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Globally Regular Solutions to the u^5 Klein-Gordon Equation

MICHAEL STRUWE

1. - Introduction

Consider the non-linear wave equation

$$(1.1) \quad u_{tt} - \Delta u + u^p = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+$$

with initial data

$$(1.2) \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1.$$

In 1961 K. Jörgens proved that, for $p < 5$, equation (1.1) admits a unique regular solution $u \in C^2$ for any Cauchy data $u_0 \in C^3$, $u_1 \in C^2$, see [1, Satz 2, p. 298].

The case $p = 5$ was later investigated by Rauch, who obtained global regularity for small initial energies

$$(1.3) \quad E_0 = \int_{\mathbb{R}^3} \left(\frac{1}{2} (|\nabla u_0|^2 + |u_1|^2) + \frac{1}{6} |u_0|^6 \right) dx < \frac{\pi}{\sqrt{3}},$$

see [3, Theorem, p. 347].

Rauch's approach, moreover, reveals that $p = 5$ arises as a limiting exponent for a Sobolev embedding relevant for problem (1.1), see [3, estimate (14), p. 346]. This and recent progress in elliptic equations involving limiting non-linearities has been our motivation for studying problem (1.1-2).

The supercritical case $p > 5$ seems to be open.

In this paper we show that Rauch's smallness assumption actually is unnecessary and that Jörgen's result continues to hold - at least for radially symmetric solutions - at the limiting exponent $p = 5$, which will be fixed from now on throughout this paper.

THEOREM 1.1. *For any radially symmetric initial data $u_0 \in C^3(\mathbb{R}^3)$, $u_1 \in C^2(\mathbb{R}^3)$, $u_0(x) = u_0(|x|)$, $u_1(x) = u_1(|x|)$, there exists a unique, global, radially symmetric solution $u \in C^2(\mathbb{R}^3 \times [0, \infty])$, $u(x, t) = u(|x|, t)$ to the Cauchy problem (1.1-2), with $p = 5$.*

The proof involves a blow-up analysis of possible singularities of equation (1.1). Thereby we heavily exploit “conformal invariance” of (1.1), i.e. invariance of (1.1) under scaling

$$(1.5) \quad u \rightarrow u_R(x, t) = R^{1/2} u(Rx, Rt).$$

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I wish to thank Sergiu Klainerman for his interest and stimulating discussions at an early stage of this work and for bringing Jeffrey Rauch’s paper [3] to my attention.

2. - Some fundamental estimates

In this section we recall Rauch’s result for equation (1.1) and prove two basic integral estimates which result from testing (1.1) with suitable functions φ . Besides the standard choice $\varphi = u_t$ - which gives rise to the well-known “energy inequality”, see Lemma 2.1 -, we will also use u and its radial derivative $x \cdot \nabla u$ as testing functions: the remaining components of the generator

$$\frac{d}{dR} u_R|_{R=1} = tu_t + x \cdot \nabla u + \frac{1}{2} u$$

of the family (1.5). This will give rise to the crucial “Pohožaev-type identity” Lemma 2.2 (see [2] for a related result in an elliptic setting).

2.1 Notations

Denote $z = (x, t)$ a generic point in space-time. The negative light-cone through $z_0 = (x_0, t_0)$ is given by

$$C(z_0) = \{(x, t) \mid t \leq t_0, |x - x_0| \leq t_0 - t\}.$$

Its mantle and space-like sections are denoted by

$$M(z_0) = \{(x, t) \in C(z_0) \mid |x - x_0| = t_0 - t\},$$

resp. by

$$D(z_0, t) = C(z_0) \cap (\mathbb{R}^3 \times \{t\}).$$

If $z_0 = (0, 0)$ the point z_0 will be omitted from this notation.

Truncated cones will be denoted

$$C_s^t = C \cap (\mathbb{R}^3 \times [s, t]), \quad C_s^0 = C_s, \quad C_{-\infty}^t = C^t.$$

$B_R(x_0)$ denotes the Euclidean ball

$$B_R(x_0) = \{x \in \mathbb{R}^3 \mid |x - x_0| < R\}.$$

Again, if $x_0 = 0$ we simply write $B_R(0) = B_R$.

Finally, for a C^1 -function u and a space-like region $\Omega(t) \subset (\mathbb{R}^3 \times \{t\})$,

$$E(u; \Omega(t)) = \int_{\Omega(t)} e(u) dx$$

denotes the energy of u in $\Omega(t)$, with density

$$e(u) = \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6.$$

The letters c, C will denote generic positive constants, occasionally numbered for clarity.

2.2 The energy inequality

Let $u \in C^2(\mathbb{R}^3 \times]-\infty, 0])$ be a solution to (1.1). Actually, by finiteness of propagation speed, all estimates only require u to be C^2 near suitable sections of cones.

Multiply (1.1) by u_t . This gives the identity

$$(2.1) \quad \left(\frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right)_t - \operatorname{div}(\nabla u \cdot u_t) = 0.$$

If we integrate this expression over a section C_s^t of the negative light-cone, we obtain the following result:

$$(2.2) \quad E(u; D(t)) - E(u; D(s)) + \frac{1}{\sqrt{2}} \int_{M_s^t} \left(\frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 - \frac{x}{|x|} \cdot \nabla u \, u_t \right) d\sigma = 0.$$

Note that the outward normal to M_s^t is given by $n(x, t) = \frac{1}{\sqrt{2}} \left(\frac{x}{|x|}, 1 \right)$; moreover, we recognize the energy density $e(u)$ inside the left bracket of (2.1).

Rauch [3, p. 345] interprets the boundary integrand as follows:

$$\begin{aligned}
 (2.3) \quad & \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 - \frac{x}{|x|} \cdot \nabla u \cdot u_t \\
 &= \frac{1}{2} \left| \frac{x}{|x|} u_t - \nabla u \right|^2 + \frac{1}{6} |u|^6 = \frac{1}{2} |\nabla_y v|^2 + \frac{1}{6} |v|^6,
 \end{aligned}$$

where

$$(2.4) \quad v(y) = u(y, -|y|).$$

Thus we may state:

LEMMA 2.1. *For any $s < t < 0$ there holds the energy estimate*

$$E(u; D(t)) + \int_{B_{|s|} \setminus B_{|t|}} \left\{ \frac{1}{2} |\nabla v|^2 + \frac{1}{6} |v|^6 \right\} dy = E(u; D(s)),$$

where v is given by (2.4).

2.3 A Pohožaev-type identity

The next result apparently is new. This and Lemma 3.3 are the crucial ingredients in the proof of Theorem 1.1.

LEMMA 2.2. *For u as above there holds*

$$\begin{aligned}
 & \frac{1}{3} \int_{C_{-1}} |u|^6 dx dt + E(u; D(-1)) \\
 & \leq \int_{D(-1)} u_t (x \cdot \nabla u + u) dx + \int_{B_1} \{ |y| |\nabla v|^2 + |\nabla v| |v| \} dy,
 \end{aligned}$$

where v is given by (2.4).

PROOF. Multiply (1.1) by $tu_t + x \cdot \nabla u + u$. By (2.1) the contribution from the first term is

$$\begin{aligned}
 0 &= \left(t \left[\frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right] \right)_t - \operatorname{div}(\nabla u \cdot tu_t) \\
 &\quad - \left(\frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right).
 \end{aligned}$$

Similarly, we compute

$$\begin{aligned} 0 &= (u_{tt} - \Delta u + u^5)(x \cdot \nabla u) \\ &= \operatorname{div} \left(-x \frac{|u_t|^2}{2} - \nabla u(x \cdot \nabla u) + x \frac{|\nabla u|^2}{2} + x \frac{|u|^6}{6} \right) \\ &\quad + \{x \cdot \nabla u \ u_t\}_t + \frac{3}{2} |u_t|^2 - \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |u|^6. \end{aligned}$$

Finally,

$$\begin{aligned} 0 &= (u_{tt} - \Delta u + u^5) u \\ &= \{u_t u\}_t - \operatorname{div}(\nabla u \cdot u) - |u_t|^2 + |\nabla u|^2 + |u|^6. \end{aligned}$$

Adding, we obtain that

$$\begin{aligned} 0 &= \frac{1}{3} |u|^6 + \left(t \left[\frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right] \right)_t + \left((x \cdot \nabla u + u) u_t \right)_t \\ &\quad - \operatorname{div} \left(x \frac{|u_t|^2}{2} - x \frac{|\nabla u|^2}{2} + \nabla u(x \cdot \nabla u + u) + \nabla u \cdot t u_t - x \frac{|u|^6}{6} \right). \end{aligned}$$

Thus, when we integrate this expression over the cone $C_{-1}^{-\varepsilon}$, we obtain

$$\begin{aligned} (2.5) \quad &\frac{1}{3} \int_{C_{-1}^{-\varepsilon}} |u|^6 dx dt + E(u; D(-1)) - \varepsilon E(u; D(-\varepsilon)) \\ &= \int_{D(-1)} u_t(x \cdot \nabla u + u) dx - \int_{D(-\varepsilon)} u_t(x \cdot \nabla u + u) dx + BI, \end{aligned}$$

with BI denoting the following boundary integral

$$\begin{aligned} BI &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[|t| \left(|u_t|^2 + \frac{|x \cdot \nabla u|^2}{|x|^2} \right) - 2x \cdot \nabla u \ u_t - \left(u_t - \frac{x}{|x|} \cdot \nabla u \right) u \right] \, d\sigma \\ &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[|t| \left| u_t - \frac{x}{|x|} \cdot \nabla u \right|^2 - \left(u_t - \frac{x}{|x|} \cdot \nabla u \right) u \right] \, d\sigma \\ &\leq \int_{\{y: |y| \leq 1\}} [|y| |\nabla v|^2 + |\nabla v| |v|] \, dy. \end{aligned}$$

By Lemma 2.1,

$$E(u; D(-\varepsilon)) \leq E(u; D(-1))$$

uniformly. Moreover, by Young's and Hölder's inequalities

$$\begin{aligned} \int_{D(-\varepsilon)} u_t(x \cdot \nabla u + u) dx &\leq \varepsilon \int_{D(-\varepsilon)} [|u_t|^2 + |\nabla u|^2 + \varepsilon^{-2}|u|^2] dx \\ &\leq \varepsilon \int_{D(-\varepsilon)} (|u_t|^2 + |\nabla u|^2) dx + \varepsilon \left(\int_{D(-\varepsilon)} |u|^6 dx \right)^{1/3} \leq C \varepsilon. \end{aligned}$$

Hence we may pass to the limit $\varepsilon \rightarrow 0$ in (2.5) and the proof is complete.

qed

2.4 Small energy

Finally, we recall the integral representation

$$(2.6) \quad u(0, 0) = \underline{u}(0, 0) - \frac{1}{4\pi} \int_{M_{t_0}} |t|^{-1} u^5(x, t) d\sigma$$

for the value $u(0, 0)$ of a solution u of (1.1) in terms of the solution $\underline{u}$ of the homogeneous wave equation

$$(2.7) \quad \underline{u}_{tt} - \Delta \underline{u} = 0,$$

sharing the Cauchy data u_0, u_1 of u at a time $t_0 < 0$:

$$(2.8) \quad \underline{u} = u_0 = u, \quad \frac{\partial}{\partial t} \underline{u} = u_1 = \frac{\partial}{\partial t} u \quad \text{at } t = t_0,$$

see [3, (10), p. 342].

We will only apply this formula for functions u which are of class C^2 in a neighborhood of C_t , for some $t < 0$.

Following Rauch [3], we turn (2.6) into a linear inequality for $\sup_{C_{t_0}^1} |u|$:

Suppose

$$\sup_{C_{t_0}^1} |u| = |u(0, 0)|$$

is achieved at the origin. Then, if we let

$$\mu(s) = \frac{1}{4\pi} \int_{M_s} |t|^{-1} u^4(x, t) d\sigma = \frac{\sqrt{2}}{4\pi} \int_{B_{|s|}} |y|^{-1} v^4(y) dy,$$

(2.6) implies the inequality, for any $s > t_0$,

$$(2.9) \quad (1 - \mu(s)) \sup_{C_{t_0}^1} |u| \leq |\underline{u}(0, 0)| - \frac{1}{4\pi} \int_{M_{t_0}^s} |t|^{-1} u^5(x, t) d\sigma.$$

By Hölder's inequality

$$(2.10) \quad \mu(s) = C \int_{B_{|s|}} |y|^{-1} v^4(y) dy \leq C \left(\int_{B_{|s|}} v^6 dy \right)^{1/2} \left(\int_{B_{|s|}} \frac{v^2(y)}{|y|^2} dy \right)^{1/2}.$$

Rauch now invokes Hardy's inequality

$$(2.11) \quad \int_{\mathbb{R}^3} \frac{\psi^2(y)}{|y|^2} dy \leq 4 \int_{\mathbb{R}^3} |\nabla \psi|^2 dy, \quad \forall \psi \in C_0^\infty(\mathbb{R}^3)$$

to estimate the last integral in (2.10).

If integration extends only over a bounded domain B_{2R} , (2.11) is not immediately applicable. However, if we truncate with a smooth localizing function $\eta \in C_0^\infty(\mathbb{R}^3)$ satisfying the conditions: $\eta \equiv 1$ on B_R , $\eta \equiv 0$ off B_{2R} , $0 \leq \eta \leq 1$, $|\nabla \eta| \leq C/R$, then with absolute constants C there holds

$$(2.12) \quad \begin{aligned} \int_{B_{2R}} \frac{\psi^2(y)}{|y|^2} dy &\leq \int_{B_{2R}} \frac{(\psi\eta)^2}{|y|^2} dy + \int_{B_{2R} \setminus B_R} \frac{\psi^2}{|y|^2} dy \\ &\leq 4 \int_{B_{2R}} |\nabla(\psi\eta)|^2 dy + R^{-2} \int_{B_{2R}} \psi^2 dy \\ &\leq C \int_{B_{2R}} |\nabla \psi|^2 dy + C R^{-2} \int_{B_{2R}} \psi^2 dy \\ &\leq C \int_{B_{2R}} |\nabla \psi|^2 dy + C \left(\int_{B_{2R}} \psi^6 dy \right)^{1/3}, \\ &\quad \forall \psi \in C^1(B_{2R}). \end{aligned}$$

Using (2.12) and the energy inequality, Lemma 2.1, the number $\mu(s)$ from (2.10) may now be estimated

$$(2.13) \quad \begin{aligned} \mu(s) &\leq C \left(\int_{B_{|s|}} v^6 dy \right)^{1/2} \left(\int_{B_{|s|}} |\nabla v|^2 dy + \left(\int_{B_{|s|}} v^6 dy \right)^{1/3} \right)^{1/2} \\ &\leq C_0 [E(u; D(s)) + E(u; D(s))^{2/3}]. \end{aligned}$$

The remaining terms in (2.9) are easily bounded using Hölder's inequality

$$\begin{aligned}
 (2.14) \quad & \int_{M_{t_0}^s} |t|^{-1} |u|^5 d\sigma \leq C \int_{B_{|t_0|} \setminus B_{|s|}} |y|^{-1} |v|^5 dy \\
 & \leq C |s|^{-1/2} \left(\int_{B_{|t_0|}} v^6 dy \right)^{5/6} \leq C_1 |s|^{-1/2} E(u; D(t_0))^{5/6}.
 \end{aligned}$$

From (2.9), (2.13) we immediately obtain Rauch's regularity result for small initial energies - while the more refined estimate (2.14) will be useful later on.

THEOREM 2.3. (Rauch [3]). *Suppose u is a C^2 -solution of (1.1) in a neighborhood of C_{t_0} with initial data $u = u_0$, $u_t = u_1$ on $D(t_0)$. There exists an absolute constant $\varepsilon_0 > 0$ with the property:*

If $E(u; D(t_0)) \leq \varepsilon_0$, then

$$|u(x, t)| \leq 2 \sup_{C_{t_0}} |\underline{u}| \text{ for all } (x, t) \in C_{t_0},$$

where $\underline{u}$ denotes the solution to the Cauchy-problem (2.7-8) for the homogeneous equation.

PROOF. By passing to a smaller cone $\tilde{C} \subset C_{t_0}$, if necessary, we may assume that

$$\sup_{C_{t_0}} |u| = |u(0, 0)|.$$

Determine $\varepsilon_0 > 0$ such that $C_0(\varepsilon_0 + \varepsilon_0^{2/3}) = \frac{1}{2}$. Applying (2.9), (2.13) with $s = t_0$ the Theorem follows.

qed

Note that there is a converse result to Theorem 2.3:

PROPOSITION 2.4. *Suppose u is a C^2 -solution of (1.1) in a neighborhood of $C_{t_0} \setminus \{0\}$ with initial data $u = u_0 \in C^3$, $u_t = u_1 \in C^2$ on $D(t_0)$, and suppose that $|u(z)| \rightarrow \infty$ as $z \rightarrow 0$, $z \in C_{t_0} \setminus \{0\}$. Then for any $t \in [t_0, 0[$ there holds*

$$E(u; D(t)) \geq \varepsilon_0 > 0,$$

where ε_0 is the constant of Theorem 2.3.

PROOF. Suppose $E(u; D(t)) \leq \varepsilon_0$ for some $t \in [t_0, 0[$. Note that, since by assumption $|u(z)| \rightarrow \infty$ as $z \rightarrow 0$, there exists a sequence $\delta_m \rightarrow 0$, $\delta_m > 0$, such that $\sup_{C_{t_0}^{-\delta_m}} |u|$ is attained in $D(-\delta_m)$. Hence (2.9), (2.13-14) are applicable

in suitable cones $C_{t_0}(z_m) \subset C_{t_0}$, with $z_m \in D(-\delta_m)$, and we obtain

$$(2.15) \quad \frac{1}{2} \sup_{C_{t_0}^{-\delta_m}} |u| \leq \sup_{C_{t_0}} |\underline{u}| + C|t|^{-1/2}.$$

Since (2.15) holds for arbitrarily small $\delta_m > 0$, there results a contradiction, and the proof is complete.

qed

3. - Proof of Theorem 1.1

Let $u_0 \in C^3(\mathbb{R}^3)$, $u_1 \in C^2(\mathbb{R}^3)$ with $u_0(x) = u_0(|x|)$, $u_1(x) = u_1(|x|)$ be given radial functions. By [1; Satz 1, p. 297] the Cauchy problem (1.1-2) admits a unique (and hence radially symmetric) C^2 -solution $u(x, t) = u(|x|, t)$ locally, i.e. in a neighborhood of $\mathbb{R}^3 \times \{0\}$.

Suppose (by contradiction) that u is not globally regular. Then there is a singularity $\bar{z} = (\bar{x}, \bar{t})$ such that $|u(z)| \rightarrow \infty$ as $z \rightarrow \bar{z}$, $z \in C_0(\bar{z})$, see [1, p. 301]. Replacing $\bar{z}_0$ by another singular point in $S = \{(x, t) : 0 \leq t < \bar{t}, |x| \leq |\bar{x}| + \bar{t} - t\}$, if necessary, we may assume that $u \in C^2$ in S .

Radial symmetry implies

LEMMA 3.1. $\bar{x} = 0$, in particular $u \in C^2(\mathbb{R}^3 \times [0, \bar{t}])$.

PROOF. By Proposition 2.4 and since $u(x, t) = u(|x|, t)$:

$$E(u; D(z, t)) \geq \varepsilon_0$$

for any $z = (x, \bar{t})$, $|x| = |\bar{x}|$, and any $t < \bar{t}$.

Now, if $|\bar{x}| > 0$, for any given $K \in \mathbb{N}$ we can choose points $x_1, \dots, x_K$ satisfying $|x_k| = |\bar{x}|$, $1 \leq k \leq K$, and $t < \bar{t}$ such that with $z_k = (x_k, \bar{t})$ we have:

$$D(z_j, t) \cap D(z_k, t) = \emptyset, \quad j \neq k.$$

But then, letting $T = |\bar{x}| + \bar{t}$, $Z = (0, T)$, by Lemma 2.1:

$$\begin{aligned} K \varepsilon_0 &\leq \sum_{k=1}^K E(u; D(z_k, t)) = E\left(u; \bigcup_{k=1}^K D(z_k, t)\right) \\ &\leq E(u; D(Z, t)) \leq E(u; D(Z, 0)) < \infty, \end{aligned}$$

uniformly in K , and for large K we obtain a contradiction.

Hence a singularity is first encountered on the line $\{x = 0\}$.

qed

For convenience we shift coordinates such that $\bar{z} = 0$ in our new coordinate frame, and denote $-\bar{t} = t_0$. Thus our solution u is transformed into a solution (indiscriminately denoted by u) of (1.1), of class C^2 in a neighborhood of $C_{t_0} \setminus \{0\}$, which becomes unbounded as $z \rightarrow 0$, $z \in C_{t_0} \setminus \{0\}$.

Denote by $\underline{u}$ the solution of the homogeneous wave equation (2.7) sharing the Cauchy data of u at t_0 . $\underline{u}$ is uniformly bounded in a closed neighborhood of C_{t_0} .

For $R_m = 2^{-m}$, $m \in \mathbb{N}$, define the blown-up functions

$$u_m(x, t) = R_m^{1/2} u(R_m x, R_m t).$$

Each u_m is of class C^2 in a neighborhood of a deleted cone $C_{t_m} \setminus \{0\}$, $t_m = t_0/R_m$.

As in (2.4) we denote the trace of u_m on M_{t_m} by

$$v_m(y) = u_m(y, -|y|) = R_m^{1/2} v(R_m y).$$

Relabelling $\{u_m\}$, if necessary, we may assume that $t_0 \leq -1$.

Note that for any m , any $t \in [t_m, 0]$, by Lemma 2.1:

$$(3.1) \quad E(u_m; D(t)) \leq E(u_m; D(t_m)) = E(u; D(t_0)) =: E_0 < \infty.$$

On the other hand, since u_m becomes unbounded at 0, by Proposition 2.4:

$$(3.2) \quad E(u_m; D(t)) \geq \varepsilon_0, \quad \text{for all } t \in [t_m, 0],$$

for any $m \in \mathbb{N}$.

By Lemma 2.1 the energy $E(u, D(t))$ is non-increasing in t , hence tends to a positive (by (3.2)) limit as $t \rightarrow 0$. But then, by Lemma 2.1 again,

$$\begin{aligned} & \int_{B_1 \setminus B_{|t|}} \left(\frac{1}{2} |\nabla v_m|^2 + \frac{1}{6} |v_m|^6 \right) dy \\ (3.3) \quad &= \int_{B_{R_m} \setminus B_{|t|R_m}} \left(\frac{1}{2} |\nabla v|^2 + \frac{1}{6} |v|^6 \right) dy \\ &\leq E(u; D(-R_m)) - E(u; D(tR_m)) \rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$, uniformly in $t < 0$.

LEMMA. 3.2. *There exists $t < 0$ such that, for some $s_m \in [-1, t]$, there holds*

$$(3.4) \quad \int_{D(s_m)} (u_m)_t u_m \, dx \leq o(1),$$

where $o(1) \rightarrow 0$ as $m \rightarrow \infty$.

PROOF. We may assume that

$$\int_{D(-1)} u_m^2 dx \geq C_2 > 0.$$

(Otherwise we can choose $s_m = -1$ to achieve our claim). Choose $t \in]-1, 0[$ such that

$$\begin{aligned} \int_{D(t)} u_m^2 dx &\leq ct^2 \left(\int_{D(t)} u_m^6 dx \right)^{1/3} \\ &\leq c t^2 E(u_m; D(t))^{1/3} \leq c t^2 < C_2. \end{aligned}$$

Suppose by contradiction that

$$\int_{D(s)} (u_m)_t u_m dx \geq C_3 > 0$$

uniformly in $m \in \mathbb{N}$, for all $s \in [-1, t]$.

Then by (3.3) we obtain

$$\begin{aligned} &\int_{D(t)} u_m^2 dx - \int_{D(-1)} u_m^2 dx \\ &= 2 \int_{C_{-1}^t} (u_m)_t u_m dx dt - \frac{1}{\sqrt{2}} \int_{M_{-1}^t} u_m^2 d\sigma \geq 2(1+t)C_3 - o(1), \end{aligned}$$

which, for large m , is in conflict with our choice of t .

qed

Since $\{s_m\}$ is bounded away from 0, we may scale with s_m to achieve (3.4) with $s_m = -1$ for all m . Note that with this change of scale the ratio R_m/R_{m+1} remains uniformly bounded, i.e. there exists $R > 0$ such that

$$(3.5) \quad 0 < R^{-1} \leq |R_m/R_{m+1}| \leq R < \infty, \text{ for all } m.$$

Now apply Lemma 2.2:

$$\begin{aligned} &\frac{1}{3} \int_{C_{-1}} u_m^6 dx dt + E(u_m; D(-1)) \\ &\leq \int_{D(-1)} (u_m)_t x \cdot \nabla u_m dx + o(1). \end{aligned}$$

It follows that

$$\begin{aligned}
 (3.6) \quad & \int_{C_{-1}} |u_m|^6 dx dt \\
 & + \int_{D(-1)} \left\{ (1 - |x|) [|u_m|_t|^2 + |\nabla u_m|^2] \right. \\
 & \left. + |x| \left| \frac{x}{|x|} (u_m)_t - \nabla u_m \right|^2 + |u_m|^6 \right\} dx \rightarrow 0,
 \end{aligned}$$

as $m \rightarrow \infty$.

LEMMA 3.3. *There exists a sequence $\Lambda \subset \mathbb{N}$ such that*

$$\liminf_{m \rightarrow \infty, m \in \Lambda} \sup_{C_{t_m}^{-1}} |u_m| > 0.$$

PROOF. Suppose by contradiction that

$$\begin{aligned}
 \sup_{C_{t_m}^{-1}} |u_m| &= R_m^{1/2} \sup_{C_{t_0}^{-R_m}} |u| \\
 &\geq R^{-1/2} \sup_{s \in [-R_{m-1}, -R_m]} \left(|s|^{1/2} \sup_{C_{t_0}^s} |u| \right) \rightarrow 0, \text{ as } m \rightarrow \infty.
 \end{aligned}$$

I.e. if we let

$$g(t) = |t|^{1/2} \sup_{C_{t_0}^t} |u|,$$

g is continuous and satisfies

$$g(t) \rightarrow 0 \quad (t \rightarrow 0).$$

Also denote

$$h(t) = \sup_{s \geq \max\{t, -1\}} g(s).$$

Then h is continuous, non-increasing, and satisfies $h(t) \equiv h(-1)$, for $t \leq -1$, and $h(t) \rightarrow 0$ ($t \rightarrow 0$).

Now the proof proceeds as follows: first we establish that $h(t)$ decays with a certain power of $|t|$: $h(t) \leq c|t|^\epsilon$, ($t \rightarrow 0$).

In a second step we use this decay estimate to prove that u is uniformly bounded near 0 - which will yield the desired contradiction.

- i) Suppose $h(t) = g(s)$ for some $s \geq t \geq -1$ and that $g(s)$ is attained at $\tilde{z} = (\tilde{x}, \tilde{t})$, where $s \geq \tilde{t} = \lambda t$, $|u(\tilde{z})| = \sup_{C_{t_0}^s} |u| = \sup_{C_{t_0}^t} |u|$. Note that

if $\lambda = \lambda(t) > 1$:

$$h(t) = g(s) = |s|^{1/2} |u(\tilde{z})| \leq |t|^{1/2} |u(\tilde{z})| \leq g(t) \leq h(t)$$

i.e.

$$s = t; \quad h(t) = |t|^{1/2} |u(\tilde{z})|.$$

Similarly, for $\bar{t} \in]\tilde{t}, t]$,

$$h(\bar{t}) = \sup_{\tilde{t} \leq s} g(s) = |\bar{t}|^{1/2} |u(\tilde{z})| = \left| \frac{\bar{t}}{\tilde{t}} \right|^{1/2} h(\tilde{t}).$$

In particular, $\lambda(\bar{t}) = \frac{\tilde{t}}{\bar{t}} > 1$.

Denote

$$J = \{t \in]-1, 0[: \lambda(t) > 1\}.$$

Remark that J consists of a union of left open intervals I and, for any pair $s \leq t$ belonging to such an interval I , there holds

$$h(t) = \left| \frac{t}{s} \right|^{1/2} h(s).$$

In particular, for any $\varepsilon \in]0, \mu]$, $0 < \mu \leq \frac{1}{2}$, there holds

$$(3.7) \quad (h(t) + |t|^\mu) \leq \left| \frac{t}{s} \right|^\varepsilon (h(s) + |s|^\mu), \quad \text{for all } s \leq t \in I.$$

On the other hand, if $\lambda \leq 1$, by (2.6)

$$\begin{aligned} (3.8) \quad h(t) &= g(s) = |s|^{1/2} |u(\tilde{z})| \\ &\leq |s|^{1/2} |\underline{u}(\tilde{z})| + c|s|^{1/2} \int_{M_{t_0}(\tilde{z})} |\tilde{t} - \tau|^{-1} |u(y, \tau)|^5 \, d\mathbf{o} \\ &\leq c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| \sup_{C_{t_0}^{\tau}} |u|^5 \, d\tau \\ &= c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| g^5(\tau) |\tau|^{-5/2} \, d\tau \\ &\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^{\tilde{t}} \lambda^{1/2} g^5(\tau) |\tau|^{-3/2} \, d\tau \end{aligned}$$

$$\begin{aligned}
&\leq c|t|^{1/2} + c|t|^{1/2} \int_{-\lambda^{-1}}^t g^5(\lambda\tau) |\tau|^{-3/2} d\tau \\
&\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^t h^5(\tau) |\tau|^{-3/2} d\tau.
\end{aligned}$$

Now choose $\mu = \frac{1}{5}$, and denote $t_1 = t^\mu := t|t|^{\mu-1} < t$. Since h is non-increasing and bounded we obtain from (3.8):

$$\begin{aligned}
&(h(t) + |t|^\mu) \\
&\leq C|t|^{1/2} + |t|^\mu + C|t|^{1/2} \int_{t^\mu}^t h^5(\tau) |\tau|^{-3/2} d\tau + C|t|^{1/2} \int_{-1}^{t^\mu} h^5(\tau) |\tau|^{-3/2} d\tau \\
&\leq C|t|^{1/2} + |t|^\mu + C h^5(t^\mu) + C|t|^{1/2} \sup_{\tau \leq 0} h^5(\tau) (|t|^{-\mu/2} - 1) \\
&\leq C|t|^{1/2} + |t|^\mu + C|t|^{(1-\mu)/2} + C h^5(t^\mu) \\
&\leq C (|t|^\mu + h^5(t^\mu)) \leq C_4 (h(t_1) + |t_1|^\mu)^5.
\end{aligned}$$

Iteratively define $t_k = t_{k-1}^\mu < t_{k-1}$, $k = 1, \dots, K$. Suppose $\lambda(t_k) \leq 1$ for all $k = 1, \dots, K-1$. Then

$$\begin{aligned}
h(t) &\leq (h(t) + |t|^\mu) \leq C_4 (h(t_1) + |t_1|^\mu)^5 \\
&\leq C_4^6 (h(t_2) + |t_2|^\mu)^{25} \leq \dots \leq C_4^{\sum_{k=0}^{K-1} 5^k} (h(t_K) + |t_K|^\mu)^{5^K} \\
&= \left[C_4^{1/4} (h(t_K) + |t_K|^\mu) \right]^{5^{K-1}} \cdot (h(t_K) + |t_K|^\mu).
\end{aligned}$$

I.e., if for some $\varepsilon > 0$:

$$(3.9) \quad C_4^{1/4} (h(t_K) + |t_K|^\mu) \leq |t_K|^\varepsilon,$$

it follows that

$$\begin{aligned}
(3.10) \quad h(t) + |t|^\mu &\leq \left(\left| \frac{t_K^{5^K}}{t_K} \right| \right)^\varepsilon (h(t_K) + |t_K|^\mu) \\
&= \left| \frac{t}{t_K} \right|^\varepsilon (h(t_K) + |t_K|^\mu).
\end{aligned}$$

Note that, since $h(t) \rightarrow 0$ ($t \rightarrow 0$), there exist $T \in]-1, 0[$, $\varepsilon \in]0, \mu[$ such that (3.9) holds whenever $t_K \in [T^\mu, T]$. But then also (3.10) holds for all such t, t_K , provided $\lambda(t), \lambda(t_k) \leq 1, k = 1, \dots, K-1$.

Now choose any $\tau = \tau_0 > T$ and define a sequence $\tau_1, \tau_2, \dots, \tau_K$ as follows:

$$\tau_{k+1} = \begin{cases} \tau_k^\mu, & \text{if } \lambda(\tau_k) \leq 1 \\ \tilde{\tau}_k, & \text{if } \lambda(\tau_k) > 1 \end{cases}, \quad k \in \mathbb{N}_0,$$

where $\tilde{\tau}_k$ denotes the left end-point of the interval $I \subset J$ containing τ_k , if $\lambda(\tau_k) > 1$, and where

$$K = \sup \{k \in \mathbb{N} \mid \tau_{k-1} > T\}.$$

Note that

$$|\tau_{k+2}/\tau_k| \geq |\tau_k|^{\mu-1} \geq |T|^{\mu-1} > 1,$$

if $\tau_k \geq \tau_{k+2} > T$. Hence K exists and is finite, for every $\tau < 0$.

Combining (3.7) and (3.10) we see that

$$h(\tau) \leq (h(\tau) + |\tau|^\mu) \leq \left| \frac{\tau}{\tau_K} \right|^\varepsilon (h(\tau_K) + |\tau_K|^\mu) \leq C|\tau|^\varepsilon,$$

i.e.

$$\sup_{C_{t_0}^+} |u| = g(t) |t|^{-1/2} \leq h(t) |t|^{-1/2} \leq C|t|^{\varepsilon-\frac{1}{2}}.$$

ii) Denote

$$\bar{\gamma} = \inf \left\{ \gamma > 0 : |t|^\gamma \sup_{C_{t_0}^+} |u| \leq C < \infty \text{ uniformly in } t \right\}.$$

By part i) $\bar{\gamma} < \frac{1}{2}$ and we may choose $\gamma > \bar{\gamma}$ such that $\mu := 5\gamma - 2 < \bar{\gamma}$, $\mu \neq 0$.

Define

$$f(t) = |t|^\gamma \sup_{C_{t_0}^+} |u|.$$

Note that $f(t)$ is uniformly bounded, continuous and satisfies $f(t) \rightarrow 0$ as $t \rightarrow 0$.

By (2.6), for all $z = (x, t) \in C_{t_0} \setminus \{0\}$:

$$\begin{aligned} |u(z)| &\leq |\underline{u}(z)| + \frac{1}{4\pi} \int_{M_{t_0}(z)} |t - \tau|^{-1} |u(\xi, \tau)|^5 \, d\sigma(\xi, \tau) \\ (3.11) \quad &\leq C + C \int_{t_0}^t |t - \tau| f^5(\tau) |\tau|^{-5\gamma} \, d\tau \\ &\leq C + C \sup_{\tau < 0} f^5(\tau) \int_{t_0}^t |\tau|^{1-5\gamma} \, d\tau \leq C + C|t|^{2-5\gamma}. \end{aligned}$$

First suppose $\mu > 0$. Then from (3.11) we obtain

$$|t|^\mu |u(x, t)| \leq C |t|^\mu + C \leq C,$$

uniformly for all $z = (x, t) \in C_{t_0} \setminus \{0\}$, which contradicts the definition of $\bar{\gamma}$.

Thus $\mu \leq 0$. But then by (3.11) u is uniformly bounded in $C_{t_0} \setminus \{0\}$, contrary to hypothesis.

qed

To proceed with the proof of Theorem 1.1, let $z_m = (x_m, s_m) \in C_{t_m}^{-1}$, $m \in \Lambda$, satisfy

$$(3.12) \quad |u_m(z_m)| = \sup_{C_{t_m}^{s_m}} |u_m| = \min\{1, \sup_{C_{t_m}^{-1}} |u_m|\} = R_m^{1/2} u(R_m x_m, R_m s_m).$$

Note that by Lemma 3.3

$$(3.13) \quad \liminf_{m \rightarrow \infty, m \in \Lambda} |u_m(z_m)| = 2 \ c_5 > 0;$$

in particular, by (3.12)

$$(3.14) \quad R_m s_m \rightarrow 0, \quad s_m \leq -1.$$

Now by (2.9), (2.13), (2.14), if $E(u_m; D(z_m, s)) < \varepsilon_0$ for some $s < s_m$:

$$\begin{aligned} c_5 \leq |u_m(z_m)| &\leq 2 \left(R_m^{1/2} \underline{u}(R_m z_m) + c |s - s_m|^{-1/2} \right) \\ &\leq o(1) + 2 \ c |s - s_m|^{-1/2} \leq o(1) + \frac{1}{2} \ c_5, \end{aligned}$$

provided $s \leq s_m - c_6$ for some $c_6 > 0$.

Since $c_5 > 0$, this is impossible for large m , and it follows that

$$E(u_m; D(z_m, s)) \geq \varepsilon_0,$$

for $s \in [t_m, s_m - c_6]$, $m \geq m_0$.

By radial symmetry

$$(3.15) \quad E(u_m; D(z, s)) \geq \varepsilon_0$$

for such $s, m \geq m_0$, for all $z = (x, s_m)$ with $|x| = |x_m|$.

LEMMA 3.4. *For any $c > 0$, any family $\{x^k\}_{1 \leq k \leq K}$ in $\mathbb{R}^3$, with $|x^k| = r \geq 0$, $|x^j - x^k| \geq c^{-1}r$, $j \neq k$, there exists $\sigma_m \in [t_m, s_m - c_6]$ such that*

$$(3.16) \quad E\left(u_m; \bigcup_{j \neq k} D(z^j, \sigma_m) \cap D(z^k, \sigma_m)\right) \rightarrow 0$$

as $m \rightarrow \infty$, $m \in \Lambda$, where $z^i = (x^i, s_m)$.

PROOF. Since $|s_m| \geq 1$, by uniform convexity of balls in $\mathbb{R}^3$, there exists $\varepsilon > 0$ such that

$$D(z^j, s) \cap D(z^k, s) \subset \{x \in D(s) : |x| < (1 - \varepsilon)|s|\} =: D^\varepsilon(s)$$

for all $j \neq k$, $s \in [R(s_m - c_6), s_m - c_6] =: I_m$.

(Note that

$$|Rs_m| \leq \frac{R}{R_m} |R_m s_m| \leq c|t_m| |R_m s_m| = o(|t_m|),$$

by (3.14). Hence $R(s_m - c_6) \geq t_m$, for $m \geq m_0$).

Now by (3.5), there exists $k(m)$ such that

$$\sigma_m := -\frac{R_{k(m)}}{R_m} \in I_m.$$

Observe that by (3.14) again:

$$R_{k(m)} = -\sigma_m R_m \leq c|s_m| R_m \rightarrow 0,$$

hence $k(m) \rightarrow \infty$, ($m \rightarrow \infty, m \in \Lambda$).

But then by (3.6)

$$\begin{aligned} E(u_m; D^\varepsilon(\sigma_m)) &= E(u_{k(m)}; D^\varepsilon(-1)) \\ &\leq c \cdot \varepsilon^{-1} \int_{D(-1)} \left\{ (1 - |x|) (|(u_{k(m)})_t|^2 + |\nabla u_{k(m)}|^2) \right. \\ &\quad \left. + |x| \left| \frac{x}{|x|} (u_{k(m)})_t - \nabla u_{k(m)} \right|^2 + |u_{k(m)}|^6 \right\} dx \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, $m \in \Lambda$.

This proves the claim.

qed

We can now complete the *proof of Theorem 1.1*.

Given $K \in \mathbb{N}$, we can find $c > 0$ such that, for any $m \in \mathbb{N}$, there are K points x_m^j , $1 \leq j \leq K$ such that $|x_m^j| = |x_m|$, $|x_m^j - x_m^k| \geq c^{-1} |x_m|$ for all $1 \leq j \neq k \leq K$. Denote $z_m^j = (x_m^j, s_m)$.

Let $\sigma_m \in [t_m, s_m]$ denote the number determined in Lemma 3.4 for the family $\{x_m^j\}$.

By (3.15-16) and (3.1)

$$\begin{aligned}
 K\varepsilon_0 &\leq \sum_{j=1}^K E(u_m; D(z_m^j, \sigma_m)) \\
 &\leq E(u_m; \bigcup_{j=1}^K D(z_m^j, \sigma_m)) \\
 &\quad + \sum_{j \neq k} E(u_m; D(z_m^j, \sigma_m) \cap D(z_m^k, \sigma_m)) \\
 &\leq E(u_m; D(\sigma_m)) + o(1) \leq E_0 + o(1),
 \end{aligned}$$

where $o(1) \rightarrow 0$ as $m \rightarrow \infty$, $m \in \Lambda$.

For K sufficiently large we obtain a contradiction, and the proof is complete.

qed

4. - A Remark on the non-symmetric case

Estimate (3.6) suggests that, also in the non-symmetric case, singularities tend to build up in a rotationally symmetric pattern. Using this observation, it is possible to extend our results to arbitrary initial data $u \in C^3$, $u_1 \in C^2$, provided the modulus of continuity of the blow-up functions u_m , restricted to $C_{t_m}^{s_m}$ (where u_m is uniformly bounded by 1), can be uniformly bounded.

Added in proof

Generalizations of (1.1-2) to higher dimensions were studied for instance by Brenner and von Wahl [4] or Pecher [5], where results analogous to those found by Jörgens in dimension 3 were obtained. See [4] for further references.

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