4 of [5]. From (4.1),  $w_1\sigma_2 \cong \sigma_2$ . Thus  $\sigma$  extends to the group generated by  $M$ ,  $w_1$ , and  $w_2$ , as asserted.

Invoking Lemma 4.1, we fix an extension of  $\sigma$  and hence consistent definitions of  $\sigma(w_1)$  and  $\sigma(w_2)$ . We use unnormalized intertwining operators as follows.

If  $S = MAN$  is our parabolic subgroup and  $S' = MAN'$  is an associated parabolic and if the induced representations act by left translations, we define a formal expression by

$$(4.2) \quad A(S' : S : \sigma : \nu)f(x) = \int_{\overline{N} \cap N'} f(x\overline{n})d\overline{n},$$

where  $\overline{N}$  is opposite to  $N$ . This operator formally satisfies

$$U(S', \sigma, \nu)A(S' : S : \sigma : \nu) = A(S' : S : \sigma : \nu)U(S, \sigma, \nu)$$

and is made rigorous in [9]. For  $w$  in  $N_K(\mathcal{A})$ , let

$$(4.3) \quad R(w)f(x) = f(xw)$$

and

$$(4.4) \quad A_S(w, \sigma, \nu) = R(w)A(w^{-1}Sw : S : \sigma : \nu).$$

This operator formally satisfies

$$U(S, w\sigma, w\nu)A_S(w, \sigma, \nu) = A_S(w, \sigma, \nu)U(S, \sigma, \nu).$$

When  $\nu$  is real-valued and is in the positive Weyl chamber,

$$(4.5) \quad \sigma(w_1 w_2)A_S(w_1 w_2, \sigma, \nu)$$

is the operator that decides the unitarity of the Langlands quotient of  $U(S, \sigma, \nu)$ , according to Theorem 16.6 of [6].

LEMMA 4.2. *In situation (1.1a),*

$$\sigma(w_1 w_2)A_S(w_1 w_2, \sigma, \nu) = ABCD,$$

where

$$A^* = R(w_{12})A(w_{12}^{-1}Sw_{12} : S : \sigma : \bar{\nu})$$

$$B = (w_{12}\sigma)(w_2)A_S(w_2, w_{12}\sigma, w_{12}w_2\nu)$$

$$C = R(w_{12})A(w_{12}^{-1}Sw_{12} : S : \sigma : w_2\nu)$$

$$D = \sigma(w_2)A_S(w_2, \sigma, \nu).$$

Here  $(w_{12}\sigma)(w_2) = \sigma(w_{12}^{-1}w_2w_{12})$ , and the adjoint is taken  $K$  type by  $K$  type relative to the  $L^2(K)$  inner product for the induced space.

PROOF. The elements  $w_1$  and  $w_{12}^{-1}w_2w_{12}$  both represent  $s_{2e_1}$  in  $W(\mathcal{A})$ . Since (4.5) is independent of the representative of  $s_{2e_1}s_{2e_2}$ , (4.5) is

$$\begin{aligned} &= \sigma[(w_{12}^{-1}w_2w_{12})w_2]A_S[(w_{12}^{-1}w_2w_{12})w_2, \sigma, \nu] \\ &= [\sigma(w_{12}^{-1}w_2w_{12})A_S(w_{12}^{-1}w_2w_{12}, \sigma, \nu)]D \end{aligned}$$

by Lemma 13.1 of [9].

If we expand the factor in brackets first by (4.4), then by Theorem 7.6 of [9], and finally by Proposition 7.1 of [9], we find that it equals  $ABC$  above. It is easy to check that  $E = \sigma(w_{12}^{-1}w_2w_{12})$  satisfies

$$w_2(w_{12}\sigma) = E^{-1}(w_{12}\sigma)E$$

and that

$$E^2 = (w_{12}\sigma)(w_2^2),$$

and then Lemma 7.9 of [9] says we may take  $(w_{12}\sigma)(w_2) = E$ . This proves the lemma.

LEMMA 4.3. *In situation (1.1a), the operators  $B$  and  $D$  of Lemma 4.2 are Hermitian (on each  $K$  type).*

PROOF. Proposition 7.10 of [9] gives

$$[\sigma(w_2)A_S(w_2, \sigma, \nu)]^* = \sigma(w_2^{-1})A_S(w_2^{-1}, \sigma, -w_2\bar{\nu}).$$

Since  $w_2$  and  $w_2^{-1}$  both represent  $s_{2e_2}$ , the right side is

$$\begin{aligned} &= \sigma(w_2)A_S(w_2, \sigma, -w_2\bar{\nu}) \\ (4.6) \quad &= \sigma(w_2)R(w_2)A(s_{2e_2}^{-1}Ss_{2e_2} : S : \sigma : w_1\bar{\nu}). \end{aligned}$$

In a suitable open subset of  $(\mathcal{A}')^{\mathbb{C}}$ , Theorem 6.6 of [9] makes sense of (4.2) as

$$(4.7) \quad A(s_{2e_2}^{-1}Ss_{2e_2} : S : \sigma : \nu)f(k) = \int_{N_{-2e_2}} e^{-(\nu+\rho)H(\bar{n})} \sigma[\mu(\bar{n})]^{-1} f[kK(\bar{n})] d\bar{n}.$$

Here  $\rho$  is half the sum of the positive  $\mathcal{A}$  roots (with multiplicities) and  $\bar{n} = K(\bar{n})\mu(\bar{n})e^{H(\bar{n})}n$  is a decomposition of  $\bar{n}$  according to  $KMAN$ . In (4.7),  $H(\bar{n})$  lies in  $\mathbb{R}H_{2e_2}$ , and (4.7) is unchanged if we replace  $\nu$  by  $w_1\nu$ . Thus (4.6) on an open set of  $(\mathcal{A}')^{\mathbb{C}}$  is

$$= \sigma(w_2)A_S(w_2, \sigma, \bar{\nu}),$$

and this equality must extend to other  $\nu$ 's by analytic continuation. Hence  $D$  is Hermitian. A similar argument proves  $B$  is Hermitian.

The above discussion handles the first sense in which Section 1 oversimplifies matters. To handle the second sense, we rewrite the intertwining operators using Frobenius reciprocity as in Section 5 of [7]. Let  $\tau$  be an irreducible representation of  $K$  acting in a space  $V^\tau$ , let  $V^\sigma$  be the space on which  $\sigma$  operates, and let  $V_\sigma^\tau$  be the subspace of the induced space transforming under  $K$  according to  $\tau$ . If  $v$  is in  $V^\tau$  and  $E$  is in  $Hom_{K \cap M}(V^\tau, V^\sigma)$ , we define

$$\varphi_{v,E}(k) = E(\tau(k)^{-1}v), \quad k \in K.$$

Then  $\varphi_{v,E}$  is in  $V_\sigma^\tau$ , and the resulting bilinear mapping on pairs  $(v, E)$  gives us a linear map

$$(4.8) \quad \Phi : V^\tau \otimes_{\mathbb{C}} Hom_{K \cap M}(V^\tau, V^\sigma) \rightarrow V_\sigma^\tau$$

that is known to be a linear isomorphism onto. (See Wallach [13], p. 270).

The space  $V^\sigma$  comes equipped with an inner product  $\langle \cdot, \cdot \rangle_\sigma$ . Let  $\langle \cdot, \cdot \rangle_\tau$  be a  $K$ -invariant inner product on  $V^\tau$ . If  $E$  is in  $Hom_{K \cap M}(V^\sigma, V^\tau)$ , we then have a well defined adjoint  $E^*$  in  $Hom_{K \cap M}(V^\sigma, V^\tau)$ , and we define an inner product on  $Hom_{K \cap M}(V^\tau, V^\sigma)$  by  $\langle E', E \rangle = d_\tau^{-1} \text{Tr}(E^* E')$ , where  $d_\tau$  is the degree of  $\tau$ . The inner products on  $V^\tau$  and  $Hom_{K \cap M}(V^\tau, V^\sigma)$  yield a canonical inner product on  $V^\tau \otimes_{\mathbb{C}} Hom_{K \cap M}(V^\tau, V^\sigma)$ .

LEMMA 4.4. *The mapping  $\Phi$  of (4.8) respects inner products.*

PROOF. Let  $\{v_i\}$  be an orthonormal basis of  $V^\tau$ . Then

$$\begin{aligned} \langle \varphi_{v', E'}, \varphi_{v, E} \rangle_{L^2(K, V^\sigma)} &= \int_K \langle E' \tau(k)^{-1} v', E \tau(k)^{-1} v \rangle_\sigma \, dk \\ &= \int_K \langle (E^* E') \tau(k)^{-1} v', \tau(k)^{-1} v \rangle_\tau \, dk \\ &= \sum_{i,j} \int_K \langle (E^* E') v_i, v_j \rangle_\tau \langle \tau(k)^{-1} v', v_i \rangle_\tau \overline{\langle \tau(k)^{-1} v, v_j \rangle_\tau} \, dk \\ &= \sum_{i,j} d_\tau^{-1} \langle (E^* E') v_i, v_j \rangle_\tau \langle v', v \rangle_\tau \overline{\langle v_i, v_j \rangle_\tau} \end{aligned}$$

by Schur orthogonality

$$\begin{aligned} &= d_\tau^{-1} \text{Tr}(E^* E) \langle v', v \rangle_\tau \\ &= \langle v' \otimes E', v \otimes E \rangle \end{aligned}$$

as required.

The next lemma moves our intertwining operators from the image of  $\Phi$  to the domain. If the parabolic subgroup  $S = MAN$  is minimal, the formula simplifies to the one in Wallach [13, p. 270].

LEMMA 4.5. *If  $\beta$  is an  $S$ -simple  $\mathcal{A}$  root, then the map  $\Phi$  in (4.8) satisfies*

$$(4.9a) \quad \Phi^{-1} A(s_\beta^{-1} S s_\beta : S : \sigma : \nu) \Phi = I \otimes a(\tau : \sigma : \beta : \nu),$$

where  $a(\tau : \sigma : \beta : \nu)$  is analytically continued from

$$(4.9b) \quad a(\tau : \sigma : \beta : \nu)(E) = \int_{\overline{N}_\beta} e^{-(\nu + \rho_\beta)H(\bar{n})} \sigma(\mu(\bar{n}))^{-1} E \tau(K(\bar{n}))^{-1} d\bar{n}.$$

Here  $\rho_\beta$  is half the sum of the  $\mathcal{A}$  roots  $t\beta$  with  $t > 0$  (counting multiplicities) and  $g = K(g)\mu(g)e^{H(g)}n$  is a decomposition of  $g$  according to  $G = KMAN$ .

REMARK. The operator  $a(\tau : \sigma : \beta : \nu)$  is in

$$\text{End}_{\mathbb{C}}(\text{Hom}_{K \cap M}(V^\tau, V^\sigma)).$$

Identity (4.9b) can be written more concretely as

$$A(s_\beta^{-1} S s_\beta : S : \sigma : \nu) \varphi_{v, E} = \varphi_{v, a(\tau : \sigma : \beta : \nu)(E)}.$$

The proof of the lemma is a routine calculation starting from (4.2) with  $f = \varphi_{v,E}$  and taking into account that  $\rho = \rho_\beta$  on  $\mathbb{R}H_\beta$  (see Proposition 1.2 of [9]).

Next we relate the operators  $A, B, C, D$  of Lemma 4.2 to operators in a parabolic-rank-one setting. For each  $A$  root  $\beta$ , let  $G_0^{(\beta)}$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{g}^{(\beta)}$ , and let  $G^{(\beta)} = G_0^{(\beta)}M$ . If  $\beta$  is  $S$ -positive, put  $S^{(\beta)} = S \cap G^{(\beta)}$ , and form the induced representations

$$U^{(\beta)}(S^{(\beta)}, \sigma, \nu) = \text{ind}_{S^{(\beta)}}^{G^{(\beta)}} (\sigma \otimes \exp(\nu|_{\mathbb{R}H_\beta}) \otimes 1).$$

These representations depend on  $\nu$  only through  $\nu|_{\mathbb{R}H_\beta}$ .

Let us study the intertwining operator  $D$  of Lemma 4.2 given by

$$D = \sigma(w_2)A_S(w_2, \sigma, \nu) \quad \text{for } G.$$

Let  $D_2$  be the corresponding operator for  $G^{(2e_2)}$  given by

$$D_2 = \sigma(w_2)A_{S^{(2e_2)}}(w_2, \sigma, \nu) \quad \text{for } G^{(2e_2)}.$$

Fix an irreducible representation  $\tau$  of  $K$ . Under the correspondence  $\Phi$  in (4.8), Lemma 4.5 and arguments in [7] give us

$$D \text{ on } V^\tau \longleftrightarrow [E \rightarrow \sigma(w_2)a(\tau : \sigma : 2e_2 : \nu)(E)\tau(w_2)^{-1}];$$

we have dropped from the notation the tensor product with the identity. Put  $K_2 = K \cap G^{(2e_2)}$ , and let  $\tau|_{K_2}$  decompose into irreducible representations as  $\tau|_{K_2} = \sum \oplus \tau_j$ . Let  $V_j^\tau$  be the subspace of  $V^\tau$  in which  $\tau_j$  operates, and let  $i_j : V^{\tau_j} \rightarrow V^\tau$  be the  $j^{\text{th}}$  injection. For each  $\tau_j$  we have, in obvious notation,

$$(4.10) \quad D_2 \text{ on } V^{\tau_j} \longleftrightarrow [E_2 \rightarrow \sigma(w_2)a_2(\tau_j : \sigma : 2e_2 : \nu)(E_2)\tau_j(w_2)^{-1}].$$

LEMMA 4.6. *If  $\nu|_{\mathbb{R}H_{2e_2}}$  is real and is in the positive Weyl chamber for  $G^{(2e_2)}$  and is such that the Langlands quotient of  $U^{(2e_2)}(S^{(2e_2)}, \sigma, \nu)$  is unitary, then the operator  $D$  of Lemma 4.2 is semidefinite. Similarly if  $w_2\nu|_{\mathbb{R}H_{2e_1}}$  is real and is in the positive Weyl chamber for  $G^{(2e_1)}$  and is such that the Langlands quotient of  $U^{(2e_1)}(S^{(2e_1)}, \sigma, w_2\nu)$  is unitary, then the operator  $B$  of Lemma 4.2 is semidefinite.*

PROOF FOR  $D$ . The assumed unitarity means that  $D_2$  is semidefinite, say positive semidefinite. By Lemma 4.4 the operators on the right side of (4.10) are positive semidefinite for all  $j$ . Let  $E$  be given in  $\text{Hom}_{K \cap M}(V^\tau, B^\sigma)$ , and put  $E_j = E \circ i_j$ . Then  $E_j$  is in  $\text{Hom}_{K_2 \cap M}(V^{\tau_j}, V^\sigma)$ , and the given positivity means that

$$(\sigma(w_2)a_2(\tau_j : \sigma : 2e_2 : \nu)(E_j)\tau_j(w_2)^{-1}, E_j) \geq 0,$$

i.e.,

$$(4.11) \quad \text{Tr} [E_j^* \sigma(w_2) a_2(\tau_j : \sigma : 2e_2 : \nu) (E_j) \tau_j(w_2)^{-1}] \geq 0.$$

If  $j \neq k$ , then

$$(4.12) \quad \text{Tr} [E_k^* \sigma(w_2) a_2(\tau_j : \sigma : 2e_2 : \nu) (E_j) \tau_j(w_2)^{-1}] = 0,$$

since the operator in question maps  $V^{\tau_j}$  to  $V^{\tau_k}$ . Summing (4.11) on  $j$ , taking (4.12) into account, and sorting matters out, we obtain

$$\text{Tr} [E^* \sigma(w_2) a(\tau : \sigma : 2e_2 : \nu) (E) \tau(w_2)^{-1}] \geq 0.$$

By Lemma 4.4,  $D$  is positive semidefinite.

PROOF FOR  $B$ . The assumptions imply that  $w_{12}w_2\nu|_{\mathbb{R}H_{2e_2}}$  is real, is in the positive Weyl chamber for  $G^{(2e_2)}$ , and is such that  $U^{(2e_2)}(S^{(2e_2)}, w_{12}\sigma, w_{12}w_2\nu)$  is unitary. Then the proof for  $D$  shows that  $B$  is semidefinite.

Next let us study the operators  $A$  and  $C$  of Lemma 4.2. The condition we seek for Proposition 1.1 is that there must be a positive semidefinite  $P$  with  $A = PC^*$ , i.e.,  $A^* = CP$ . Taking the formulas of Lemma 4.2 into account, we are looking for circumstances when

$$A(w_{12}^{-1}Sw_{12} : S : \sigma : \nu) = A(w_{12}^{-1}Sw_{12} : S : \sigma : w_2\nu)P$$

with  $\nu$  real-valued. The operators here all carry  $V_\sigma^\tau$  into itself if  $\tau$  is a  $K$  type of  $U(S, \sigma, \nu)$ . Translating this equation by  $\Phi$  in (4.8) and applying Lemma 4.5, we see that we are seeking an operator  $d(\nu)$  carrying  $\text{Hom}_{K \cap M}(V^\tau, V^\sigma)$  into itself such that

$$(4.13) \quad a(\tau : \sigma : e_1 - e_2 : \nu) = a(\tau : \sigma : e_1 - e_2 : w_2\nu)d(\nu).$$

Let  $\alpha = e_1 - e_2$ . First we study equation (4.13) in  $G^{(\alpha)}$ . Let  $K^{(\alpha)} = K \cap G^{(\alpha)}$ , etc., and let  $\tau_0$  be a  $K^{(\alpha)}$  type of  $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$ . Recall the decomposition  $\mathcal{G}^{(\alpha)} = \mathcal{SO} \oplus \mathcal{SO}'$ , with  $\mathcal{SO}$  and  $\mathcal{SO}'$  ideals in  $\mathcal{G}^{(\alpha)}$ ; here  $\mathcal{SO} \cong \mathcal{SO}$  (odd, 1). Since  $\mathcal{SO}' \subseteq \mathcal{M} \subseteq \mathcal{G}^{(\alpha)}$ , we have

$$\mathcal{M} = (\mathcal{M} \cap \mathcal{SO}) \oplus \mathcal{SO}'.$$

On the level of connected groups, we write correspondingly

$$(4.14) \quad M_0 = M^{SO} M^{SO'}$$

as a commuting product, possibly not direct. The group  $M$  normalizes each factor of (4.14) since  $M$  normalizes  $\mathcal{G}_\alpha$  and  $\mathcal{G}_{-\alpha}$ , which generate  $\mathcal{M} \cap \mathcal{SO}$ . Theorem 1.1 of [10] allows us to write

$$(4.15) \quad \sigma|_{M_0} = \sigma_1 \oplus \cdots \oplus \sigma_L$$

with each  $\sigma_j$  irreducible on  $M_0$  and the various  $\sigma_j$  inequivalent. Since  $\sigma$  is irreducible and (4.15) is the primary decomposition of  $\sigma|_{M_0}$ , it follows that

$$(4.16) \quad \sigma_j \cong m_j \sigma_1 \quad \text{for some } m_j \in M.$$

Meanwhile the irreducibility of  $\sigma_j$  and the compactness of  $M_0^{SO}$  imply

$$(4.17) \quad \sigma_j \cong \sigma_j^{SO} \otimes \sigma_j^{SO'}$$

with  $\sigma_j^{SO}$  and  $\sigma_j^{SO'}$  irreducible on  $M^{SO}$  and  $M^{SO'}$ , respectively. Since the element  $m_j$  in (4.16) normalizes  $M^{SO}$  and  $M^{SO'}$ , (4.17) gives

$$(4.18) \quad \sigma_j^{SO} \cong m_j \sigma_1^{SO} \quad \text{and} \quad \sigma_j^{SO'} \cong m_j \sigma_1^{SO'}.$$

LEMMA 4.7. *Up to inner automorphisms, the only nontrivial outer automorphism of  $\mathcal{SO}$  (odd, 1) is the Cartan involution.*

PROOF. Let  $\varphi$  be an automorphism of  $\mathcal{SO} = \mathcal{SO}$  (odd, 1), and let  $\mathcal{SO} = \mathcal{T} \oplus \mathcal{P}$  be a Cartan decomposition. Since any two  $\mathcal{T}$ 's are conjugate, we may assume  $\varphi(\mathcal{T}) = \mathcal{T}$ . If  $B$  is the Killing form, it follows that  $\varphi$  carries the orthocomplement (with respect to  $B$ ) of  $\mathcal{T}$  into itself. Thus  $\varphi(\mathcal{P}) = \mathcal{P}$ . Let  $B_\theta(X, Y) = -B(X, \theta Y)$ , where  $\theta$  is the Cartan involution. Since  $\varphi$  leaves  $\mathcal{T}$  and  $\mathcal{P}$  stable,  $\varphi$  is an isometry of  $\mathcal{SO}$  with respect to  $B_\theta$ . In particular,  $\varphi|_{\mathcal{P}}$  is in the orthogonal group  $O(\mathcal{P})$ . Since  $\mathcal{SO}$  is  $\mathcal{SO}$  (odd, 1),  $\theta$  has determinant  $-1$  on  $\mathcal{P}$ . Possibly composing  $\varphi$  with  $\theta$ , we may assume  $\varphi|_{\mathcal{P}}$  is in  $SO(\mathcal{P})$ . Since  $\text{Ad}(K)|_{\mathcal{P}} = SO(\mathcal{P})$ , we can choose  $k \in K$  with  $\text{Ad}(k)|_{\mathcal{P}} = \varphi|_{\mathcal{P}}$ . Then  $\text{Ad}(k)^{-1}\varphi$  is 1 on  $\mathcal{P}$  and maps  $\mathcal{T}$  to itself. Since  $[\mathcal{P}, \mathcal{P}] = \mathcal{T}$ ,  $\text{Ad}(k)^{-1}\varphi = 1$  on  $\mathcal{SO}$ . Thus  $\varphi = \text{Ad}(k)$ .

In terms of (4.18), Lemma 4.7 implies that each  $\sigma_j^{SO}$  is equivalent with  $\sigma_1^{SO}$  or with its Weyl group transform  $s_\alpha \sigma_1^{SO}$ . Thus the space  $V^\sigma$  on which  $\sigma$  operates splits as

$$(4.19) \quad V^\sigma = (V_1 \otimes V'_1) \oplus (V_2 \otimes V'_2),$$

with  $M^{SO}$  operating on  $V_1$  and  $V_2$  by  $\sigma_1^{SO}$  and  $s_\alpha \sigma_1^{SO}$ , respectively, and with  $M^{SO'}$  operating in  $V'_1$  and  $V'_2$ . (When  $s_\alpha \sigma_1^{SO} \cong \sigma_1^{SO}$ , we just write  $V^\sigma = V_1 \otimes V'_1$ ).

Next consider the irreducible  $K^{(\alpha)}$  type  $\tau_0$ . Intersecting  $\mathcal{G}^{(\alpha)} = \mathcal{SO} \oplus \mathcal{SO}'$  with  $\mathcal{T}^{(\alpha)}$  gives us

$$\mathcal{T}^{(\alpha)} = \mathcal{T}^{SO} \oplus \mathcal{T}^{SO'}.$$

On the level of connected groups, we have a commuting product decomposition

$$(4.20) \quad K_0^{(\alpha)} = K^{SO} K^{SO'},$$

possibly not direct. The group  $K^{(\alpha)}$  normalizes each factor of (4.20) since  $K^{(\alpha)}$  is generated by  $K^{(\alpha)} \cap M$  and  $K_0^{(\alpha)}$ . We write

$$(4.21) \quad \tau_0|_{K_0^{(\alpha)}} = \sum n_j \tau_{0,j}$$

with the  $\tau_{0,j}$  irreducible and inequivalent, and (4.20) gives us

$$\tau_{0,j} = \tau_{0,j}^{SO} \otimes \tau_{0,j}^{SO'}.$$

Since  $\tau_0$  is irreducible and (4.21) is a primary decomposition, we obtain

$$\tau_{0,j} \cong k_j \tau_{0,1} \quad \text{for some} \quad k_j \in K^{(\alpha)}.$$

Since  $k_j$  normalizes each factor of (4.20), we conclude

$$(4.22) \quad \tau_{0,j}^{SO} \cong k_j \tau_{0,1}^{SO} \quad \text{and} \quad \tau_{0,j}^{SO'} \cong k_j \tau_{0,1}^{SO'}.$$

LEMMA 4.8. *Every automorphism of  $\mathcal{SO}$  (odd) is inner.*

PROOF. The argument is in the same style as Lemma 4.7. Write  $\mathcal{SO}(2n+1) = \mathcal{SO}(2n) \oplus i\mathcal{P}$  with  $i\mathcal{P}$  built from the last row and column. Given an automorphism  $\varphi$ , we can compose with an inner automorphism so that  $\varphi$  leaves  $\mathcal{SO}(2n)$  and  $i\mathcal{P}$  stable. Moreover  $\varphi$  will act as an orthogonal transformation of  $i\mathcal{P}$ . The matrix  $\text{diag}(1, \dots, 1, -1, -1)$  gives an inner automorphism on  $\mathcal{SO}(2n+1)$  and acts on  $i\mathcal{P}$  with determinant  $-1$  since  $\dim \mathcal{P}$  is even. If  $\varphi$  has determinant  $-1$  on  $i\mathcal{P}$ , we can compose with this inner automorphism and make it have determinant  $+1$ . Once  $\varphi$  acts as a rotation on  $i\mathcal{P}$ ,  $\varphi$  has to be inner by the same argument as in Lemma 4.8.

In terms of (4.22), Lemma 4.8 implies that each  $\tau_{0,j}^{SO}$  is equivalent with  $\tau_{0,1}^{SO}$ . Thus the space  $V^{\tau_0}$  on which  $\tau_0$  operates splits as

$$(4.23) \quad V^{\tau_0} \cong V_0 \otimes V'_0$$

with  $K^{SO}$  operating by  $\tau_{0,1}^{SO}$  in  $V_0$  and with  $K^{SO'}$  operating in  $V'_0$ .

LEMMA 4.9. *Within  $G^{(\alpha)}$ , let  $\tau_0$  be a  $K^{(\alpha)}$  type of  $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$ . Then the endomorphism  $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$  of  $\text{Hom}_{K^{(\alpha)}}(\widetilde{V^{\tau_0}}, V^\sigma)$  is scalar, and the scalar is obtained as follows: Within the group  $\widetilde{\mathcal{SO}}(\text{odd}, 1)$ , let  $\sigma_1^{SO}$  be the representation of  $M^{SO}$  in (4.17), and let  $\tau_{0,1}^{SO}$  be the representation of  $K^{SO}$  in (4.22). Then the scalar is the scalar by which the operator  $A(s_\alpha^{-1} S^{SO} s_\alpha : S^{SO} : \sigma_1^{SO} : \nu)$  operates on the  $K^{SO}$  type  $\tau_{0,1}^{SO}$ . (Recall property (i) in Section 1.)*

REMARK. Note that  $\sigma_1^{SO}$  is determined only up to the operation of the Weyl group reflection  $s_\alpha$ .

PROOF. For  $E$  in  $Hom_{K^{(\alpha)}}(V^{\tau_0}, V^{\sigma})$ , we compute  $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$  from (4.9b). The group  $\overline{N}_\alpha$  is contained in  $K_0^{(\alpha)} A^{(\alpha)} N^{(\alpha)}$  (with no need for  $M$ ), and so  $\sigma(\mu(\overline{n}))$  can be taken to be 1. Thus

$$(4.24) \quad a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu) = \text{right by } \int_{\overline{N}_\alpha} e^{-(\nu + \rho_\alpha)H(\overline{n})} \tau_0(K(\overline{n}))^{-1} d\overline{n}.$$

By (4.23), this expression is conjugate (by a transformation of  $V^{\tau_0}$ ) to

$$\text{right by } \left[ \int_{\overline{N}_\alpha} e^{-(\nu + \rho_\alpha)H(\overline{n})} \tau_{0,1}^{SO}(K(\overline{n}))^{-1} d\overline{n} \otimes 1 \right].$$

The integral here commutes with  $\tau_{0,1}^{SO}(M^{SO})$ . Since the  $M^{SO}$  types have multiplicity one, the integral is diagonal and is scalar on each  $M^{SO}$  type. On the other hand,  $E$  is in  $Hom_{K_0^{(\alpha)}}(V^{\tau_0}, V^{\sigma})$  and is 0 on the  $M^{SO}$  types that are incompatible with  $\sigma$ . By (4.19), the operator acts on the direct sum of two spaces as a scalar in each, the two spaces corresponding to  $\sigma_1^{SO}$  and  $s_\alpha \sigma_1^{SO}$ . The scalar in each case is the value of the  $\widetilde{SO}$  (odd, 1) intertwining operator on the  $K^{SO}$  type  $\tau_{0,1}^{SO}$ , by Lemma 4.5. But these two scalars are the same, as it is seen from the bottom display on p. 22 of [9]. This proves the lemma.

COROLLARY 4.10. *Within  $G^{(\alpha)}$ , let  $\tau_0, \tau_{0,1}^{SO}$  and  $\sigma_1^{SO}$  be as in Lemma 4.9. Suppose that  $\nu = t_0 \alpha$ , with  $t_0 > 0$ , is a reducibility point for  $U^{SO}(S^{SO}, \sigma_1^{SO}, t_0 \alpha)$ . Then there is a meromorphic complex-valued function  $d_0(z)$  with the following properties:*

- (i)  $a^{(\alpha)}(\tau_0 : \sigma : \alpha : t_0 \alpha) = d_0(z) a^{(\alpha)}(\tau_0 : \sigma : \alpha : z \alpha)$
- (ii)  $d_0(t)$  takes values in  $\mathbb{R} \cup \{\infty\}$  for  $t$  real
- (iii)  $d_0(t)$  is finite and  $\geq 0$  for those  $t > t_0$  such that there do not exist  $t_1$  and  $t_2$ , with  $t_0 < t_1 < t_2 < t$ , for which  $U^{SO}(S^{SO}, \sigma_1^{SO}, t_1 \alpha)$  has singular infinitesimal character and  $U^{SO}(S^{SO}, \sigma_1^{SO}, t_2 \alpha)$  is reducible.

PROOF. This is immediate from Lemma 4.9 and the properties of  $\widetilde{SO}$  (odd, 1) listed in Section 1.

Corollary 4.10 addresses equation (4.13) in the group  $G^{(\alpha)}$ . We return to equation (4.13) in  $G$ . Let  $\tau|_{K^{(\alpha)}} = \sum \tau_j$  be a decomposition into irreducibles, let  $\tau_j$  act on  $V^{\tau_j}$ , and let  $i_j$  and  $p_j$  be the  $j^{th}$  injection and  $j^{th}$  projection for the direct sum decomposition  $V^\tau = \sum V^{\tau_j}$ . The functional  $\nu$  varies over a two-parameter family, and we freeze one of the parameters by insisting that the projection of  $\nu$ , in the direction of  $\alpha$ , be  $t_0 \alpha$ , with  $t_0$  as in Corollary 4.10. Let  $d_j$  be the function provided by Corollary 4.10 for the  $K^{(\alpha)}$  type  $\tau_j$ , and write, in obvious notation,  $d_j(\nu)$  in place of  $d_j(z)$ . Then the corollary gives us

$$a^{(\alpha)}(\tau_j : \sigma : \alpha : \nu) = a^{(\alpha)}(\tau_j : \sigma : \alpha : w_2 \nu) d_j(w_2 \nu).$$

For  $E$  in  $\text{Hom}_{K \cap M}(V^\tau, V^\sigma)$ ,  $E \circ i_j$  is in  $\text{Hom}_{K^{(\alpha)} \cap M}(V^{\tau_j}, V^\sigma)$ . Then we have

$$a(\tau : \sigma : \alpha : \nu)(E) = \sum_j a^{(\alpha)}(\tau_j : \sigma : \alpha : \nu)(E \circ i_j) \circ p_j.$$

Define  $\tilde{d}(\nu)$  in  $\text{Hom}_{\mathbb{C}}(V^\tau, V^\tau)$  by

$$\tilde{d}(\nu) = \sum_j i_j d_j(w_2 \nu) p_j.$$

Then we find

$$\begin{aligned} & a(\tau : \sigma : \alpha : w_2 \nu)^{-1} a(\tau : \sigma : \alpha : \nu)(E) \\ &= \sum_{j,k} a^{(\alpha)}(\tau_j : \sigma : \alpha : w_2 \nu)^{-1} \left( [a^{(\alpha)}(\tau_k : \sigma : \alpha : \nu)(E \circ i_k) \circ p_k] \circ i_j \right) \circ p_j \\ &= \sum_j a^{(\alpha)}(\tau_j : \sigma : \alpha : w_2 \nu)^{-1} a^{(\alpha)}(\tau_j : \sigma : \alpha : \nu)(E \circ i_j) \circ p_j \\ &= \sum_j d_j(w_2 \nu)(E \circ i_j) \circ p_j \\ &= (\text{right by } \tilde{d}(\nu))(E). \end{aligned}$$

Consequently  $d(\nu) = (\text{right by } \tilde{d}(\nu))$  is in  $\text{Hom}_{K \cap M}(V^\tau, V^\sigma)$  and satisfies (4.13). Evidently  $d(\nu)$  is Hermitian when it is finite, and (iii) in Corollary 4.10 tells us conditions under which it is positive semidefinite. We summarize as follows.

**THEOREM 4.11.** *In situation (1.1a), as defined at the beginning of Section 4, let  $\sigma$  be a discrete series or (nonzero) limit of discrete series representation, and let  $\sigma_1^{SO}$  be associated to  $\sigma$  and  $\widetilde{SO}$  (odd, 1) by (4.17). With  $\nu$  in the closed positive Weyl chamber, write  $\nu = ae_1 + be_2$ , and assume that  $U(S, \sigma, 0)$  is irreducible in  $G$  and*

$$U^{SO} \left( S^{SO}, \sigma_1^{SO}, \frac{1}{2}(a-b)(e_1 - e_2) \right)$$

*is reducible in  $\widetilde{SO}$  (odd, 1). Further assume that  $\nu$  has the property that there do not exist  $t_1$  and  $t_2$ , with  $\frac{1}{2}(a-b) < t_1 < t_2 < \frac{1}{2}(a+b)$ , such that  $U^{SO}(S^{SO}, \sigma_1^{SO}, t_1(e_1 - e_2))$  has singular infinitesimal character and  $U^{SO}(S^{SO}, \sigma_1^{SO}, t_2(e_1 - e_2))$  is reducible. Then the operators  $A$  and  $C$  of Lemma 4.2 satisfy  $A = PC^*$  with  $P$  positive semidefinite. If, in addition, the Langlands quotients of  $U^{(2e_2)}(S^{(2e_2)}, \sigma, be_2)$  and  $U^{(2e_1)}(S^{(2e_1)}, \sigma, ae_1)$  are unitary, then the Langlands quotient of  $U(S, \sigma, \nu)$  is unitary.*

**PROOF.** The assertions about  $A$  and  $C$  have just been proved above in the setting of the domain of  $\Phi$ . Lemma 4.4 allows us to transfer them to the

induced space. The hypotheses about  $U^{(2e_2)}$  and  $U^{(2e_1)}$  imply that  $B$  and  $D$  in Lemma 4.2 are semidefinite, according to Lemma 4.6. By Proposition 1.1,  $ABCD$  is semidefinite. But Theorem 16.6 of [6] and Lemma 4.2 show that  $ABCD$  is the operator that decides the unitarity of  $U(S, \sigma, \nu)$ .

## 5. - Configuration (1.1b)

Previous sections have dealt largely with configuration (1.1a) for the roots of  $(\mathcal{G}, \mathcal{A})$ . The above arguments need some changes - and some additional hypotheses - in the case of (1.1b). In (1.1b),  $SI(2, \mathbb{R})$  replaces  $SO$  (odd, 1). The list of properties of intertwining operators and nonunitary principal series for  $SL(2, \mathbb{R})$  is almost the same as for  $SO$  (odd, 1); this time the properties are more elementary.

- (i) The  $K$  types have multiplicity one.
- (ii) The total operator has a kernel at  $\nu$  if and only if  $(0, 1)$  is reducible at  $\nu$ .
- (iii) The representation  $\sigma$  is one-dimensional, and we write it as  $I$  or  $\text{sgn}$ . If  $\sigma = I$ , then reducibility occurs at odd multiples of the restricted root  $\alpha$ . If  $\sigma = \text{sgn}$ , then reducibility occurs at even multiples of  $\alpha$ .
- (iv) The infinitesimal character is singular only at  $\nu = 0$ .
- (v) On any interval in the positive Weyl chamber, the kernel of the intertwining operator at reducibility points decreases as one moves to the right.

Lemma 4.1 does not extend to (1.1b) without modification. For one thing,  $e_1$  and  $e_2$  need not be real roots, and for another thing,  $e_1 - e_2$  and  $e_1 + e_2$  have become real roots. Thus we shall incorporate the conclusion of Lemma 4.1 into our hypotheses.

Then everything through Lemma 4.6 works without further change. Again we study (4.13). Taking  $\alpha = e_1 - e_2$ , we first study equation (4.13) in  $G^{(\alpha)}$ . Since  $\alpha$  is real, the element  $\gamma_\alpha$  lies in the center of  $M$ . Thus  $\sigma(\gamma_\alpha)$  is scalar, necessarily  $\pm 1$ . Since  $\{1, \gamma_\alpha\}$  is the  $M$  group for the  $SL(2, \mathbb{R})$  subgroup,  $\sigma$  yields a well defined one-dimensional representation  $\sigma^{SL}$  of  $\{1, \gamma_\alpha\}$  by restriction. We do not need Lemma 4.7.

Next we trace through what happens to the  $K^{(\alpha)}$  type  $\tau_0$ . On the level of connected groups, we have a commuting product decomposition  $K_0^{(\alpha)} = K^{SL} K^{SL'}$  in obvious notation. The group  $K^{SL}$  is a circle, with one-dimensional irreducible representations parametrized by integers  $N$ . An outer automorphism of  $K^{SL}$  interchanges  $N$  and  $-N$ . Now the  $SL(2, \mathbb{R})$  intertwining operators are the same for  $N$  as for  $-N$ , by (5.8) of [7], for example. So we get an analog of Lemma 4.9 as follows.

**LEMMA 5.1.** *Within  $G^{(\alpha)}$ , let  $\tau_0$  be a  $K^{(\alpha)}$  type of  $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$ . Then the endomorphism  $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$  of  $\text{Hom}_{K^{(\alpha)} \cap M}(V^{\tau_0}, V^\sigma)$  is scalar, and the scalar is obtained as follows: Within the group  $SL(2, \mathbb{R})$ , let  $\sigma^{SL}$  be the*

representation of  $\{1, \gamma_\alpha\}$  obtained by restriction from  $\sigma$ , and let  $N$  or  $(N, -N)$  be the parameter(s) of the representation(s) of  $K^{SL}$  obtained from  $\tau_0|_{K_0^{(\alpha)}}$ . Then the scalar is the scalar by which the operator  $A(s_\alpha^{-1} S^{SL} s_\alpha : S^{SL} : \sigma^{SL} : \nu)$  operates on the  $K^{SL}$  type(s) with parameter(s)  $N$  or  $(N, -N)$ .

Consequently we obtain a function  $d_0(z)$  just as in Corollary 4.10. Property (iii) simplifies in the corollary to say that  $d_0(t)$  is finite and  $\geq 0$  for  $t > t_0$ , since singular infinitesimal character cannot occur in  $SL(2, \mathbb{R})$  for  $t > 0$ .

The remainder of the argument requires no change, and we arrive at the following result.

**THEOREM 5.2.** *In situation (1.1b), let  $\sigma$  be a discrete series or (nonzero) limit of discrete series representation of  $M$ . Let  $w_1$  and  $w_2$  be representatives in  $N_K(\mathcal{A})$  of  $s_{e_1}$  and  $s_{e_2}$ , and assume that  $w_1\sigma \cong \sigma \cong w_2\sigma$  in such a way that  $\sigma$  extends from  $M$  to the group generated by  $M$ ,  $w_1$ , and  $w_2$ . Let  $\sigma^{SL}$  be the restriction of  $\sigma$  to  $\{1, \gamma_{e_1-e_2}\}$ . With  $\nu$  in the closed positive Weyl chamber, write  $\nu = ae_1 + be_2$ , and assume that  $U(S, \sigma, 0)$  is irreducible in  $G$  and*

$$U^{SL} \left( S^{SL}, \sigma^{SL}, \frac{1}{2}(a-b)(e_1 - e_2) \right)$$

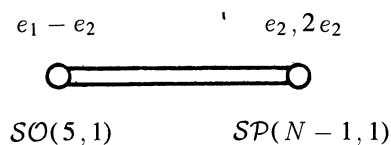
*is reducible in  $SL(2, \mathbb{R})$ . Then the operators  $A$  and  $C$  of Lemma 4.2 satisfy  $A = PC^*$  with  $P$  positive semidefinite. If, in addition, the Langlands quotients of  $U^{(e_2)}(S^{(e_2)}, \sigma, be_2)$  and  $U^{(e_1)}(S^{(e_1)}, \sigma, ae_1)$  are unitary, then the Langlands quotient of  $U(S, \sigma, \nu)$  is unitary.*

## 6. - Examples

1)  $SU(N, 2)$ . The diagram of roots of  $(\mathcal{G}, \mathcal{A})$  appears at the beginning of Section 1, and a qualitative picture of unitarity appears in Figure 1. There is reducibility along the diagonal lines below the diagonal, and Theorem 4.11 addresses what happens along the upward-sloping diagonals. At the end of Section 1, we observed that the numbers  $t_1$  and  $t_2$  cannot exist in the theorem, and thus  $A = PC^*$  all the way along the upward-sloping diagonals. The theorem goes on to say that we get unitarity at a point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points in subgroups that are essentially  $SU(N-1, 1)$ .

2)  $\widetilde{SO}(N, 2)$ . The diagram of roots of  $(\mathcal{G}, \mathcal{A})$  is as in (1.1b) with  $m = N-1$  and  $n = 1$ , and a qualitative picture of unitarity is as in Figure 1, except that the outlined rectangle is now square. Theorem 5.2 implies that  $A = PC^*$  all the way along the upward-sloping diagonals. The theorem goes on to say that we get unitarity at any point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points in subgroups that are essentially  $SO(N-1, 1)$ .

3)  $Sp(N, 2)$ . The diagram of roots of  $(\mathcal{G}, \mathcal{A})$  is



Figures 3 and 4 give two illustrations of what can happen.

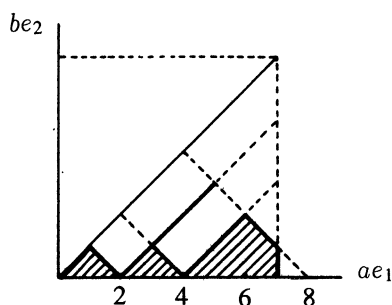


Figure 3: Unitary points in  $Sp(6, 2)$  with minimal  $K$  type  $2e_8 + 2e_7$

In Figure 3, a downward-sloping diagonal terminates unitarity along the upward-sloping diagonals. Imagine inserting the missing downward-sloping diagonal that meets the  $e_1$  axis at  $6e_1$ . This diagonal meets the upward-sloping diagonals at parameters that correspond to  $t_1$  in Theorem 4.11. The downward-sloping diagonal that terminates unitarity crosses where the parameters are  $t_2$ . So the theorem gives no further unitarity along the upward-sloping diagonals. It is known that this picture is sharp.

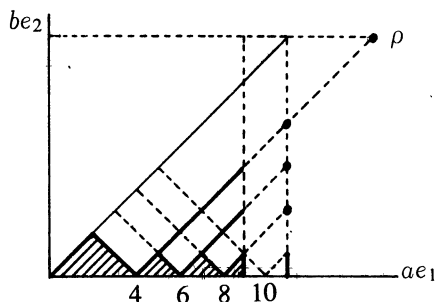


Figure 4: Known unitary points in  $Sp(6, 2)$  for  $\sigma$  trivial

In Figure 4, one has  $A = PC^*$  all the way along the upward-sloping diagonals. Thus Theorem 4.11 says we get unitarity at any point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points for the spherical case in  $Sp(5, 1)$ . Since the trivial representation is isolated as a unitary representation in  $Sp(5, 1)$ , the effect is that Theorem 4.11 gives us several isolated unitary representations in Figure 4. At our urging, H. Schlichtkrull has obtained this unitarity in the spherical case by other methods [11].

Figure 4 illustrates something else. The relationship  $A = PC^*$  is valid within the triangles along the  $e_1$  axis. So once again we get unitarity when the horizontal and vertical coordinates correspond to unitary points in  $Sp(5, 1)$ . The result is an interval of unitarity at the bottom of the vertical line of isolated unitary representations.

## 7. - Concluding remarks

Not all parabolic-rank-two situations, even with  $(BC)_2$  or  $B_2$  as  $(\mathcal{G}, \mathcal{A})$  root diagram, lead to configurations as in (1.1). Such parabolic-rank-two situations perhaps exhibit totally new phenomena. We do not know. What we have investigated is whether there is *some* parabolic-rank- $n$  generalization of our theorems. All of our investigations suggest that the multiplicity-one behavior that we used for  $\widetilde{SO}$  (odd, 1) or  $SL(2, \mathbb{R})$  is vital. Meanwhile Lemma 1.2 is special to three Hermitian operators and does not extend to arbitrarily long products.

Thus any generalization of our results whose techniques are not too far removed from ours ought to use Proposition 1.1 in a situation where  $A$  and  $C$  correspond to a real-rank-one group. When  $B$  or  $D$  corresponds to a parabolic-rank-*two* situation, the generic sets of unitary points, for large  $\nu$ , are one-dimensional, and we find that Proposition 1.1 is at best giving us one-dimensional sets of unitary points for large  $\nu$ . This phenomenon persists if we induct on the parabolic rank. When all coordinates are large, we get only one-dimensional sets of unitary points (as well as isolated points).

Now one might except from Figure 1 that there are  $(n - 1)$ -dimensional sets of unitary points in parabolic rank  $n$  if all coordinates are large. Yet our construction gives only one-dimensional sets. This situation suggests the question:

In parabolic rank  $n$  (say with  $(\mathcal{G}, \mathcal{A})$  roots of type  $(BC)_n$ ), are there sets of unitary points with dimension  $> 1$  and with all coordinates large?

No examples come to mind.

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