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# Asymptotic Behaviour of Generalized Poisson Integrals in Rank One Symmetric Spaces and in Trees

PETER SJÖGREN

## 1. - Introduction

Let  $X = G/K$  be a Riemannian symmetric space of the noncompact type and of real rank 1, with boundary  $K/M$ . Some standard notation used here is explained in Section 2. Take  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . Any  $f \in L^1(K/M)$  has a  $\lambda$ -Poisson integral

$$(1.1) \quad P_\lambda f(g \cdot \circ) = \int f(kM) \exp(-\langle \rho + \lambda, H(g^{-1}k) \rangle) dkM$$

for  $g \cdot \circ \in X$ . Here  $H(\cdot)$  comes from the Iwasawa decomposition of  $G$ , whereas  $H$  will be generic in  $\mathfrak{a}$ .

When the real part of  $\lambda$  is positive, it is known that  $f$  can be recovered as the limit of the normalized  $\lambda$ -Poisson integral  $\mathcal{P}_\lambda f = P_\lambda f / P_\lambda 1$  at the boundary. Here 1 is the constant function. Indeed,

$$\mathcal{P}_\lambda f(k_1 \exp H \cdot \circ) \rightarrow f(k_1 M) \text{ as } H \rightarrow +\infty \text{ in } \mathfrak{a}_+$$

for a.a.  $k_1 M$  in  $K/M$ . In terms of the  $\overline{N}A$  model for  $X$ , this reads

$$\mathcal{P}_\lambda f(\overline{n}_1 \exp H \cdot \circ) \rightarrow f(k(\overline{n}_1)M) \text{ as } H \rightarrow +\infty$$

for a.a.  $\overline{n}_1 \in \overline{N}$ . More generally, one can use an admissible approach here, which means replacing  $\circ$  by a point  $x$  staying in a compact subset of  $X$ . Such results are known to hold also for  $\lambda = 0$ , see Sjögren [9].

In this paper, we shall consider the case  $\text{Re } \lambda = 0$ ,  $\lambda \neq 0$ , i.e.,  $\lambda = i\gamma\rho$  with  $0 \neq \gamma \in \mathbb{R}$ .

The normalizing factor  $P_\lambda 1$  is a spherical function, in particular it is biinvariant under  $K$ . For  $\text{Re } \lambda > 0$ , it behaves like  $e^{\langle \lambda - \rho, H \rangle}$  at  $\exp H \cdot \circ$  for large  $H \in \mathfrak{a}_+$ . But in our case the dominating term of its asymptotic

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expansion is  $2 \operatorname{Re} c(\lambda) \exp\langle \lambda - \rho, H \rangle$ , which has zeroes for arbitrarily large  $H$ . To examine the asymptotic behaviour of  $P_\lambda f$ , it is therefore reasonable to divide by  $2c(\lambda)e^{\langle \lambda - \rho, H \rangle} \neq 0$ , or simply by  $e^{-\langle \rho, H \rangle}$ , instead of  $P_\lambda 1$ .

The usual transformation of (1.1) to  $\bar{N}$  gives

$$(1.2) \quad \begin{aligned} e^{\langle \rho, H \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot \circ) \\ = e^{\langle 2\rho + \lambda, H \rangle} \int f(k(\bar{n}_1 \bar{n})M) \exp(\langle \lambda - \rho, H(\bar{n}_1 \bar{n}) \rangle) \\ \exp(-\langle \lambda + \rho, H(\bar{n}^{-H}) \rangle) d\bar{n} \end{aligned}$$

where  $\bar{n}^{-H} = \exp(-H)\bar{n}\exp H$ . The last factor in the integrand is

$$\exp(-\langle \lambda + \rho, H(\bar{n}^{-H}) \rangle) = P(\bar{n}^{-H})^{(1+i\gamma)/2},$$

where  $P(\bar{n})$  is the Poisson kernel in  $\bar{N}$ . The expression

$$e^{\langle 2\rho, H \rangle} P(\bar{n}^{-H})^{1/2}$$

has a limit as  $H \rightarrow +\infty$  which can be written as  $|\bar{n}|^{-Q}$ . Here  $|\cdot|$  is a homogeneous gauge in  $\bar{N}$  and  $Q$  the corresponding dimension of  $\bar{N}$ .

If we could let  $H \rightarrow +\infty$  under the integral sign in (1.2), the conclusion would be that

$$e^{\langle \lambda + \rho, H \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot \circ)$$

tends to

$$I = \int f(k(\bar{n}_1 \bar{n})M) \exp(\langle \lambda - \rho, H(\bar{n}_1 \bar{n}) \rangle) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n},$$

a divergent integral. In fact, the asymptotic behaviour of  $P_\lambda f$  is given by

$$\begin{aligned} e^{\langle \rho, H \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot \circ) &= c(\lambda) e^{\langle \lambda, H \rangle} \exp(\langle \lambda - \rho, H(\bar{n}_1) \rangle) f(k(\bar{n}_1)M) \\ &+ e^{-\langle \lambda, H \rangle} I + o(1), \quad H \rightarrow +\infty \end{aligned}$$

with a suitable evaluation of the integral. Notice that  $e^{\pm\langle \lambda, H \rangle}$  are oscillating factors.

This can be written in a neater way if we extend  $f$  to all of  $G$  by means of

$$(1.3) \quad f(k \exp(H)n) = f(kM) e^{\langle \lambda - \rho, H \rangle}$$

for  $k \in K$ ,  $H \in \mathfrak{a}$ ,  $n \in N$ . This extension is used in connection with the representation of the principal series of  $G$  corresponding to  $-\lambda \in \mathfrak{a}$ , the parabolic subgroup  $MAN$ , and the trivial representation of  $M$ . Also notice that the singular integral we obtained defines an intertwining operator from this representation to that corresponding to  $+\lambda$ .

The result of the main part of this paper gives the asymptotic behaviour of  $P_\lambda f$  for admissible approach to the boundary. The paper [1] by van den Ban and Schlichtkrull contains an asymptotic expansion of  $P_\lambda f$ .

Our result means that for the values of  $\lambda$  considered here, the principal terms of this expansion are determined explicitly. We also obtain a pointwise estimate of the difference between  $P_\lambda f$  and the principal terms. This estimate holds at all boundary points for Hölder functions  $f$  and almost everywhere for  $f \in L^1(K/M)$ . For  $K$ -finite functions  $f$ , the expansion was already known, with explicit formulae for the terms, see Helgason [4, §4]. Our proofs are more concrete. The only asymptotic behaviour we use is that of the spherical function  $P_\lambda 1$ . To prove our results for  $L^1$  functions, we go via a maximal function estimate.

The last part of this paper gives an analogous result for a homogeneous tree of branching number  $q + 1 \geq 3$ . The  $z$ -Poisson integral is defined for integrable functions  $f$  on the boundary by means of the  $z$ th power of the Poisson kernel,  $z \in \mathbb{C}$ . For  $\operatorname{Re} z > 1/2$  and for  $z = 1/2$  and  $z = 1/2 + \pi i / \log q$ , the normalized  $z$ -Poisson integral  $K_z f$  converges to  $f$  almost everywhere on the boundary. This was proved by Korányi and Picardello [8]. We shall deal with the remaining values of  $\operatorname{Im} z$  when  $\operatorname{Re} z = 1/2$ . The result is an asymptotic formula

$$K_z f(x) = \operatorname{const} \cdot e^{\operatorname{const} \cdot i|x|} f(\omega_1) + I_{\bar{z}} f(\omega_1) + o(1)$$

as  $x$  approaches the boundary point  $\omega_1$ . As in the case of symmetric spaces,  $I_{\bar{z}}$  turns out to be an intertwining operator between representations of a related group. The proof is rather straightforward.

## 2. - Preliminaries

We write the symmetric space as  $X = G/K$  in the standard way. For more details, see [5] or [9]. Here  $K$  is a maximal compact subgroup of the connected semi-simple Lie group  $G$ . The Iwasawa decomposition  $G = KAN$  means that any  $g \in G$  can be written uniquely as  $g = k(g) \exp(H(g))n(g)$ . Here  $k(g)$  is in  $K$ ,  $n(g)$  in the nilpotent group  $N$  and  $H(g)$  in  $\mathfrak{a}$ , the Lie algebra of the abelian group  $A$ . Since  $\operatorname{rank} X = 1$ , both  $A$  and  $\mathfrak{a}$  are isomorphic to  $\mathbb{R}$ . The positive Weyl chamber  $\mathfrak{a}_+ \subset \mathfrak{a}$  then corresponds to  $\mathbb{R}_+$ . By  $\mathfrak{a}_{\mathbb{C}}$  we denote the complexification of  $\mathfrak{a}$ , and  $\mathfrak{a}^*$  and  $\mathfrak{a}_{\mathbb{C}}^*$  are the duals of these spaces.

The exponential map gives a diffeomorphism between  $N$  and its Lie algebra  $\mathfrak{n}$ . Further,  $\mathfrak{n}$  is the direct sum of the root spaces  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_{2\alpha}$ , which are subspaces of the Lie algebra  $\mathfrak{g}$  of  $G$ . The positive restricted roots  $\alpha$  and  $2\alpha$  are elements of  $\mathfrak{a}^*$ . Their multiplicities are  $m_\alpha = \dim \mathfrak{g}_\alpha > 0$  and  $m_{2\alpha} = \dim \mathfrak{g}_{2\alpha} \geq 0$ . Write  $n^H$  for  $\exp(H)n \exp(-H)$  when  $n \in N$ ,  $H \in \mathfrak{a}$ . Then  $n = \exp(Y_1 + Y_2)$ ,  $Y_1 \in \mathfrak{g}_\alpha$ ,  $Y_2 \in \mathfrak{g}_{2\alpha}$  implies

$$n^H = \exp(e^{\langle \alpha, H \rangle} Y_1 + e^{\langle 2\alpha, H \rangle} Y_2).$$

These properties of  $N$  are shared by its image  $\bar{N}$  under the Cartan involution  $\theta$ , except that  $\alpha$  and  $2\alpha$  are replaced by the negative roots  $-\alpha$  and  $-2\alpha$ . Since  $\theta$  is an isomorphism, the multiplicities verify  $m_{-\alpha} = m_\alpha$  and  $m_{-2\alpha} = m_{2\alpha}$ .

Both  $\alpha$  and  $\rho = (m_\alpha + 2m_{2\alpha})\alpha/2 \in \mathfrak{a}^*$  are positive in the sense that they belong to the polar  $\mathfrak{a}_+^*$  of  $\mathfrak{a}_+$ . Let  $M$  be the centralizer of  $A$  in  $K$ , with Lie algebra  $\mathfrak{m}$ . The root space  $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$  is abelian.

Let  $\circ = eK \in X$ , and write  $g \cdot \circ$  for  $gK \in X$ . Because of the Cartan decomposition, any point  $x \in X$  can be written as  $x = k \exp(H_C(x)) \cdot \circ$ , where  $k \in K$  and  $H_C(x)$  is a uniquely determined point in the closure of  $\mathfrak{a}_+$ . In fact,  $H_C(x)$  is proportional to the distance from  $\circ$  to  $x$ . Because of the modified Iwasawa decomposition  $G = \bar{N}AK$ , one can also write  $x = \bar{n}(x) \exp A(x) \cdot \circ$ , with uniquely determined  $\bar{n}(x) \in \bar{N}$  and  $A(x) \in \mathfrak{a}$ .

The boundary of  $X$  is  $K/M$ , and a point  $k_1M \in K/M$  is the limit of  $k_1 \exp H \cdot \circ \in X$  as  $H \in \mathfrak{a}$  tends to  $+\infty$ , i.e., as  $\alpha(H) \rightarrow +\infty$ . Letting  $\bar{n} \in \bar{N}$  correspond to  $k(\bar{n})M \in K/M$ , one can also realize the boundary as  $\bar{N}$ , except for one point. Then  $\bar{n}$  is the limit of  $\bar{n} \exp H \cdot \circ$  as  $H \rightarrow +\infty$ .

Any function  $f$  in  $K/M$  will be defined in  $G$  by means of (1.3). We say that  $f$  is Hölder if it satisfies a Hölder condition in terms of any local coordinate system in  $K/M$ , with exponent in  $]0, 1]$ . Then its values in  $\bar{N}$  verify a local Hölder condition.

The  $\lambda$ -Poisson integral of any  $f \in L^1(K/M)$  can now be defined via (1.1), and (1.2) follows. The Poisson kernel  $P(\bar{n})$  was defined in the introduction. With  $\bar{n} = \exp(Y_1 + Y_2)$ ,  $Y_j \in \mathfrak{g}_{-j\alpha}$ , it is given by

$$(2.1) \quad P(\bar{n}) = \frac{1}{(1 + 2c|Y_1|^2 + c^2|Y_1|^4 + 4c|Y_2|^2)^{(m_\alpha + 2m_{2\alpha})/2}},$$

see [5, Theorem IX.3.8]. Here  $c = (m_\alpha + 4m_{2\alpha})^{-1}/4$ , and  $|Y| = -B(Y, \theta Y)^{1/2}$  is for any  $Y \in \mathfrak{g}$  the norm coming from the Killing form  $B$ .

The last two terms in the denominator form a homogeneous gauge

$$|\bar{n}| = (c^2|Y_1|^4 + 4c|Y_2|^2)^{1/4},$$

where the exponent  $1/4$  is a matter of convenience. One has  $|\bar{n}^H| = e^{-\alpha(H)}|\bar{n}|$ , for  $H \in \mathfrak{a}$ , and  $|\bar{n} \bar{n}'| \leq \text{const}(|\bar{n}| + |\bar{n}'|)$ . The Haar measure of the ball  $B(r) = \{\bar{n} \in \bar{N} : |\bar{n}| < r\}$  is proportional to  $r^Q$ , where  $Q = m_\alpha + 2m_{2\alpha}$  is the homogeneous dimension of  $\bar{N}$ .

It is now clear that

$$e^{\langle 2\rho, H \rangle} P(\bar{n}^{-H})^{1/2} \rightarrow |\bar{n}|^{-Q} \text{ as } H \rightarrow +\infty.$$

We remark that this limit can also be written  $|\bar{n}|^{-Q} = e^{\langle \rho, B(m^* \bar{n}) \rangle}$ , see [5, Theorem IX.3.8]. Here  $m^* \in K$  defines the nontrivial element of the Weyl group, and  $B(g) \in \mathfrak{a}$  is determined for a.a.  $g \in G$  by the Bruhat decomposition

$g = \bar{n}m \exp(B(g))n$ ,  $\bar{n} \in \bar{N}$ ,  $m \in M$ ,  $n \in N$ . This is why the singular integrals in our result define intertwining operators, cf. Knapp and Stein [7, I.3].

We next discuss integrals containing the oscillating singular kernel  $|\bar{n}|^{-Q(1+i\gamma)}$ . If  $F$  is an integrable Hölder function in  $\bar{N}$ , one has

$$(2.2) \quad \int_{|\bar{n}| > \epsilon} F(\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} = A + B\epsilon^{-iQ\gamma} + o(1)$$

as  $\epsilon \rightarrow 0$ , with complex constants  $A$  and  $B$ . This can be seen by means of polar coordinates in  $\bar{N}$  as in [3, Ch. 1.A]. Since (2.2) says that the values of the integral approximate a circle in  $\mathbb{C}$  centred at  $A$ , we then call  $A$  a central principal value and write

$$A = \text{cpv} \int F(\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n}.$$

This value is also the limit, or the analytic continuation, of the convergent integrals obtained by using exponents  $z$ ,  $\text{Re } z > -Q$ , instead of  $-Q(1+i\gamma)$ .

Now replace  $F(\bar{n})$  by  $F(\bar{n}_1\bar{n})$  for  $\bar{n}_1 \in \bar{N}$  so that  $A = A(\bar{n}_1)$  and we have a convolution. It is well known that the operator  $F \rightarrow A$  is of weak type (1, 1) and thus defined for all  $F \in L^1(\bar{N})$ , see [3, Ch. 6].

Moreover, the corresponding maximal operator

$$M_\gamma F(\bar{n}_1) = \sup_{\epsilon > 0} \left| \int_{|\bar{n}| > \epsilon} F(\bar{n}_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} \right|$$

is of weak type (1, 1), as one can see by extending the well-known proof in  $\mathbb{R}^n$ . It follows that  $F(\bar{n}_1\bar{n})$  satisfies (2.2) for a.a.  $\bar{n}_1$  when  $F \in L^1$ .

Hence,

$$\text{cpv} \int F(\bar{n}_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n}$$

is defined almost everywhere and coincides with  $A(\bar{n}_1)$ .

We denote by  $c(\lambda)$  Harish-Chandra's  $c$ -function.

### 3. - The result for symmetric spaces

**THEOREM 3.1.** *Let  $f \in L^1(K/M)$  and assume  $\lambda = i\gamma\rho \in \mathfrak{a}_\mathbb{C}^*$  with  $0 \neq \gamma \in \mathbb{R}$ .*

*Then for a.a.  $\bar{n}_1 \in \bar{N}$*

$$(3.1) \quad e^{\langle \rho, H+A(x) \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot x) = c(\lambda) e^{\langle \lambda, H+A(x) \rangle} f(\bar{n}_1) \\ + e^{-\langle \lambda, H+A(x) \rangle} \text{cpv} \int_{\bar{N}} f(\bar{n}_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} + o(1)$$

and for a.a.  $k_1M \in K/M$

$$(3.2) \quad e^{\langle \rho, H+A(x) \rangle} P_\lambda f(k_1 \exp H \cdot x) = c(\lambda) e^{\langle \lambda, H+A(x) \rangle} f(k_1) \\ + e^{-\langle \lambda, H+A(x) \rangle} \text{cpv} \int_{\overline{N}} f(k_1 \overline{n}) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n} + o(1),$$

both as  $H \rightarrow +\infty$  and  $x$  stays in a compact subset of  $X$ . When  $f$  is Hölder in an open set  $\Omega \subset K/M$ , these formulas hold for all  $\overline{n}_1$  with  $k(\overline{n}_1)M \in \Omega$  and all  $k_1M \in \Omega$ , respectively.

Notice that it is natural that  $H + A(x)$  appears here, because  $H + A(x) = A(\overline{n}_1 \exp H \cdot x) = A(\exp H \cdot x)$ .

LEMMA 3.2. For any  $\overline{n}_1 \in \overline{N}$ ,

$$\text{cpv} \int_{\overline{N}} \exp(\langle \lambda - \rho, H(\overline{n}_1 \overline{n}) \rangle) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n} = c(-\lambda) e^{-\langle \lambda + \rho, H(\overline{n}_1) \rangle}.$$

PROOF. We first claim that the left-hand side here is the limit of the convergent integrals

$$(3.3) \quad \int_{\overline{N}} \exp(\langle \lambda - \rho - \eta\rho, H(\overline{n}_1 \overline{n}) \rangle) |\overline{n}|^{-Q(1-\eta+i\gamma)} d\overline{n}$$

as  $\eta \rightarrow 0$ ,  $\eta > 0$ . The only difficulty is near  $\overline{n} = e$ , so consider the integrals in  $\{|\overline{n}| < 1\}$ . If in (3.3) we subtract from the exponential factor its value at  $\overline{n} = e$ , dominated convergence will allow us to let  $\eta \rightarrow 0$  under the integral sign. It then remains to consider the integral of  $|\overline{n}|^{-Q(1-\eta+i\gamma)}$  in  $|\overline{n}| < 1$ . By means of polar coordinates, this integral is seen to tend to

$$\text{cpv} \int_{|\overline{n}| < 1} |\overline{n}|^{-Q(1+i\gamma)} d\overline{n}.$$

The claim follows.

From (2.1), we know that  $|\overline{n}|^{-Q}$  is the increasing limit of  $e^{\langle 2\rho, H \rangle} e^{-\langle \rho, H(\overline{n}^{-H}) \rangle}$  as  $H \rightarrow +\infty$ . Thus by dominated convergence

$$\int \exp(\langle \lambda - \rho - \eta\rho, H(\overline{n}_1 \overline{n}) \rangle) |\overline{n}|^{-Q(1-\eta+i\gamma)} d\overline{n} \\ = \lim_{H \rightarrow +\infty} e^{2\langle \rho - \eta\rho + \lambda, H \rangle} \\ \int \exp(\langle \lambda - \rho - \eta\rho, H(\overline{n}_1 \overline{n}) \rangle) \exp(-\langle \rho - \eta\rho + \lambda, H(\overline{n}^{-H}) \rangle) d\overline{n} \\ = \lim_{H \rightarrow +\infty} e^{\langle \rho - \eta\rho + \lambda, H \rangle} P_{\lambda - \eta\rho} 1(\overline{n}_1 \exp H \cdot \circ),$$

where the last equality comes from (1.2). Now  $P_{\lambda-\eta\rho}1 = P_{\eta\rho-\lambda}1$  because of [6, Theorem IV.4.3]. This allows us to use the known asymptotic behaviour of spherical functions. Applying the Iwasawa decomposition to  $\bar{n}_1$ , we get

$$\bar{n}_1 \exp H \cdot \circ = k(\bar{n}_1) \exp(H + H(\bar{n}_1))n(\bar{n}_1)^{-H} \cdot \circ.$$

Here  $n(\bar{n}_1)^{-H} \rightarrow e$  as  $H \rightarrow +\infty$ . Considering the distance to  $\circ$ , we conclude that

$$(3.4) \quad H_C(\bar{n}_1 \exp H \cdot \circ) = H + H(\bar{n}_1) + o(1), \quad H \rightarrow +\infty.$$

Since  $\operatorname{Re}(\eta\rho - \lambda) \in \mathfrak{a}_+$ , Lemma IV.6.2 of [6] shows that

$$P_{\eta\rho-\lambda}1(\bar{n}_1 \exp H \cdot \circ) = c(\eta\rho - \lambda) \exp(\langle \eta\rho - \lambda - \rho, H + H(\bar{n}_1) + o(1) \rangle) \\ + \text{smaller terms}$$

for large  $H$ . Hence,

$$\int \exp(\langle \lambda - \rho - \eta\rho, H(\bar{n}_1 \bar{n}) \rangle) |\bar{n}|^{-Q(1-\eta+i\gamma)} d\bar{n} = c(\eta\rho - \lambda) e^{\langle \eta\rho - \lambda - \rho, H(\bar{n}_1) \rangle}.$$

The lemma follows if we let  $\eta \rightarrow 0$ .

PROOF OF THEOREM 3.1. We start with (3.1). In the case when  $f \equiv 1$  in  $K/M$ , we use (3.4) and the asymptotic formula for  $P_\lambda f$ , see [6, Theorem IV.5.5]. We find

$$P_\lambda 1(\bar{n}_1 \exp H \cdot \circ) = c(\lambda) \exp(\langle \lambda - \rho, H + H(\bar{n}_1) \rangle) \\ + c(-\lambda) \exp(\langle -\lambda - \rho, H + H(\bar{n}_1) \rangle) \\ + o(1), \quad H \rightarrow +\infty.$$

Because of Lemma 3.2 and (1.3), this implies (3.1) with  $f \equiv 1$ ,  $x = \circ$ .

Now consider an  $f \in L^1(K/M)$  which is Hölder in a neighbourhood of a point  $k_0M = k(\bar{n}_0)M$ . Write

$$(3.5) \quad e^{\langle \rho, H \rangle} P_\lambda f = e^{\langle \rho, H \rangle} P_\lambda (f - f_0) + e^{\langle \rho, H \rangle} P_\lambda f_0,$$

where  $f_0$  is the constant function  $f(k(\bar{n}_1)M)$  and the Poisson integrals are evaluated at  $\bar{n}_1 \exp H \cdot \circ$ . In the first term to the right, we pass to the limit as in (1.2) in the introduction, for  $\bar{n}_1$  near  $\bar{n}_0$ . This is justified by dominated convergence, because

$$e^{\langle 2\rho + \lambda, H \rangle} \exp(\langle \lambda - \rho, H(\bar{n}_1 \bar{n}) \rangle) \exp(-\langle \lambda + \rho, H(\bar{n}^{-H}) \rangle)$$

is  $O(|\bar{n}|^{-Q})$  near  $\bar{n} = e$  and  $O(|\bar{n}|^{-2Q})$  at infinity, uniformly in  $H$ . Moreover,  $f \in L^1(K/M)$  translates to  $\int |f(k(\bar{n}))| (1 + |\bar{n}|)^{-2Q} d\bar{n} < \infty$ . We know the



behaviour of the last term in (3.5) and need only add to obtain (3.1) with  $x = \circ$ . This is easily seen to be uniform in  $\bar{n}_1$  near  $\bar{n}_0$ .

With the same  $f$ , we now let  $x$  be arbitrary in a compact subset of  $X$ . Writing  $x = \bar{n}(x) \exp A(x) \cdot \circ$ , we have

$$\bar{n}_1 \exp H \cdot x = \bar{n}_1 \bar{n}(x)^H \exp(H + A(x)) \cdot \circ,$$

with  $A(x)$  bounded and  $\bar{n}(x)^H \rightarrow e$  as  $H \rightarrow +\infty$ .

This allows us to apply the case  $x = \circ$ . Notice that the expressions  $f(\bar{n}_1)$  and  $\text{cpv} \int \dots$  occurring in the right-hand side of (3.1) depend continuously on  $\bar{n}_1$  near  $\bar{n}_0$ . From this and the uniformity mentioned above, (3.1) follows for  $\bar{n}_1$  near  $\bar{n}_0$ .

To get (3.2) when  $f$  is Hölder near  $k_1 M$ , we use  $K$ -invariance and let  $f_1(k) = f(k_1 k)$ . Then

$$P_\lambda f(k_1 \exp H \cdot x) = P_\lambda f_1(\exp H \cdot x),$$

and it is enough to apply (3.1) with  $f$  replaced by  $f_1$  and  $\bar{n}_1 = e$ .

We remark that one can also find the behaviour of  $P_\lambda f(k_1 \exp H \cdot \circ)$  in another way for these  $f$ . Assume  $k_1 = k(\bar{n}_1)$  for some  $\bar{n}_1 \in \bar{N}$ , which is true for almost all  $k_1 M$ . Then

$$k_1 \exp H \cdot x = \bar{n}_1 \exp(H - H(\bar{n}_1)) \cdot x',$$

where

$$x' = (\bar{n}(\bar{n}_1)^{-1})^{H(\bar{n}_1) - H} \cdot x \rightarrow x$$

as  $H \rightarrow +\infty$ . Now (3.1) yields

$$(3.6) \quad e^{\langle \rho, H + A(x) \rangle} P_\lambda f(k_1 \exp H \cdot x) = c(\lambda) e^{\langle \lambda, H + A(x) \rangle} f(k_1 M) \\ + e^{\langle \lambda + \rho, H(\bar{n}_1) \rangle} e^{-\langle \lambda, H + A(x) \rangle} \\ \text{cpv} \int f(\bar{n}_1 \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} + o(1), \quad H \rightarrow +\infty.$$

Comparing this with (3.2), we conclude that

$$(3.7) \quad \text{cpv} \int f(k(\bar{n}_1) \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} \\ = e^{\langle \lambda + \rho, H(\bar{n}_1) \rangle} \text{cpv} \int f(\bar{n}_1 \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n},$$

when  $f \in L^1$  is Hölder near  $k(\bar{n}_1)M$ . In the special case  $f \equiv 1$  in  $K/M$ , this is a consequence of Lemma 3.2.

Next we let  $f \in L^1(K/M)$ . We shall prove (3.1) for a.a.  $\bar{n}_1$  in an arbitrary compact subset of  $\bar{N}$ . Because of the case just treated, we can assume that the

support of  $f$ , considered as a function in  $K/M$ , is contained in  $\{k(\bar{n})M : \bar{n} \in L\}$  for some compact set  $L \subset \bar{N}$ . For a fixed compact set  $D \subset X$ , we shall prove that the maximal operator

$$Mf(\bar{n}_1) = \sup_{H \in \mathfrak{a}_+, x \in D} |e^{\langle \rho, H+A(x) \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot x)|$$

is of weak type  $(1, 1)$  in  $L$ . This is enough by standard density arguments, since the expressions in the right-hand side of (3.1) define operators of weak type  $(1, 1)$ .

We write  $\bar{n}_1 \exp H \cdot x = \bar{n}_1 \bar{n}(x)^H \exp H' \cdot \circ$  as before, with  $H' = H + A(x) = H + 0(1)$ . With  $\alpha = \alpha(H')$ , we observe that  $|\bar{n}(x)^H| \leq Ce^{-\alpha}$ . If  $\bar{n} = \exp(Y_1 + Y_2)$  as in Section 2, we have

$$\begin{aligned} & e^{\langle \rho, H' \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot x) \\ &= e^{-\lambda(H')} \int \frac{f(\bar{n}_1 \bar{n}(x)^H \bar{n}) d\bar{n}}{(e^{-4\alpha} + 2ce^{-2\alpha}|Y_1|^2 + |\bar{n}|^4)^{Q(1+i\gamma)/4}}. \end{aligned}$$

In this integral, we take  $\bar{n}(x)^H \bar{n}$  as a new variable, still denoted  $\bar{n} = \exp(Y_1 + Y_2)$ . Then the kernel will be evaluated at the point

$$(\bar{n}(x)^H)^{-1} \bar{n} = \bar{n}' = \exp(Y_1' + Y_2').$$

Multiplying, we see that  $|Y_1' - Y_1| < Ce^{-\alpha}$ . Moreover,  $|\bar{n}|$  and  $|\bar{n}'|$  differ by at most  $Ce^{-\alpha}$  because of formula (1.9) p. 12 of [3]. We obtain

$$\begin{aligned} & e^{\langle \rho, H' \rangle} P_\lambda f(\bar{n}_1 \exp H \cdot x) \\ &= e^{-\lambda(H')} \int \frac{f(\bar{n}_1 \bar{n}) d\bar{n}}{(e^{-4\alpha} + 2ce^{-2\alpha}|Y_1'|^2 + |\bar{n}'|^4)^{Q(1+i\gamma)/4}}. \end{aligned}$$

Here we first integrate over the ball  $B(C_1 e^{-\alpha})$  for a large constant  $C_1$ . Clearly,

$$\left| \int_{B(C_1 e^{-\alpha})} \right| \leq Ce^{Q\alpha} \int_{B(C_1 e^{-\alpha})} |f(\bar{n}_1 \bar{n})| d\bar{n} \leq CM_0 f(\bar{n}_1),$$

where  $C = C(C_1)$  and  $M_0$  is the standard maximal operator for the gauge in  $\bar{N}$ .

For  $|\bar{n}| > C_1 e^{-\alpha}$ , we compare the kernel with  $|\bar{n}|^{-Q(1+i\gamma)}$ . If  $C_1$  is large, it is elementary to verify that

$$\left| \frac{1}{(e^{-4\alpha} + 2c|Y_1'|^2 + |\bar{n}'|^4)^{Q(1+i\gamma)/4}} - \frac{1}{|\bar{n}|^{Q(1+i\gamma)}} \right| \leq C \frac{e^{-\alpha}}{|\bar{n}|^{Q+1}}.$$

The integral

$$\int_{|\bar{n}| > C_1 e^{-\alpha}} f(\bar{n}_1 \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n}$$

is controlled by the maximal operator  $M_\gamma$  introduced in Section 2, which is of weak type (1, 1). It remains to estimate

$$\int_{|\bar{n}| > C_1 e^{-\alpha}} |f(\bar{n}_1 \bar{n})| e^{-\alpha} |\bar{n}|^{-Q-1} d\bar{n},$$

which is dominated by  $CM_0 f(\bar{n}_1)$ . This gives the weak type (1, 1) estimate.

Finally, we must verify (3.2) for  $f \in L^1$ . Observe first that (3.6) holds for a.a.  $k_1$ ,  $k_1 = k(\bar{n}_1)$ , by the same argument as before. The following lemma will therefore end the proof of Theorem 1.

LEMMA 3.3. *Let  $f \in L^1(K/M)$ . Then*

$$\text{cpv} \int f(k(\bar{n}_1) \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n}$$

*exists and (3.7) holds for almost all  $\bar{n}_1 \in \bar{N}$ .*

PROOF. Take  $\bar{n}_1 \in \bar{N}$  and write  $\bar{n}_1 = k_1 \exp(H_1) n_1 \in KAN$ . Let  $f_\epsilon = f \chi_\epsilon$ , where  $\epsilon > 0$  is small and  $\chi_\epsilon$  is the characteristic function of the set  $\{k_1 k(\bar{n}) M : |\bar{n}| \geq \epsilon\} \subset K/M$ . Since  $f_\epsilon$  vanishes near  $k_1 = k(\bar{n}_1)$ , equation (3.7) applies to  $f_\epsilon$ . This means that

$$(3.8) \quad \int_{|\bar{n}| > \epsilon} f(k_1 \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} = e^{\langle \lambda + \rho, H_1 \rangle} \int_U f(\bar{n}_1 \bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n},$$

where  $U$  is the set of  $\bar{n}$  for which  $k(\bar{n}_1 \bar{n}) M \notin \{k_1 k(\bar{n}') M : \bar{n}' \in B(\epsilon)\}$  or equivalently  $k(k_1^{-1} \bar{n}_1 \bar{n}) M \neq k(\bar{n}') M$  for all  $\bar{n}' \in B(\epsilon)$ . Clearly,  $U$  is all of  $\bar{N}$  except a small neighbourhood of  $e$ .

To determine this neighbourhood, assume that  $r = |\bar{n}|$  is small and write

$$k_1^{-1} \bar{n}_1 \bar{n} = \exp(H_1) n_1 \bar{n} = (n_1 \bar{n} n_1^{-1})^{H_1} \exp(H_1) n_1.$$

Let  $\bar{n} = \exp(Y_{-2} + Y_{-1})$  with  $Y_{-j} \in \mathfrak{g}_{-j\alpha}$ , so that  $|Y_{-j}| \leq Cr^j$ . If  $n_1 = \exp(X_1 + X_2)$ ,  $X_j \in \mathfrak{g}_{j\alpha}$ , we have

$$\begin{aligned} n_1 \bar{n} n_1^{-1} &= \exp(e^{ad(X_1+X_2)}(Y_{-2} + Y_{-1})) \\ &= \exp(Y_{-2} + Y_{-1} + [X_1, Y_{-2}] + R), \text{ with } R \in \bigoplus_{j=0}^2 \mathfrak{g}_{j\alpha}. \end{aligned}$$

Further,  $|R| \leq Cr$ . There is a unique decomposition

$$n_1 \bar{n} n_1^{-1} = \exp(Z_{-2} + Z_{-1}) \exp(Z_0) \exp(Z_1 + Z_2)$$

with small  $Z_j \in \mathfrak{g}_{j\alpha}$ . Multiplying by means of the Campbell-Hausdorff formula, one easily finds  $Z_{-2} = Y_{-2} + O(r^3)$  and  $Z_{-1} = Y_{-1} + O(r^2)$ . Now  $Z_0 \in \mathfrak{m} \oplus \mathfrak{a}$  and  $Z_1 + Z_2 \in \mathfrak{n}$ . Thus,  $k_1^{-1}\bar{n}_1\bar{n} = \bar{n}'m'a'n'$  with  $m' \in M$ ,  $a' \in A$ ,  $n' \in N$  and

$$\bar{n}' = \exp(e^{-2\alpha(H_1)}Z_{-2} + e^{-\alpha(H_1)}Z_{-1}) \in \bar{N}.$$

Since  $M$  normalizes  $N$ , this gives  $k(k_1^{-1}\bar{n}_1\bar{n})M = k(\bar{n}')M$ , and  $\bar{n}'$  is the only element of  $\bar{N}$  with this property. Notice that

$$|\bar{n}'|/|\bar{n}^{H_1}| = 1 + O(r), \quad r \rightarrow 0,$$

and

$$|\bar{n}^{H_1}| = e^{-\alpha(H_1)}r.$$

We conclude that the symmetric difference

$$U\Delta(\bar{N} \setminus B(\epsilon e^{-\alpha(H_1)}))$$

is contained in the annulus

$$R_\epsilon = B((1 + C\epsilon)\epsilon e^{-\alpha(H_1)}) \setminus B((1 - C\epsilon)\epsilon e^{-\alpha(H_1)}).$$

But if  $\bar{n}_1$  is a Lebesgue point of  $|f|$  with respect to the gauge, one easily gets

$$\int_{R_\epsilon} |f(\bar{n}_1\bar{n})| |\bar{n}|^{-Q} d\bar{n} \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Now (3.8) implies that for a.a.  $\bar{n}_1$

$$(3.9) \quad \int_{\bar{N} \setminus B(\epsilon)} f(k_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} \\ = e^{\langle \lambda + \rho, H_1 \rangle} \int_{\bar{N} \setminus B(\epsilon e^{-\alpha(H_1)})} f(\bar{n}_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n} + o(1), \quad \epsilon \rightarrow 0.$$

For almost all  $\bar{n}_1$ , we know that

$$\text{cpv} \int f(\bar{n}_1\bar{n}) |\bar{n}|^{-Q(1+i\gamma)} d\bar{n}$$

exists, which means that the value of the integral in the right-hand side of (3.9) describes an approximate circle in  $\mathbb{C}$  as  $\epsilon$  approaches 0. The same must then be true of the left-hand integral. Since the centres of these circles are the central principal values of (3.7), the lemma follows.

#### 4. - The result for trees

Let  $T$  be a homogeneous tree with branching degree  $q + 1 \geq 3$ . We essentially follow the notation from [8], see also Figà-Talamanca and Picardello [2]. In particular, we fix a vertex  $\circ \in T$  and identify any  $x \in T$  with the shortest (geodesic) path from  $\circ$  to  $x$ . The boundary  $\Omega$  of  $T$  then consists of all infinite geodesic paths. If  $x, x' \in T$ , we denote by  $N(x, x')$  the number of edges that  $x$  and  $x'$  have in common, and similarly for  $N(x, \omega)$  and  $N(\omega, \omega')$  with  $\omega, \omega' \in \Omega$ . One sets  $|x| = N(x, \circ)$ .

The sets

$$E_n(\omega) = \{\omega' \in \Omega : N(\omega, \omega') \geq n\}$$

define a basis of a topology in  $\Omega$ . Similarly, one lets  $E_n(x) = \{\omega \in \Omega : N(x, \omega) \geq n\}$ . In particular,  $E_n(x) = \emptyset$  for  $n > |x|$ . The disjoint union  $T \cup \Omega$  also has a natural topology. If  $\alpha$  is a nonnegative integer, an admissible approach region at  $\omega \in \Omega$  is defined as

$$\Gamma_\alpha(\omega) = \{x \in T : N(x, \omega) \geq |x| - \alpha\}.$$

A complex-valued function  $f$  in  $\Omega$  is said to be Hölder if it satisfies  $|f(\omega) - f(\omega')| \leq \text{const } e^{-\epsilon N(\omega, \omega')}$  for some  $\epsilon > 0$  and all  $\omega, \omega' \in \Omega$ .

The standard normalized measure  $\nu$  in  $\Omega$  satisfies  $\nu(E_n(\omega)) = q^{1-n}/(q+1)$  for  $n \geq 1$  and  $\omega \in \Omega$ .

The Poisson kernel of  $T$  is

$$K(x, \omega) = q^{2N(x, \omega) - |x|}, \quad x \in T, \quad \omega \in \Omega.$$

Let  $z \in \mathbb{C}$ . Any  $f \in L^1(\nu)$  (and any martingale in  $\Omega$ ) has a  $z$ -Poisson integral

$$K_z f(x) = \int_{\Omega} K(x, \omega)^z f(\omega) d\nu(\omega), \quad x \in T.$$

The function  $K_z f$  is an eigenfunction of the isotropic transition operator  $P$  in  $T$ , with eigenvalue  $\gamma(z) = (q^z + q^{1-z})/(q+1)$ .

Korányi and Picardello [8] study the convergence of normalized  $z$ -Poisson integrals, defined for  $\text{Re } z > 1/2$  and for  $z = 1/2$  and  $z = 1/2 + i\pi/\log q$  by  $K_z f = K_z f / K_z 1$ . For these values of  $z$ , they prove that  $K_z f$  converges admissibly to  $f$  almost everywhere in  $\Omega$  for  $f \in L^1(\nu)$ . When  $f$  is continuous in  $\Omega$ , one can extend  $K_z f$  by  $f$  to a continuous functions in  $T \cup \Omega$ . Because of the properties of the expression for  $\gamma(z)$ , this takes care of all eigenvalues of  $P$  except those corresponding to  $\text{Re } z = 1/2$ ,  $0 < |\text{Im } z| < \pi/\log q$ .

Therefore, we shall have  $z = (1 + i\tau)/2$  in the sequel, with  $0 < |\tau| < 2\pi/\log q$ . The “spherical function”  $K_z 1(x)$  then equals  $\text{Re}(c_z q^{-z|x|})$ , where

$$c_z = \frac{q}{q+1} \frac{1 - q^{i\tau-1}}{1 - q^{i\tau}},$$

see [8] and [2, §3.2]. To avoid the zeroes of  $K_z 1$ , we define now  $K_z f(x) = K_z f(x)/(c_z q^{-z|x|})$ . We shall obtain a formula for the asymptotic behaviour of  $K_z f(x)$  like (3.2). For this we need intertwining operators.

The mean values of a function  $f \in L^1(\nu)$  are

$$E_n f(\omega) = \frac{1}{\nu(E_n(\omega))} \int_{E_n(\omega)} f \, d\nu, \quad \omega \in \Omega, \quad n = 0, 1, \dots$$

The differences of  $f$  are  $\Delta_n f(\omega) = E_n f(\omega) - E_{n-1} f(\omega)$ ,  $n \geq 0$ , where  $E_{-1} f \equiv 0$ . Clearly,

$$f = \sum_0^{\infty} \Delta_n f \quad \text{a.e.}$$

An operator  $I_z$  is defined for  $z \in \mathbb{C}$  by

$$I_z f = \sum_0^{\infty} c(n, z) \Delta_n f,$$

where  $c(0, z) = 1$  and

$$c(n, z) = \frac{1 - q^{2(z-1)}}{1 - q^{-2z}} q^{(1-2z)n}, \quad n > 0.$$

For  $\text{Re } z = 1/2$  we see that  $I_z$  is unitary in  $L^2(\nu)$  and of weak type  $(1, 1)$  for  $\nu$ . When  $q$  is odd,  $T$  has a natural free group structure.

Then  $I_z$  intertwines representations  $\pi_z$  and  $\pi_{1-z}$  of  $T$ , see [2, §4.4]. These representations are unitary and belong to the principal series of  $T$  when  $\text{Re } z = 1/2$ .

**THEOREM 4.1.** *Let  $z = (1+i\tau)/2$ ,  $0 < |\tau| < 2\pi/\log q$ , and take  $f \in L^1(\nu)$  and  $\alpha > 0$ . Then for a.a.  $\omega_1 \in \Omega$*

$$(4.1) \quad K_z f(x) = (c_z^{-1} - 1) q^{i\tau|x|} f(\omega_1) + I_{\bar{z}} f(\omega_1) + o(1)$$

as  $x \rightarrow \omega_1$ ,  $x \in \Gamma_\alpha(\omega_1)$ . If  $f$  is Hölder, (4.1) holds as  $x \in T$  approaches  $\omega_1$  in the topology of  $T \cup \Omega$ , uniformly for  $\omega_1 \in \Omega$ .

**PROOF.** We have

$$\begin{aligned} K_z f(x) &= c_z^{-1} \int_{\Omega} q^{2N(x,\omega)z} f(\omega) \, d\nu(\omega) \\ &= c_z^{-1} \sum_{n=0}^{|x|} q^{(1+i\tau)n} \int_{E_n(x) \setminus E_{n+1}(x)} f(\omega) \, d\nu(\omega). \end{aligned}$$

If to begin with  $x \in \Gamma_0(\omega_1)$ , one gets

$$\int_{E_n(\omega)} f d\nu(\omega) = \frac{q}{q+1} q^{-n} E_n f(\omega_1)$$

for  $n \leq |x|$ , except that the factor  $q/q+1$  must be deleted when  $n = 0$ .

It follows that

$$\begin{aligned} \mathcal{K}_z f(x) &= c_z^{-1} \left( \frac{q}{q+1} \left( \sum_{n=1}^{|x|} q^{i\tau n} E_n f(\omega_1) - \sum_{n=0}^{|x|-1} q^{i\tau n-1} E_{n+1} f(\omega_1) \right) \right. \\ &\quad \left. + E_0 f(\omega_1) \right) = c_z^{-1} \left( \frac{q}{q+1} (1 - q^{-1-i\tau}) \sum_1^{|x|} q^{i\tau n} E_n f(\omega_1) + E_0 f(\omega_1) \right). \end{aligned}$$

Now we write  $E_n f$  as  $\sum_0^n \Delta_m f$  and change the order of summation:

$$\begin{aligned} \mathcal{K}_z f(x) &= \sum_{m=1}^{|x|} c(m, \bar{z}) \Delta_m f(\omega_1) - \frac{1 - q^{-1-i\tau}}{1 - q^{-1+i\tau}} q^{i\tau} q^{i\tau|x|} \sum_{m=0}^{|x|} \Delta_m f(\omega_1) \\ &\quad + \left( \frac{1 - q^{-1-i\tau}}{1 - q^{-1+i\tau}} q^{i\tau} + c_z^{-1} \right) \Delta_0 f(\omega_1). \end{aligned}$$

The coefficients of  $\Delta_0 f(\omega_1)$  here equals  $1 = c(0, \bar{z})$ , and we conclude that

$$(4.2) \quad \mathcal{K}_z f(x) = (c_z^{-1} - 1) q^{i\tau|x|} E_{|x|} f(\omega_1) + \sum_0^{|x|} c(m, \bar{z}) \Delta_m f(\omega_1)$$

for  $x \in \Gamma_0(\omega_1)$ .

By standard martingale theory, this implies (4.1) for a.a.  $\omega_1$  when  $\alpha = 0$ . If  $f$  is Hölder, the differences  $\Delta_m f$  decrease exponentially in  $m$ , uniformly in  $\Omega$ . Then (4.1) holds as  $x \rightarrow \omega_1$  staying in  $\Gamma_0(\omega_1)$ , uniformly in  $\omega_1$ . Since  $f$  and  $I_{\bar{z}} f$  are now continuous in  $\Omega$ , the last statement of Theorem 4.1 follows.

It remains to consider  $f \in L^1$  and  $\alpha > 0$ . Approximating with Hölder or locally constant functions, we see that it is enough to show that the maximal operator

$$M_\alpha f(\omega_1) = \sup_{x \in \Gamma_\alpha(\omega_1)} |\mathcal{K}_z f(x)|$$

is of weak type  $(1, 1)$  in  $\Omega$ . Letting  $x \in \Gamma_\alpha(\omega_1)$ , we choose  $\omega_2 \in \Omega$  with  $x \in \Gamma_0(\omega_2)$ . Then we apply (4.2) with  $\omega_2$  instead of  $\omega_1$  and estimate the two terms. Since  $E_{|x|}(\omega_2) \subset E_{|x|-\alpha}(\omega_1)$ , one has

$$|E_{|x|}f(\omega_2)| \leq C E_{|x|-\alpha}|f(\omega_1)|.$$

All constants  $C$  may depend on  $\alpha$ . Further,  $\Delta_m f(\omega_2)$  equals  $\Delta_m f(\omega_1)$  for  $m \leq |x| - \alpha$  and is dominated by  $C E_{|x|-\alpha}|f(\omega_1)|$  for  $|x| - \alpha < m \leq |x|$ . As a result,

$$|\mathcal{K}_z f(x)| \leq \left| \sum_0^{|x|-\alpha} c(m, \bar{z}) \Delta_m f(\omega_1) \right| + C E_{|x|-\alpha}|f(\omega_1)|.$$

Thus  $M_\alpha f$  is dominated by the maximal function of the martingale  $\sum c(m, \bar{z}) \Delta_m f$  and the standard maximal function of  $f$ . This gives the weak type  $(1, 1)$  estimate which ends the proof of Theorem 4.1.

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