ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

L. CHIERCHIA
E. ZEHNDER

Asymptotic expansions of quasiperiodic solutions

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 16, nº 2 (1989), p. 245-258
<http://www.numdam.org/item?id=ASNSP_1989_4_16_2_245_0>

Asymptotic Expansions of Quasiperiodic Solutions

L. CHIERCHIA - E. ZEHNDER

1. - Introduction

We first describe the existence problem of quasiperiodic solutions in a general setting and consider a Lagrangian function $F = F(t, x, p)$,

$$F(t, x, p) \text{ defined on } T^{n+1} \times \mathbb{R}^n, \tag{1.1}$$

i.e. periodic in $(t, x) \in \mathbb{R}^{n+1}$ with periodic 1, $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$. The aim is to find special solutions of the associated Euler-equations

$$\frac{\mathrm{d}}{\mathrm{d}t} F_p(t, x(t), \dot{x}(t)) = F_x(t, x(t), \dot{x}(t)). \tag{1.2}$$

We shall call, in the following, a solution $x(t)$ quasiperiodic with frequencies ω, if it is of the form

$$x(t) = U(t, \omega t), \tag{1.3}$$

where $\omega \in \mathbb{R}^n$ is a given vector with rationally independent components, and where

$$U(t, \vartheta) - \vartheta =: u(t, \vartheta) \quad \text{is defined on } T^{n+1} \tag{1.4}$$

i.e. is periodic in (t, ϑ). Inserting (1.3) into (1.2), one obtains the nonlinear partial differential equation for U:

$$\mathrm{D}F_p(t, U, \mathrm{D}U) = F_x(t, U, \mathrm{D}U), \tag{1.5}$$

where

$$\mathrm{D} = \mathrm{D}(\omega) = \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j} + \frac{\partial}{\partial t}. \tag{1.6}$$

Pervenuto alla Redazione il 19 Aprile 1989.

The differential operator D depends on the frequencies $\omega = (\omega_1, \cdots, \omega_n)$. It is the differentiation in the direction $(\omega, 1)$. Restricting our attention to functions of the special form

$$F(t,x,p) = \frac{1}{2}|p|^2 + f(t,x), \tag{1.7}$$

the equation to be solved becomes

$$\mathrm{D}^2 u = f_x(t, \vartheta + u) \tag{1.8}$$

for $u(t,\vartheta) = U(t,\vartheta) - \vartheta$ being a function on T^{n+1}. In order to solve (1.8) we shall assume f to be analytic and the frequencies ω to satisfy the diophantine conditions:

$$|\langle \omega, j\rangle + m| \geq \gamma(|j|)^{-\tau} \tag{1.9}$$

for two constants $\gamma > 0$ and $\tau \geq n$ and for all $(j,m) \in \mathbb{Z}^n \times \mathbb{Z}\setminus\{0\}$.

It is well know that under these conditions on f and ω the equation (1.8) has a solution, provided f is sufficiently small (in an appropriate sense). This is a consequence of the KAM theory, and we refer to [CC], [SZ] and [M1]. However, if f is not small, then (1.8) may not admit any solutions for frequencies contained in a compact region of $\mathbb{R}^n$, see [Ma]. We shall not impose any smallness conditions on f in the following. Instead we shall construct quasiperiodic solutions having sufficiently large frequencies. We point out, that the system under consideration, decribed by a Lagrangian function in the special form of (1.7), can be viewed as beeing "close to an integrable system" in the region in which $|p|$ is large. Introducing

$$\omega(\alpha) = \frac{1}{\alpha}\omega, \tag{1.10}$$

we look for quasiperiodic solutions having frequencies $\omega(\alpha)$ for sufficiently small $\alpha \in R$ with $\alpha \neq 0$. We shall abbreviate

$$E(u) =: \mathrm{D}^2 u - f_x(t, \vartheta + u), \tag{1.11}$$

with $\mathrm{D} = \mathrm{D}(\omega(\alpha))$.

In the second section we shall prove that there is unique formal powerseries expansion in α:

$$\tilde{u} \sim \sum_{j=2}^{\infty} \alpha^j u_j(\vartheta, t), \tag{1.12}$$

with analytic functions u_j on T^{n+1}, which solves the equation $E(\tilde{u}) = 0$ formally, and satisfies

$$\int_{T^{n+1}} u_j \mathrm{d}t\mathrm{d}\vartheta = 0, \text{ for all } j.$$

However, in general, the series diverges as it is well known, and our aim is to show that the formal series can be interpreted as an asymptotic expansion for the true quasiperiodic solutions u_α, as α tends to zero. For this purpose α is required to belong to the subset

$$A(\omega) = \left\{\alpha \in \mathbb{R} : \left|\frac{1}{\alpha}\langle\omega, j\rangle + m\right| \geq \gamma|j|^{-\tau}, \text{ for all } (j, m) \in \mathbb{Z}^n \times \mathbb{Z}\backslash 0\right\}. \tag{1.13}$$

If γ is sufficiently small and $\tau > n + 1$, we will see that the set $\{\alpha \in A(\omega) : |\alpha| \leq \epsilon\}$ has positive Lebesgue measure for every $1 > \epsilon > 0$. Setting now for every $N \geq 2$

$$\tilde{u}_N := \sum_{j=2}^{N} \alpha^j u_j(t, \vartheta), \tag{1.14}$$

one concludes that, in proper norms,

$$|E(\tilde{u}_N)| \leq C_N |\alpha|^{N-1} \tag{1.15}$$

for all $|\alpha| \leq 1$, with a constant C_N independent of α. Consequently, $\tilde{u}$ can be interpreted as an approximate solution of $E(u) = 0$, if only α is small. Moreover, $\tilde{u}_N$ is stable in the sense that the matrixfunction on T^{n+1},

$$V_\vartheta^T F_{pp}(t, V, \mathrm{D}V) V_\vartheta, \tag{1.16}$$

with $V =: \vartheta + \tilde{u}_N(\vartheta, t)$, is close to the identity matrix. Thus the assumptions of the KAM theory are met and one concludes that there is an $\alpha^* = \alpha^*(N)$, such that for $\alpha \in A(\omega)$ satisfying $|\alpha| < \alpha^*$ there is a unique analytic solution u_α of (1.8) having frequencies $\omega(\alpha)$, hence solving

$$E(u_\alpha) = 0, \tag{1.17}$$

moreover

$$\int_{T^{n+1}} u_\alpha = 0.$$

In addition, one has an estimate of the form

$$|u_\alpha - \tilde{u}_N| \leq C_N |E(u_N)|. \tag{1.18}$$

This establishes the existence of uncountably many quasiperiodic solutions for every analytic f. We point out again, that f is not assumed to be small. Moreover, on account of (1.18) and (1.15) one concludes that for every $N \geq 2$ there are constants $C_N > 0$ and $\alpha^* = \alpha^*(N)$ such that

$$\left| u_\alpha - \sum_{j=2}^{N} \alpha^k u_j \right|_\infty \leq C_N |\alpha|^{N+1} \tag{1.19}$$

for all $\alpha \in A(\omega)$ satisfying $|\alpha| \leq \alpha^*$. This shows that indeed the formal series (1.12) serves as an asymptotic expansion for the solutions having large frequencies $\omega(\alpha)$. The precise statement and the details of this argument are given in section 3. For simplicity we shall only treat the case in which f is analytic. We point out that the asymptotic expansion holds true also for $f \in C^\infty(T^{n+1})$, in which case also the solutions u_α belong to $C^\infty(T^{n+1})$.

It should be mentioned that in the special case $n = 1$ the existence of quasiperiodic solutions having large frequencies can be used in order to prove that all solutions of

$$\ddot{x} - f_x(t,x) = 0, \quad (t,x) \in T^2 \tag{1.20}$$

are bounded, i.e.

$$\sup_t |\dot{x}(t)| < \infty.$$

This has already been pointed out in [M1] and we shall recall the argument. We shall write (1.20) as a system

$$\dot{x} = y, \;\; \dot{y} = f_x(t,x), \;\; \dot{t} = 1, \tag{1.21}$$

which is considered as a vectorfield on the phase space $T^2 \times \mathbb{R}$. Assume now that U is a solutions of

$$\begin{aligned} &\mathrm{D}^2 U = f_x(t,U) \\ &U(t,\vartheta) - \vartheta = u(t,\vartheta) \quad \text{on } T^2, \\ &\mathrm{D} = \frac{\omega}{\alpha}\frac{\partial}{\partial\vartheta} + \frac{\partial}{\partial t}. \end{aligned} \tag{1.22}$$

Then the map $\psi : T^2 \to T^2 \times \mathbb{R}$, defined by $(t,\vartheta) \to (t, x = U(t,\vartheta), y = \mathrm{D}U(t,\vartheta))$, describes an embedding of the torus T^2 into the phase space. In view of (1.22), the vectorfield (1.21) is tangential to $\psi(T^2) \subset T^2 \times \mathbb{R}$ so that its flow leaves this embedded torus invariant. If now $a_1 = \min \mathrm{D}U \leq \mathrm{D}U \leq a_2 = \max \mathrm{D}U$, then $\psi(T^2) \subset T^2 \times [a_1, a_2]$, and since $\psi(T^2)$ is invariant under the flow we conclude, for every solution $(t, x(t), y(t))$ satisfying $y(t^*) < a_1$ for some $t^* \in \mathbb{R}$, that $y(t) < a_2$ for all $t \in \mathbb{R}$. Since $\mathrm{D}U = \frac{\omega}{\alpha} + O(\alpha)$, we can construct for every $C > 0$ a quasiperiodic solution U satisfying $\mathrm{D}U > C$ by choosing α sufficiently small. This proves the claim, that all solutions are bounded. One can show that the analyticity of f is not necessary for the argument. It is sufficient to assume f to be sufficiently smooth, e.g. $f \in C^6(T^2)$, for the smooth case we refer to [M2]. Similar arguments allow to prove the boundedness of solutions of other equations, for example for the Euler equation associated to

$$F(t,x,p) = \frac{1}{2}p^2 + \sqrt{1+p^2}\, f(t,x)$$

on $T^2 \times \mathbb{R}$. The above argument was used also in the more subtle proof in [DZ] of the boundedness of solutions for a nonlinear Duffing equation on $\mathbb{R}^2 \times \mathbb{R}$.

Observe that this note deals only with systems of very restricted nature and it is desirable to have asymptotic expansion for a more general class of Euler equations associated to

$$F(t,x,p) = g(p) + f(t,x,p)$$

on $T^{n+1} \times \mathbb{R}^n$, with

$$\frac{|f(t,x,p)|}{|g(p)|} \to 0, \text{ as } |p| \to \infty.$$

2. - The formal expansion

In order to solve $E(u) = 0$ we set formally

$$u =: \sum_{j=0}^{\infty} \alpha^j u_j(\vartheta, t), \tag{2.1}$$

and recall that

$$E(u) := D^2 u - f_x(t, \vartheta + u) \tag{2.2}$$

contains the parameter α also in the differential operator D. Introducing the operator

$$\partial =: \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j}$$

we can write

$$\mathrm{D}^2 = \frac{1}{\alpha^2}\partial^2 + \frac{2}{\alpha}\partial \mathrm{D}_t + \mathrm{D}_t^2, \tag{2.3}$$

where D_t denotes partial derivative with respect to t. Expanding $\alpha^2 E(u) = 0$ into powers of α we find the following equations to be solved for the functions u_j:

$$\begin{aligned} &\partial^2 u_0 = 0 \\ &\partial^2 u_1 + 2\partial \mathrm{D}_t u_0 = 0 \\ &\partial^2 u_j + 2\partial \mathrm{D}_t u_{j-1} + \mathrm{D}_t^2 u_{j-2} = \varphi_{j-2}, \end{aligned} \tag{2.4}$$

for $j \geq 2$, where

$$\varphi_j = \varphi_j(u_0, \cdots, u_j) = \frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j f_x\left(t, \vartheta + \sum_{s=0}^{j} \alpha^s u_s\right)\Bigg|_{\alpha=0} \tag{2.5}$$

is a polynomial in $u_1, \cdots, u_j$.

We shall show that there are unique analytic solutions u_j defined on T^{n+1}, if we normalize

$$\int_{T^{n+1}} u_j \mathrm{d}\vartheta \mathrm{d}t = 0.$$

We first observe that the linear equation $\partial u = g$ on T^{n+1} admits a unique analytic solution u with meanvalue zero, provided g is analytic and has vanishing meanvalue. Since we will need it we formulate this well know result in quantitative terms. Denote by H_σ the space of holomorphic functions $g(t,x)$ defined in the complex strip $\sum_\sigma = \{(x,t) \in C^{n+1} : |\mathrm{Im}\ x_i| < \sigma, |\mathrm{Im}\ t| < \sigma\}$ and periodic in all its variables, and abbreviate

$$|g|_\sigma =: \sup_{\Sigma_\sigma} |g|.$$

LEMMA 1. *Let ω satisfy the diophantine conditions* (1.9). *Assume $g \in H_\sigma$ satisfies $|g|_\sigma < \infty$ and $\int g \mathrm{d}x = 0$. Then there is a unique analytic and periodic solution u satisfying*

$$\partial u = g \text{ on } \sum\nolimits_\sigma, \text{ and } \int_{T^n} u \mathrm{d}x = 0. \tag{2.6}$$

Moreover, there is a constant $C = C(n, \tau)$ such that

$$|u|_{\sigma-\delta} \le \frac{1}{\gamma}\delta^{-\tau} C |f|_\sigma, \text{ for all } 0 < \delta \le \sigma. \tag{2.7}$$

For a proof we refer e.g. to [R]. We notice that here the variable t is only a parameter. To construct the solutions one proceeds inductively.

a) First we show that $u_0 = u_1 = 0$. Indeed from the first two equations in (2.4) we conclude, in view of Lemma 1, that $u_0 = u_0(t)$ and $u_1 = u_1(t)$ are independent of the ϑ-variable. Integration of

$$\partial^2 u_2 + 2\partial \mathrm{D}_t u_1 + \mathrm{D}_t u_1 + \mathrm{D}_t^2 u_0 = f_x(t, \vartheta + u_0) \tag{2.8}$$

in the ϑ-variable gives $\mathrm{D}_t^2 u_0(t) = 0$ and hence $u_0 = 0$, if the meanvalue should vanish. Integrating now

$$\partial^2 u_3 + 2\partial \mathrm{D}_t u_2 + \mathrm{D}_t^2 u_1 = f_{xx}(t, \vartheta) u_1(t)$$

in the ϑ-variable over T^n we find $\mathrm{D}_t^2 u_1(t) = 0$ and hence $u_1(t) = 0$.

b) Next we proceed by induction and assume that

$$\text{(2.9)} \qquad \begin{gathered} \partial^2 u_j + 2\partial \mathrm{D}_t u_{j-1} + \mathrm{D}_t^2 u_{j-1} = \varphi_{j-2}, \\ \int_{T^n} (\mathrm{D}_t^2 u_j - \varphi_j) \mathrm{d}\vartheta = 0 \end{gathered}$$

hold true for $0 \le j \le n$, where quantities with negative subscripts are defined to be zero. In order to prove the statement for $j = n+1$ we first solve

$$\text{(2.10)} \qquad \partial^2 u_{n+1} = \varphi_{n-1} - 2\partial \mathrm{D}_t u_n - \mathrm{D}_t^2 u_{n-1}.$$

On account of the induction assumption the meanvalue over T^n of the right hand side vanishes, and by Lemma 1 there is a solution

$$\text{(2.11)} \qquad u_{n+1} = a + b,$$

where $a = a(\vartheta, t)$ is uniquely determined, if we set

$$\text{(2.12)} \qquad \int_{T^n} a(\vartheta, t) \mathrm{d}\vartheta = 0,$$

$b = b(t)$ is arbitrary. It will be determined by the condition

$$\int_{T^n} (\mathrm{D}_t^2 u_{n+1} - \varphi_{n+1}) \mathrm{d}\vartheta = 0$$

or

$$\text{(2.13)} \qquad \mathrm{D}_t^2 b = \int_{T^n} \left(\varphi_{n+1} - \mathrm{D}_t^2 a\right) \frac{\mathrm{d}\vartheta}{(2\pi)^n} = \int_{T^n} \varphi_{n+1} \frac{\mathrm{d}\vartheta}{(2\pi)^n}.$$

Observe that the average over T^n of φ_{n+1} does *not* depend on b. Indeed φ_{n+1} is, in view of (2.5), of the form

$$\varphi_{n+1} = f_{xx}(t, \vartheta) u_{n+1} + \tilde{\varphi},$$

where $\tilde{\varphi}$ depends on $u_n, u_{n-1}, \cdots, u_1$ only. Therefore, since the meanvalue of $f_{xx}(t, \vartheta) b(t)$ is zero, the meanvalue of φ_{n+1} is independent of b.

Now the necessary and sufficient condition for a solution of (2.13) is the vanishing of the meanvalue in the t variable:

$$\text{(2.14)} \qquad \int_{T^{n+1}} \varphi_{n+1} \mathrm{d}\vartheta \mathrm{d}t = 0.$$

Assuming (2.14) to hold true there is a unique solution b of (2.13) having meanvalue zero and the induction is completed. It remains to prove (2.14).

c) For the proof of (2.14) we need

LEMMA 2. *For every* $u \in C^2(T^{n+1})$.

$$\int_{T^{n+1}} (1+u_\vartheta)^T E(u) d\vartheta dt = 0, \tag{2.15}$$

where u_ϑ *is the Jacobian matrix in the* ϑ*-variable.*

PROOF. Set $1+u_\vartheta = V$, by integration:

$$\begin{aligned} \int_{T^{n+1}} V^T E(u) d\vartheta dt &= \int_{T^{n+1}} \left(V^T \mathrm{D} F_p - V^T F_x\right) d\vartheta dt \\ &= -\int_{T^{n+1}} \left(\mathrm{D} V^T F_p + V^T F_x\right) d\vartheta dt \\ &\quad -\int_{T^{n+1}} \frac{\partial}{\partial \vartheta} F(t, \vartheta + u, \mathrm{D}(\vartheta + u)) d\vartheta dt = 0. \end{aligned}$$

Inserting the expansion for $\alpha^2 E(u)$ into (2.15) one finds the identities

$$\int_{T^{n+1}} \varphi_j = \int_{T^{n+1}} \sum_{s+\ell=j+2} u_{s,\vartheta}^T \Phi_\ell d\vartheta dt,$$

where $$\Phi_\ell =: \partial^2 u_\ell + 2\partial \mathrm{D}_t u_{\ell-1} + \mathrm{D}_T^2 u_{\ell-2} - \varphi_{\ell-2},$$

for all $j \geq 0$, and for every formal series u. The claim (2.14) follows immediately if we set $j = n+1$, since the integrand on the right hand side vanishes: indeed if $s=0$ and $s=1$, then $u_0 = u_1 = 0$. If $s \geq 2$, then by the induction assumption and by (2.10), $\Phi_\ell = 0$ for all $\ell \leq n+1$. This finishes the proof of the unique formal power series.

3. - Existence and asymptotic character

In this section we give the necessary details in order to prove (1.17)-(1.19). First we observe that the set

$$A := \left\{ \alpha > 0 : \left| \frac{1}{\alpha} \langle \omega, j \rangle + m \right| \geq \gamma |j|_\cdot^{-\tau} \text{ for all } j, m \in \mathbb{Z}^{n+1}, j \neq 0 \right\}$$

has positive Lebesgue measure μ provided the constant γ is sufficiently small. Here ω is a fixed vector with rationally independent components and τ is a constant satysfying $\tau > n+1$. More precisely:

LEMMA 3. *Fix* $0 < \lambda < 1$. *Then there is a constant* $\gamma^* = \gamma^*(\lambda)$ *such that for* $0 < \gamma \leq \gamma^*$

$$\mu\{\alpha \in A | 0 < \alpha \leq \epsilon\} \geq \epsilon(1-\lambda), \tag{3.1}$$

for every $0 < \epsilon \leq 1$.

PROOF. Assume $\gamma \leq \frac{1}{2}$, we prove that $\mu(B_\epsilon) \leq \epsilon\lambda$ if γ is sufficiently small, where $B_\epsilon = (0,\epsilon)\backslash A$ is the complement. We have

$$\mu(B_\epsilon) \leq \mu\left(\bigcup_{(j,m)\neq 0} A_{jm}\right)$$

where

$$A_{jm} = \left\{0 < \alpha \leq \epsilon : \left|\frac{1}{\alpha} - \frac{m}{\langle\omega, j\rangle}\right| < \frac{\gamma}{|\langle\omega, j\rangle||j|^\tau}\right\}.$$

In view of $\gamma \leq \frac{1}{2}$ one verifies readily that

$$\sum_{(j,m)} \mu(A_{jm}) \leq 4\gamma \sum_{j\neq 0} \frac{|\langle\omega, j\rangle|}{|j|^\tau} \sum_{|m|\geq \frac{1}{\epsilon}|\langle\omega, j\rangle|} \frac{1}{m^2}.$$

Since the sum over m is dominated by $\frac{2\epsilon}{|\langle\omega, j\rangle|}$ we conclude that

$$\mu(B_\epsilon) \leq 8\gamma\epsilon \sum_{j\neq 0} \frac{1}{|j|^\tau}.$$

In view of $\tau > n+1$, the right hand side is equal to $8\gamma\epsilon C$. Therefore, defining $\gamma^*(\lambda) = \min\left\{\frac{1}{2}, \frac{\lambda}{8C}\right\}$, one concludes that $\mu(B_\epsilon) \leq \lambda\epsilon$ as claimed.

Now, we can state our main result.

THEOREM. *Assume* $\gamma < \gamma^*$. *Assume* f *is real analytic in the (closure of the) complex strip* $\sum_\sigma$ *for some* $1 \geq \sigma > 0$. *For every* $N \geq 2$, *there exist positive constants* $\alpha^* = \alpha^*(N)$ *and* C_N *with the following properties:*

For $\alpha \in A(\omega)$ *satisfying* $|\alpha| < \alpha^*$ *there is a unique* u_α *real-analytic in, say,* $\sum_{\sigma/8}$ *and of mean value* 0 *such that*

$$E(u_\alpha) = 0 \quad in \sum_{\sigma/8} \tag{3.2}$$

and

$$\left| u_\alpha - \sum_{j\geq 2}^{N} \alpha^j u_j \right|_{\sigma/8} \leq C_N |\alpha|^{N+1}. \tag{3.3}$$

The proof rests on the discussion in section 2 and on the following KAM result, for which we refer to [SZ] (Theorem 1) and [CC] (Lemma 6).

LEMMA 4. *Let f be as in the above Theorem. Let ω satisfy* (1.9) *and let $v \in H_\sigma$ with $|v|_\sigma \leq \sigma, |v_\vartheta|_\sigma \leq \frac{1}{2}$. There exists a costant $C = C(n, f, \sigma, \gamma, \tau)$ such that if*

$$C|E(v)|_\sigma \leq 1, \tag{3.4}$$

then there is a unique real analytic $u \in U_{\sigma/2}$ satisfying

$$E(u) = 0, \qquad \int (u - v) = 0, \qquad |u - v|_{\sigma/2} < C|E(v)|_\sigma. \tag{3.5}$$

PROOF OF THE THEOREM. Applying iteratively Lemma 1 and the Cauchy estimates (to control derivatives in terms of functions) to the u_i's constructed in section 2, one finds estimates of the form

$$|u_i|_{\sigma/2} \leq K_i, 2 \leq i \leq N,$$

with constants K_i depending on n, f and γ, τ. Thus one can find an α_0^* so small, that for $|\alpha| < \alpha_0^*$ one has

$$|\tilde{u}_N|_{\sigma/4} < \frac{\sigma}{4}, |\tilde{u}_{N,\vartheta}|_{\sigma/4} \leq \frac{1}{2}, \tag{3.6}$$

where, as above, $\tilde{u}_N =: \sum_{j=2}^{N} \alpha^j u_j$. Moreover, Taylor's formula leads to the bound

$$|E(\tilde{u}_N)|_{\sigma/4} \leq K_N^* |\alpha|^{N-1}. \tag{3.7}$$

Now, if we set

$$\alpha^* = \min\left\{\alpha_0^*, (CK_N^*)^{\frac{1}{1-N}}\right\},$$

the Theorem follows from Lemma 4 simply replacing ω by $\frac{\omega}{\alpha}$ $(\alpha \epsilon A(\omega))$, σ by $\frac{\sigma}{4}$ and v by $\tilde{u}_N$. In this case (3.3) holds with $C_N =: CK_N^*$.

This theorem gives a precise meaning to the asymptotic character of the series $\sum \alpha^i u_i$ which, as mentioned in the introduction, is in general divergent. It would, therefore, also be desirable to have good estimates for the functions

u_j. In the special case in which $n = 1$ the operator ∂ is simply the differential operator $\omega\frac{\partial}{\partial\vartheta}$. We may therefore assume $\omega = 1$ and find the following estimates:

PROPOSITION. *Assume $n = 1 = \omega$, and assume that f is analytic and bounded on the strip $\sum_\sigma$ with $0 < \sigma \leq 1$. Then the unique formal power series in section 2 satisfies*

$$|u_{j+2}|_{\sigma/2} \leq B^{j+2} j^{2j} \text{ for all } j \geq 0.$$

Here $B = \left(\frac{30M}{\sigma}\right)^2$ *with* $M = \max\left\{|f|_\sigma, |f_x|_\sigma, 1\right\}$.

We shall use the following

LEMMA 4. *For all $j \geq 1$:*

$$\sum_{k_1+2k_2+\cdots+jk_j=j} \prod_{s=1}^{j} \frac{1}{k_s!} < e^4.$$

PROOF. Using the generating functions, the left hand side of the inequality is equal to

$$\frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j \exp\left(\sum_1^\infty \alpha^s\right)\Bigg|_{\alpha=0}$$

$$= \frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j \exp\left(\frac{\alpha}{1-\alpha}\right)\Bigg|_{\alpha=0} = \frac{e^{-1}}{j!}\sum_{n=1}^\infty \frac{(n+j-1)(n+j-2)\cdots n}{n!},$$

so that the claim follows from

$$\frac{(n+j-1)(n+j-2)\cdots n}{j!} < 4^n \text{ for all } n, j \geq 1.$$

PROOF OF THE PROPOSITION. Recall that $u_0 = u_1 = 0$, and

$$u_j = a_j + b_j, \quad j \geq 2, \tag{3.8}$$

is determined by

$$\int a_j(\vartheta, t)\mathrm{d}\vartheta = 0, \int b_j(t)\mathrm{d}t = 0, \tag{3.9}$$

$$\partial_\vartheta^2 a_{j+2} = -2\partial_\vartheta\partial_t a_{j+1} - \partial^2 u_j - \varphi_j \tag{3.10}$$

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2}\mathrm{d}\vartheta, \tag{3.11}$$

where $\varphi_0 = f_x(\vartheta, t), \varphi_1 = 0$ and where, for $j \geq 2$

$$\varphi_j = \sum_{k \in P_j} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j} \frac{u_s^{k_s}}{k_s!} \tag{3.12}$$

$$= \frac{1}{j!} \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j f_x \left(t, \vartheta + \sum_{n \geq 2} \alpha^n u_n\right)\Bigg|_{\alpha=0} .$$

here

$$P_j = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j\} .$$

and $|k| = k_2 + k_3 + \cdots + k_j$. Setting

$$P_{j+2}^* = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j + 2\}$$

we can rewrite equation (3.11) as

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2} \mathrm{d}\vartheta = \int \left\{ f_{xx} a_{j+2} + \sum_{P_{j+2}^*} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j+2} \frac{u_s^{k_s}}{k_s!} \right\} .$$

Integrating the first term by parts and inserting the equation (3.10) for a_{j+2} gives

$$\partial_t^2 b_{j+2} = -\int f \left(2\partial_\vartheta \partial_t a_{j+1} + \partial_t^2 u_j - \varphi_j\right) + \int \Psi_j \mathrm{d}\vartheta, \tag{3.13}$$

where

$$\Psi_j = \Psi_j(u_j, u_{j-1}, \cdots, u_2) = \sum_{P_{j+2}^*} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j} \frac{u_s^{k^s}}{k_s!} . \tag{3.14}$$

We proof first the Lemma for $j = 0$. From

$$\partial_t^2 b_2 = \int f f_x \mathrm{d}\vartheta = 0$$

we conclude that $b_2 = 0$ so that $u_2 = a_2$. Since the meanvalue of a_2 vanishes we conclude that

$$|u_2|_\sigma \leq |\partial_\vartheta^2 u_2|_\sigma = |f_x|_\sigma \leq M,$$

which proves the Lemma for $j = 0$.

Assume now $j \geq 1$. We shall show that

$$|u_{i+2}|_{\sigma_i} \leq B^{i+1} j^{2i} \text{ for all } 0 \leq i \leq j, \tag{3.15}$$

where

$$\sigma_i = \sigma\left(1 - \frac{i}{2j}\right). \tag{3.16}$$

The Lemma then follows by setting $i = j$. The estimate (3.15) will be proved by induction in i. In the case $i = 0$, (3.15) is already proved above for $\sigma_0 = \sigma$ and we shall assume now that

$$|u_{s+2}|_{\sigma_s} \leq B^{s+1} j^{2s}, 0 \leq s \leq i-1, \tag{3.17}$$

where, of course, $1 \leq i \leq j$. From (3.9), (3.10) and (3.13) we conclude

$$\begin{aligned} |u_{i+2}|_{\sigma_i} &\leq |a_{i+2}|_{\sigma_i} + |b_{i+2}|_{\sigma_i} \\ &\leq |\partial_\vartheta^2 a_{i+2}|_{\sigma_i} + |\partial_t^2 b_{i+2}|_{\sigma_i} \\ &\leq 4M|\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} + 2M|\partial_t^2 u_i|_{\sigma_i} + 2M|\varphi_i|_{\sigma_i} + |\Psi_i|_{\sigma_i}. \end{aligned} \tag{3.18}$$

We estimate each term separately. Using the Cauchy estimates and the induction hypothesis (3.17) one finds

$$\begin{aligned} |\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} &\leq \frac{1}{(\sigma_{i-1} - \sigma_i)^2} |u_{i+1}|_{\sigma_{i-1}} = \left(\frac{2j}{\sigma}\right)^2 |u_{i+1}|_{\sigma_i} \\ &\leq \frac{4}{\sigma^2} B^i j^{2i}, \end{aligned} \tag{3.19}$$

similarly

$$|\partial_t^2 u_i|_{\sigma_i} \leq \left(\frac{j}{\sigma}\right)^2 |u_i|_{\sigma_{i-2}} \leq \frac{1}{\sigma^2} B^{i-1} j^{2(i-1)}. \tag{3.20}$$

Observe now that $\sigma_{s-2} \geq \sigma_s \geq \sigma_i$, and $B \geq \frac{2}{\sigma}, i \geq 1$ and that $|k| \geq 1$ for $k \in P_i$, then

$$\begin{aligned} |\varphi_i|_{\sigma_i} &\leq M \sum_{P_i} \frac{1}{(\sigma - \sigma_i)^{|k|}} \prod_{s=2}^{i} \frac{|u_s|_{\sigma_{s-2}}^{k_s}}{k_s!} \\ &\leq M \sum_{P_i} \left(\frac{2j}{i\sigma}\right)^{|k|} \frac{B^i j^{2i}}{B^{|k|} j^{4|k|}} \prod_{s=2}^{i} \frac{1}{k_s!} \\ &\leq \frac{2M}{\sigma} B^{i-1} j^{2i-3} \sum_{P_i} \prod_{s=1}^{i} \frac{1}{k_s!}, \end{aligned}$$

so that, by Lemma 4,

$$|\varphi_i|_{\sigma_i} \leq \frac{1}{\sigma} 2e^4 M B^{i-1} j^{2i-3}. \tag{3.21}$$

Observing that, if $k \in P^*_{i+2}$, then $2k_2 + \cdots + jk_j = j + 2$ and $|k| \geq 2$, one concludes similarly

$$|\Psi_i|_{\sigma_i} \leq 4e^4 M B^i j^{2(i-1)}. \tag{3.22}$$

Adding up we find from (3.18)-(3.22) that

$$\begin{aligned} |u_{i+2}|_{\sigma_i} &\leq \frac{1}{\sigma^2}(16 + 2 + 8e^4) M^2 B^i j^{2i}, \\ &< \frac{900 M^2}{\sigma^2} B^i j^{2i} = B^{i+1} j^{2i}, \end{aligned}$$

where we have used the definition of the constant B. This finishes the proof of the proposition.

REFERENCES

[CC] A. Celletti - L. Chierchia, *Construction of analytic KAM surfaces and effective stability bounds*, Commun. Math. Phys. **118** (1988) pp. 119-161.

[DZ] R. Dieckerhoff - E. Zehnder, *Boundedness of Solutions via the Twist-Theorem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987) pp. 79-95.

[MA] J. Mather, *Nonexistence of invariant circles*, Ergodic Theory Dynamical Systems **4**, (1984) pp. 301-309.

[M1] J. Moser, *Quasiperiodic solutions of nonlinear elliptic partial differential equations*, To be published in the Bulletin of the Brazilian Math. Soc.

[M2] J. Moser, *A Stability Theorem for Minimal Foliations on a Torus*, Ergodic Theory Dynamical Systems, **8***, (1988) pp. 251-281.

[R] H. Rüssmann, *Konvergente Reihenentwicklungen in der Störungstheorie der Himmelsmechanik*, Selecta Mathematica V, Heidelberger Taschenbücher, **201**, (1979) pp. 93-260.

[SZ] S. Salamon - E. Zehnder, *KAM theory in configuration space*, Comment. Math. Helv. **64** (1989) 84-132.

Dipartimento di Matematica
II Università di Roma
Via Orazio Raimondo
00173 Roma

Mathematik
ETH-Zentrum
8092 Zürich
Switzerland