ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 16, nº 2 (1989), p. 165-224

<http://www.numdam.org/item?id=ASNSP_1989_4_16_2_165_0>

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Improved Theory for a Nonlinear Degenerate Parabolic Equation

B.H. GILDING

1. - Introduction

The subject of this paper is the nonlinear equation

$$u_t = (a(u))_{xx} + (b(u))_x \tag{1.1}$$

in which subscripts denote partial differentiation. The functions a and b are hypothesized to belong to $C([0,\infty)) \cap C^2(0,\infty)$ and be such that $a'(s) > 0$ for $s > 0$, and a'' and b'' are locally Hölder continuous on $(0,\infty)$. Without any loss of generality, it will also be assumed that $a(0) = 0$ and $b(0) = 0$.

If the functions a and b in equation (1.1) are members of $C^1([0,\infty))$ and moreover $a'(0) > 0$, then equation (1.1) is a model quasilinear parabolic equation which is covered by established theories [14]. Of particular interest here will be those cases which fall out of the scope of standard theories. Specifically, we leave open the possibilities that a' and b' may oscillate wildly, that b' may be unbounded above and below, that a' may be unbounded above, and that $a'(s)$ is not bounded away from zero as $s \downarrow 0$. In this latter case, equation (1.1) may be classified as being of degenerate parabolic type.

Some twelve years ago, the present author [11] established the existence of a weak solution of the Cauchy problem, the Cauchy-Dirichlet problem, and the first boundary-value problem for equation (1.1) under a number of regularity assumptions on the boundary data and under the hypotheses

$$a,\ b \in C^1([0,\infty)) \tag{1.2}$$

and

$$sa''(s),\ sb''(s) \in L^1(0,\varepsilon) \qquad \text{for any } \varepsilon > 0; \tag{1.3}$$

Pervenuto alla Redazione il 17 Settembre 1986.

and the uniqueness of a weak solution for these problems under hypothesis (1.2) and the hypothesis

$$(b'(s))^2/a'(s) \in L^\infty(0,\varepsilon) \qquad \text{for any } \varepsilon > 0. \tag{1.4}$$

Recently, this work has been superseded by the researches of Bénilan and Touré [4], and, Diaz and Kersner [8]. By casting the right-hand side of equation (1.1) in the form of an accretive operator, Bénilan and Touré [4] have shown, under extremely weak conditions on the functions a and b, that the boundary-value problems with homogeneous Dirichlet conditions on the lateral boundary have unique mild solutions. In their work, Diaz and Kersner [8] followed a less abstract approach, and also considered the problems with nonhomogeneous lateral boundary conditions. Among the striking improvements on the results in [11] obtained, they have shown that, for existence, the assumptions on the boundary data for the different problems could be weakened, and that condition (1.2) was superfluous. The *trs* most noteworthy achievement though has been indicating that conditions (1.2) and (1.4) could be disposed of in the verification of uniqueness. Unfortunately the proofs for the Cauchy problem and the Cauchy-Dirichlet problem presented in [8] contain a step which is not entirely justified.

The objective of the present paper is to combine the strongest features of the arguments in [8] and [11], and, in the light of recent work of Bénilan [2] and Bénilan and Diaz [3], re-establish improved existence and uniqueness theorems for the Cauchy problem, the Cauchy-Dirichlet problem, and the first boundary-value problem for equation (1.1). In particular, we shall let it be seen in this manner, that the uniqueness results stated in [8] can be proven avoiding the step in the proofs in [8] which is open to question, and that a number of technical conditions still required for existence and uniqueness in [8] can be removed.

The approach which we maintain is to construct a generalized solution of equation (1.1) as the limit of a sequence of positive classical solutions of the equation. This approach, which was also previously followed in [8] and [11], propagates from the now classic work of Oleinik, Kalashnikov, and Chzhou [15] in which it was applied to the corresponding problems for the equation

$$u_t = (a(u))_{xx}. \tag{1.5}$$

A consequence of this approach is that we will not be able to obtain quite such remarkable results for the problems with homogeneous Dirichlet lateral boundary conditions as Bénilan and Touré [4] have done. This comes about purely because we require a certain additional minimal regularity on the functions a and b to imply that the sequence of positive classical solutions of equation (1.1) exists. Beyond this restriction though, our results are equivalent, leading in fact to a relaxation of the regularity assumptions which need to be imposed on the initial data functions. The motivation for adhering to the indicated approach is that it enables us to formulate pointwise comparison principles for

the generalized solutions of equation (1.1) with which their properties can be investigated. In this respect, we mention two contemporary publications [12,13] in which this construction is put to good effect.

Equation (1.1) is not only of intrinsic mathematical interest. The equation is significant in describing a number of physical diffusion-advection processes. For instance, unsaturated soil-moisture flow and the movement of a thin viscous film under the influence of gravity can both be described by this equation [5,7]. Owing to its resemblance to the celebrated Fokker-Planck equation of statistical mechanics [6], equation (1.1) is often termed the nonlinear Fokker-Planck equation.

The structure of the remainder of this paper is as follows. In the next section we shall indicate precisely what we mean by the Cauchy problem, the Cauchy-Dirichlet problem, and the first boundary-value problem for equation (1.1); and clarify the definition of a generalized solution of these problems. We shall also introduce some other basic concepts and notation which will be frequently used in the subsequent analysis, for purposes of easy reference. As has already been mentioned, the existence and uniqueness theorems which we establish in this article are based upon the construction of generalized solutions of equation (1.1) as the limit of sequences of positive classical solutions of the equation. To be able to perform such constructions, an obvious prerequisite is that suitable classical solutions exist. Furthermore, some a priori estimates of the regularity of these solutions will be required. These a priori estimates form the key to our analysis, and can be claimed to be of some interest in their own right. Notwithstanding, so as not to encumber the proofs of existence and uniqueness with too many technical details, we treat the existence of the classical solutions and these a priori estimates as preliminaries. This we do in Section 3. This clears the field for proving our existence theorems in Section 4, and, thereafter, our uniqueness theorems in Section 5. Section 6 subsequently summarizes a number of regularity results for the constructed generalized solutions of equation (1.1) which follow immediately from the existence proofs. In Section 7, we establish the previously-mentioned comparison principles for generalized solutions of equation (1.1). The paper is concluded in Section 8 by reviewing the relationship between our results and those in earlier publications in comparative detail.

2. - Statement of problems

This paper is specifically concerned with the following three boundary-value problems, where $0 < T < \infty$.

PROBLEM 1 (The Cauchy problem). *To find a solution of equation* (1.1) *in the strip*

$$S = (-\infty, \infty) \times (0, T]$$

satisfying the initial condition

$$u(x,0) = u_0(x) \qquad \textit{for } -\infty < x < \infty, \tag{2.1}$$

where u_0 is a given real function which is defined, nonnegative, bounded, and continuous on $(-\infty,\infty)$.

PROBLEM 2 (The Cauchy-Dirichlet problem). *To find a solution of equation* (1.1) *in the half-strip*

$$H = (0,\infty) \times (0,T]$$

satisfying the conditions

$$u(x,0) = u_0(x) \qquad \textit{for } 0 \le x < \infty, \tag{2.2}$$

$$u(0,t) = \psi(t) \qquad \textit{for } 0 < t \le T, \tag{2.3}$$

where u_0 is a given real function which is defined, nonnegative, bounded, and continuous on $[0,\infty)$, and ψ is a given real function which is defined, nonnegative, and continuous on $[0,T]$, and satisfies the compatibility condition $\psi(0) = u_0(0)$.

PROBLEM 3 (The first boundary-value problem). *To find a solution of equation* (1.1) *in the rectangle*

$$Q = (-1,1) \times (0,T]$$

satisfying the conditions

$$u(x,0) = u_0(x) \qquad \textit{for } -1 \le x \le 1, \tag{2.4}$$

$$u(-1,t) = \psi^-(t) \qquad \textit{for } 0 < t \le T, \tag{2.5}$$

$$u(1,t) = \psi^+(t) \qquad \textit{for } 0 < t \le T, \tag{2.6}$$

where u_0 is a given real function which is defined, nonnegative, and continuous on $[-1,1]$, and, ψ^- and ψ^+ are given real functions which are defined, nonnegative, and continuous on $[0,T]$, and satisfy the compatibility conditions $\psi^-(0) = u_0(-1)$ and $\psi^+(0) = u_0(1)$.

In defining generalized solutions to these problems, we follow [8]. Let D denote the domain

$$D = (\eta_1,\eta_2) \times (\tau_1,\tau_2] \tag{2.7}$$

with

$$-\infty \le \eta_1 < \eta_2 \le \infty \quad \text{and} \quad 0 \le \tau_1 < \tau_2 < \infty. \tag{2.8}$$

DEFINITION 1. *A function $u(x,t)$ is said to be a generalized supersolution of equation* (1.1) *in D if* (i) *u is defined, real, nonnegative, bounded, and continuous in $\overline{D}$; and* (ii) *satisfies the integral inequality*

$$\begin{aligned}(2.9)\qquad &\int_{t_1}^{t_2}\int_{x_1}^{x_2}\{u\phi_t + a(u)\phi_{xx} - b(u)\phi_x\}\,dxdt \\ &\le \int_{x_1}^{x_2}\{u(x,t_2)\phi(x,t_2) - u(x,t_1)\phi(x,t_1)\}\,dx \\ &+ \int_{t_1}^{t_2}\{(a(u(x_2,t))\phi_x(x_2,t) - a(u(x_1,t))\phi_x(x_1,t)\}\,dt\end{aligned}$$

for all non-empty bounded rectangles $R = (x_1,x_2)\times(t_1,t_2] \subseteq D$ and nonnegative functions $\phi \in C^{2,1}(\overline{R})$ such that $\phi(x_1,t) = \phi(x_2,t) = 0$ for all $t \in [t_1,t_2]$.

DEFINITION 2. *A function $u(x,t)$ is said to be a generalized subsolution of equation* (1.1) *in D if it meets the requirements of parts* (i) *and* (ii) *of the definition of a generalized supersolution of equation* (1.1) *in D with the inequality sign in* (2.9) *reversed.*

DEFINITION 3. *A function $u(x,t)$ is said to be a generalized solution of equation* (1.1) *in D if it is a generalized supersolution and a generalized subsolution of equation* (1.1) *in D.*

DEFINITION 4. *A function $u(x,t)$ is said to be a generalized solution of Problem* 1 *if it is a generalized solution of equation* (1.1) *in S, and satisfies* (2.1).

DEFINITION 5. *A function $u(x,t)$ is said to be a generalized solution of Problem* 2 *if it is a generalized solution of equation* (1.1) *in H, and satisfies* (2.2) *and* (2.3).

DEFINITION 6. *A function $u(x,t)$ is said to be a generalized solution of Problem* 3 *if it is a generalized solution of equation* (1.1) *in Q, and satisfies* (2.4)-(2.6).

Throughout the remainder of the paper it will be supposed that the coefficients of equation (1.1) satisfy the following basic hypothesis.

HYPOTHESIS 1. *The functions $a, b \in C([0,\infty)) \cap C^2(0,\infty)$ and are such that*

$$(2.10)\qquad a'(s) > 0 \qquad \textit{for } s > 0$$

and a'' and b'' are locally Hölder continuous on $(0,\infty)$. Moreover,

$$(2.11)\qquad a(0) = 0 \quad \textit{and} \quad b(0) = 0.$$

In certain parts, we shall also require the following additional hypothesis.

HYPOTHESIS 2. *The functions* $sa''(s)$, $sa'(s)b'(s) \in L^1(0,\varepsilon)$ *for any* $\varepsilon > 0$.

As examples of pairs of functions a and b which satisfy Hypothesis 1, but do not satisfy Hypothesis 2, we may take

$$a(s) = \int_0^s r\{2 + \sin(r^{-2})\}dr \quad \text{and} \quad b(s) \equiv 0,$$

or,

$$a(s) \equiv s \qquad \text{and } b(s) = s\{2 + \sin(s^{-2})\}.$$

Cf. [11]. The first pair of functions have the property $sa''(s) \notin L^1(0,\varepsilon)$ although trivially $sa'(s)b'(s) \in L^1(0,\varepsilon)$ for any $\varepsilon > 0$. Whereas the second pair of functions satisfy $sa'(s)b'(s) \notin L^1(0,\varepsilon)$ albeit $sa''(s) \in L^1(0,\varepsilon)$ for any $\varepsilon > 0$.

We set

$$\chi = \sup\{a(s) : 0 \le s < \infty\},$$

and define

$$A(a(s)) = s \qquad \text{for all } s \in [0,\infty), \tag{2.12}$$

$$a(A(r)) = r \qquad \text{for all } r \in [0,\chi). \tag{2.13}$$

In view of (2.10) and (2.11), the inverse of the function a on $[0,\infty)$, A, is well-defined on $[0,\chi)$. Finally, given any variable y, we adopt the notational convention

$$[y]_+ = \max\{0, y\}.$$

3. - The existence of classical solutions and a priori estimates of their regularity

Throughout this section, we shall denote by R the rectangle

$$R = (\eta_1, \eta_2) \times (0, T]$$

where

$$-\infty < \eta_1 < \eta_2 < \infty \text{ and } 0 < T < \infty,$$

and consider the following problem

$$u_t = (a(u))_{xx} + (b(u))_x \qquad \text{in } \overline{R}, \tag{3.1}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in [\eta_1, \eta_2], \tag{3.2}$$

$$u(\eta_i, t) = \psi_i(t) \qquad \text{for } t \in [0,T] \text{ and } i = 1,2. \tag{3.3}$$

We recall the following result from [11].

LEMMA 1. *Suppose that the functions a and b satisfy Hypothesis* 1 *and that there exist real constants $\alpha \in (0,1]$, $\varepsilon > 0$, and $M \geq \varepsilon$ such that:*

$$u_0 \in C^{2+\alpha}([\eta_1, \eta_2]), \quad \psi_i \in C^{1+\alpha}([0,T]),$$
$$\varepsilon \leq u_0(x) \leq M \text{ for all } x \in [\eta_1, \eta_2], \quad \varepsilon \leq \psi_i(t) \leq M \text{ for all } t \in [0,T],$$
$$\psi_i(0) = u_0(\eta_i), \quad \psi_i'(0) = (a(u_0))''(\eta_i) + (b(u_0))'(\eta_i),$$

for $i = 1, 2$. Then there exists a unique classical solution of problem (3.1)-(3.3), *$u(x,t)$, with the properties*

$$u \in C^{2,1}(\overline{R}), \quad (a(u))_x \in C^{2,1}(R),$$

and

$$\varepsilon \leq u(x,t) \leq M \qquad \text{for all } (x,t) \in \overline{R}.$$

Henceforth in this section, without further mention, u will denote the solution of problem (3.1)-(3.3) referred to in Lemma 1,

$$M = \sup\{u(x,t) : (x,t) \in \overline{R}\}$$

and

$$v(x,t) = \{(a(u))_x + b(u)\}(x,t).$$

The objective is now to acquire a priori estimates for v in R. The technique which we shall use to obtain these estimates is a modification of the Bernstein technique due to Bénilan [2]. Indeed, in completing Lemma 2 below, we make extensive use of ideas contained in [2] and unpublished notes [3] on this subject.

The basic idea is the following. Set

$$p(x,t) = v(x,t)/\theta(u(x,t)),$$

or alternatively

$$u_x = \{p\ \theta(u) - b(u)\}/a'(u), \tag{3.4}$$

in R, where θ is a positive twice continuously-differentiable function on $(0, M]$. Equation (3.1) may then be rewritten in the form

$$u_t = (p\ \theta(u))_x. \tag{3.5}$$

Differentiating (3.5) with respect to x and subsequently using (3.4) and (3.5) to eliminate all derivatives of u in the resulting expression, one finds that

$$N(p) = 0 \quad \text{in} \quad R$$

where N is the nonlinear parabolic differential operator

$$
\begin{aligned}
N(p) = a'(u)\ p_{xx} &+ \{2\theta'(u) + \frac{a''(u)\theta(u)}{a'(u)}\}\ pp_x \\
&+ \{b'(u) - \frac{a''(u)b(u)}{a'(u)} - \frac{2b(u)\theta'(u)}{\theta(u)}\}\ p_x \\
&+ \frac{\theta(u)\theta''(u)\ p^3}{a'(u)} - \frac{2b(u)\theta''(u)\ p^2}{a'(u)} \\
&+ \frac{b^2(u)\theta''(u)\ p}{a'(u)\theta(u)} - p_t.
\end{aligned} \tag{3.6}
$$

One seeks now combinations of functions θ and $z^{\pm}(x,t)$ such that

$$z^-(x,t) \le 0 \le z^+(x,t) \qquad \text{for all } (x,t) \in R,$$

$$z^-(x,0) \le p(x,0) \le z^+(x,0) \qquad \text{for all } x \in [\eta_1, \eta_2], \tag{3.7}$$

$$z^-(\eta_i,t) \le p(\eta_i,t) \le z^+(\eta_i,t) \qquad \text{for all } t \in [0,T], \tag{3.8}$$

$i = 1, 2$, and

$$N(z^+) \le 0 \le N(z^-) \qquad \text{in } R. \tag{3.9}$$

Given such a combination of functions, by the maximum principle for nonlinear parabolic equations [16], one can conclude that

$$z^-(x,t) \le p(x,t) \le z^+(x,t) \qquad \text{for all } (x,t) \in R.$$

Whence, if

$$\Theta = \sup\{\theta(s) : 0 < s \le M\} < \infty \tag{3.10}$$

one has the estimate

$$\Theta\ z^-(x,t) \le v(x,t) \le \Theta\ z^+(x,t) \qquad \text{in } R.$$

We shall apply this technique to obtain estimates for v in the lemma below. The major difficulty in the proof is the fabrication of suitable functions θ and $z^{\pm}$, and we beg the forbearance of the reader in that the construction of these functions is extremely technical.

LEMMA 2. (a) *Suppose that there exists a constant $K > 0$ such that*

$$
\begin{aligned}
|v(x,0)| &\le K \qquad \textit{for all } x \in [\eta_1, \eta_2], \\
|v(\eta_i,t)| &\le K \qquad \textit{for all } t \in (0,T],
\end{aligned} \tag{3.11}
$$

$i = 1, 2$. *Then*

$$|v(x,t)| \le K \qquad \textit{for all } (x,t) \in R.$$

(b) *Suppose that there exists constants* $K > 0$ *and* $\iota > 0$ *such that*

$$|v(\eta_i, t)| \le K \ \max\{t^{-1/2},\ \iota^{-1/2}\} \qquad \textit{for all } t \in (0, T],$$

$i = 1, 2$. *Then there exists constants* $C > 0$ *and* $\tau > 0$ *which depend only on* K, ι *and* M *such that*

$$|v(x,t)| \le C \ \max\{t^{-1/2},\ \tau^{-1/2}\} \qquad \textit{for all } (x,t) \in R.$$

(c) *Suppose that there exists a constant* $K > 0$ *such that* (3.11) *holds, and that*

$$|\eta_2 - \eta_1| > \rho \qquad \textit{for some } \rho > 0. \tag{3.12}$$

Then, under Hypothesis 2, *there exists constants* $C > 0$ *and* $\delta > 0$ *which depend only on* K, M *and* ρ *such that*

$$|v(x,t)| \le C \ \max\{(x-\eta_1)^{-1}, (\eta_2 - x)^{-1}, \delta^{-1}\} \qquad \textit{for all } (x,t) \in R.$$

(d) *Suppose that* (3.12) *holds. Then, under Hypothesis* 2, *there exist constants* $C > 0$, $\tau > 0$ *and* $\delta > 0$ *which depend only on* M *and* ρ *such that*

$$|v(x,t)| \le C \ \max\{t^{-1/2}, \tau^{-1/2}\} \ \max\{(x-\eta_1)^{-1}, (\eta_2 - x)^{-1}, \delta^{-1}\} \qquad \textit{for all } (x,t) \in R.$$

PROOF. Part (a) is easily proved by choosing $\theta(s) \equiv 1$ and $z^{\pm}(x,t) \equiv \pm K$. We shall therefore concentrate on the remaining cases.

Choose

$$\begin{aligned}\theta(s) = (\mu &+ \int_0^s \{a(M) - a(r)\}dr \\ &+ \lambda[\int_0^s r\{|a''(r)| + |a'(r)b'(r)|\}dr \\ &+ s\int_s^M \{|a''(r)| + |a'(r)b'(r)|\}dr + sa'(M)])^{1/2}\end{aligned}$$

where

$$\mu = 1 + \sup\{b^2(s) : 0 \le s \le M\}$$

and λ is a nonnegative real constant which will be specified later. Cf. [11]. We assert that θ is twice continuously-differentiable and that

$$\begin{aligned}
&0 < \theta(s),\ \ \theta''(s) < 0,\\
&\lambda^2|a'(s)|^2 \leq -4\ \theta''(s)\ \theta^3(s),\\
&\lambda|2\ a'(s)\ \theta'(s) + a''(s)\ \theta(s)| \leq -6\ \theta''(s)\ \theta^2(s),\\
&\lambda|a'(s)b'(s) - a''(s)b(s) - 2a'(s)b(s)\theta'(s)\theta^{-1}(s)|\\
&\leq -8\ \theta''(s)\ \theta^2(s),\\
&|b(s)\theta''(s)| \leq -\theta''(s)\theta(s),
\end{aligned}$$

and

$$a'(s) \leq -2\ \theta''(s)\ \theta(s),$$

for all $s \in (0, M]$. By the definition of μ,

$$1,\ \ |b(s)| \leq \theta(s)$$

for all $s \in (0, M]$. Moreover, differentiating θ,

$$\theta'(s) = \frac{1}{2}(a(M) - a(s) + \lambda[\int_s^M \{|a''(r)| + |a'(r)b'(r)|\}dr + a'(M)])/\theta(s).$$

Hence, for all $s \in (0, M]$,

$$0 < \theta'(s),$$

and, since

$$|a'(M) - a'(s)| \leq \int_s^M |a''(r)|dr,$$

there holds

$$\lambda\ a'(s) \leq 2\ \theta'(s)\ \theta(s).$$

Differentiating a second time,

$$\theta''(s) = -(\frac{1}{2}[a'(s) + \lambda\{|a''(s)| + |a'(s)b'(s)|\}] + (\theta'(s))^2)/\theta(s).$$

Whence,

$$\begin{aligned}
&0 < a'(s) \leq -2\ \theta''(s)\theta(s),\\
&\lambda|a''(s)| \leq -2\ \theta''(s)\theta(s),\\
&\lambda|a'(s)b'(s)| \leq -2\ \theta''(s)\theta(s),
\end{aligned}$$

and

$$(\theta'(s))^2 \leq -\theta''(s)\theta(s),$$

for all $s \in (0, M]$. Gathering all this information together, one can verify that the assertions attributed to the function θ are indeed justified.

To continue, suppose that we can construct a function $f \in C^1(0,\infty)$ with the property

$$1 \le f(\xi), \quad 0 \le -f'(\xi) \le f^3(\xi) \tag{3.13}$$

for all $\xi \in (0,\infty)$; and; where $\gamma = \frac{1}{2}(\eta_2 - \eta_1)/\delta$, for small enough $\delta > 0$, a function $g \in C^2(-\gamma,\gamma)$ with the property

$$1 \le g(\xi), \ |g'(\xi)| \le \lambda g^2(\xi), \ 0 \le g''(\xi) \le \lambda^2 g^3(\xi) \tag{3.14}$$

for all $\xi \in (-\gamma,\gamma)$. Then setting

$$z^{\pm}(x,t) = \pm\tau^{-1/2}\ \delta^{-1} f(t/\tau)\ g(\{x - (\eta_1+\eta_2)/2\}/\delta)$$

direct substitution in (3.6) yields

$$\begin{aligned}
\pm N(z^{\pm}) &= \tau^{-3/2}\ \delta^{-3}\ f^3 g^3\ (a'(u))^{-1}\ [(a'(u))^2\ f^{-2}g^{-3}g''\ \tau \\
&\pm \{2\ a'(u)\theta'(u) + a''(u)\theta(u)\}\ f^{-1}g^{-2}g'\tau^{1/2} \\
&+ \{a'(u)b'(u) - a''(u)b(u) - 2a'(u)b(u)\theta'(u)\theta^{-1}(u)\}\ f^{-2}g^{-3}g'\ \tau\ \delta \\
&+ \theta(u)\theta''(u) \mp 2b(u)\theta''(u)\ f^{-1}g^{-1}\tau^{1/2}\delta \\
&+ b^2(u)\theta''(u)\theta^{-1}(u)\ f^{-2}g^{-2}\tau\ \delta^2 - a'(u)\ f^{-3}f'g^{-2}\delta^2] \\
&\le \tau^{-3/2}\delta^{-3}\ f^3g^3\ (a'(u))^{-1}\ [\lambda^2(a'(u))^2\ \tau \\
&+ \lambda|2a'(u)\theta'(u) + a''(u)\theta(u)|\tau^{1/2} \\
&+ \lambda|a'(u)b'(u) - a''(u)b(u) - 2a'(u)b(u)\theta'(u)\theta^{-1}(u)|\tau\delta \\
&+ \theta(u)\theta''(u) + 2|b(u)\theta''(u)|\ \tau^{1/2}\delta + a'(u)\ \delta^2] \\
&\le -\tau^{-3/2}\delta^{-3}\ f^3g^3\ \{\theta(u)\theta''(u)/a'(u)\} \\
&\quad \cdot \{4\theta^2(u)\tau + 6\theta(u)\tau^{1/2} + 8\theta(u)\tau\delta - 1 + 2\tau^{1/2}\delta + 2\delta^2\}.
\end{aligned}$$

Whence, if θ is bounded on $(0,M]$, i.e. if (3.10) holds, by choosing τ and δ small enough, one can conclude that (3.9) holds.

To complete the proof of the lemma, it therefore suffices to show that one can execute the following steps.

Step 1. Choose λ so that (3.10) holds.

Step 2. Choose a function $f \in C^1(0,\infty)$ such that (3.13) holds.

Step 3. For small enough δ choose a function $g \in C^2(-\gamma,\gamma)$ with $\gamma = \frac{1}{2}(\eta_2 - \eta_1)/\delta$, such that (3.14) holds.

Step 4. Choose τ and δ sufficiently small so that not only (3.9), but also (3.7) and (3.8) hold.

We carry out this strategy considering each part (b), (c) and (d) of the lemma in turn.

(b) Set $\lambda = 0$, and let

$$f(\xi) = \begin{cases} 4\ \xi^{-1/2} & \text{if } 0 < \xi \leq 1, \\ 3 + (2-\xi)^2 & \text{if } 1 < \xi \leq 2, \\ 3 & \text{if } 2 < \xi, \end{cases} \tag{3.15}$$

and

$$g(\xi) \equiv 1.$$

With $\lambda = 0$, (3.10) clearly holds. Furthermore (3.13) and (3.14) can be verified by elementary calculus. Hence, observing that $f(\xi) \sim 4\ \xi^{-1/2}$ as $\xi \downarrow 0$, if $\tau \leq \iota$ and $\delta \leq 1/K$, conditions (3.7) and (3.8) are automatically fulfilled. Plainly then, τ and δ may be chosen so small that (3.9), and (3.7) and (3.8) hold.

(c) Set $\lambda = 1$. Take

$$f(\xi) \equiv 1,$$

and, for $\delta \leq \rho/4$, take

$$g(\xi) = \begin{cases} 2 & \text{if } |\xi| \leq \gamma - 2, \\ 2 - (\gamma - 2 - |\xi|)^3 & \text{if } \gamma - 2 < |\xi| \leq \gamma - 1, \\ 3\ (\gamma - |\xi|)^{-1} & \text{if } \gamma - 1 < |\xi| < \gamma. \end{cases} \tag{3.16}$$

With this choice of λ, (3.10) holds under the provision that Hypothesis 2 is evoked. Moreover, once more (3.13) and (3.14) can be verified by elementary calculus. Observing now that $g(\xi) \sim 3\ (\gamma - |\xi|)^{-1}$ as $|\xi| \uparrow \gamma$, it can be verified that (3.7) and (3.8) are fulfilled if $K \leq \tau^{-1/2}\ \delta^{-1}$. Subsequently, again by choosing δ and τ small enough, (3.9) may also be satisfied.

(d) In this case, we can straightforwardly combine the features of the strategies of parts (b) and (c). Taking $\lambda = 1$ and defining f and g by (3.15) and (3.16) respectively, the necessary steps may be completed by analogy.

The four different parts of Lemma 2 are distinguishable by the different assumptions regarding the a priori boundedness of v on the parabolic boundary of R. Given a bound for v on the initial boundary segment or on the lateral boundary segments of R, an estimate is forthcoming which bounds v in the interior of R up to and including the respective boundary segments. Simultaneously, the lack of a priori information on the behaviour of v on the lateral boundary of R exacts a price in the imposition of Hypothesis 2. Consequentially, the most favourable situation is that in which substantive a priori bounds for v on the parabolic boundary of R exist. Unfortunately, in general, such bounds are hard to come by. This motivates the consideration of all four alternatives in Lemma 2. Notwithstanding, under certain conditions, suitable bounds for v may be obtained. Specifically, the following lemma concerns appropriate estimates.

LEMMA 3. *Suppose that* (3.12) *holds and that there exists a constant* $K > 0$ *such that*

$$|(a(\psi_1))'(t)| \leq K \qquad \text{for all } t \in [0, T]. \tag{3.17}$$

Then given any $\tau > 0$ *there exists a constant* $C > 0$ *which depends only on* ρ, K, M *and* τ *such that*

$$|v(\eta_1, t)| \leq C \ \max\{t^{-1/2}, \tau^{-1/2}\} \qquad \text{for all } t \in (0, T].$$

If, in addition,

$$|(a(u_0))'(x)| \leq K \qquad \text{for all } x \in [\eta_1, \eta_1 + \rho], \tag{3.18}$$

then there exists a constant $C > 0$ *which depends only on* ρ, K *and* M *such that*

$$|v(\eta_1, t)| \leq C \qquad \text{for all } t \in (0, T].$$

PROOF. We use a barrier-function argument.

Pick $\tau > 0$ and choose $\mu > 0$ so small that

$$\mu \ \tau^{1/2} \leq \rho.$$

Subsequently, setting

$$B = \sup\{|b(s)| : 0 \leq s \leq M\} \geq 0,$$

choose

$$\alpha > \tau^{1/2} B + K\tau^{1/2} + \tau K/\mu \tag{3.19}$$

so large that

$$\int_0^M a'(s)/\{\alpha - \tau^{1/2} \ B\} ds \leq \mu. \tag{3.20}$$

From (3.19) and (3.20) it follows that for any $t \in (0, \tau]$, for all $s \in (0, M]$ there holds

$$\alpha + \mu s + t^{1/2} b(s) \geq \alpha - t^{1/2} \ B \geq \alpha - \tau^{1/2} \ B > 0,$$

and consequently

$$0 < \int_0^M a'(s)/\{\alpha + \mu s + t^{1/2} b(s)\} ds \leq \mu.$$

For our present purposes, this last expression can be more conveniently reformulated as

$$\int_0^M a'(s)/\{\alpha\ t^{-1/2} + \mu t^{-1/2} s + b(s)\} ds \leq \mu t^{1/2} \leq \rho \tag{3.21}$$

for any $t \in (0, \tau]$.

Now, pick $t_0 \in (0, T]$. If (3.18) holds or $t_0 \geq \tau$, set

$$\nu = \alpha\ \tau^{-1/2} \quad \text{and} \quad \sigma = \mu\ \tau^{-1/2}.$$

On the other hand, if (3.18) does not hold and $t_0 < \tau$, set

$$\nu = \alpha\ t_0^{-1/2} \quad \text{and} \quad \sigma = \mu\ t_0^{-1/2}.$$

Consider the function $z(x, t)$ defined by

$$\int_{z(x.t)}^{\psi_1(t_0)} a'(s)/\{\nu + \sigma s + b(s)\} ds = x - \eta_1 + \sigma(t_0 - t).$$

in the closure of the domain

$$\Omega = \{(x, t) : \eta_1 < x < \varsigma(t),\ 0 < t \leq t_0\}$$

where

$$\varsigma(t) = \eta_1 + \sigma(t - t_0) + \int_0^{\psi_1(t_0)} a'(s)/\{\nu + \sigma s + b(s)\} ds. \tag{3.22}$$

One can check by differentiation that z is a classical solution of equation (3.1) in Ω. Actually, z is a travelling-wave solution of the type z is a function of $(x - \sigma t)$. Moreover, combining (3.22) with (3.21), $\Omega \subseteq R$. In addition,

$$u(\varsigma(t), t) \geq 0 = z(\varsigma(t), t) \qquad \text{for all } t \in [0, t_0]$$

such that $\varsigma(t) \geq \eta_1$. Whilst, by construction

$$u(\eta_1, t_0) = \psi_1(t_0) = z(\eta_1, t_0) \tag{3.23}$$

and using (3.19)

$$\begin{aligned} (a(z))_t(\eta_1, t) &= \sigma\{\nu + \sigma z(\eta_1, t) + b(z(\eta_1, t))\} \geq \sigma\{\nu - B\} \\ &\geq \mu\tau^{-1/2}\{\alpha\tau^{-1/2} - B\} > K \text{ for all } t \in [0, t_0] \end{aligned} \tag{3.24}$$

such that $z(\eta_1, t) > 0$. Similarly,

$$(3.25)\quad (a(z))_x(x,0) = -\{\nu + \sigma z(x,0) + b(z(x,0))\} \leq -K \text{ for all } x \in (\eta_1, \varsigma(0))$$

such that $z(x,0) > 0$. If now (3.18) holds, if follows immediately from (3.17) and (3.23)-(3.25) that

$$(3.26)\qquad u(x,t) \geq z(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}\backslash\Omega.$$

On the other hand, if (3.18) does not hold, the construction (3.21), (3.22) is such that $\varsigma(0) \leq \eta_1$. Therefore, (3.26) also holds even in this case. Applying the maximum principle to equation (3.1) in Ω [16] it can subsequently be concluded that

$$u(x,t) \geq z(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}.$$

However, in view of (3.23), this means that necessarily

$$u_x(\eta_1, t_0) \geq z_x(\eta_1, t_0).$$

Consequently,

$$\begin{aligned} v(\eta_1, t_0) &= a'(\psi_1(t_0))\ u_x(\eta_1, t_0) + b(\psi_1(t_0)) \\ &\geq a'(\psi_1(t_0))\ z_x(\eta_1, t_0) + b(\psi_1(t_0)) \\ &= a'(z(\eta_1, t_0))\ z_x(\eta_1, t_0) + b(z(\eta_1, t_0)) \\ &= -\{\nu + \sigma\ z(\eta_1, t_0)\} \geq -\{\nu + \sigma M\}. \end{aligned}$$

Since t_0 was arbitrary this establishes a lower bound on $v(\eta_1, t)$ of the type sought. To obtain the corresponding upper bound we may proceed analogously. Alternatively, we can merely apply the above technique to the equation

$$\tilde{u}_t = (\tilde{a}(\tilde{u}))_{xx} + (\tilde{b}(\tilde{u}))_x \qquad \text{in } R$$

where

$$\tilde{u}(x,t) = 2M - u(x,t),\ \ \tilde{a}(s) = -a(2M - s) \text{ and } \tilde{b}(s) = -b(2M - s).$$

We have now almost amassed sufficient munition to tackle our existence theorems. The only implement still wanting is one to convert the estimates on $v(x,t)$ in R into continuity estimates for the function u with respect to x and t. We realize this implement with the lemma below which constitutes the last in this section. Here, we enlist the additional notation:

$$R' = (\xi_1, \xi_2) \times (\tau, T],$$

with

$$\eta_1 \leq \xi_1 < \xi_2 \leq \eta_2 \quad \text{and} \quad 0 \leq \tau < T,$$

and

$$R'_\delta = (\xi_1 + \delta, \xi_2 - \delta) \times (\tau, T]$$

for $\delta \in (0, (\xi_2 - \xi_1)/2)$. For completeness, we recall (2.12), (2.13).

LEMMA 4. *Suppose that there exists a constant $K > 0$ such that*

$$|v(x,t)| \le K \qquad \text{for all } (x,t) \in \overline{R}'. \tag{3.27}$$

Then, given any $\delta \in (0, (\xi_2 - \xi_1)/2)$ there exists a constant $C > 0$ which depends only on K, δ and M, such that

$$\begin{aligned} &A([a(u(x_1,t_1)) - C|x_1 - x_2| - C|t_1 - t_2|^{1/2}]_+) \\ &\le A(\min[a(M), a(u(x_2,t_2)) + C|x_1 - x_2| + C|t_1 - t_2|^{1/2}]) \\ &+ C|t_1 - t_2|^{1/2} \end{aligned} \tag{3.28}$$

for all (x_1,t_1), $(x_2,t_2) \in \overline{R}'_\delta$.

PROOF. Fix (x_1,t_1), $(x_2,t_2) \in \overline{R}'_\delta$, and let $x^* = (x_1 + x_2)/2$. Suppose in the first instance that $0 < |t_1 - t_2| \le \delta^2$. Set $t^- = \min\{t_1, t_2\}$, $t^+ = \max\{t_1, t_2\}$, and $\mu = |t_1 - t_2|^{1/2}$. Then, integrating equation (3.1) over the rectangle $[x^* - \mu, x^* + \mu] \times [t^-, t^+]$,

$$\int_{x^*-\mu}^{x^*+\mu} \{u(x,t^+) - u(x,t^-)\}dx = \int_{t^-}^{t^+} \{v(x^* + \mu, t) - v(x^* - \mu, t)\}dt.$$

Thus, by (3.27)

$$\left| \int_{x^*-\mu}^{x^*+\mu} \{u(x,t^+) - u(x,t^-)\}dx \right| \le 2K|t^+ - t^-|. \tag{3.29}$$

Whence

$$\int_{x^*-\mu}^{x^*+\mu} u(x,t_1)dx \le \int_{x^*-\mu}^{x^*+\mu} u(x,t_2)dx + 2K|t_1 - t_2|. \tag{3.30}$$

Next, setting

$$L = K + \sup\{|b(s)| : 0 \le s \le M\},$$

observe that by (3.27),

$$|(a(u))_x|(x,t) \le |v(x,t)| + |b(u(x,t))| \le L \tag{3.31}$$

for all $(x,t) \in \overline{R}'$. Hence, for $i = 1, 2$,

$$|a(u(x,t_i)) - a(u(x^*,t_i))| \leq L|x - x^*| \leq L\mu$$

for all $x \in [x^* - \mu, x^* + \mu]$. It follows that

$$u(x,t_1) \geq A([a(u(x^*,t_1)) - L\mu]_+)$$

and

$$u(x,t_2) \leq A(\min[a(M), a(u(x^*,t_2)) + L\mu])$$

for all $x \in [x^* - \mu, x^* + \mu]$. Substituting these inequalities in (3.30) and recalling that $\mu = |t_2 - t_1|^{1/2}$ yields

$$\begin{aligned} &A([a(u(x^*,t_1)) - L|t_1 - t_2|^{1/2}]_+) \\ &\leq A(\min[a(M), a(u(x^*,t_2)) + L|t_1 - t_2|^{1/2}]) + K|t_1 - t_2|^{1/2}. \end{aligned}$$

However, in view of the continuity and boundedness of u, setting $C = \max\{L, K, M/\delta\}$, this implies

$$\begin{aligned} &A([a(u(x^*,t_1)) - C|t_1 - t_2|^{1/2}]_+) \\ &\leq A(\min[a(M), a(u(x^*,t_2)) + C|t_1 - t_2|^{1/2}]) + C|t_2 - t_1|^{1/2} \end{aligned}$$

for all $t_1, t_2 \in [\tau, T]$. Inequality (3.28) follows from the observation that by a second application of (3.31)

$$|a(u(x_i,t_i)) - a(u(x^*,t_i))| \leq L|x_i - x^*| \leq C|x_1 - x_2|,$$

for $i = 1, 2$.

Inspiration to use the integral inequality (3.29) in Lemma 4 to obtain the continuity estimate (3.28) from the bound (3.27) was acquired from the application of this tool for a related problem studied by van Duyn and Peletier [9]. Indeed, if $1/a'(s) \in L^\infty(0,M)$, then (3.31) implies that $u(x,t)$ is uniformly Lipschitz continuous with respect to x in $\overline{R}'$ with coefficient $L\|1/a'\|_{L^\infty(0,M)}$. Thus, combining (3.29) with a lemma of van Duyn and Peletier [9], for any $\delta \in (0, (\xi_2 - \xi_1)/2)$ one can obtain a $C^{1,1/2}(\overline{R}')$ estimate of the continuity of u which depends only on K, M and δ. On the other hand, if $a'(s) \in L^\infty(0,M)$ and $b'(s) \in L^\infty(0,M)$, using a result of Gilding [10], one can derive a $C^{1,1/2}(\overline{R}'_\delta)$ estimate of the continuity of $a(u)$ which depends only on K, M and δ, cf. [11]. However, to give credit where credit is due, the rudiment of using (3.29) to obtain the continuity of u can be found in [15].

4. - Existence

The existence result for the Cauchy problem for equation (1.1) which we are able to prove using the a priori estimates derived in the previous section is the following.

THEOREM 1. *Under Hypothesis* 1, *Problem* 1 *admits at least one generalized solution.*

PROOF. Pick

$$M > \sup\{u_0(x) : x \in \mathbb{R}\}$$

and a positive integer k_0 so large that

$$2^{-k_0+1} \leq a(M) - a(u_0(x)) \qquad \text{for all } x \in \mathbb{R}.$$

For $k \geq k_0$, define the function $v_k \in C(\mathbb{R})$ by

$$(4.1) \quad v_k(x) = \begin{cases} 0 & \text{for } |x| \leq k-1 \\ 3\{a(M) - 2^{-k}\}\{|x| - k + 1\} & \text{for } k-1 < |x| \leq k - 2/3 \\ a(M) - 2^{-k} & \text{for } |x| > k - 2/3. \end{cases}$$

Set

$$w_k(x) = \max\{a(u_0(x)), v_k(x)\}.$$

Let

$$(4.2) \qquad J(s) = \begin{cases} \exp\{-1/(1-s^2)\} & \text{for } |s| < 1 \\ 0 & \text{for } |s| \geq 1, \end{cases}$$

and for all $\mu > 0$ define

$$(4.3) \qquad J_\mu(s) = \mu^{-1} J(s/\mu)/\|J\|_{L^1(\mathbb{R})}.$$

Next, for all $k \geq k_0$, choose $\mu_k \in (0, \min\{1/k, 1/6\})$ so small that

$$(4.4) \qquad |\int_{\mathbb{R}} J_{\mu_k}(x-y) w_k(y) dy - w_k(x)| \leq 2^{-k-2}$$

for all $x \in \mathbb{R}$. Such a choice is possible by standard properties of the mollifier J [1]. Finally, for all $k \geq k_0$, define $u_{0,k}(x)$ by

$$(4.5) \qquad u_{0,k}(x) = A(2^{-k} + \int_{\mathbb{R}} J_{\mu_k}(x-y) w_k(y) dy)$$

where A denotes the inverse of a defined by (2.12), (2.13).

By the particular choice of k_0, $\{v_k\}$, $\{w_k\}$, and $\{\mu_k\}$; the sequence of functions $\{u_{0,k}\}$ has been so contrived that it has the following properties:

(i) $u_{0,k}(x) \leq M$ for all $x \in \mathbb{R}$ and $k \geq k_0$;

(ii) there exists an $\varepsilon_k > 0$ such that $\varepsilon_k \leq u_{0,k}(x)$ for all $x \in \mathbb{R}$ and $k \geq k_0$;

(iii) there exists an $\alpha_k \in (0,1]$ such that $u_{0,k} \in C^{2+\alpha_k}(\mathbb{R})$ for all $k \geq k_0$;

(iv) $u_{0,k+1}(x) \leq u_{0,k}(x)$ for all $x \in \mathbb{R}$ and $k \geq k_0$;

(v) $u_{0,k} \downarrow u_0$ as $k \uparrow \infty$ uniformly on compact subsets of $\mathbb{R}$;

(vi) $u_{0,k}(x) = M$ for all $|x| \geq k - 1/2$ and $k \geq k_0$;

(vii) if $a(u_0)$ is uniformly Lipschitz continuous on $\mathbb{R}$ there exists a constant $K > 0$ which does not depend on k such that $|(a(u_{0,k}))' + b(u_{0,k})|(x) \leq K$ for all $x \in \mathbb{R}$ and $k \geq k_0$.

For arbitrary $k \geq k_0$, let

$$S_k = (-k, k) \times (0, T],$$

and consider the following boundary-value problem:

$$u_t = (a(u))_{xx} + (b(u))_x \qquad \text{in } \overline{S}_k, \tag{4.6}$$

$$u(x,0) = u_{0,k}(x) \qquad \text{for all } |x| \leq k, \tag{4.7}$$

$$u(\pm k, t) = M \qquad \text{for all } t \in [0,T]. \tag{4.8}$$

By Lemma 1, problem (4.6)-(4.8) has a unique classical solution $u_k(x,t)$ which satisfies $\varepsilon_k \leq u_k(x,t) \leq M$ for all $(x,t) \in \overline{S}_k$. Moreover, in view of property (iv) of the sequence $\{u_{0,k}\}$, by the maximum principle for equation (4.6),

$$u_{k+1}(x,t) \leq u_k(x,t) \qquad \text{for all } (x,t) \in \overline{S}_k$$

and $k \geq k_0$. Hence, we can define

$$u(x,t) = \lim_{k \uparrow \infty} u_k(x,t) \qquad \text{for all } (x,t) \in \overline{S}.$$

We assert that u is the generalized solution of Problem 1 which we seek.

Since u is the pointwise limit of the sequence $\{u_k\}$, u is defined, real, nonnegative, and bounded on $\overline{S}$. Moreover, because each u_k is a classical solution of (4.6), it satisfies part (ii) of the definitions of a generalized supersolution and subsolution of equation (1.1) in S_k. Hence, in the limit, u satisfies part (ii) of the definitions of a generalized supersolution and subsolution of equation (1.1) in S. Furthermore, we note that by construction, u satisfies (2.1). To show that u is indeed a generalized solution of Problem 1, it therefore remains to prove that u is continuous in $\overline{S}$.

By Lemma 3, condition (4.8) and property (vi) of the sequence $\{u_{0,k}\}$, there exists a constant $C_0 > 0$ which depends only on M such that

$$|((a(u_k))_x + b(u_k)|(-k,t) \leq C_0 \text{ for all } t \in (0,T].$$

Equally well, applying the change of variables $x \to -x$, $b \to -b$, there holds

$$|(a(u_k))_x + b(u_k)|(k,t) \leq C_0 \text{ for all } t \in (0,T].$$

Consequently, by Lemma 2(b), for any $\tau \in (0,T)$, there exists a constant $C_1 > 0$ which depends only on M and τ such that

$$|(a(u_k))_x + b(u_k)|(x,t) \leq C_1 \text{ for all } (x,t) \in [-k,k] \times [\tau,T].$$

But then, by Lemma 4, there exists a further constant C which again depends on M and τ but not on k, such that for all $k \geq k_0$,

$$\begin{aligned} &A([a(u_k(x_1,t_1)) - C|x_1 - x_2| - C|t_1 - t_2|^{1/2}]_+) \\ &\leq A(\min[a(M), a(u_k(x_2,t_2)) + C|x_1 - x_2| + C|t_1 - t_2|^{1/2}]) \\ &+ C|t_1 - t_2|^{1/2} \end{aligned} \tag{4.9}$$

for all (x_1,t_1), $(x_2,t_2) \in [-k+1, k-1] \times [\tau,T]$. Taking the limit $k \uparrow \infty$, (4.9) will also hold with u_k replaced by u, for all (x_1,t_1), $(x_2,t_2) \in (-\infty,\infty) \times [\tau,T]$. Whence, u is continuous in $(-\infty,\infty) \times [\tau,T]$. Since $\tau \in (0,T]$ was arbitrary, it follows that u is continuous in S.

Suppose now that $a(u_0)$ is uniformly Lipschitz continuous on $\mathbb{R}$. Then, utilizing property (vii) of the sequence $\{u_{0,k}\}$ in combination with Lemma 2(a), by analogy to the previous argument it can be concluded that u is continuous in $\overline{S}$. The final outstanding detail in the proof of Theorem 1 is therefore to confirm that for any $x_0 \in \mathbb{R}$

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}}} u(x,t) \leq u_0(x_0) \tag{4.10}$$

and

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}}} u(x,t) \geq u_0(x_0) \tag{4.11}$$

in the event that $a(u_0)$ is not uniformly Lipschitz continuous on $\mathbb{R}$.

Fix $x_0 \in \mathbb{R}$. Observe that by the monotonicity of the sequence $\{u_k\}$, for any $k > \max\{k_0, |x_0|\}$

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}}} u(x,t) \leq \limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}_k}} u_k(x,t) = u_{0,k}(x_0).$$

Hence, in the limit $k \uparrow \infty$, (4.10) must hold. On the other hand, if $u_0(x_0) = 0$, since u is nonnegative in $\overline{S}$, trivially (4.11) holds. All that therefore remains is to verify (4.11) in the event that $u_0(x_0) > 0$. For confirmation in this case, pick $\varepsilon \in (0, u_0(x_0))$, and choose $\delta > 0$ such that $u_0(x) > u_0(x_0) - \varepsilon$ for all $|x - x_0| < \delta$. Set

$$\tilde{u}_0(x) = \begin{cases} 0 & \text{for } |x - x_0| \geq \delta \\ A(\{a(u_0(x_0)) - \varepsilon\}\{1 - |x - x_0|/\delta\}) & \text{for } |x - x_0| < \delta. \end{cases}$$

Next, set $\tilde{M} = M$, and for $k \geq \tilde{k}_0 = k_0$ construct a sequence of functions $\tilde{u}_{0,k}$ approximating $\tilde{u}_0$ in the identical way to the sequence $\{u_{0,k}\}$ approximating u_0. Denote by $\{\tilde{u}_k\}$ the sequence of solutions of the corresponding boundary-value problems, and its limit by $\tilde{u}$. Note that $\tilde{u}_0$ has been so fabricated that $a(\tilde{u}_0)$ is uniformly Lipschitz continuous on $\mathbb{R}$. Thus by the preceeding argument, $\tilde{u}$ can be shown to be a continuous generalized solution of equation (1.1) in $\overline{S}$. However, the fabrication of $\tilde{u}_0$ is also such that

$$\tilde{u}_{0,k+1}(x) \leq u_{0,k}(x) \qquad \text{for all } x \in [-k, k]$$

and

$$\tilde{u}_{0,k+1}(\pm k, t) \leq \tilde{M} = u_k(\pm k, t) \qquad \text{for all } t \in (0, T]$$

for any $k \geq k_0$. Hence, by the maximum principle for classical solutions of equation (1.1),

$$\tilde{u}_{k+1}(x, t) \leq u_k(x, t) \qquad \text{for all } (x, t) \in \overline{S}_k$$

for any $k \geq k_0$. Whence, letting $k \uparrow \infty$, it follows that

$$\tilde{u}(x, t) \leq u(x, t) \qquad \text{for all } (x, t) \in \overline{S}.$$

However, since $\tilde{u}$ is known to be continuous in $\overline{S}$, this implies that

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}}} u(x, t) \geq \liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in\overline{S}}} \tilde{u}(x, t) = \tilde{u}_0(x_0) = u_0(x_0) - \varepsilon,$$

by definition. Recalling that $\varepsilon \in (0, u_0(x))$ was chosen arbitrarily, this confirms that (4.11) also holds in this last outstanding case in the proof of Theorem 1.

The proof of the theorem has therefore been completed.

Let us turn now to the corresponding result for the Cauchy-Dirichlet problem. For this problem, an additional hypothesis will be useful.

HYPOTHESIS 3. *The function $a(\psi)$ is uniformly Lipschitz continuous on* $[0, T]$.

THEOREM 2. *Under Hypotheses* 1 *and* 3, *and/or, under Hypotheses* 1 *and* 2, *Problem* 2 *admits at least one generalized solution.*

PROOF. The strategy for proving Theorem 2 is similar to that for Theorem 1.

Pick

$$M > \sup\{u_0(x) : 0 \le x < \infty\}, \ \sup\{\psi(t) : 0 \le t \le T\}.$$

and a positive integer k_0 so large that

$$2^{-k_0+1} \le a(M) - a(u_0(x)) \qquad \text{for all } x \in [0, \infty),$$

and

$$2^{-k_0+1} \le a(M) - a(\psi(t)) \qquad \text{for all } t \in [0, T].$$

For $k \ge k_0$, pick $\nu_k \in (0, 1/k) \cap (0, T]$ so small that

$$|a(u_0(x)) - a(u_0(0))| \le 2^{-k-7} \qquad \text{for all } x \in [0, \nu_k]$$

and

$$|a(\psi(t)) - a(\psi(0))| \le 2^{-k-7} \qquad \text{for all } t \in [0, \nu_k],$$

and set

$$\sigma_k = a(\psi(\nu_k)) - a(u_0(\nu_k)),$$

$$w_k(x) = \begin{cases} a(u_0(\nu_k)) + [\sigma_k]_+ & \text{for } x \le \nu_k \\ \max\{a(u_0(x)) + [\sigma_k]_+, v_k(x)\} & \text{for } x > \nu_k, \end{cases}$$

where $v_k(x)$ is defined by (4.1), and

$$\theta_k(t) = \begin{cases} a(\psi(\nu_k)) + [-\sigma_k]_+ & \text{for } t \le \nu_k \\ a(\psi(t)) + [-\sigma_k]_+ & \text{for } \nu_k < t \le T \\ a(\psi(T)) + [-\sigma_k]_+ & \text{for } t > T. \end{cases}$$

Next, for all $k \ge k_0$, choose $\mu_k \in (0, \min\{\nu_k/2, 1/6\})$ so small that (4.4) holds for all $x \in \mathbb{R}$ and

$$\left| \int_{\mathbb{R}} J_{\mu_k}(t-s)\theta_k(s)ds - \theta_k(t) \right| \le 2^{-k-2}$$

for all $t \in \mathbb{R}$. For reference, the function J_μ is given by (4.2), (4.3). Finally, for all $k \ge k_0$ define $u_{0.k}$ by (4.5) and set

$$\psi_k(t) = A(2^{-k} + \int_{\mathbb{R}} J_{\mu_k}(t-s)\theta_k(s)ds),$$

where by convention A denotes the inverse of a defined in (2.12), (2.13).

The constructed sequence of functions $\{u_{0,k}, \psi_k\}$ can be shown to possess the following properties:

(i) $u_{0,k}(x) \leq M$ for all $x \in [0, \infty)$, and $\psi_k(t) \leq M$ for all $t \in [0, T]$, for all $k \geq k_0$;

(ii) there exists an $\varepsilon_k > 0$ such that $\varepsilon_k \leq u_{0.k}(x)$ for all $x \in [0, \infty)$, and $\varepsilon_k \leq \psi_k(t)$ for all $t \in [0, T]$, for all $k \geq k_0$;

(iii) there exists an $\alpha_k \in (0, 1]$ such that $u_{0,k} \in C^{2+\alpha_k}([0, \infty))$ and $\psi_k \in C^{1+\alpha_k}([0, T])$ for all $k \geq k_0$;

(iv) $u_{0.k}(0) = \psi_k(0)$ and $(a(u_{0.k}))''(0) + (b(u_{0,k}))'(0) = \psi_k'(0)$ for all $k \geq k_0$;

(v) $u_{0.k+1}(x) \leq u_{0,k}(x)$ for all $x \in [0, \infty)$, and $\psi_{k+1}(t) \leq \psi_k(t)$ for all $t \in [0, T]$, for all $k \geq k_0$;

(vi) $u_{0.k} \downarrow u_0$ as $k \uparrow \infty$ uniformly on compact subsets of $[0, \infty)$, and $\psi_k \downarrow \psi$ as $k \uparrow \infty$ uniformly on $[0, T]$;

(vii) $u_{0.k}(x) = M$ for all $x \geq k - 1/2$ and $k \geq k_0$;

(viii) if $a(u_0)$ is uniformly Lipschitz continuous on $[0, \infty)$ there exists a constant $K > 0$ which does not depend on k such that

$$|(a(u_{0,k}))' + b(u_{0,k})|(x) \leq K \text{ for all } x \in [0, \infty) \text{ and } k \geq k_0;$$

(ix) if Hypothesis 3 holds there exists a constant $L > 0$ which does not depend on k such that

$$|(a(\psi_k))'|(t) \leq L \text{ for all } t \in [0, T] \text{ and } k \geq k_0.$$

Subsequently, for all $k \geq k_0$, let

$$H_k = (0, k) \times (0, T]$$

and consider the solution $u_k(x, t)$ of the following boundary-value problem:

$$\begin{aligned} &u_t = (a(u))_{xx} + (b(u))_x && \text{in } \overline{H}_k, \\ &u(x, 0) = u_{0,k}(x) && \text{for } x \in [0, k], \\ &u(0, t) = \psi_k(t) && \text{for } t \in [0, T], \\ &u(k, t) = M && \text{for } t \in [0, T]. \end{aligned} \tag{4.12}$$

The function u_k exists by Lemma 1. Moreover, in view of property (v) of the sequence $\{u_{0,k}, \psi_k\}$ and the maximum principle; $u_{k+1}(x, t) \leq u_k(x, t)$ for all $(x, t) \in \overline{H}_k$ and $k \geq k_0$. Thus, one can define the limit

$$u(x, t) = \lim_{k \uparrow \infty} u_k(x, t) \text{ for all } (x, t) \in \overline{H}.$$

We assert, just as we did in the proof of Theorem 1, that the function u is the sought-after generalized solution of equation (1.1). Similarly to in the proof of Theorem 1, the critical step in verifying this assertion is proving the continuity of u in $\overline{H}$.

Applying the change of variables $x \to -x$, $b \to -b$ in Lemma 3, condition (4.12) and property (vii) of the sequence $\{u_{0.k}, \psi_k\}$ imply the existence of a constant $C_0 > 0$ which depends only on M such that

$$|(a(u_k))_x + b(u_k)|(k,t) \leq C_0 \qquad \text{for all } t \in (0,T].$$

Whilst, if Hypothesis 3 holds; by property (ix) of the sequence $\{u_{0.k}, \psi_k\}$, one can conclude that given any $\tau > 0$ there holds

$$|(a(u_k))_x + b(u_k)|(0,t) \leq C_0 \ \max\{t^{-1/2}, \tau^{-1/2}\} \qquad \text{for all } t \in (0,T],$$

where now C_0 also depends on τ and L. Consequently, by Lemma 2(b), there exists a constant $C_1 > 0$ which depends only on M, L and τ such that

$$|(a(u_k))_x + b(u_k)|(x,t) \leq C_1 \qquad \text{for all } (x,t) \in [0,k] \times [\tau, T].$$

On the other hand, if Hypothesis 2 holds, Lemma 2(d) implies that given any $\delta \in (0,1)$ there exists a constant $C_1 > 0$ which depends only on M, δ and τ such that

$$|(a(u_k))_x + b(u_k)|(x,t) \leq C_1 \text{ for all } (x,t) \in [\delta, k-\delta] \times [\tau, T].$$

In this way, analogously to in the proof of Theorem 1, the continuity of u in H may be demonstrated. Similarly, if $a(u_0)$ is uniformly Lipschitz continuous on $[0,\infty)$, utilizing property (viii) of the sequence $\{u_{0,k}, \psi_k\}$, and applying Lemma 2(a) or 2(c), the continuity of u in $(0,\infty) \times [0,T]$ may be established. In fact, by extension, if Hypothesis 3 holds and $a(u_0)$ is uniformly Lipschitz continuous on $[0,\infty)$, the continuity of u in $\overline{H}$ follows. However, given this information, the continuity of u in $\overline{H}$ in the remaining cases can be established using exactly the same tricks as those completing the proof of Theorem 1.

The final result in this section is existence for the first boundary-value problem for equation (1.1), similar to that established for the Cauchy problem and the Cauchy-Dirichlet problem in Theorems 1 and 2 respectively. We do not intend to burden the reader with the details of the proof. Suffice to note that it may be completed by extension of the ideas used in the proofs of the preceeding theorems.

HYPOTHESIS 4. *The functions $a(\psi^-)$ and $a(\psi^+)$ are uniformly Lipschitz continuous on* $[0,T]$.

THEOREM 3. *Under Hypotheses* 1 *and* 4, *and/or, under Hypotheses* 1 *and* 2, *Problem* 3 *admits at least one generalized solution.*

5. - Uniqueness

The goal of this section is to establish that, under the hypotheses of the theorems in the previous section, the constructed generalized solutions are the only admissible generalized solutions of the respective problems. To achieve this goal we refine arguments of Diaz and Kersner [8]. We use five lemmata which follow.

LEMMA 5. *Suppose that Hypothesis* 1 *holds and let D denote a domain of the form* (2.7), (2.8). *Let $\underline{u}$ denote a generalized subsolution of equation* (1.1) *in D, and $\overline{u}$ denote a generalized supersolution of equation* (1.1) *in D for which*

$$0 < \mu \leq \max\{\underline{u}(x,t), \overline{u}(x,t)\} \leq M < \infty \qquad \textit{for all } (x,t) \in \overline{D} \tag{5.1}$$

with some real constants μ and M. Set

$$\alpha(x,t) = \int_0^1 a'(\lambda \underline{u}(x,t) + (1-\lambda)\overline{u}(x,t))\, d\lambda \tag{5.2}$$

and

$$\beta(x,t) = \int_0^1 b'(\lambda \underline{u}(x,t) + (1-\lambda)\overline{u}(x,t))\, d\lambda.$$

Then there exist constants α, $\overline{\alpha}$ and $\overline{\beta}$ which depend only on μ and M such that

$$0 < \underline{\alpha} \leq \alpha(x,t) \leq \overline{\alpha} < \infty \qquad \textit{for all } (x,t) \in \overline{D} \tag{5.3}$$

and

$$|\beta(x,t)| \leq \overline{\beta} < \infty \qquad \textit{for all } (x,t) \in \overline{D}. \tag{5.4}$$

Moreover α and β are continuous in $\overline{D}$.

PROOF. We prove (5.3) only. The proof of (5.4) is similar.
Set

$$u^+(x,t) = \max\{\underline{u}(x,t), \overline{u}(x,t)\}$$

and

$$u^-(x,t) = \min\{\underline{u}(x,t), \overline{u}(x,t)\}.$$

Suppose firstly that $u^-(x,t) \geq \mu/2$. Then, by (5.1)

$$\mu/2 \leq \lambda \underline{u}(x,t) + (1-\lambda)\overline{u}(x,t) \leq M$$

for any $\lambda \in (0,1)$. Therefore

$$\inf\{a'(s) : \mu/2 \le s \le M\} \le \alpha(x,t) \le \sup\{a'(s) : \mu/2 \le s \le M\}.$$

Supposing, on the other hand, that $u^-(x,t) \le \mu/2$, then in view of (5.1) we can rewrite (5.2) as

$$\alpha(x,t) = \{a(u^+(x,t)) - a(u^-(x,t))\}/\{u^+(x,t) - u^-(x,t)\},$$

inferring the estimate

$$\{a(\mu) - a(\mu/2)\}/M \le \alpha(x,t) \le a(M)/\{\mu - \mu/2\}.$$

In the light of the verification of (5.3) and (5.4), the continuity of α and β is a natural consequence of the continuity of $\underline{u}$ and $\overline{u}$.

LEMMA 6. *Let D denote a domain of the form* (2.7), (2.8) *with*

$$-\infty < \eta_1 < \eta_2 < \infty, \tag{5.5}$$

let $\alpha \in C^\infty(\overline{D})$ and $\beta \in C^\infty(\overline{D})$ be such that

$$0 < \underline{\alpha} \le \alpha(x,t) \le \overline{\alpha} < \infty \qquad \textit{for all } (x,t) \in \overline{D}$$

and

$$|\beta(x,t)| \le \overline{\beta} < \infty \qquad \textit{for all } (x,t) \in \overline{D}$$

for some real constants $\underline{\alpha}, \overline{\alpha}$ and $\overline{\beta} > 0$, and let $\omega \in C^\infty(\eta_1, \eta_2)$ be such that

$$0 \le \omega(x) \le 1 \qquad \textit{for all } x \in (\eta_1, \eta_2) \tag{5.6}$$

and

$$\omega(x) = 0 \qquad \textit{for all } x \in (\eta_1, \eta_2)\backslash(\rho_1, \rho_2) \tag{5.7}$$

where

$$\eta_1 < \rho_1 \le \rho_2 < \eta_2. \tag{5.8}$$

Define the intervals

$$I_1(\rho) = (\eta_1, \rho] \textit{ and } I_2(\rho) = [\rho, \eta_2), \tag{5.9}$$

and for $i = 1, 2$, let

$$\alpha_i = \sup\{\alpha(x,t) : (x,t) \in I_i(\rho_i) \times [\tau_1, \tau_2]\}, \tag{5.10}$$

$$\beta_i = \sup\{[(-1)^i \beta(x,t)]_+ : (x,t) \in I_i(\rho_i) \times [\tau_1, \tau_2]\}, \tag{5.11}$$

and

$$Z_i(x,t) = erfc\left(\frac{(-1)^i(x-\rho_i) - \beta_i(\tau_2 - t)}{\sqrt{4\alpha_i(\tau_2 - t)}}\right) \tag{5.12}$$

where $erfc$ *denotes the complementary error function. Set*

$$Z(x,t) = \min\{Z_1(x,t), 1, Z_2(x,t)\}. \tag{5.13}$$

Finally, for $i = 1, 2$, *let* $\xi_i \in I_i(\rho_i)$ *and set*

$$\gamma_i = 1 + \sup\{[(-1)^i\beta(x,t)/\alpha(x,t)]_+ : (x,t) \in I_i(\xi_i) \times [\tau_1, \tau_2]\} \tag{5.14}$$

and

$$C_i = Z(\xi_i, \tau_1)\ \gamma_i/[1 - \exp\{-\gamma_i|\xi_i - \eta_i|\}]. \tag{5.15}$$

Then, there exists a unique solution $z \in C^\infty(\overline{D})$ *to the problem*

$$0 = \alpha\ z_{xx} - \beta\ z_x + z_t \quad \text{in } \overline{D}, \tag{5.16}$$

$$z(x,\tau_2) = \omega(x) \quad \text{for } x \in (\eta_1, \eta_2), \tag{5.17}$$

$$z(\eta_1, t) = z(\eta_2, t) = 0 \quad \text{for } t \in [\tau_1, \tau_2]. \tag{5.18}$$

Furthermore,

$$0 \le z(x,t) \le Z(x,t) \quad \text{for all } (x,t) \in \overline{D}, \tag{5.19}$$

$$0 \le z_x(\eta_1, t) \le C_1 \quad \text{for all } t \in [\tau_1, \tau_2], \tag{5.20}$$

$$0 \ge z_x(\eta_2, t) \ge -C_2 \quad \text{for all } t \in [\tau_1, \tau_2], \tag{5.21}$$

and there exists a constant K *which depends only on* $\underline{\alpha}, \overline{\beta}$,

$$\omega^* = \int_{-\infty}^{\infty} \{\omega'(x)\}^2 dx,$$

and $|\tau_2 - \tau_1|$, *such that*

$$\int\int_D (z_x)^2\ dxdt \le K \quad \text{and} \quad \int\int_D (z_{xx})^2\ dxdt \le K. \tag{5.22}$$

PROOF. Observing that (5.16)-(5.18) is a backward parabolic problem, the existence, uniqueness and regularity of z follow from the classical theory of linear parabolic equations with smooth coefficients [14]. Furthermore, by (5.6) and a standard application of the maximum principle [16],

$$0 \le z(x,t) \le 1 \quad \text{for all } (x,t) \in \overline{D}. \tag{5.23}$$

Hence, noting that $Z(x,t)$ is a classical supersolution of (5.16) in the domain $\Omega = \{(x,t) \in (\rho_1, \rho_2) \times [\tau_1, \tau_2) : x < \rho_1 - \beta_1(\tau_2 - t) \text{ or } x > \rho_2 + \beta_2(\tau_2 - t)\}$ with the properties $Z(x,t) = 1 \geq z(x,t)$ or $Z(x,t) \geq 0 = z(x,t)$ for all $(x,t) \in (\overline{\Omega} \backslash \Omega)$, by a second application of the maximum principle, we have $Z(x,t) \geq z(x,t)$ for all $(x,t) \in \overline{\Omega}$. Whence, since by the definition of Z and (5.23), $Z(x,t) = 1 \geq z(x,t)$ for all $(x,t) \in \overline{D} \backslash \overline{\Omega}$, (5.19) follows.

The left-hand inequality in (5.20) is an immediate consequence of (5.18) and (5.19). To verify the right-hand inequality in (5.20) we use a barrier function argument. Consider the function

$$w(x,t) = C_1[1 - \exp\{-\gamma_1(x - \eta_1)\}]/\gamma_1$$

defined in the domain

$$\Omega_1 = (\eta_1, \xi_1) \times [\tau_1, \tau_2).$$

The function w is a classical supersolution of (5.16) in Ω_1 with the properties

$$\begin{aligned} &w(x, \tau_2) \geq 0 = z(x, \tau_2) && \text{for all } x \in (\eta_1, \xi_1), \\ &w(\eta_1, t) = 0 = z(\eta_1, t) && \text{for all } t \in [\tau_1, \tau_2], \end{aligned} \tag{5.24}$$

and

$$w(\xi_1, t) = Z(\xi_1, \tau_1) \geq Z(\xi_1, t) \geq z(\xi_1, t) \qquad \text{for all } t \in [\tau_1, \tau_2].$$

Hence, by the classical maximum principle for our backward parabolic problem,

$$w(x,t) \geq z(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}_1.$$

However, in view of (5.24) this implies

$$w_x(\eta_1, t) \geq z_x(\eta_1, t) \qquad \text{for all } t \in [\tau_1, \tau_2],$$

which is equivalent to the right-hand inequality of (5.20). The proof of (5.21) is similar.

To justify (5.22), multiply (5.16) by z_{xx} and integrate by parts over $[\eta_1, \eta_2] \times [\tau, \tau_2]$ for any $\tau \in [\tau_1, \tau_2)$. This yields

$$\begin{aligned} &2 \int_{\tau}^{\tau_2} \int_{\eta_1}^{\eta_2} \alpha (z_{xx})^2 \, dx dt + \int_{\eta_1}^{\eta_2} \{z_x(x, \tau)\}^2 \, dx \\ &= 2 \int_{\tau}^{\tau_2} \int_{\eta_1}^{\eta_2} \beta \, z_x \, z_{xx} \, dx dt + \omega^*. \end{aligned}$$

Therefore,

$$2 \, \underline{\alpha} \int_{\tau}^{\tau_2} \int_{\eta_1}^{\eta_2} (z_{xx})^2 \, dx dt + \int_{\eta_1}^{\eta_2} \{z_x(x, \tau)\}^2 \, dx \tag{5.25}$$

$$\leq 2\,\overline{\beta}\int_{\tau}^{\tau_2}\int_{\eta_1}^{\eta_2} |z_x z_{xx}|\; dxdt + \omega^*.$$

However, by Young's inequality

$$|z_x z_{xx}| \leq (\underline{\alpha}/2\overline{\beta})\;(z_{xx})^2 + (\overline{\beta}/2\underline{\alpha})\;(z_x)^2. \tag{5.26}$$

Hence, substituting (5.26) in (5.25),

$$\underline{\alpha}\int_{\tau}^{\tau_2}\int_{\eta_1}^{\eta_2} (z_{xx})^2\; dxdt + \int_{\eta_1}^{\eta_2} \{z_x(x,\tau)\}^2\; dx \leq \omega^* + (\overline{\beta}^2/\underline{\alpha})\int_{\tau}^{\tau_2}\int_{\eta_1}^{\eta_2} (z_x)^2\; dxdt. \tag{5.27}$$

Ignoring the first term in the left-hand side of (5.27) and applying Gronwall's lemma we deduce

$$\int_{\eta_1}^{\eta_2} \{z_x(x,\tau)\}^2 dx \leq \omega^* \exp\{(\overline{\beta}^2/\underline{\alpha})\;(\tau_2 - \tau)\} \tag{5.28}$$

for any $\tau \in [\tau_1, \tau_2]$. Subsequently, integration of (5.28) with respect to τ from τ_1 to τ_2 confirms the first assertion in (5.22), and therewith by substitution in (5.27), the second also.

LEMMA 7. *Suppose that the hypotheses of Lemma* 5 *hold, and let* ω *denote a* $C^\infty(\eta_1, \eta_2)$ *function with the properties* (5.6)-(5.8). *Then, if* (5.5) *holds, given any* $\xi_1 \in (\eta_1, \rho_1]$ *and* $\xi_2 \in [\rho_2, \eta_2)$,

$$\begin{aligned}\int_{\eta_1}^{\eta_2} \omega(x)\;\{\underline{u}(x,\tau_2) - \overline{u}(x,\tau_2)\}dx &\leq \int_{\eta_1}^{\eta_2} [\underline{u}(x,\tau_1) - \overline{u}(x,\tau_1)]_+ Z(x,\tau_1)dx \\ &+ C_1\int_{\tau_1}^{\tau_2} [a(\underline{u}(\eta_1,t)) - a(\overline{u}(\eta_1,t))]_+ dt \\ &+ C_2\int_{\tau_1}^{\tau_2} [a(\underline{u}(\eta_2,t)) - a(\overline{u}(\eta_2,t))]_+ dt\end{aligned} \tag{5.29}$$

where $Z(x,\tau_1)$, C_1 *and* C_2 *are defined by* (5.9)-(5.15).

PROOF. Define the extension of α and β onto $\mathbb{R}^2$ by

$$\tilde{\alpha}(x,t) = \alpha(x^*,t^*) \text{ and } \tilde{\beta}(x,t) = \beta(x^*,t^*)$$

where

$$x^* = \begin{cases} \eta_1 & \text{if } x \le \eta_1 \\ x & \text{if } \eta_1 < x < \eta_2 \\ \eta_2 & \text{if } \eta_2 \le x \end{cases} \quad \text{and} \quad t^* = \begin{cases} \tau_1 & \text{if } t \le \tau_1 \\ t & \text{if } \tau_1 < t < \tau_2, \\ \tau_2 & \text{if } \tau_2 \le t \end{cases}$$

and for every positive integer n, set

$$\alpha^{(n)}(x,t) = \int\int_{\mathbb{R}^2} J_n(x-y,t-s)\ \tilde{\alpha}(y,s)\ dy\ ds$$

and

$$\beta^{(n)}(x,t) = \int\int_{\mathbb{R}^2} J_n(x-y,t-s)\ \tilde{\beta}(y,s)\ dy\ ds$$

with

$$J_n(y,s) = n^2\ J(ny,ns)/\|J\|_{L^1(\mathbb{R}^2)}$$

and

$$J(y,s) = \begin{cases} \exp\{-1/(1-y^2-s^2)\} & \text{if } y^2+s^2 < 1 \\ 0 & \text{if } y^2+s^2 \ge 1. \end{cases}$$

By the definition of $\tilde{\alpha}$ $\tilde{\beta}$ and by and the standard theory of mollifiers [1]

$$\underline{\alpha} \le \alpha^{(n)}(x,t) \le \overline{\alpha} \qquad \text{for all } (x,t) \in \mathbb{R}^2$$

and

$$|\beta^{(n)}(x,t)| \le \overline{\beta} \qquad \text{for all } (x,t) \in \mathbb{R}^2$$

for all positive integers n. Furthermore $\alpha^{(n)}(x,t) \to \alpha(x,t)$ and $\beta^{(n)}(x,t) \to \beta(x,t)$ as $n \uparrow \infty$ uniformly on $\overline{D}$. Now, let $z^{(n)} \in C^\infty(\overline{D})$ denote the solution to the problem

$$\begin{aligned} &0 = \alpha^{(n)}\ z^{(n)}_{xx} - \beta^{(n)}\ z^{(n)}_x + z^{(n)}_t && \text{in } \overline{D}, \\ &z^{(n)}(x,\tau_2) = \omega(x) && \text{for } x \in (\eta_1,\eta_2), \\ &z^{(n)}(\eta_1,t) = z^{(n)}(\eta_2,t) = 0 && \text{for } t \in [\tau_1,\tau_2]. \end{aligned} \tag{5.30}$$

The function $z^{(n)}$ exists by Lemma 6, and furthermore satisfies the estimates

$$0 \le z^{(n)}(x,t) \le Z^{(n)}(x,t) \qquad \text{for all } (x,t) \in \overline{D}, \tag{5.31}$$

$$0 \le z^{(n)}_x(\eta_1,t) \le C_1^{(n)} \qquad \text{for all } t \in [\tau_1,\tau_2], \tag{5.32}$$

$$0 \ge z^{(n)}_x(\eta_2,t) \ge -C_2^{(n)} \qquad \text{for all } t \in [\tau_1,\tau_2] \tag{5.33}$$

and

$$\int\int_D (z_x^{(n)})^2 \, dxdt \le K \quad \text{and} \quad \int\int_D (z_{xx}^{(n)})^2 \, dxdt \le K \tag{5.34}$$

where $Z^{(n)}(x,t)$, $C_1^{(n)}$ and $C_2^{(n)}$ are defined by (5.9)-(5.15) with α replaced by $\alpha^{(n)}$ and β replaced by $\beta^{(n)}$, and where the constant K is independent of n.

Applying inequality (2.9) of the definition of $\underline{u}$ as a generalized subsolution of equation (1.1) in D and of $\overline{u}$ as a generalized supersolution of equation (1.1) in D there holds

$$\begin{aligned} &\int_{\eta_1}^{\eta_2} \phi(x,\tau_2)\{\underline{u}(x,\tau_2) - \overline{u}(x,\tau_2)\}dx \\ &- \int_{\eta_1}^{\eta_2} \phi(x,\tau_1)\{\underline{u}(x,\tau_1) - \overline{u}(x,\tau_1)\}dx \\ &- \int_{\tau_1}^{\tau_2} \phi_x(\eta_1,t)\{a(\underline{u}(\eta_1,t)) - a(\overline{u}(\eta_1,t))\}dt \\ &+ \int_{\tau_1}^{\tau_2} \phi_x(\eta_2,t)\{a(\underline{u}(\eta_2,t)) - a(\overline{u}(\eta_2,t))\}dt \\ &\le \int\int_D (\underline{u} - \overline{u})\{\alpha \ \phi_{xx} - \beta \ \phi_x + \phi_t\}dxdt \end{aligned} \tag{5.35}$$

for all $\phi \in C^{2,1}(\overline{D})$. Therefore, inserting $\phi = z^{(n)}$ in (5.35) and incorporating (5.30)-(5.33), there holds

$$\begin{aligned} &\int_{\eta_1}^{\eta_2} \omega(x)\{\underline{u}(x,\tau_2) - \overline{u}(x,\tau_2)\} \ dx \\ &- \int_{\eta_1}^{\eta_2} [\underline{u}(x,\tau_1) - \overline{u}(x,\tau_1)]_+ Z^{(n)}(x,\tau_1) \ dx \\ &- C_1^{(n)} \int_{\tau_1}^{\tau_2} [a(\underline{u}(\eta_1,t) - a(\overline{u}(\eta_1,t))]_+ \ dt \\ &- C_2^{(n)} \int_{\tau_1}^{\tau_2} [a(\underline{u}(\eta_2,t) - a(\overline{u}(\eta_2,t))]_+ \ dt \end{aligned} \tag{5.36}$$

$$\leq \iint_D (\underline{u} - \overline{u})\{(\alpha - \alpha^{(n)})z_{xx}^{(n)} - (\beta - \beta^{(n)})z_x^{(n)}\}\ dxdt$$

$$\leq \{K \iint_D (\underline{u} - \overline{u})^2(\alpha - \alpha^{(n)})^2\ dxdt\}^{1/2}$$

$$+ \{K \iint_D (\underline{u} - \overline{u})^2(\beta - \beta^{(n)})^2\ dxdt\}^{1/2}$$

by Hölder's inequality and (5.34), for any n. Letting $n \uparrow \infty$ in (5.36) one obtains (5.29).

LEMMA 8. *Suppose that the assumptions of Theorem* 1, 2 *or* 3 *hold, and let* Ω *denote the domain* S, H *or* Q *accordingly. Suppose that there exists a domain* D *of the form* (2.7), (2.8) *such that* $D \subseteq \Omega$. *Let* U_1 *and* U_2 *denote generalized subsolutions of equation* (1.1) *in* D. *Then, Problem* 1, 2 *or* 3, *depending on the case in hand, admits a generalized solution* u *with the following property. If*

$$u(x,t) \geq U_i(x,t) \text{ for all } (x,t) \in \overline{D} \setminus (\eta_1, \eta_2) \times [\tau_1, \tau_2] \tag{5.37}$$

then

$$\int_{\eta_1}^{\eta_2} [U_i(x,\tau) - u(x,\tau)]_+ dx \leq \int_{\eta_1}^{\eta_2} [U_i(x,\tau_1) - u(x,\tau_1)]_+ dx \tag{5.38}$$

for all $\tau \in (\tau_1, \tau_2]$; *for* $i = 1$ *and* 2.

PROOF. Pick M as an upper bound which is not the least upper bound for the initial and lateral boundary data of the problem under consideration so large that

$$M \geq U_1(x,t),\ U_2(x,t) \qquad \text{for all } (x,t) \in \overline{D}.$$

Then, following the proofs of Theorems 1, 2 and 3, we can construct a generalized solution of Problem 1, 2 or 3, $u(x,t)$, as the limit as $k \uparrow \infty$ of a sequence of functions $\{u_k\}_{k=1}^{\infty}$ with the behaviour:

(i) $u_k \in C^{2,1}(\overline{\Omega}_k)$ where

$$\Omega_k = (-k,k) \times (0,T] \cap \Omega$$

for all $k \geq 1$;

(ii) there exists a $\mu_k > 0$ such that $\mu_k \leq u_k(x,t) \leq M$ for all $(x,t) \in \overline{\Omega}_k$ and $k \geq 1$;

(iii) u_k is a classical solution of equation (1.1) in Ω_k for all $k \geq 1$;

(iv) $u_k(x,t) = M$ for all $(x,t) \in (\overline{\Omega}_k \setminus \Omega_k) \cap \Omega$;

(v) $u_k \downarrow u$ as $k \uparrow \infty$ uniformly on compact subsets of $\overline{\Omega}$.

We assert that u possesses the required property.

Pick $i = 1$ or 2, and suppose that (5.37) holds. Let $\tau \in (\tau_1, \tau_2]$ and $\omega \in C^\infty(\eta_1, \eta_2)$ satisfying (5.6)-(5.8) be arbitrary, and set

$$D_k = \Omega_k \cap (\eta_1, \eta_2) \times (\tau_1, \tau],$$
$$\eta_{1,k} = \inf\{x : (x, \tau) \in D_k\}$$

and

$$\eta_{2,k} = \sup\{x : (x, \tau) \in D_k\}.$$

Then, applying Lemma 7 in the domain D_k with $\underline{u} = U_i$ and $\overline{u} = u_k$ there holds

$$\int_{\eta_{1,k}}^{\eta_{2,k}} \omega(x)\{U_i(x,\tau) - u_k(x,\tau)\}dx \le \int_{\eta_{1,k}}^{\eta_{2,k}} [U_i(x,\tau_1) - u_k(x,\tau_1)]_+ dx$$

in view of (5.37) and the monotonicity of the sequence $\{u_k\}_{k=1}^\infty$, for all k such that $\eta_{1,k} < \rho_1$ and $\eta_{2,k} > \rho_2$. Whence, taking the limit $k \uparrow \infty$,

$$\int_{\eta_1}^{\eta_2} \omega(x)\{U_i(x,\tau) - u(x,\tau)\}dx \le \int_{\eta_1}^{\eta_2} [U_i(x,\tau_1) - u(x,\tau_1)]_+ dx.$$

This implies (5.38), because of the arbitrariness of the choice of ω.

We have subsequently shown that if (5.37) holds for $i = 1, 2$, then (5.38) also holds for the corresponding i. Thus, the generalized solution u has the asserted behaviour.

LEMMA 9. *Suppose that the assumptions of Theorem* 1, 2 *or* 3 *hold and that the corresponding problem - Problem* 1, 2 *or* 3 *- has two generalized solutions* U_1 *and* U_2. *Let* Ω *denote* S, H *or* Q *depending on the problem in hand. Then there exists a generalized solution to the given problem,* $u(x,t)$, *with the property*

$$u(x,t) \ge U_i(x,t) \qquad \textit{for all } (x,t) \in \overline{\Omega} \tag{5.39}$$

and

$$\int_0^T \int_{x_1}^{x_2} |u(x,t) - U_i(x,t)|\phi(x)\ dxdt \tag{5.40}$$
$$\le T \int_0^T \int_{x_1}^{x_2} |b(u(x,t)) - b(U_i(x,t))|\ |\phi'(x)|\ dxdt$$

$$+ T \int_0^T \int_{x_1}^{x_2} [a(u(x,t)) - a(U_i(x,t))]\ [\phi''(x)]_+\ dxdt$$

$$+ T\phi'(x_1) \int_0^T |a(u(x_1,t)) - a(U_i(x_1,t))|\ dt$$

$$- T\phi'(x_2) \int_0^T |a(u(x_2,t)) - a(U_i(x_2,t))|\ dt$$

for any x_1 *and* x_2 *such that* $(x_1,T),(x_2,T) \in \overline{\Omega}$ *with* $x_1 < x_2$ *and any nonnegative function*

$$\phi \in C^2([x_1,x_2]) \text{ such that } \phi(x_1) = \phi(x_2) = 0; \tag{5.41}$$

for $i = 1,2$.

PROOF. Mobilizing Lemma 8 with $D = \Omega$ there exists a generalized solution $u(x,t)$ which satisfies (5.39) for $i = 1,2$.

To verify that u also satisfies (5.40) we now utilize the fact that every generalized solution fulfils (2.9) with equality with any bounded rectangle

$$R = (x_1,x_2) \times (0,\tau] \subseteq \Omega$$

and nonnegative function ϕ satisfying (5.41). Substituting R and ϕ in (2.9) for both u and U_i, with $i = 1$ or 2, subtracting, and using (5.39), we deduce

$$\int_{x_1}^{x_2} |u(x,\tau) - U_i(x,\tau)|\phi(x)\ dx \tag{5.42}$$

$$\leq \int_0^\tau \int_{x_1}^{x_2} |b(u(x,t)) - b(U_i(x,t))|\ |\phi'(x)|\ dxdt$$

$$+ \int_0^\tau \int_{x_1}^{x_2} |a(u(x,t)) - a(U_i(x,t))|\ [\phi''(x)]_+\ dxdt$$

$$+ \phi'(x_1) \int_0^\tau |a(u(x_1,t)) - a(U_i(x_1,t))|\ dt$$

$$- \phi'(x_2) \int_0^\tau |a(u(x_2,t)) - a(U_i(x_2,t))|\ dt.$$

Finally, integrating (5.42) with respect to τ from 0 to T verifies (5.40).

We are now in a position to prove our uniqueness theorems.

THEOREM 4. *Suppose that the assumptions of Theorem* 1 *hold, then Problem* 1 *admits at most one generalized solution.*

THEOREM 5. *Suppose that the assumptions of Theorem* 2 *hold, then Problem* 2 *admits at most one generalized solution.*

THEOREM 6. *Suppose that the assumptions of Theorem* 3 *hold, then Problem* 3 *admits at most one generalized solution.*

The basic strategy behind the proof of all three theorems is the same. However, we have to make a distinction between whether the domain has no, one or two lateral boundaries in the details of the proof. In the interest of brevity, we present the proofs of Theorems 4 and 6 only. The proof of Theorem 5 may be completed by a combination of the arguments used in the proofs of the other two theorems.

PROOF OF THEOREM 4. Suppose that Problem 1 has two generalized solutions U_1 and U_2. Then, by Lemma 9, there exists a (possible third) generalized solution u such that

$$u(x,t) \geq U_i(x,t) \qquad \text{for all } (x,t) \in S$$

and such that (5.40) holds for any $-\infty < x_1 < x_2 < \infty$ and nonnegative function ϕ satisfying (5.41), for $i = 1, 2$. Setting $x_1 = -j$, $x_2 = j$ and

$$\phi_j(x) = (1-y)^2(3y^2 + 2y + 1) \text{ with } y = [|x| - j + 1]_+$$

in (5.40), and noting that $\phi_j(x) \equiv 1$ for $|x| \leq j-1$, $|\phi_j'(x)| \leq 2$ for all $|x| \leq j$, $|\phi_j''(x)| \leq 12$ for all $|x| \leq j$, and $\phi'(-j) = \phi'(j) = 0$, we deduce

$$\begin{aligned} (5.43) \qquad & \int_0^T \int_{-j+1}^{j-1} |u(x,t) - U_i(x,t)| \, dxdt \\ & \leq 2\, T \iint_{\Omega_j} |b(u(x,t)) - b(U_i(x,t))| \, dxdt \\ & + 12\, T \iint_{\Omega_j} |a(u(x,t)) - a(U_i(x,t))| \, dxdt \end{aligned}$$

where

$$\Omega_j = \{(x,t) \in S : j-1 < |x| < j\},$$

for all positive integers j and $i = 1, 2$. Now, fix $i = 1$ or 2 and $\epsilon > 0$, and let $\delta > 0$ be such that

$$|b(s)| < \epsilon \qquad \text{for all } s < \delta.$$

Set

$$\Omega_j^{(1)} = \{(x,t) \in \Omega_j : u(x,t) \leq \delta\}$$
$$\Omega_j^{(2)} = \{(x,t) \in \Omega_j : U_i(x,t) \geq \delta/2\},$$

and

$$\Omega_j^{(3)} = \{(x,t) \in \Omega_j : u(x,t) \geq \delta > \delta/2 \geq U_i(x,t)\}.$$

By definition

$$|b(u(x,t)) - b(U_i(x,t))| \leq 2\epsilon \qquad \text{for all } (x,t) \in \Omega_j^{(1)}.$$

Whilst, by the Mean-Value Theorem

$$|(b(u(x,t)) - b(U_i(x,t))| \leq K|u(x,t) - U_i(x,t)| \text{ for all } (x,t) \in \Omega_j^{(2)},$$

where

$$K = \sup\{b'(s) : \delta/2 \leq s \leq M\}$$

and

$$M = \sup\{u(x,t) : (x,t) \in S\}.$$

Finally, defining

$$L = \sup\{b(s) : 0 \leq s \leq M\},$$

there holds

$$\begin{aligned} |b(u(x,t)) - b(U_i(x,t))| &\leq 2L \\ &\leq (4L/\delta)|u(x,t) - U_i(x,t)| \text{ for all } (x,t) \in \Omega_j^{(3)} \end{aligned}$$

by contrivance. Thus, since $\Omega_j^{(1)} \cup \Omega_j^{(2)} \cup \Omega_j^{(3)} = \Omega_j$, we obtain the estimate

$$\begin{aligned} (5.44) \qquad &\iint_{\Omega_j} |b(u(x,t)) - b(U_i(x,t))| \, dxdt \\ &\leq 4\epsilon T + C_b \iint_{\Omega_j} |u(x,t) - U_i(x,t)| \, dxdt \end{aligned}$$

where $C_b = (K + 4L/\delta)$ does not depend on j. Similarly, we can obtain an estimate of the type (5.44) with C_b replaced by C_a for the second term in (5.43). Substituting both in (5.43) yields

$$\begin{aligned} (5.45) \quad &\int_0^T \int_{-j+1}^{j-1} |u(x,t) - U_i(x,t)| \, dxdt \\ &\leq 56 \ \epsilon T^2 + (12C_a + 2C_b)T \iint_{\Omega_j} |u(x,t) - U_i(x,t)| \, dxdt \\ &\leq 56 \ \epsilon T^2 + 2(12C_a + 2C_b) \ T^2 M. \end{aligned}$$

It follows, comparing the first and third expressions in (5.45) that

$$\iint_S |u(x,t) - U_i(x,t)| \, dxdt < \infty.$$

Whence, comparing the first and second expressions in (5.45) and letting $j \uparrow \infty$,

$$\iint_S |u(x,t) - U_i(x,t)| \, dxdt \leq 56 \ \epsilon T^2.$$

However, in view of the arbitrariness of ϵ, this means that

$$U_i(x,t) = u(x,t) \qquad \text{for all } (x,t) \in S.$$

Here $i = 1, 2$. Hence $U_1 \equiv U_2$ and Problem 1 cannot have two distinct generalized solutions.

PROOF OF THEOREM 6. Supposing that Problem 3 has two solutions U_1 and U_2, by Lemma 9 there exists a (possible third) generalized solution of the problem, u, such that

$$u(x,t) \geq U_i(x,t) \qquad \text{for all } (x,t) \in Q$$

and such that (5.40) holds for any nonnegative function ϕ satisfying (5.41) with $-1 \leq x_1 < x_2 \leq 1$; for $i = 1, 2$. Setting $x_1 = -1$, $x_2 = 1$ and

$$\phi_j(x) = 1 - [1 + j|x| - j]_+^3$$

where j is a positive integer in (5.40), and observing that $\phi_j(x) \equiv 1$ for $|x| \leq 1 - 1/j$, $|\phi_j'(x)| \leq 3j$ for all $|x| \leq 1$, and $\phi_j''(x) \leq 0$ for all $|x| \leq 1$, there must hold

$$\begin{aligned} (5.46) \qquad & \int_0^T \int_{-1+1/j}^{1-1/j} |u(x,t) - U_i(x,t)| \, dxdt \\ & \leq 3j \ T \iint_{\Omega_j} |b(u(x,t)) - b(U_i(x,t))| \, dxdt \\ & \leq 6T^2 \ \sup\{|b(u(x,t)) - b(U_i(x,t))| : (x,t) \in \Omega_j\} \end{aligned}$$

where

$$\Omega_j = \{(x,t) \in Q : |x| > 1 - 1/j\}.$$

Thus, bearing in mind that $b(u(x,t)) = b(U_i(x,t))$ for all $(x,t) \in \overline{Q} \backslash Q$, letting $j \uparrow \infty$ in (5.46) one obtains

$$\iint_Q |u(x,t) - U_i(x,t)| \, dxdt = 0$$

for $i = 1, 2$. Whence $U_1 \equiv U_2$.

6. - Regularity

The following regularity results for generalized solutions of Problems 1-3 are a direct consequence of the proofs of Theorems 1-3.

THEOREM 7. *Suppose that the assumptions of Theorem* 1 *hold, and let* u *denote the corresponding generalized solution of Problem* 1. *Then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $(-\infty, \infty) \times [\tau, T]$, $\tau \in (0, T)$. *Moreover, if* $a(u_0)$ *is uniformly Lipschitz continuous on* $(-\infty, \infty)$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $\overline{S} = (-\infty, \infty) \times [0, T]$.

THEOREM 8. *Suppose that the assumptions of Theorem* 2 *hold, and let* u *denote the corresponding generalized solution of Problem* 2. *Then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $[\delta, \infty) \times [\tau, T]$, $\delta \in (0, \infty)$, $\tau \in (0, T)$. *Moreover; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[0, \infty)$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $[\delta, \infty) \times [0, T]$ *for any* $\delta \in (0, \infty)$; *if Hypothesis* 3 *holds, then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $[0, \infty) \times [\tau, T]$ *for any* $\tau \in (0, T)$; *and; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[0, \infty)$, *and, Hypothesis* 3 *holds, then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $\overline{H} = [0, \infty) \times [0, T]$.

THEOREM 9. *Suppose that the assumptions of Theorem* 3 *hold, and let* u *denote the corresponding generalized solution of Problem* 3. *Then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $[-\delta, \delta] \times [\tau, T]$, $\delta \in (0, 1)$, $\tau \in (0, T)$. *Moreover; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[-1, 1]$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $[-\delta, \delta] \times [0, T]$ *for any* $\delta \in (0, 1)$; *if* $a(\psi^-)$ *is uniformly Lipschitz continuous on* $[0, T]$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $[-1, \delta] \times [\tau, T]$ *for any* $\delta \in (0, 1)$ *and* $\tau \in (0, T)$; *and; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[-1, 1]$, *and,* $a(\psi^-)$ *is uniformly Lipschitz continuous on* $[0, T]$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in* $[-1, \delta] \times [0, T]$ *for any* $\delta \in (0, 1)$. *Similarly; if* $a(\psi^+)$ *is uniformly Lipschitz continuous on* $[0, T]$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $[-\delta, 1] \times [\tau, T]$, $\delta \in (0, 1)$, $\tau \in (0, T)$; *whilst; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[-1, 1]$, *and,* $a(\psi^+)$ *is uniformly Lipschitz continuous on* $[0, T]$, *then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $[-\delta, 1] \times [0, T]$, $\delta \in (0, 1)$. *Finally; if Hypothesis* 4 *holds, then, in the sense of distributions,* $(a(u))_x$ *exists and is bounded in any set of the form* $[-1, 1] \times [\tau, T]$, $\tau \in (0, T)$; *whilst; if* $a(u_0)$ *is uniformly Lipschitz continuous on* $[-1, 1]$, *and, Hypothesis* 4 *holds, then, in*

the sense of distributions, $(a(u))_x$ *exists and is bounded in* $\overline{Q} = [-1,1] \times [0,T]$.

Note that, in view of the nature of the estimates in Lemmata 2 and 3, as much as the bounds in Theorems 7-9 depend on the Lipschitz continuity of any relevant lateral boundary-data functions, they are independent of the magnitude of T.

Using a standard boot-strap argument, cf. [11], the following results may also be obtained from the analysis of the previous sections.

THEOREM 10. *Suppose that the assumptions of Theorem* 1 *hold, and let* u *denote the corresponding generalized solution of Problem* 1. *Then* u, $(a(u))_x \in C^{2,1}(P)$, *where* $P = \{(x,t) \in S : u(x,t) > 0\}$. *Moreover,* u *is a classical solution of equation* (1.1) *in* P.

THEOREM 11. *Suppose that the assumptions of Theorem* 2 *hold, and let* u *denote the corresponding generalized solution of Problem* 2. *Then* u, $(a(u))_x \in C^{2,1}(P)$ *where* $P = \{(x,t) \in H : u(x,t) > 0\}$. *Moreover,* u *is a classical solution of equation* (1.1) *in* P.

THEOREM 12. *Suppose that the assumptions of Theorem* 3 *hold, and let* u *denote the corresponding generalized solution of Problem* 3. *Then* u, $(a(u))_x \in C^{2,1}(P)$ *where* $P = \{(x,t) \in Q : u(x,t) > 0\}$. *Moreover,* u *is a classical solution of equation* (1.1) *in* P.

7. - Comparison principles

THEOREM 13. *Suppose that the assumptions of Theorem* 1, 2 *or* 3 *hold, and let* u *denote the corresponding generalized solution of Problem* 1, 2 *or* 3. *Let* D *denote a domain of the form* (2.7), (2.8) *such that* $D \subset S$, H *or* Q *respectively, and let* U *be a generalized subsolution of equation* (1.1) *in* D *such that*

$$U(x,t) \le u(x,t) \qquad \textit{for all } (x,t) \in \overline{D} \setminus (\eta_1,\eta_2) \times [\tau_1,\tau_2].$$

Then

$$\int_{\eta_1}^{\eta_2} [U(x,t) - u(x,t)]_+ \, dx \le \int_{\eta_1}^{\eta_2} [U(x,\tau_1) - u(x,\tau_1)]_+ \, dx$$

for all $t \in [\tau_1,\tau_2]$.

This theorem is a direct consequence of Lemma 8 since uniqueness of the solutions of Problems 1-3 has been established in Theorems 4-6.

COROLLARY TO THEOREM 13. *If*

$$U(x,t) \le u(x,t) \qquad \textit{for all } (x,t) \in \overline{D} \setminus D$$

then

$$U(x,t) \le u(x,t) \qquad \textit{for all } (x,t) \in \overline{D}.$$

The remainder of this section is devoted to the proof of a complementary result for comparing a generalized solution of Problem 1, 2 or 3 with a generalized supersolution of equation (1.1). However, this result is not to be obtained so straightforwardly as Theorem 13, and correspondingly involves some limitations on the character of the generalized supersolution or of the coefficients in equation (1.1) which may be considered.

THEOREM 14. *Suppose that the assumptions of Theorem* 1, 2 *or* 3 *hold, and let* u *denote the corresponding generalized solution of Problem* 1, 2 *or* 3. *Let* D *denote a domain of the form* (2.7), (2.8) *such that* $D \subseteq S$, H *or* Q *respectively, and let* U *be a generalized supersolution of equation* (1.1) *in* D *such that*

$$U(x,t) \ge u(x,t) \qquad \textit{for all } (x,t) \in \overline{D} \setminus (\eta_1,\eta_2)\times[\tau_1,\tau_2]. \tag{7.1}$$

Suppose too that one of the following hold.

(a) *In the classical sense,* $(a(U))_x$ *exists and is continuous in a neighbourhood of any point* $(x,t) \in D$ *where* $U(x,t) = 0$, *and* $\{t \in [\tau_1,\tau_2] : U(x,t) > 0\}$ *has a finite number of connected components for all* $x \in \overline{(\eta_1,\eta_2)}$.

(b) *When* $(-1)^i\eta_i < \infty$,

$$a(s)[(-1)^i\ b'(s)]_+ = o(a'(s)) \ \textit{as}\ s \downarrow 0, \tag{7.2}$$

and when $(-1)^i\eta_i = \infty$, (7.2) *holds or*

$$s^2a'(s) = o(1) \ \textit{and}\ s[(-1)^ib'(s)]_+ = o(1) \ \textit{as}\ s \downarrow 0, \tag{7.3}$$

for $i = 1,2$.

Then, in both cases (a) *and* (b),

$$\int_{\eta_1}^{\eta_2} [u(x,t) - U(x,t)]_+\ dx \le \int_{\eta_1}^{\eta_2} [u(x,\tau_1) - U(x,\tau_1)]_+\ dx \tag{7.4}$$

for all $t \in [\tau_1,\tau_2]$.

COROLLARY TO THEOREM 14. *If*

$$U(x,t) \ge u(x,t) \qquad \textit{for all } (x,t) \in \overline{D}\backslash D$$

then

$$U(x,t) \ge u(x,t) \qquad \textit{for all } (x,t) \in \overline{D}.$$

We stress that Theorem 14 is not the sharpest result that we can obtain. It is an excerpt from the most general obtainable result which is convenient for application [12,13]. We formulate and prove all the steps leading up to the proof of Theorem 14 comprehensively, and leave it to the devoted reader to determine the most general set of conditions under which the conclusion of Theorem 14 may be reached.

The key to the proof of Theorem 14 is the following lemma.

LEMMA 10. *Suppose that Hypothesis* 1 *holds. Let D denote a domain of the form* (2.7), (2.8), *let u denote a nonnegative $C(\overline{D}) \cap L^\infty(D)$ function which is a classical subsolution of equation* (1.1) *in $\{(x,t) \in D : u(x,t) > 0\}$, and let U denote a generalized supersolution of equation* (1.1) *in D such that* (7.1) *holds. For any $(x,t) \in \overline{D}$ define*

$$u_\epsilon(x,t) = \max\{u(x,t), \epsilon\},$$

$$\alpha_\epsilon(x,t) = \int_0^1 a'(\lambda u_\epsilon(x,t) + (1-\lambda)U(x,t))\ d\lambda$$

and

$$\beta_\epsilon(x,t) = \int_0^1 b'(\lambda u_\epsilon(x,t) + (1-\lambda)U(x,t))\ d\lambda.$$

For, any $\rho \in (\eta_1, \eta_2)$, let $I_1(\rho)$ and $I_2(\rho)$ be defined by (5.9), *and for $i = 1, 2$ set*

$$\alpha_i(\epsilon,\rho) = \sup\{\alpha_\epsilon(x,t) : (x,t) \in I_i(\rho) \times [\tau_1, \tau_2]\},$$
$$\beta_i(\epsilon,\rho) = \sup\{[(-1)^i \beta_\epsilon(x,t)]_+ : (x,t) \in I_i(\rho) \times [\tau_1, \tau_2]\},$$
$$\gamma_i(\epsilon,\rho) = 1 + \sup\{[(-1)^i \beta_\epsilon(x,t)/\alpha_\epsilon(x,t)]_+ : (x,t) \in I_i(\rho) \times [\tau_1, \tau_2]\},$$
$$C_i(\epsilon,\rho) = \gamma_i(\epsilon,\rho)/[1 - \exp\{-\gamma_i(\epsilon,\rho)|\rho - \eta_i|\}],$$

and

$$\sigma_i(\epsilon,\rho) = \inf\{C_i(\epsilon,\xi) : \xi \in I_i(\rho)\}. \tag{7.5}$$

Suppose that the following hold.

I. *When $(-1)^i \eta_i < \infty$ for $i = 1, 2$:*

$$\sigma_i(\epsilon,\rho) \int_{\tau_1}^{\tau_2} [a(\epsilon) - a(U(\eta_i,t))]_+ dt \to 0 \qquad \text{as } \epsilon \downarrow 0$$

for any $\rho \in (\eta_1, \eta_2)$.

II. *When* $(-1)^i\eta_i = \infty$ *for* $i = 1, 2$:
There exists a p_i *and* q_i *with* $1 \le p_i \le \infty$, $1 \le q_i \le \infty$ *and*

$$1/p_i + 1/q_i = 1 \tag{7.6}$$

such that

$$\limsup_{\epsilon \downarrow 0} \{\alpha_i(\epsilon, \rho)^{1/2} + \beta_i(\epsilon, \rho)\}^{1/p_i} \, \|[\epsilon - U(\cdot, \tau_1)]_+\|_{L^{q_i}(I_i(\rho))} \to 0 \text{ as } \rho \to \eta_i.$$

Then, (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

PROOF. Strictly speaking, to prove the lemma we have to distinguish four different cases; viz. $-\infty = \eta_1 < \eta_2 < \infty$, $-\infty = \eta_1 < \eta_2 = \infty$, $-\infty < \eta_1 < \eta_2 < \infty$, and $-\infty < \eta_1 < \eta_2 = \infty$. We shall nevertheless restrict the proof to the last case. The proof of this case is sufficient to illustrate how the proof of the remaining cases proceeds.

Suppose then that $-\infty < \eta_1 < \eta_2 = \infty$. Let ω be an arbitrary $C^\infty(\eta_1, \eta_2)$ function which satisfies (5.6)-(5.8) and let $\tau \in [\tau_1, \tau_2]$ be arbitrary. Note that for every $\epsilon > 0$, u_ϵ is a generalized subsolution of equation (1.1) in D. Hence, $\underline{u} = u_\epsilon$ and $\overline{u} = U$ are admissible candidates for substitution in Lemma 7 in any domain $(\eta_1, \varsigma) \times (\tau_1, \tau]$ with

$$\rho_2 < \rho < \xi < \varsigma < \eta_2 = \infty.$$

Moreover, carrying out this substitution, by Lemma 7, we have

$$\begin{aligned}
&\int_{\eta_1}^{\varsigma} \omega(x)\{u_\epsilon(x, \tau) - U(x, \tau)\}dx \\
&\le \sigma_1(\epsilon, \rho_1) \int_{\tau_1}^{\tau} [a(u_\epsilon(\eta_1, t)) - a(U(\eta_1, t))]_+ dt \\
&+ Z(\xi)\gamma_2(\epsilon, \xi)[1 - \exp\{-\gamma_2(\epsilon, \xi)|\varsigma - \xi|\}]^{-1} \int_{\tau_1}^{\tau} [a(u_\epsilon(\varsigma, t)) - a(U(\varsigma, t))]_+ dt \\
&+ \int_{\eta_1}^{\varsigma} [u_\epsilon(x, \tau_1) - U(x, \tau_1)]_+ Z(x)dx
\end{aligned}$$

where

$$Z(x) = \min\left\{ erfc\left(\frac{x - \rho - \beta_2(\epsilon, \rho)\Delta}{\sqrt{4\alpha_2(\epsilon, \rho)\Delta}}\right), 1\right\}$$

and

$$\Delta = \tau - \tau_1.$$

Whence, setting $\xi = \varsigma - 1$, and letting $\varsigma \uparrow \infty$, we deduce

$$
\begin{aligned}
(7.7)\qquad & \int_{\eta_1}^{\infty} \omega(x)\{u_\epsilon(x,\tau) - U(x,\tau)\}dx \\
& \le \sigma_1(\epsilon,\rho_1) \int_{\tau_1}^{\tau} [a(u_\epsilon(\eta_1,t)) - a(U(\eta_1,t))]_+ dt \\
& + \int_{\eta_1}^{\infty} [u_\epsilon(x,\tau_1) - U(x,\tau_1)]_+ Z(x)dx \\
& \le \sigma_1(\epsilon,\rho_1) \int_{\tau_1}^{\tau} [a(\epsilon) - a(U(\eta_1,t))]_+ dt \\
& + \int_{\eta_1}^{\infty} [u(x,\tau_1) - U(x,\tau_1)]_+ Z(x)dx + \int_{\eta_1}^{\infty} [\epsilon - U(x,\tau_1)]_+ Z(x)dx \\
& \le \sigma_1(\epsilon,\rho_1) \int_{\tau_1}^{\tau} [a(\epsilon) - a(U(\eta_1,t))]_+ dt \\
& + \int_{\eta_1}^{\infty} [u(x,\tau_1) - U(x,\tau_1)]_+ dx + \int_{\eta_1}^{\rho_2} [\epsilon - U(x,\tau_1)]_+ dx \\
& + \|Z(\cdot)\|_{L^{p_2}(\rho,\infty)} \; \|[\epsilon - U(\cdot,\tau_1)]_+\|_{L^{q_2}(\rho,\infty)}
\end{aligned}
$$

by (7.1), the definition of u_ϵ, and Hölders inequality. Hence computing that

$$\int_{\rho}^{\infty} Z^{p_2}(x)dx = \beta_2(\epsilon,\rho)\Delta + \{4\alpha_2(\epsilon,\rho)\Delta\}^{1/2} \int_0^{\infty} erfc^{p_2}(y)dy$$

when $p_2 < \infty$; letting $\epsilon \downarrow 0$ and then $\rho \uparrow \infty$ in (7.7) one derives

$$\int_{\eta_1}^{\infty} \omega(x)[u(x,\tau) - U(x,\tau)]_+ dx \le \int_{\eta_1}^{\infty} [u(x,\tau_1) - U(x,\tau_1)]_+ dx.$$

Since though ω and τ were arbitrary this confirms (7.4) with any $t \in [\tau_1, \tau_2]$.

The crucial point in the application of Lemma 10 to prove Theorem 14 is to determine suitable conditions under which the hypotheses I and II in the statement of the lemma hold. The next four lemmata are concerned with this.

LEMMA 11. *Let f and g be $C([0,\infty)) \cap C^1(0,\infty)$ functions such that*

$$f'(s) > 0 \qquad \textit{for all } s > 0$$

and

$$f(0) = g(0) = 0.$$

Let $p \geq 1$ and $M < \infty$ be fixed. For any $\epsilon > 0$, set

$$\Upsilon(\epsilon) = [\epsilon, M] \times [0, M] \tag{7.8}$$

and for $(v,w) \in \Upsilon(\epsilon)$ define

$$\phi(v,w) = \int_0^1 g'(\lambda v + (1-\lambda)w)\ d\lambda \ / \int_0^1 f'(\lambda v + (1-\lambda)w)\ d\lambda$$

and

$$\Phi(\epsilon) = \sup\{\phi(v,w) : (v,w) \in \Upsilon(\epsilon)\}.$$

Then

$$\limsup_{\epsilon \downarrow 0}\ f^p(\epsilon)\Phi(\epsilon) \leq \limsup_{s \downarrow 0}\ f^p(s)[g'(s)]_+/f'(s). \tag{7.9}$$

PROOF. Pick $\nu \in (0,1)$ and $\delta \in (0,M)$. For fixed $\epsilon > 0$ let

$$\begin{aligned}
\Upsilon_1 &= \{(v,w) \in \Upsilon(\epsilon) : f(w) \leq \nu f(v),\ v \leq \delta\},\\
\Upsilon_2 &= \{(v,w) \in \Upsilon(\epsilon) : f(w) \leq \nu f(v),\ \delta \leq v\},\\
\Upsilon_3 &= \{(v,w) \in \Upsilon(\epsilon) : w \leq v,\ \nu f(v) \leq f(w),\ v \leq \delta\}\\
\Upsilon_4 &= \{(v,w) \in \Upsilon(\epsilon) : w \leq v,\ \nu f(v) \leq f(w),\ \delta \leq v\}
\end{aligned}$$

and

$$\Upsilon_5 = \{(v,w) \in \Upsilon(\epsilon) : v \leq w\}.$$

For $(v,w) \in \Upsilon_1 \cup \Upsilon_2$ we observe that automatically $w < v$ and therefore

$$\phi(v,w) = \{g(v) - g(w)\}/\{f(v) - f(w)\} \leq (1-\nu)^{-1}|g(v) - g(w)|/f(v).$$

Hence, if $(v,w) \in \Upsilon_1$

$$\phi(v,w) \leq 2(1-\nu)^{-1} \sup\{|g(s)| : 0 \leq s \leq \delta\}/f(\epsilon), \tag{7.10}$$

whereas if $(v,w) \in \Upsilon_2$

$$\phi(v,w) \leq 2(1-\nu)^{-1} \sup\{|g(s)| : 0 \leq s \leq M\}/f(\delta). \tag{7.11}$$

On the other hand for $(v,w) \in \Upsilon_3 \cup \Upsilon_4$ by the Mean Value Theorem there exists an $r \in [w,v]$ such that

$$\phi(v,w) = g'(r)/f'(r).$$

So, for $(v,w) \in \Upsilon_3$

$$\begin{aligned}(7.12)\qquad \phi(v,w) &\le f(\epsilon)^{-p} f^p(v)[g'(r)]_+/f'(r)\\ &\le f(\epsilon)^{-p}\nu^{-p} f^p(w)[g'(r)]_+/f'(r)\\ &\le f(\epsilon)^{-p}\nu^{-p} f^p(r)[g'(r)]_+/f'(r)\\ &\le f(\epsilon)^{-p}\nu^{-p}\sup\{f^p(s)[g'(s)]_+/f'(s) : 0 \le s \le \delta\},\end{aligned}$$

whilst for $(v,w) \in \Upsilon_4$

$$(7.13)\qquad \phi(v,w) \le \sup\{[g'(s)]_+/f'(s) : \nu f(\delta) \le f(s) \text{ and } s \le M\}.$$

Noting that if $(v,w) \in \Upsilon_5$ then $(w,v) \in \bigcup_{j=1}^{4} \Upsilon_j$ and that $\Upsilon(\epsilon) = \bigcup_{j=1}^{5} \Upsilon_j$ it follows from (7.10)-(7.13) that

$$\begin{aligned}\limsup_{\epsilon\downarrow 0}\ f^p(\epsilon)\Phi(\epsilon) &\le 2(1-\nu)^{-1}\sup\{|g(s)| : 0 \le s \le \delta\}\\ &\quad + \nu^{-p}\sup\{f^p(s)[g'(s)]_+/f'(s) : 0 \le s \le \delta\}.\end{aligned}$$

Whence, first letting $\delta \downarrow 0$ and then letting $\nu \uparrow 1$, (7.9) is proved.

LEMMA 12. *Suppose that the assumptions of Lemma* 10 *up to and including the definition* (7.5) *hold. Suppose too that* $(-1)^i\eta_i < \infty$ *for* $i = 1,2$. *Then if:*

(a) $U(\eta_i,t) > 0$ *for all* $t \in [\tau_1,\tau_2]$;

(b) $U(\eta_i,t) > 0$ *for almost all* $t \in [\tau_1,\tau_2]$ *and* $a(s)[(-1)^i b'(s)]_+ = o(a'(s))$ *as* $s \downarrow 0$; *or*

(c) $a(s)[(-1)^i b'(s)]_+ = o(a'(s))$ *as* $s \downarrow 0$;

hypothesis I *of Lemma* 10 *holds.*

PROOF. Choose $\rho \in (\eta_1,\eta_2)$ arbitrarily, and let $M < \infty$ be so large that

$$(7.14)\qquad M \ge \max\{u(x,t), U(x,t)\} \qquad \text{for all } (x,t) \in \overline{D}.$$

Next, for any $\epsilon > 0$ define

$$\Phi(\epsilon) = \sup\{[(-1)^i\beta(v,w)/\alpha(v,w)]_+ : (v,w) \in \Upsilon(\epsilon)\}$$

where $\Upsilon(\epsilon)$ is defined by (7.8),

$$(7.15)\qquad \alpha(v,w) = \int_0^1 a'(\lambda v + (1-\lambda)w)\ d\lambda$$

and

$$\beta(v,w) = \int_0^1 b'(\lambda v + (1-\lambda)w)\ d\lambda. \tag{7.16}$$

Then, by definition

$$\begin{aligned}\sigma_i(\epsilon,\rho) \le C_i(\epsilon,\rho) &\le \gamma_i(\epsilon,\rho)/[1-\exp\{-|\rho-\eta_i|\}]\\ &\le [1+\Phi(\epsilon)]/[1-\exp\{-|\rho-\eta_i|\}]\end{aligned}$$

and therefore

$$\begin{aligned}&\sigma_i(\epsilon,\rho)\int_{\tau_1}^{\tau_2}[a(\epsilon)-a(U(\eta_i,t))]_+dt\\ &\le \int_{\tau_1}^{\tau_2}[1-a(U(\eta_i,t))/a(\epsilon)]_+dt\ \ [a(\epsilon)+a(\epsilon)\Phi(\epsilon)]/[1-\exp\{-|\rho-\eta_i|\}].\end{aligned}$$

The result now follows from Lemma 11.

LEMMA 13. *Suppose that the assumptions of Lemma* 10 *up to and including the definition* (7.5) *hold. Suppose too that* $(-1)^i\eta_i = \infty$ *for* $i = 1,2$. *Then if:*

(a) *there exists a* $\nu > 0$ *and an* $\eta \in (\eta_1,\eta_2)$ *such that* $[\nu - U(\cdot,\tau_1)]_+ \in L^1(I_i(\eta))$ *where* $I_i(\eta)$ *is given by* (5.9)*;*

(b) *there exists a* $\nu > 0$, *an* $\eta \in (\eta_1,\eta_2)$, *a* $p_i \in (1,\infty)$ *and a* q_i *defined by* (7.6) *such that* $[\nu - U(\cdot,\tau_1)]_+ \in L^{q_i}(I_i(\eta))$ *where* $I_i(\eta)$ *is given by* (5.9), *and,* $s^{2p_i}a'(s) = o(1)$ *and* $s^{p_i}[(-1)^i b'(s)]_+ = o(1)$ *as* $s \downarrow 0$*; or*

(c) $s^2a'(s) = o(1)$ *and* $s[(-1)^i b'(s)]_+ = o(1)$ *as* $s \downarrow 0$*;*

hypothesis II *of Lemma* 10 *holds.*

PROOF. The proof is similar in spirit to that of the previous lemma. Let $\rho \in (\eta_1,\eta_2)$ be arbitrary. Choose $M < \infty$ so that (7.14) is satisfied and for any $\epsilon > 0$ define

$$\Phi_a(\epsilon) = \sup\{\alpha(v,w) : (v,w) \in \Upsilon(\epsilon)\}$$

and

$$\Phi_b(\epsilon) = \sup\{[(-1)^i\beta(v,w)]_+ : (v,w) \in \Upsilon(\epsilon)\}$$

where $\Upsilon(\epsilon)$, $\alpha(v,w)$ and $\beta(v,w)$ are defined by (7.8), (7.15) and (7.16) respectively. For any p_i and q_i satisfying $1 \le p_i \le \infty$, $1 \le q_i \le \infty$ and (7.6) we then have

$$\begin{aligned}&\{\alpha_i(\epsilon,\rho)^{1/2} + \beta_i(\epsilon,\rho)\}^{1/p_1}\ \|[\epsilon - U(\cdot,\tau_1)]_+\|_{L^{q_i}(I_i(\rho))}\\ &\le \epsilon\{\Phi_a^{1/2}(\epsilon) + \Phi_b(\epsilon)\}^{1/p_1}\ \|[1 - U(\cdot,\tau_1)/\epsilon]_+\|_{L^{q_i}(I_i(\rho))}\\ &\le \nu^{-1}\epsilon\{\Phi_a^{1/2}(\epsilon) + \Phi_b(\epsilon)\}^{1/p_i}\|[\nu - U(\cdot,\tau_1)]_+\|_{L^{q_i}(I_i(\rho))}\end{aligned}$$

whenever $0 < \epsilon < \nu < \infty$. The result now follows from Lemma 11 and the Monotone Convergence Theorem.

LEMMA 14. *Suppose that the assumptions of Lemma* 10 *up to and including the definition* (7.5) *hold. Suppose too that* $(-1)^i \eta_i < \infty$ *for* $i = 1, 2$. *Then, if*

$$a(U(x,t))/|x - \eta_i| \to 0 \quad \text{as} \quad (-1)^i(\eta_i - x) \downarrow 0$$

uniformly with respect to $t \in [\tau_1, \tau_2]$, *and*

$$u(x,t) \equiv 0 \text{ for all } (x,t) \in \overline{D} \text{ such that } |x - \eta_i| \leq \delta$$

for some $\delta > 0$, *hypothesis* I *of Lemma* 10 *holds.*

PROOF. Let $\rho \in (\eta_1, \eta_2)$ and $\iota \in (0, 1)$ be fixed. Then, by assumption there exists a $\nu \in (0, \min\{\delta, |\rho - \eta_i|\})$ such that

$$a(U(x,t)) \leq \iota|x - \eta_i| \quad \text{and} \quad |b(U(x,t))| \leq \iota$$

for all $(x,t) \in \overline{D}$ with $|x - \eta_i| < \nu$. Pick an $\epsilon > 0$ such that $a(\epsilon) < \iota\nu$ and $|b(\epsilon)| < \iota$, and let

$$\xi = \eta_i - (-1)^i a(\epsilon)/(2\iota).$$

For any $(x,t) \in \overline{D}$ with $|x - \eta_i| \leq |\xi - \eta_i|$ there holds

$$|\beta_\epsilon(x,t)/\alpha_\epsilon(x,t)| = |b(\epsilon) - b(U(x,t))|/\{a(\epsilon) - a(U(x,t))\} \leq 4\iota\ a(\epsilon)^{-1}$$

So that, one may deduce the estimates

$$\gamma_i(\epsilon, \xi) \leq 1 + 4\iota\ a(\epsilon)^{-1}$$

and

$$\sigma_i(\epsilon, \rho) \leq C_i(\epsilon, \xi) \leq [1 + 4\iota\ a(\epsilon)^{-1}][1 - \exp\{-2 - a(\epsilon)/(2\iota)\}]^{-1}.$$

Therefore,

$$\begin{aligned}
&\limsup_{\epsilon \downarrow 0}\ \sigma_i(\epsilon, \rho) \int_{\tau_1}^{\tau_2} [a(\epsilon) - a(U(\eta_i, t))]_+\ dt \\
&\leq \limsup_{\epsilon \downarrow 0}\ [a(\epsilon) + 4\iota][1 - \exp\{-2 - a(\epsilon)/(2\iota)\}]^{-1}(\tau_2 - \tau_1) \\
&= 4\iota[1 - \exp\{-2\}]^{-1}(\tau_2 - \tau_1).
\end{aligned}$$

Since though ρ and ι were arbitrary, this proves the lemma.

LEMMA 15. *Suppose that the assumptions of Theorem* 14 *up to and including* (7.1) *hold.*

(a) *Suppose that*

$$\eta_1 = -\infty, \text{ or, } \eta_1 > -\infty \text{ and } u(\eta_1, t) = 0 \text{ for all } t \in [\tau_1, \tau_2], \tag{7.17}$$

and let $\{\eta_{1.k}\}_{k=1}^{\infty}$ *denote an arbitrary sequence of values such that* $\eta_{1.1} < \eta_2$ *and* $\eta_{1.k} \downarrow \eta_1$ *as* $k \uparrow \infty$. *Define* $\eta_{2.k} = \eta_2$ *for any* $k \geq 1$.

(b) *Suppose that*

$$\eta_2 = \infty, \text{ or, } \eta_2 < \infty \text{ and } u(\eta_2, t) = 0 \text{ for all } t \in [\tau_1, \tau_2], \tag{7.18}$$

and let $\{\eta_{2.k}\}_{k=1}^{\infty}$ *denote an arbitrary sequence of values such that* $\eta_{2.1} > \eta_1$ *and* $\eta_{2.k} \uparrow \eta_2$ *as* $k \uparrow \infty$. *Define* $\eta_{1.k} = \eta_1$ *for any* $k \geq 1$.

(c) *Suppose that* (7.17) *and* (7.18) *hold, and let* $\{\eta_{1.k}\}_{k=1}^{\infty}$ *and* $\{\eta_{2.k}\}_{k=1}^{\infty}$ *denote arbitrary sequences of values such that* $\eta_{1.1} < \eta_{2.1}$ *and* $\eta_{1.k} \downarrow \eta_1$ *and* $\eta_{2.k} \uparrow \eta_2$ *as* $k \uparrow \infty$.
Then, in all three cases, setting

$$D_k = (\eta_{1.k}, \eta_{2.k}) \times (\tau_1, \tau_2], \tag{7.19}$$

there exists a sequence of functions $\{u_k\}_{k=1}^{\infty}$ *with the properties*:

(i) u_k *is a generalized solution of equation* (1.1) *in* D_k *which may be constructed under conditions equivalent to those of Theorem* 1, 2 *or* 3;

(ii) $U(x,t) \geq u_k(x,t)$ *for all* $(x,t) \in \overline{D}_k \setminus (\eta_{1.k}, \eta_{2.k}) \times [\tau_1, \tau_2]$;

(iii) $u_k(x,t) \uparrow u(x,t)$ *as* $k \uparrow \infty$ *for all* $(x,t) \in \overline{D}$.

PROOF. Denote by Ω the domain S, H, or Q depending on the problem in hand. In case (a) define $\Omega_k = (\eta_{1,k}, \infty) \times (\tau_1, \tau_2] \cap \Omega$ for any $k \geq 1$ and let $\{\chi_k\}_{k=1}^{\infty}$ denote an arbitrary sequence of nonnegative $C(\eta_1, \infty)$ functions such that the support of χ_k is contained in $[\eta_{1.k}, \infty)$ and $\chi_k(x) \uparrow 1$ as $k \uparrow \infty$ for all $x \in (\eta_1, \infty)$. In case (b) define $\Omega_k = (-\infty, \eta_{2,k}) \times (\tau_1, \tau_2] \cap \Omega$ and let $\{\chi_k\}_{k=1}^{\infty}$ denote an arbitrary sequence of nonnegative $C(-\infty, \eta_2)$ functions such that the support of χ_k is contained in $(-\infty, \eta_{2.k}]$ and $\chi_k(x) \uparrow 1$ as $k \uparrow \infty$ for all $x \in (-\infty, \eta_2)$. Finally, in case (c) define $\Omega_k = D_k$ and let $\{\chi_k\}_{k=1}^{\infty}$ denote an arbitrary sequence of nonnegative $C(\eta_1, \eta_2)$ functions such that the support of χ_k is contained in $[\eta_{1,k}, \eta_{2,k}]$ and $\chi_k(x) \uparrow 1$ as $k \uparrow \infty$ for all $x \in (\eta_1, \eta_2)$.

Let u_k denote the generalized solution of equation (1.1) in the domain Ω_k with boundary data

$$u_k(x,t) = A(\chi_k(x) a(u(x,t))) \text{ for all } (x,t) \in \overline{\Omega}_k \setminus \Omega_k.$$

We remark that if $\chi_k a(u)$ is not identically zero on any segment of the lateral boundary of Ω_k, then this segment necessarily constitutes part of the lateral boundary of the original domain Ω. Consequently, any assumptions regarding

the regularity of the lateral boundary data for the problem for u are bequeathed to the problem for u_k. Therefore, retracing the proofs of Theorems 1-6 it can be shown that each u_k exists and is unique. Moreover, uniform continuity estimates of the type (4.9) can be obtained for the sequence $\{u_k\}_{k=1}^{\infty}$.

For $(x,t) \in \overline{D}\backslash\overline{\Omega}_k$, we define $u_k(x,t) = 0$.

Now, we note that the argument used to justify the existence of each u_k also supports application of Theorem 13 to infer the monotonicity of the sequence $\{u_k\}_{k=1}^{\infty}$. Recalling the uniform continuity estimates for the sequence, setting

$$\Omega_\infty = \bigcup_{k=1}^{\infty} \Omega_k$$

and

$$u_\infty(x,t) = \lim_{k\uparrow\infty} u_k(x,t) \qquad \text{for } (x,t) \in \overline{\Omega}_\infty$$

it can subsequently be shown that u_∞ is a generalized solution of (1.1) in Ω_∞ with boundary data

$$u_\infty(x,t) = u(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}_\infty\backslash\Omega_\infty \tag{7.20}$$

(just as Theorems 1, 2 and 3 were proved). However, seeing that the argument invoked to justify the unique existence of each u_k for $k < \infty$ can be repeated to infer the unique existence of a generalized solution of (1.1) in Ω_∞ satisfying (7.20), and u itself is such a solution, we must have $u_\infty \equiv u$ in $\overline{\Omega}_\infty$.

To confirm that the sequence $\{u_k\}_{k=1}^{\infty}$ has all the properties cited, we finally note that by construction each $u_k(x,t) = 0$ at all points (x,t) on the lateral boundary of D_k which do not lie on the boundary of D itself and that $\Omega_\infty \supseteq D$.

COROLLARY TO LEMMA 15. *From the proof of the lemma it can be ascertained that the following may be added to the list of properties of the sequence* $\{u_k\}_{k=1}^{\infty}$, *for each* $k \geq 1$:

(iv) $u_k \in C(\overline{D}) \cap L^\infty(D)$;

(v) $u_k(x,t) = 0$ *for all* $(x,t) \in \overline{D\backslash D_k}$;

(vi) u_k *is a classical solution of equation* (1.1) *at any point* $(x,t) \in D$ *where* $u_k(x,t) > 0$.

Combining Lemmata 12-15 with Lemma 10, we are able to prove the next result which is a contender for the position occupied by Theorem 14 in its own right.

PROPOSITION 1. *Suppose that the assumptions of Theorem* 14 *up to and including* (7.1) *hold. Suppose too that the following hold.*

I. *When* $(-1)^i\eta_i < \infty$ *for* $i = 1,2$:

(a) $U(\eta_i,t) > 0$ *for all* $t \in [\tau_1,\tau_2]$;

(b) $U(\eta_i, t) > 0$ *for almost all* $t \in [\tau_1, \tau_2]$ *and* $a(s)[(-1)^i b'(s)]_+ = o(a'(s))$ *as* $s \downarrow 0$;

(c) $a(s)[(-1)^i b'(s)]_+ = o(a'(s))$ *as* $s \downarrow 0$; *or*

(d) $a(U(x,t))/|x - \eta_i| \to 0$ *as* $(-1)^i(\eta_i - x) \downarrow 0$ *uniformly with respect to* $t \in [\tau_1, \tau_2]$.

II. *When* $(-1)^i \eta_i = \infty$ *for* $i = 1, 2$:

(a) *there exists a* $\nu > 0$ *and an* $\eta \in (\eta_1, \eta_2)$ *such that* $[\nu - U(\cdot, \tau_1)]_+ \in L^1(I_i(\eta))$ *where* $I_i(\eta)$ *is given by* (5.9);

(b) *there exists a* $\nu > 0$, *an* $\eta \in (\eta_1, \eta_2)$, *a* $p_i \in (1, \infty)$ *and a* q_i *defined by* (7.6) *such that* $[\nu - U(\cdot, \tau_1)]_+ \in L^{q_i}(I_i(\eta))$ *where* $I_i(\eta)$ *is given by* (5.9), *and,* $s^{2p_i} a'(s) = o(1)$ *and* $s^{p_i}[(-1)^i b'(s)]_+ = o(1)$ *as* $s \downarrow 0$; *or*

(c) $s^2 a'(s) = o(1)$ *and* $s[(-1)^i b'(s)]_+ = o(1)$ *as* $s \downarrow 0$.
Then (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

PROOF. If conditions I(a)-(c) or II(a)-(c) are satisfied for $i = 1, 2$, the proposition is a mere amalgamation of Lemma 10 with Lemmata 12 and 13. Suppose therefore for arguments sake that condition I(d) applies for $i = 1$ or 2. In this case, by Lemma 15 and its corollary there exists a sequence of nonnegative functions $\{u_k\}_{k=1}^{\infty} \subseteq C(\overline{D}) \cap L^\infty(D)$ such that each u_k is a classical solution of equation (1.1) at any point $(x,t) \in D$ where $u_k(x,t) > 0$, $u_k(x,t) = 0$ for all $(x,t) \in \overline{D}$ with $|x - \eta_i| \leq 1/k$, and, $u_k(x,t) \uparrow u(x,t)$ as $k \uparrow \infty$ for all $(x,t) \in \overline{D}$. Thus amalgamating Lemmata 12-14 with Lemma 10 for the sequence $\{u_k\}$,

$$\int_{\eta_1}^{\eta_2} [u_k(x,t) - U(x,t)]_+ dx \leq \int_{\eta_1}^{\eta_2} [u_k(x,\tau_1) - U(x,\tau_1)]_+ dx$$

for all $k \geq 1$ and $t \in [\tau_1, \tau_2]$. Letting $k \uparrow \infty$ subsequently yields (7.4) in this instance also.

The scope of the applicability of Proposition 1 may be enhanced by the following propositions.

PROPOSITION 2. *Suppose that the assumptions of Theorem* 14 *up to and including* (7.1) *hold. Suppose that* (7.4) *holds for all* $t \in [\tau_1, \tau_2)$. *Then* (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

PROOF. If the right-hand integral in (7.4) is unbounded, the result is trivially true. Whereas, if the right-hand integral in (7.4) is bounded one can apply Fatou's Lemma to obtain the result.

PROPOSITION 3. *Suppose that the assumptions of Theorem* 14 *up to and*

including (7.1) *hold. Suppose that*

$$\int_{\eta_1}^{\eta_2} [u(x,t) - U(x,t)]_+ \, dx \le \int_{\eta_1}^{\eta_2} [u(x,\tau) - U(x,\tau)]_+ \, dx \tag{7.21}$$

for all $\tau \in (\tau_1, \tau_2]$ *and* $t \in [\tau, \tau_2]$. *Then* (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

PROOF. To confirm the proposition it suffices to show that

$$\limsup_{\tau \downarrow \tau_1} \int_{\eta_1}^{\eta_2} [u(x,\tau) - U(x,\tau)]_+ dx \le \int_{\eta_1}^{\eta_2} [u(x,\tau_1) - U(x,\tau_1)]_+ dx. \tag{7.22}$$

However,

$$\limsup_{\tau \downarrow \tau_1} \int_I [u(x,\tau) - U(x,\tau)]_+ dx = \int_I [u(x,\tau_1) - U(x,\tau_1)]_+ dx$$

for any bounded interval $I \subseteq (\eta_1, \eta_2)$ by the Dominated Convergence Theorem. Thus to verify (7.22) it is enough, in the event that $(-1)^i \eta_i = \infty$ for $i = 1$ or 2, to demonstrate that

$$\limsup_{\tau \downarrow \tau_1} \int_{I_i(\rho)} [u(x,\tau) - U(x,\tau)]_+ dx \le \int_{I_i(\rho)} [u(x,\tau_1) - U(x,\tau_1)]_+ dx \tag{7.23}$$

where $I_i(\rho)$ is defined by (5.9), for some $\rho \in (\eta_1, \eta_2)$. We shall prove (7.23) with $i = 1$ only, for the proof with $i = 2$ is similar.

Fix $\rho_2 \in (-\infty, \eta_2)$ and let $M = \sup\{u(x,t) : (x,t) \in \overline{D}\}$. Choose $\varsigma \in (\rho_2, \eta_2)$ and consider the functions $u^{\pm}(x,t)$ define respectively as the generalized solutions of equation (1.1) in the domain

$$\Omega_0 = (\rho_2, \varsigma) \times (\tau_1, \tau_2]$$

with data

$$\begin{aligned} &u^+(\rho_2, t) = 0 && \text{for } t \in (\tau_1, \tau_2] \\ &u^+(x, \tau_1) = M(x - \rho_2)/(\varsigma - \rho_2) && \text{for } x \in [\rho_2, \varsigma] \\ &u^+(\varsigma, t) = M && \text{for } t \in (\tau_1, \tau_2] \end{aligned}$$

and

$$\begin{aligned} &u^-(\rho_2, t) = M && \text{for } t \in (\tau_1, \tau_2] \\ &u^-(x, \tau_1) = M(\varsigma - x)/(\varsigma - \rho_2) && \text{for } x \in [\rho_2, \varsigma] \\ &u^-(\varsigma, t) = 0 && \text{for } t \in (\tau_1, \tau_2]. \end{aligned}$$

These generalized solutions can be shown to exist by linearly transforming the domain Ω_0 to Q and applying Theorem 3. Furthermore because these generalized solutions are continuous, there must be a $\mu > 0$ and a $\delta > 0$ such that

$$u^+(x,t) \geq \mu \qquad \text{for all } (x,t) \in [\varsigma - \delta, \varsigma] \times [\tau_1, \tau_2] \tag{7.24}$$

and

$$u^-(x,t) \geq \mu \qquad \text{for all } (x,t) \in [\rho_2 + \delta, \rho_2] \times [\tau_1, \tau_2]. \tag{7.25}$$

Now, for every positive integer k define u_k as the generalized solution of equation (1.1) in the domain

$$\Omega_k = (\rho_2 - k, \varsigma) \times (\tau_1, \tau_2]$$

with data

$$u_k(x,t) = \max\{u^-(x+k,t), u(x,t), u^+(x,t)\} \qquad \text{for } (x,t) \in \overline{\Omega}_k \backslash \Omega_k.$$

Here, to prevent any misunderstanding, we implicitly assume that $u^{\pm}(x,t) = 0$ if $(x,t) \notin \overline{\Omega}_0$. The solution u_k also exists by Theorem 3 upon applying a suitable rescaling of Ω_k. Moreover, by the corollary to Theorem 13 for the rescaled problem,

$$u^-(x,t) \leq u_k(x-k,t) \qquad \text{for all } (x,t) \in \overline{\Omega}_0 \tag{7.26}$$

$$u^+(x,t) \leq u_k(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}_0 \tag{7.27}$$

and

$$u(x,t) \leq u_k(x,t) \leq M \qquad \text{for all } (x,t) \in \overline{\Omega}_k. \tag{7.28}$$

In the light of (7.24)-(7.28) though, Lemma 7 subsequently tells us that for any function $\omega \in C^\infty(-\infty, \eta_2)$ satisfying (5.6)-(5.8) there holds

$$\int_{\rho_1}^{\rho_2} \omega(x)\{u_k(x,\tau) - U(x,\tau)\}dx \leq \int_{\rho_2 - k}^{\varsigma} [u_k(x,\tau_1) - U(x,\tau_1)]_+ dx$$

$$+ C \int_{\tau_1}^{\tau} [a(u_k(\rho_2 - k, t)) - a(U(\rho_2 - k, t))]_+ dt$$

$$+ C \int_{\tau_1}^{\tau} [a(u_k(\varsigma, t)) - a(U(\varsigma, t))]_+ dt$$

for all k such that $k > |\rho_2 - \rho_1|$ and any $\tau \in (\tau_1, \tau_2]$. Here C is a constant which depends only on μ, M and δ. However, recalling (7.28) and noting that $u_k(x, \tau_1) = u(x, \tau_1)$ for all $x \in (\varsigma - k, \rho_2)$, this implies

$$\int_{\rho_1}^{\rho_2} \omega(x)\{u(x,\tau) - U(x,\tau)\}dx \le \int_{\varsigma-k}^{\rho_2} [u(x,\tau_1) - U(x,\tau_1)]_+ dx + 2M|\varsigma - \rho_2| + 2Ca(M)(\tau - \tau_1).$$

Whence, letting $k \uparrow \infty$,

$$\int_{-\infty}^{\rho_2} [u(x,\tau) - U(x,\tau)]_+ dx \le \int_{-\infty}^{\rho_2} [u(x,\tau_1) - U(x,\tau_1)]_+ dx + 2M|\varsigma - \rho_2| + 2Ca(M)(\tau - \tau_1)$$

in view of the arbitrariness of ω. Consequently,

$$\limsup_{\tau \downarrow \tau_1} \int_{I_1(\rho_2)} [u(x,\tau) - U(x,\tau)]_+ dx \le \int_{I_1(\rho_2)} [u(x,\tau_1) - U(x,\tau_1)]_+ dx + 2M|\varsigma - \rho_2|.$$

Passing to the limit as $\varsigma \downarrow \rho_2$, this yields the desired inequality (7.23) with $i = 1$ and $\rho = \rho_2$.

PROPOSITION 4. *Suppose that the assumptions of Theorem* 14 *up to and including* (7.1) *hold. Suppose that* (7.4) *holds for all* $t \in [\tau_1, \tau]$ *and* (7.21) *holds for all* $t \in [\tau, \tau_2]$, *for some* $\tau \in (\tau_1, \tau_2)$. *Then* (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

This proposition requires no proof.

PROPOSITION 5. *Suppose that the assumptions* (*and therefore also the conclusions*) *of Lemma* 15 *hold. Suppose that*

$$\int_{\eta_{1,k}}^{\eta_{2,k}} [u_k(x,t) - U(x,t)]_+ \, dx \le \int_{\eta_{1,k}}^{\eta_{2,k}} [u_k(x,\tau_1) - U(x,\tau_1)]_+ \, dx \tag{7.29}$$

for all $t \in [\tau_1, \tau_2]$ *and* $k \ge 1$. *Then* (7.4) *holds for all* $t \in [\tau_1, \tau_2]$.

This proposition follows from the Monotone Convergence Theorem.

PROOF OF THEOREM 14. (a) We say that an arbitrary domain $(\tilde{\eta}_1, \tilde{\eta}_2) \times (\tilde{\tau}_1, \tilde{\tau}_2] \subseteq D$ is of class I if

$$\tilde{\eta}_1 > -\infty \text{ and } U(\tilde{\eta}_1, t) > 0 \qquad \text{for all } t \in (\tilde{\tau}_1, \tilde{\tau}_2), \tag{7.30}$$

and

$$\tilde{\eta}_2 < \infty \text{ and } U(\tilde{\eta}_2, t) > 0 \qquad \text{for all } t \in (\tilde{\tau}_1, \tilde{\tau}_2). \tag{7.31}$$

We say that it is of class II if (7.30) is satisfied and

$$\tilde{\eta}_2 = \infty, \text{ or, } \tilde{\eta}_2 < \infty \text{ and } U(\tilde{\eta}_2, t) = 0 \qquad \text{for all } t \in [\tilde{\tau}_1, \tilde{\tau}_2]. \tag{7.32}$$

We say that it is of class III if (7.31) holds and

$$\tilde{\eta}_1 = -\infty, \text{ or, } \tilde{\eta}_1 > -\infty \text{ and } U(\tilde{\eta}_1, t) = 0 \qquad \text{for all } t \in [\tilde{\tau}_1, \tilde{\tau}_2]. \tag{7.33}$$

Finally, we say that it is of class IV if (7.32) and (7.33) hold.

By assumption we can write

$$D = \bigcup_{j=1}^{N} D_j \tag{7.34}$$

where each subdomain

$$D_j = (\eta_1, \eta_2) \times (\tau_{1,j}, \tau_{2,j}]$$

is of class I, II, III or IV. Next, we let $\{\eta_{1,k}\}_{k=1}^{\infty}$ and $\{\eta_{2,k}\}_{k=1}^{\infty}$ denote two sequences of values such that $\eta_{1,1} < \eta_{2,1}$ and $\eta_{1,k} \downarrow \eta_1$ and $\eta_{2,k} \uparrow \eta_2$ as $k \uparrow \infty$. If D_j is of class I we set

$$D_{j,k} = (\eta_1, \eta_2) \times (\tau_{1,j}, \tau_{2,j}],$$

if D_j is of class II we set

$$D_{j,k} = (\eta_1, \eta_{2,k}) \times (\tau_{1,j}, \tau_{2,j}],$$

if D_j if of class III we set

$$D_{j,k} = (\eta_{1,k}, \eta_2) \times (\tau_{1,j}, \tau_{2,j}],$$

and finally if D_j is of class IV we set

$$D_{j,k} = (\eta_{1,k}, \eta_{2,k}) \times (\tau_{1,j}, \tau_{2,j}],$$

for every $k \geq 1$. Furthermore, if D_j is of class I we define $u_{j,k} = u$ for any $k \geq 1$, whereas if D_j belongs to one of the other classes we let $\{u_{j,k}\}_{k=1}^{\infty}$ denote a sequence of generalized solutions of equation (1.1) with the properties ascribed in Lemma 15 so that $u_{j,k} \uparrow u$ as $k \uparrow \infty$ on D_j. For notational convenience, we write

$$D_{j,k} = (\eta_{1,j,k}, \eta_{2,j,k}) \times (\tau_{1,j}, \tau_{2,j}]$$

irrespective of the classification of D_j. Finally, for every j and k, we define

$$D_{j,k} = \bigcup_{\ell=1}^{N_{j,k}} D_{j,k,\ell}$$

where each subdomain

$$D_{j,k,\ell} = (\eta_{1,j,k}, \eta_{2,j,k}) \times (\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell}]$$

is of class I, II, III or IV. Here, as in (7.34), the union is finite because of the assumption on the number of connected components of $\{t \in [\tau_1, \tau_2] : U(\eta, t) > 0\}$ for every $\eta \in \overline{(\eta_1, \eta_2)}$.

Now, we observe that

$$-\infty < \eta_{1,j,k} < \eta_{2,j,k} < \infty$$

for all $j = 1, 2, \ldots, N$ and $k \geq 1$. Moreover, for $i = 1, 2$, either

$$U(\eta_{i,j,k}, t) > 0 \qquad \text{for all } t \in (\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell})$$

or, $\eta_{i,j,k} \in (\eta_1, \eta_2)$ and

$$U(\eta_{i,j,k}, t) = 0 \qquad \text{for all } t \in [\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell}].$$

In this latter instance though, since $(a(U))_x$ exists and is continuous in a neighbourhood of any point $(x, t) \in D$ where $U(x, t) = 0$, it can be shown that

$$a(U(x,t)) = o(|x - \eta_{i,j,k}|) \text{ as } x \to \eta_{i,j,k}$$

uniformly on compact subsets of $(\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell}]$. Consequently, in either event,

$$\int_{\eta_{1,j,k}}^{\eta_{2,j,k}} [u_{j,k}(x, t_2) - U(x, t_2)]_+ dx \leq \int_{\eta_{1,j,k}}^{\eta_{2,j,k}} [u_{j,k}(x, t_1) - U(x, t_1)]_+ dx \tag{7.35}$$

for all $t_1 \in (\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell})$ and $t_2 \in [t_1, \tau_{2,j,k,\ell})$, by Proposition 1. However by Propositions 2 and 3, this implies (7.35) for any $t_1 \in [\tau_{1,j,k,\ell}, \tau_{2,j,k,\ell}]$ and $t_2 \in [t_1, \tau_{2,j,k,\ell}]$. Whereupon, by Proposition 4, (7.35) actually holds for any $t_1 \in [\tau_{1,j}, \tau_{2,j}]$ and $t_2 \in [t_1, \tau_{2,j}]$, and, for every j and k. Now, applying Proposition 5, this means

$$\int_{\eta_1}^{\eta_2} [u(x,t) - U(x,t)]_+ dx \leq \int_{\eta_1}^{\eta_2} [u(x, \tau_{1,j}) - U(x, \tau_{1,j})]_+ dx$$

for all $t \in [\tau_{1,j}, \tau_{2,j}]$ and $j = 1, \ldots, N$. Whence, by a further application of Proposition 4, (7.4) holds for all $t \in [\tau_1, \tau_2]$.

(b) If $(-1)^i\eta_i < \infty$ and (7.2) holds, or, $(-1)^i\eta_i = \infty$ and (7.3) holds for $i = 1, 2$, the theorem follows immediately from Proposition 1. If the aforesaid is not the case, we consider the sequence of domains D_k defined by (7.19), where $\eta_{1,k} = \eta - k$ if $\eta_1 = -\infty$ and (7.3) does not hold for $i = 1$ and $\eta_{1.k} = \eta_1$ otherwise, and where $\eta_{2.k} = \eta + k$ if $\eta_2 = \infty$ and (7.3) does not hold for $i = 2$ and $\eta_{2,k} = \eta_2$ otherwise, for some $\eta \in (\eta_1, \eta_2)$ and for any $k \geq 1$. Furthermore, we let $\{u_k\}_{k=1}^{\infty}$ denote a sequence of generalized solutions of equation (1.1) approximating u with the properties described in Lemma 15 whose existence is justified there. Since u_k, U and the coefficients of (1.1) satisfy the assumptions of Proposition 1 on the domain D_k for every $k \geq 1$, (7.29) holds for all $k \geq 1$. Whence, invoking Proposition 5, the theorem is proved.

It should be clear from the proof of Theorem 14, how using Propositions 2-5 to "boot-strap" from the basic result in Proposition 1, other results generalizing Theorem 14 may be obtained.

8. - Discussion

In [11] the existence of a weak solution of the Cauchy problem, the Cauchy-Dirichlet problem, and the first boundary-value problem for equation (1.1) was established under a number of regularity assumptions on the boundary data and under Hypothesis 1 plus the supplementary hypotheses (1.2) and (1.3). On the other hand, in [8], the existence of a generalized solution of Problems 1-3 was established under Hypothesis 1 plus the supplementary hypotheses

$$a'(s)/b(s) \in L^1(0, \varepsilon) \qquad \text{for any } \varepsilon > 0, \tag{8.1}$$

and, given any $M \in (0, \infty)$ there exist constants $0 < K_1,\ K_2 < \infty$ such that

$$b''(s)b(s) \leq -K_1 a'(s),\ -K_1(a'(s))^2,\ -K_1 a'(s)|b'(s)| \text{ for all } s \in (0, M], \tag{8.2}$$

and

$$|a''(s)| \leq K_2|b''(s)| \qquad \text{for all } s \in (0, M]. \tag{8.3}$$

The present Theorems 1-3 can be seen to be improvements on these results in view of the following two propositions.

PROPOSITION 6. *If* (1.3) *holds, then by necessity Hypothesis* 2 *holds.*

PROPOSITION 7. *If* (8.2) *and* (8.3) *hold, then by necessity* (1.3) *holds.*

PROOF OF PROPOSITION 6. It suffices to show that $sa'(s)b'(s) \in L^1(0, \varepsilon)$ for fixed $\varepsilon > 0$.

The fact that $sb''(s) \in L^1(0,\varepsilon)$ implies that there exists a constant $C_1 > 0$ such that for all $s \in (0,\varepsilon)$,

$$C_1 \geq \int_s^\varepsilon r|b''(r)|dr \geq |\int_s^\varepsilon rb''(r)dr| = |\varepsilon b'(\varepsilon) - sb'(s) - b(\varepsilon) + b(s)|.$$

Hence, because $b \in C([0,\infty))$, there exists a second constant $C_2 > 0$ such that

$$|sb'(s)| \leq C_2 \qquad \text{for all } s \in (0,\varepsilon).$$

However, this means that

$$\int_s^\varepsilon |ra'(r)b'(r)|dr \leq C_2 \int_s^\varepsilon a'(r)dr \leq C_2 a(\varepsilon) \text{ for all } s \in (0,\varepsilon).$$

PROOF OF PROPOSITION 7. Without any loss of generality, by (8.2) and (2.10), we may assume that

$$b(s) > 0 > b''(s) \qquad \text{for all } s > 0.$$

Thus, for any $s \in (0,\varepsilon)$, $\varepsilon > 0$,

$$\begin{aligned}
\int_s^\varepsilon r|b''(r)|dr &= -\int_s^\varepsilon rb''(r)dr \\
&= sb'(s) - \varepsilon b'(\varepsilon) - b(s) + b(\varepsilon) \\
&= \int_0^s b'(s)dr - \varepsilon b'(\varepsilon) - b(s) + b(\varepsilon) \\
&\leq \int_0^s b'(r)dr - \varepsilon b'(\varepsilon) - b(s) + b(\varepsilon) \\
&= -\varepsilon b'(\varepsilon) + b(\varepsilon),
\end{aligned}$$

using (2.11). However, since $s \in (0,\varepsilon)$ was arbitrary, this means that $sb''(s) \in L^1(0,\varepsilon)$. Consequently, (8.3) implies that in turn, $sa''(s) \in L^1(0,\varepsilon)$.

Problems 1, 2 and 3 were shown to have unique weak solutions in [11] when (1.2) and (1.4) hold in addition to Hypothesis 1. Uniqueness of generalized solutions of these problems was asserted in [8], when in addition to Hypothesis 1, (8.1)-(8.3) hold,

$$a \in C^1([0,\infty)) \tag{8.4}$$

with $a'(0) = 0$, and there exists a convex function $\mu \in C^1([0,\infty)) \cap C^2(0,\infty)$ such that

$$\mu(0) = 0 < \mu'(s) \leq a'(s) \qquad \text{for all } s > 0 \tag{8.5}$$

and at least one of the following hold

$$\limsup_{s\downarrow 0} |b'(s)| < \infty, \tag{8.6}$$

$$\limsup_{s\downarrow 0} [-b'(s)]_+ < \infty \text{ and } \limsup_{s\downarrow 0} [b''(s)]_+ < \infty, \tag{8.7}$$

$$\limsup_{s\downarrow 0} [b'(s)]_+ < \infty \text{ and } \limsup_{s\downarrow 0} [-b''(s)]_+ < \infty. \tag{8.8}$$

Moreover, for Problems 2 and 3,

$$a(s) \leq K\ \mu(s) \qquad \text{for all } s > 0 \tag{8.9}$$

for some constant $0 < K < \infty$, where μ is as defined above.

In Theorems 4-6 of the present paper, uniqueness of generalized solutions of Problems 1-3 has been established without any additional assumptions on the coefficients a and b in equation (1.1) over and above those required for existence.

The proofs of our existence and uniqueness theorems are founded upon the construction of a generalized solution of equation (1.1) as the limit of a sequence of positive classical solutions of the equation. The conditions on the functions a and b in Hypothesis 1 are precisely those which permit such a construction. Thus, within the confines of our method, Hypothesis 1 may be regarded as minimal.

In this respect, we note that for the Cauchy problem, the Cauchy-Dirichlet problem, and the first boundary-value problem for equation (1.1) with homogeneous lateral boundary conditions, our results are comparable with those of Bénilan and Touré [4]; leading to a slight relaxation of the conditions imposed on the initial data functions.

With regard to the comparison principles, Theorems 13 and 14 appear in [8] with the domain D restricted to the problem domain S, H or Q, with the same hypotheses as in the uniqueness theorems there. In the light of the propositions below, both of the present Theorems 13 and 14 can be seen to supersede these results even ignoring the question of the justification of every step in the arguments in [8] for the problems on the unbounded domains.

PROPOSITION 8. *If* (8.4) *holds,*

$$s^2 a'(s) = o(1) \text{ as } s \downarrow 0, \tag{8.10}$$

and if any of (8.6)-(8.8) *hold,*

$$s|b'(s)| = o(1) \text{ as } s \downarrow 0. \tag{8.11}$$

PROPOSITION 9. *If there is a convex function* $\mu \in C^1([0,\infty)) \cap C^2(0,\infty)$ *such that* (8.5) *and* (8.9) *hold, and any of* (8.6)-(8.8) *hold,*

$$a(s)|b'(s)| = o(a'(s)) \text{ as } s \downarrow 0.$$

PROOF OF PROPOSITION 8. The assertion (8.10) is a trivial consequence of (8.4). Furthermore, when (8.6) holds, (8.11) is trivial. Suppose therefore that (8.7) applies. In this event, there is a constant $C < \infty$ such that

$$b''(s) < C \qquad \text{for all } s \in (0,1].$$

Hence,

$$b'(s) < b'(r) + C(s-r) \qquad \text{for all } 0 < r < s \leq 1.$$

Whence, integrating with respect to r from 0 to s,

$$sb'(s) < b(s) + Cs^2/2 \qquad \text{for all } s \in (0,1].$$

From this, it follows that

$$s[b'(s)]_+ \to 0 \text{ as } s \downarrow 0.$$

However, recalling (8.7), plainly

$$s[-b'(s)]_+ \to 0 \text{ as } s \downarrow 0.$$

So, (8.11) must hold in this case also. If, to conclude, (8.8) is true, we can verify (8.11) by replacing b with $-b$ in the preceeding argument.

PROOF OF PROPOSITION 9. Since μ is convex and $\mu(0) = 0$,

$$\mu'(s) \geq \mu(s)/s \qquad \text{for all } s > 0.$$

Hence, combining (8.5) with (8.9),

$$a(s)/a'(s) \leq Ks \qquad \text{for all } s > 0.$$

If though any one of (8.6)-(8.8) is satisfied, by Proposition 8, (8.11) holds.

From an historical point of view, it is interesting to note that (to the best of the author's knowledge) it is only with the appearance of the present paper that existence results for the Cauchy problem for equation (1.1), comparable to those obtained in 1958 by Oleinik, Kalashnikov and Chzhou [15] for equation (1.5), have been established. In previous studies, the inclusion of the first-order term on the right-hand side of equation (1.1) had inevitably led to the imposition of additional assumptions on the function a in equation (1.1) over and above those required by Oleinik et al. for the function a in equation (1.5), cf. [11].

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