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Asymptotic Behaviour of Generalized Poisson Integrals in Rank One Symmetric Spaces and in Trees

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1. - Introduction

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type and of real rank 1, with boundary K/M. Some standard notation used here is explained in Section 2. Take $\lambda \in \mathbf{a}_{\mathbb{C}}^*$. Any $f \in L^1(K/M)$ has a λ-Poisson integral

$$(1.1) \qquad P_\lambda f(g \cdot \circ) = \int f(kM) \exp(- < \rho + \lambda,\ H(g^{-1}k) >) dkM$$

for $g \cdot \circ \in X$. Here $H(\cdot)$ comes from the Iwasawa decomposition of G, whereas H will be generic in $\mathbf{a}$.

When the real part of λ is positive, it is known that f can be recovered as the limit of the normalized λ-Poisson integral $\mathcal{P}_\lambda f = P_\lambda f / P_\lambda 1$ at the boundary. Here 1 is the constant function. Indeed,

$$\mathcal{P}_\lambda f(k_1 \exp\ H \cdot \circ) \to f(k_1 M) \text{ as } H \to +\infty \text{ in } \mathbf{a}_+$$

for a.a. $k_1 M$ in K/M. In terms of the $\overline{N}A$ model for X, this reads

$$\mathcal{P}_\lambda f(\overline{n}_1 \exp\ H \cdot \circ) \to f(k(\overline{n}_1)M) \text{ as } H \to +\infty$$

for a.a. $\overline{n}_1 \in \overline{N}$. More generally, one can use an admissible approach here, which means replacing $\circ$ by a point x staying in a compact subset of X. Such results are known to hold also for $\lambda = 0$, see Sjögren [9].

In this paper, we shall consider the case Re $\lambda = 0$, $\lambda \neq 0$, i.e., $\lambda = i\gamma\rho$ with $0 \neq \gamma \in \mathbb{R}$.

The normalizing factor $P_\lambda 1$ is a spherical function, in particular it is biinvariant under K. For Re $\lambda > 0$, it behaves like $e^{<\lambda - \rho, H>}$ at $\exp\ H \cdot \circ$ for large $H \in \mathbf{a}_+$. But in our case the dominating term of its asymptotic

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expansion is 2 Re $c(\lambda)\exp^{<\lambda-\rho,H>}$, which has zeroes for arbitrarily large H. To examine the asymptotic behaviour of $P_\lambda f$, it is therefore reasonable to divide by $2c(\lambda)e^{<\lambda-\rho,H>} \neq 0$, or simply by $e^{-<\rho,H>}$, instead of $P_\lambda 1$.

The usual transformation of (1.1) to $\overline{N}$ gives

$$(1.2) \qquad e^{<\rho,H>} P_\lambda f(\overline{n}_1 \exp H \cdot o) = e^{<2\rho+\lambda,H>} \int f(k(\overline{n}_1\overline{n})M) \exp(< \lambda - \rho, H(\overline{n}_1\overline{n}) >) \exp(- < \lambda + \rho, H(\overline{n}^{-H}) >) d\overline{n}$$

where $\overline{n}^{-H} = \exp(-H)\overline{n} \exp H$. The last factor in the integrand is

$$\exp(- < \lambda + \rho, H(\overline{n}^{-H}) >) = P(\overline{n}^{-H})^{(1+i\gamma)/2},$$

where $P(\overline{n})$ is the Poisson kernel in $\overline{N}$. The expression

$$e^{<2\rho,H>} P(\overline{n}^{-H})^{1/2}$$

has a limit as $H \to +\infty$ which can be written as $|\overline{n}|^{-Q}$. Here $|\cdot|$ is a homogeneous gauge in $\overline{N}$ and Q the corresponding dimension of $\overline{N}$.

If we could let $H \to +\infty$ under the integral sign in (1.2), the conclusion would be that

$$e^{<\lambda+\rho,H>} P_\lambda f(\overline{n}_1 \exp H \cdot o)$$

tends to

$$I = \int f(k(\overline{n}_1\overline{n})M) \exp(< \lambda - \rho, H(\overline{n}_1\overline{n}) >) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n},$$

a divergent integral. In fact, the asymptotic behaviour of $P_\lambda f$ is given by

$$e^{<\rho,H>} P_\lambda f(\overline{n}_1 \exp H \cdot o) = c(\lambda) e^{<\lambda,H>} \exp(< \lambda - \rho, H(\overline{n}_1) >) f(k(\overline{n}_1)M) + e^{-<\lambda,H>} I + o(1), \quad H \to +\infty$$

with a suitable evaluation of the integral. Notice that $e^{\pm<\lambda,H>}$ are oscillating factors.

This can be written in a neater way if we extend f to all of G by means of

$$(1.3) \qquad f(k \exp(H) n) = f(kM) e^{<\lambda-\rho,H>}$$

for $k \in K$, $H \in \mathbf{a}$, $n \in N$. This extension is used in connection with the representation of the principal series of G corresponding to $-\lambda \in \mathbf{a}$, the parabolic subgroup MAN, and the trivial representation of M. Also notice that the singular integral we obtained defines an interwining operator from this representation to that corresponding to $+\lambda$.

The result of the main part of this paper gives the asymptotic behaviour of $P_\lambda f$ for admissible approach to the boundary. The paper [1] by van den Ban and Schlichtkrull contains an asymptotic expansion of $P_\lambda f$.

Our result means that for the values of λ considered here, the principal terms of this expansion are determined explicitly. We also obtain a pointwise estimate of the difference between $P_\lambda f$ and the principal terms. This estimate holds at all boundary points for Hölder functions f and almost everywhere for $f \in L^1(K/M)$. For K-finite functions f, the expansion was already known, with explicit formulae for the terms, see Helgason [4, §4]. Our proofs are more concrete. The only asymptotic behaviour we use is that of the spherical function $P_\lambda 1$. To prove our results for L^1 functions, we go via a maximal function estimate.

The last part of this paper gives an analogous result for a homogeneous tree of branching number $q+1 \geq 3$. The z-Poisson integral is defined for integrable functions f on the boundary by means of the zth power of the Poisson kernel, $z \in \mathbb{C}$. For Re $z > 1/2$ and for $z = 1/2$ and $z = 1/2 + \pi i/\log\ q$, the normalized z-Poisson integral $\mathcal{K}_z f$ converges to f almost everywhere on the boundary. This was proved by Korányi and Picardello [8]. We shall deal with the remaining values of Im z when Re $z = 1/2$. The result is an asymptotic formula

$$\mathcal{K}_z f(x) = \text{const} \cdot e^{\text{const} \cdot i|x|} f(\omega_1) + I_{\bar{z}} f(\omega_1) + o(1)$$

as x approaches the boundary point ω_1. As in the case of symmetric spaces, $I_{\bar{z}}$ turns out to be an interwining operator between representations of a related group. The proof is rather straightforward.

2. - Preliminaries

We write the symmetric space as $X = G/K$ in the standard way. For more details, see [5] or [9]. Here K is a maximal compact subgroup of the connected semi-simple Lie group G. The Iwasawa decomposition $G = KAN$ means that any $g \in G$ can be written uniquely as $g = k(g)\exp(H(g))n(g)$. Here $k(g)$ is in K, $n(g)$ in the nilpotent group N and $H(g)$ in $\mathbf{a}$, the Lie algebra of the abelian group A. Since rank $X = 1$, both A and $\mathbf{a}$ are isomorphic to $\mathbb{R}$. The positive Weyl chamber $\mathbf{a}_+ \subset \mathbf{a}$ then corresponds to $\mathbb{R}_+$. By $\mathbf{a}_\mathbb{C}$ we denote the complexification of $\mathbf{a}$, and $\mathbf{a}^*$ and $\mathbf{a}_\mathbb{C}^*$ are the duals of these spaces.

The exponential map gives a diffeomorphism between N and its Lie algebra $\mathbf{n}$. Further, $\mathbf{n}$ is the direct sum of the root spaces $\mathbf{g}_\alpha$ and $\mathbf{g}_{2\alpha}$, which are subspaces of the Lie algebra $\mathbf{g}$ of G. The positive restricted roots α and 2α are elements of $\mathbf{a}^*$. Their multiplicities are $m_\alpha = \dim \mathbf{g}_\alpha > 0$ and $m_{2\alpha} = \dim \mathbf{g}_{2\alpha} \geq 0$. Write n^H for $\exp\ (H)n\ \exp(-H)$ when $n \in N,\ H \in \mathbf{a}$. Then $n = \exp(Y_1 + Y_2),\ Y_1 \in \mathbf{g}_\alpha,\ Y_2 \in \mathbf{g}_{2\alpha}$ implies

$$n^H = \exp(e^{<\alpha, H>} Y_1 + e^{<2\alpha, H>} Y_2).$$

These properties of N are shared by its image $\overline{N}$ under the Cartan involution θ, except that α and 2α are replaced by the negative roots $-\alpha$ and -2α. Since θ is an isomorphism, the multiplicities verify $m_{-\alpha} = m_\alpha$ and $m_{-2\alpha} = m_{2\alpha}$.

Both α and $\rho = (m_\alpha + 2m_{2\alpha})\alpha/2 \in \mathbf{a}^*$ are positive in the sense that they belong to the polar $\mathbf{a}_+^*$ of $\mathbf{a}_+$. Let M be the centralizer of A in K, with Lie algebra $\mathbf{m}$. The root space $\mathbf{g}_0 = \mathbf{a} \oplus \mathbf{m}$ is abelian.

Let $\circ = eK \in X$, and write $g \cdot \circ$ for $gK \in X$. Because of the Cartan decomposition, any point $x \in X$ can be written as $x = k\ \exp(H_C(x)) \cdot \circ$, where $k \in K$ and $H_C(x)$ is a uniquely determined point in the closure of $\mathbf{a}_+$. In fact, $H_C(x)$ is proportional to the distance from $\circ$ to x. Because of the modified Iwasawa decomposition $G = \overline{N}AK$, one can also write $x = \overline{n}(x) \exp\ A(x) \cdot \circ$, with uniquely determined $\overline{n}(x) \in \overline{N}$ and $A(x) \in \mathbf{a}$.

The boundary of X is K/M, and a point $k_1 M \in K/M$ is the limit of $k_1 \exp\ H \cdot \circ \in X$ as $H \in \mathbf{a}$ tends to $+\infty$, i.e., as $\alpha(H) \to +\infty$. Letting $\overline{n} \in \overline{N}$ correspond to $k(\overline{n})M \in K/M$, one can also realize the boundary as $\overline{N}$, except for one point. Then $\overline{n}$ is the limit of $\overline{n} \exp\ H \cdot \circ$ as $H \to +\infty$.

Any function f in K/M will be defined in G by means of (1.3). We say that f is Hölder if it satisfies a Hölder condition in terms of any local coordinate system in K/M, with exponent in $]0, 1]$. Then its values in $\overline{N}$ verify a local Hölder condition.

The λ-Poisson integral of any $f \in L^1(K/M)$ can now be defined via (1.1), and (1.2) follows. The Poisson kernel $P(\overline{n})$ was defined in the introduction. With $\overline{n} = \exp(Y_1 + Y_2)$, $Y_j \in \mathbf{g}_{-j\alpha}$, it is given by

$$P(\overline{n}) = \frac{1}{(1 + 2c|Y_1|^2 + c^2|Y_1|^4 + 4c|Y_2|^2)^{(m_\alpha + 2m_{2\alpha})/2}}, \tag{2.1}$$

see [5, Theorem IX.3.8]. Here $c = (m_\alpha + 4m_{2\alpha})^{-1}/4$, and $|Y| = -B(Y, \theta Y)^{1/2}$ is for any $Y \in \mathbf{g}$ the norm coming from the Killing form B.

The last two terms in the denominator form a homogeneous gauge

$$|\overline{n}| = (c^2|Y_1|^4 + 4c|Y_2|^2)^{1/4},$$

where the exponent $1/4$ is a matter of convenience. One has $|\overline{n}^H| = e^{-\alpha(H)}|\overline{n}|$, for $H \in \mathbf{a}$, and $|\overline{n}\ \overline{n}'| \leq \text{const}(|\overline{n}| + |\overline{n}'|)$. The Haar measure of the ball $B(r) = \{\overline{n} \in \overline{N} : |\overline{n}| < r\}$ is proportional to r^Q, where $Q = m_\alpha + 2m_{2\alpha}$ is the homogeneous dimension of $\overline{N}$.

It is now clear that

$$e^{<2\rho, H>} P(\overline{n}^{-H})^{1/2} \to |\overline{n}|^{-Q} \text{ as } H \to +\infty.$$

We remark that this limit can also be written $|\overline{n}|^{-Q} = e^{<\rho, B(m^* \overline{n})>}$, see [5, Theorem IX.3.8]. Here $m^* \in K$ defines the nontrivial element of the Weyl group, and $B(g) \in \mathbf{a}$ is determined for a.a. $g \in G$ by the Bruhat decomposition

$g = \overline{n} m \exp(B(g))n$, $\overline{n} \in \overline{N}$, $m \in M$, $n \in N$. This is why the singular integrals in our result define intertwining operators, cf. Knapp and Stein [7, I.3].

We next discuss integrals containing the oscillating singular kernel $|\overline{n}|^{-Q(1+i\gamma)}$. If F is an integrable Hölder function in $\overline{N}$, one has

$$\int_{|\overline{n}|>\epsilon} F(\overline{n})|\overline{n}|^{-Q(1+i\gamma)} d\overline{n} = A + B\epsilon^{-iQ\gamma} + o(1) \tag{2.2}$$

as $\epsilon \to 0$, with complex constants A and B. This can be seen by means of polar coordinates in $\overline{N}$ as in [3, Ch. 1.A]. Since (2.2) says that the values of the integral approximate a circle in $\mathbb{C}$ centred at A, we then call A a central principal value and write

$$A = \operatorname{cpv} \int F(\overline{n})|\overline{n}|^{-Q(1+i\gamma)} d\overline{n}.$$

This value is also the limit, or the analytic continuation, of the convergent integrals obtained by using exponents z, Re $z > -Q$, instead of $-Q(1+i\gamma)$.

Now replace $F(\overline{n})$ by $F(\overline{n}_1\overline{n})$ for $\overline{n}_1 \in \overline{N}$ so that $A = A(\overline{n}_1)$ and we have a convolution. It is well known that the operator $F \to A$ is of weak type $(1,1)$ and thus defined for all $F \in L^1(\overline{N})$, see [3, Ch. 6].

Moreover, the corresponding maximal operator

$$M_\gamma F(\overline{n}_1) = \sup_{\epsilon>0} \left| \int_{|\overline{n}|>\epsilon} F(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)} d\overline{n} \right|$$

is of weak type $(1,1)$, as one can see by extending the well-known proof in $\mathbb{R}^n$. It follows that $F(\overline{n}_1\overline{n})$ satisfies (2.2) for a.a. $\overline{n}_1$ when $F \in L^1$.

Hence,

$$\operatorname{cpv} \int F(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)} d\overline{n}$$

is defined almost everywhere and coincides with $A(\overline{n}_1)$.

We denote by $c(\lambda)$ Harish-Chandra's c-function.

3. - The result for symmetric spaces

THEOREM 3.1. *Let $f \in L^1(K/M)$ and assume $\lambda = i\gamma\rho \in \mathfrak{a}^*_{\mathbb{C}}$ with $0 \neq \gamma \in \mathbb{R}$.*

Then for a.a. $\overline{n}_1 \in \overline{N}$

$$\begin{aligned} e^{<\rho,H+A(x)>} P_\lambda f(\overline{n}_1 \exp\ H \cdot x) &= c(\lambda) e^{<\lambda,H+A(x)>} f(\overline{n}_1) \\ &+ e^{-<\lambda,H+A(x)>} \operatorname{cpv} \int_{\overline{N}} f(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)} d\overline{n} + o(1) \end{aligned} \tag{3.1}$$

and for a.a. $k_1M \in K/M$

$$e^{<\rho,H+A(x)>}P_\lambda f(k_1 \exp\ H \cdot x) = c(\lambda)e^{<\lambda,H+A(x)>}f(k_1) + e^{-<\lambda,H+A(x)>}\operatorname{cpv}\int_{\overline{N}} f(k_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n} + o(1), \tag{3.2}$$

both as $H \to +\infty$ *and* x *stays in a compact subset of* X. *When* f *is Hölder in an open set* $\Omega \subset K/M$, *these formulas hold for all* $\overline{n}_1$ *with* $k(\overline{n}_1)M \in \Omega$ *and all* $k_1M \in \Omega$, *respectively.*

Notice that it is natural that $H + A(x)$ appears here, because $H + A(x) = A(\overline{n}_1 \exp\ H \cdot x) = A(\exp\ H \cdot x)$.

LEMMA 3.2. *For any* $\overline{n}_1 \in \overline{N}$,

$$\operatorname{cpv}\int_{\overline{N}} \exp(< \lambda - \rho, H(\overline{n}_1\overline{n}) >)|\overline{n}|^{-Q(1+i\gamma)}d\overline{n} = c(-\lambda)e^{-<\lambda+\rho,H(\overline{n}_1)>}.$$

PROOF. We first claim that the left-hand side here is the limit of the convergent integrals

$$\int_{\overline{N}} \exp(< \lambda - \rho - \eta\rho, H(\overline{n}_1\overline{n}) >)|\overline{n}|^{-Q(1-\eta+i\gamma)}d\overline{n} \tag{3.3}$$

as $\eta \to 0,\ \eta > 0$. The only difficulty is near $\overline{n} = e$, so consider the integrals in $\{|\overline{n}| < 1\}$. If in (3.3) we subtract from the exponential factor its value at $\overline{n} = e$, dominated convergence will allow us to let $\eta \to 0$ under the integral sign. It then remains to consider the integral of $|\overline{n}|^{-Q(1-\eta+i\gamma)}$ in $|\overline{n}| < 1$. By means of polar coordinates, this integral is seen to tend to

$$\operatorname{cpv}\int_{|\overline{n}|<1} |\overline{n}|^{-Q(1+i\gamma)}d\overline{n}.$$

The claim follows.

From (2.1), we know that $|\overline{n}|^{-Q}$ is the increasing limit of $e^{<2\rho,H>}e^{-<\rho,H(\overline{n}^{-H})>}$ as $H \to +\infty$. Thus by dominated convergence

$$\begin{aligned}&\int \exp(< \lambda - \rho - \eta\rho, H(\overline{n}_1\overline{n}) >)|\overline{n}|^{-Q(1-\eta+i\gamma)}d\overline{n}\\ &= \lim_{H\to+\infty} e^{2<\rho-\eta\rho+\lambda,H>}\\ &\int \exp(< \lambda - \rho - \eta\rho, H(\overline{n}_1\overline{n}) >)\exp(- < \rho - \eta\rho + \lambda, H(\overline{n}^{-H}) >)d\overline{n}\\ &= \lim_{H\to+\infty} e^{<\rho-\eta\rho+\lambda,H>}P_{\lambda-\eta\rho}1(\overline{n}_1 \exp\ H \cdot o),\end{aligned}$$

where the last equality comes from (1.2). Now $P_{\lambda-\eta\rho}1 = P_{\eta\rho-\lambda}1$ because of [6, Theorem IV.4.3]. This allows us to use the known asymptotic behaviour of spherical functions. Applying the Iwasawa decomposition to $\overline{n}_1$, we get

$$\overline{n}_1 \exp\ H \cdot o = k(\overline{n}_1)\exp(H + H(\overline{n}_1))n(\overline{n}_1)^{-H} \cdot o.$$

Here $n(\overline{n}_1)^{-H} \to e$ as $H \to +\infty$. Considering the distance to o, we conclude that

$$H_C(\overline{n}_1 \exp\ H \cdot o) = H + H(\overline{n}_1) + o(1),\ \ H \to +\infty. \tag{3.4}$$

Since $\mathrm{Re}(\eta\rho - \lambda) \in \mathbf{a}_+$, Lemma IV.6.2 of [6] shows that

$$P_{\eta\rho-\lambda}1(\overline{n}_1 \exp\ H \cdot o) = c(\eta\rho - \lambda)\exp(< \eta\rho - \lambda - \rho, H + H(\overline{n}_1) + o(1) >)$$
$$+ \text{smaller terms}$$

for large H. Hence,

$$\int \exp(< \lambda - \rho - \eta\rho, H(\overline{n}_1\overline{n}) >)|\overline{n}|^{-Q(1-\eta+i\gamma)}d\overline{n} = c(\eta\rho - \lambda)e^{<\eta\rho-\lambda-\rho, H(\overline{n}_1)>}.$$

The lemma follows if we let $\eta \to 0$.

Proof of Theorem 3.1. We start with (3.1). In the case when $f \equiv 1$ in K/M, we use (3.4) and the asymptotic formula for $P_\lambda f$, see [6, Theorem IV.5.5]. We find

$$\begin{aligned} P_\lambda 1(\overline{n}_1 \exp\ H \cdot o) &= c(\lambda)\exp(< \lambda - \rho, H + H(\overline{n}_1) >) \\ &+ c(-\lambda)\exp(< -\lambda - \rho, H + H(\overline{n}_1) >) \\ &+ o(1),\ \ H \to +\infty. \end{aligned}$$

Because of Lemma 3.2 and (1.3), this implies (3.1) with $f \equiv 1,\ x = o$.

Now consider an $f \in L^1(K/M)$ which is Hölder in a neighbourhood of a point $k_0M = k(\overline{n}_0)M$. Write

$$e^{<\rho,H>}P_\lambda f = e^{<\rho,H>}P_\lambda(f - f_0) + e^{<\rho,H>}P_\lambda f_0, \tag{3.5}$$

where f_0 is the constant function $f(k(\overline{n}_1)M)$ and the Poisson integrals are evaluated at $\overline{n}_1 \exp\ H \cdot o$. In the first term to the right, we pass to the limit as in (1.2) in the introduction, for $\overline{n}_1$ near $\overline{n}_0$. This is justified by dominated convergence, because

$$e^{<2\rho+\lambda,H>}\exp(< \lambda - \rho, H(\overline{n}_1\overline{n}) >)\exp(- < \lambda + \rho, H(\overline{n}^{-H}) >)$$

is $O(|\overline{n}|^{-Q})$ near $\overline{n} = e$ and $O(|\overline{n}|^{-2Q})$ at infinity, uniformly in H. Moreover, $f \in L^1(K/M)$ translates to $\int |f(k(\overline{n}))|(1 + |\overline{n}|)^{-2Q}d\overline{n} < \infty$. We know the behaviour of the last term in (3.5) and need only add to obtain (3.1) with $x = \circ$. This is easily seen to be uniform in $\overline{n}_1$ near $\overline{n}_0$.

With the same f, we now let x be arbitrary in a compact subset of X. Writing $x = \overline{n}(x) \exp\ A(x) \cdot \circ$, we have

$$\overline{n}_1 \exp\ H \cdot x = \overline{n}_1 \overline{n}(x)^H \exp(H + A(x)) \cdot \circ,$$

with $A(x)$ bounded and $\overline{n}(x)^H \to e$ as $H \to +\infty$.

This allows us to apply the case $x = o$. Notice that the expressions $f(\overline{n}_1)$ and $\mathrm{cpv}\int \cdots$ occurring in the right-hand side of (3.1) depend continuously on $\overline{n}_1$ near $\overline{n}_0$. From this and the uniformity mentioned above, (3.1) follows for $\overline{n}_1$ near $\overline{n}_0$.

To get (3.2) when f is Hölder near $k_1 M$, we use K-invariance and let $f_1(k) = f(k_1 k)$. Then

$$P_\lambda f(k_1 \exp\ H \cdot x) = P_\lambda f_1(\exp\ H \cdot x),$$

and it is enough to apply (3.1) with f replaced by f_1 and $\overline{n}_1 = e$.

We remark that one can also find the behaviour of $P_\lambda f(k_1 \exp\ H \cdot \circ)$ in another way for these f. Assume $k_1 = k(\overline{n}_1)$ for some $\overline{n}_1 \in \overline{N}$, which is true for almost all $k_1 M$. Then

$$k_1 \exp\ H \cdot x = \overline{n}_1 \exp(H - H(\overline{n}_1)) \cdot x',$$

where

$$x' = (n(\overline{n}_1)^{-1})^{H(\overline{n}_1) - H} \cdot x \to x$$

as $H \to +\infty$. Now (3.1) yields

$$\begin{aligned} (3.6) \qquad & e^{<\rho, H + A(x)>} P_\lambda f(k_1 \exp\ H \cdot x) = c(\lambda) e^{<\lambda, H + A(x)>} f(k_1 M) \\ & + e^{<\lambda + \rho, H(\overline{n}_1)>} e^{-<\lambda, H + A(x)>} \\ & \mathrm{cpv} \int f(\overline{n}_1 \overline{n}) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n} + o(1), \quad H \to +\infty. \end{aligned}$$

Comparing this with (3.2), we conclude that

$$\begin{aligned} (3.7) \qquad & \mathrm{cpv} \int f(k(\overline{n}_1)\overline{n}) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n} \\ & = e^{<\lambda + \rho, H(\overline{n}_1)>} \mathrm{cpv} \int f(\overline{n}_1 \overline{n}) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n}, \end{aligned}$$

when $f \in L^1$ is Hölder near $k(\overline{n}_1)M$. In the special case $f \equiv 1$ in K/M, this is a consequence of Lemma 3.2.

Next we let $f \in L^1(K/M)$. We shall prove (3.1) for a.a. $\overline{n}_1$ in an arbitrary compact subset of $\overline{N}$. Because of the case just treated, we can assume that the support of f, considered as a function in K/M, is contained in $\{k(\overline{n})M : \overline{n} \in L\}$ for some compact set $L \subset \overline{N}$. For a fixed compact set $D \subset X$, we shall prove that the maximal operator

$$Mf(\overline{n}_1) = \sup_{H \in \mathfrak{a}_+; x \in D} |e^{<\rho, H+A(x)>} P_\lambda f(\overline{n}_1 \exp\ H \cdot x)|$$

is of weak type $(1,1)$ in L. This is enough by standard density arguments, since the expressions in the right-hand side of (3.1) define operators of weak type $(1,1)$.

We write $\overline{n}_1 \exp\ H \cdot x = \overline{n}_1 \overline{n}(x)^H \exp\ H' \cdot o$ as before, with $H' = H + A(x) = H + 0(1)$. With $\alpha = \alpha(H')$, we observe that $|\overline{n}(x)^H| \leq Ce^{-\alpha}$. If $\overline{n} = \exp(Y_1 + Y_2)$ as in Section 2, we have

$$\begin{aligned} &e^{<\rho, H'>} P_\lambda f(\overline{n}_1 \exp\ H \cdot x) \\ &= e^{-\lambda(H')} \int \frac{f(\overline{n}_1 \overline{n}(x)^H \overline{n}) d\overline{n}}{(e^{-4\alpha} + 2ce^{-2\alpha}|Y_1|^2 + |\overline{n}|^4)^{Q(1+i\gamma)/4}}. \end{aligned}$$

In this integral, we take $\overline{n}(x)^H \overline{n}$ as a new variable, still denoted $\overline{n} = \exp(Y_1 + Y_2)$. Then the kernel will be evaluated at the point

$$(\overline{n}(x)^H)^{-1} \overline{n} = \overline{n}' = \exp(Y_1' + Y_2').$$

Multiplying, we see that $|Y_1' - Y_1| < Ce^{-\alpha}$. Moreover, $|\overline{n}|$ and $|\overline{n}'|$ differ by at most $Ce^{-\alpha}$ because of formula (1.9) p. 12 of [3]. We obtain

$$\begin{aligned} &e^{<\rho, H'>} P_\lambda f(\overline{n}_1 \exp\ H \cdot x) \\ &= e^{-\lambda(H')} \int \frac{f(\overline{n}_1 \overline{n})\ d\overline{n}}{(e^{-4\alpha} + 2ce^{-2\alpha}|Y_1'|^2 + |\overline{n}'|^4)^{Q(1+i\gamma)/4}}. \end{aligned}$$

Here we first integrate over the ball $B(C_1 e^{-\alpha})$ for a large constant C_1. Clearly,

$$\left| \int_{B(C_1 e^{-\alpha})} \right| \leq Ce^{Q\alpha} \int_{B(C_1 e^{-\alpha})} |f(\overline{n}_1 \overline{n})| d\overline{n} \leq CM_0 f(\overline{n}_1),$$

where $C = C(C_1)$ and M_0 is the standard maximal operator for the gauge in $\overline{N}$.

For $|\overline{n}| > C_1 e^{-\alpha}$, we compare the kernel with $|\overline{n}|^{-Q(1+i\gamma)}$. If C_1 is large, it is elementary to verify that

$$\left| \frac{1}{(e^{-4\alpha} + 2c|Y_1'|^2 + |\overline{n}'|^4)^{Q(1+i\gamma)/4}} - \frac{1}{|\overline{n}|^{Q(1+i\gamma)}} \right| \leq C \frac{e^{-\alpha}}{|\overline{n}|^{Q+1}}.$$

The integral

$$\int_{|\overline{n}| > C_1 e^{-\alpha}} f(\overline{n}_1 \overline{n}) |\overline{n}|^{-Q(1+i\gamma)} d\overline{n}$$

is controlled by the maximal operator M_γ introduced in Section 2, which is of weak type $(1,1)$. It remains to estimate

$$\int_{|\overline{n}|>C_1e^{-\alpha}} |f(\overline{n}_1\overline{n})|e^{-\alpha}|\overline{n}|^{-Q-1}d\overline{n},$$

which is dominated by $CM_0f(\overline{n}_1)$. This gives the weak type $(1,1)$ estimate.

Finally, we must verify (3.2) for $f \in L^1$. Observe first that (3.6) holds for a.a. k_1, $k_1 = k(\overline{n}_1)$, by the same argument as before. The following lemma will therefore end the proof of Theorem 1.

LEMMA 3.3. *Let $f \in L^1(K/M)$. Then*

$$\text{cpv}\int f(k(\overline{n}_1)\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n}$$

exists and (3.7) *holds for almost all $\overline{n}_1 \in \overline{N}$.*

PROOF. Take $\overline{n}_1 \in \overline{N}$ and write $\overline{n}_1 = k_1\exp(H_1)n_1 \in KAN$. Let $f_\epsilon = f\chi_\epsilon$, where $\epsilon > 0$ is small and χ_ϵ is the characteristic function of the set $\{k_1k(\overline{n})M : |\overline{n}| \geq \epsilon\} \subset K/M$. Since f_ϵ vanishes near $k_1 = k(\overline{n}_1)$, equation (3.7) applies to f_ϵ. This means that

$$\int_{|\overline{n}|>\epsilon} f(k_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n} = e^{<\lambda+\rho,H_1>}\int_U f(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n}, \tag{3.8}$$

where U is the set of $\overline{n}$ for which $k(\overline{n}_1\overline{n})M \notin \{k_1k(\overline{n}')M : \overline{n}' \in B(\epsilon)\}$ or equivalently $k(k_1^{-1}\overline{n}_1\overline{n})M \neq k(\overline{n}')M$ for all $\overline{n}' \in B(\epsilon)$. Clearly, U is all of $\overline{N}$ except a small neighbourhood of e.

To determine this neighbourhood, assume that $r = |\overline{n}|$ is small and write

$$k_1^{-1}\overline{n}_1\overline{n} = \exp(H_1)n_1\overline{n} = (n_1\overline{n}n_1^{-1})^{H_1}\exp(H_1)n_1.$$

Let $\overline{n} = \exp(Y_{-2} + Y_{-1})$ with $Y_{-j} \in \mathfrak{g}_{-j\alpha}$, so that $|Y_{-j}| \leq Cr^j$. If $n_1 = \exp(X_1 + X_2)$, $X_j \in \mathfrak{g}_{j\alpha}$, we have

$$\begin{aligned} n_1\overline{n}n_1^{-1} &= \exp(e^{ad(X_1+X_2)}(Y_{-2} + Y_{-1})) \\ &= \exp(Y_{-2} + Y_{-1} + [X_1, Y_{-2}] + R), \text{ with } R \in \bigoplus_{j=0}^{2} \mathfrak{g}_{j\alpha}. \end{aligned}$$

Further, $|R| \leq Cr$. There is a unique decomposition

$$n_1\overline{n}n_1^{-1} = \exp(Z_{-2} + Z_{-1})\exp(Z_0)\exp(Z_1 + Z_2)$$

with small $Z_j \in \mathfrak{g}_{j\alpha}$. Multiplying by means of the Campbell-Hausdorff formula, one easily finds $Z_{-2} = Y_{-2} + 0(r^3)$ and $Z_{-1} = Y_{-1} + 0(r^2)$. Now $Z_0 \in \mathbf{m} \oplus \mathbf{a}$ and $Z_1 + Z_2 \in \mathbf{n}$. Thus, $k_1^{-1}\overline{n}_1\overline{n} = \overline{n}'m'a'n'$ with $m' \in M$, $a' \in A$, $n' \in N$ and

$$\overline{n}' = \exp(e^{-2\alpha(H_1)}Z_{-2} + e^{-\alpha(H_1)}Z_{-1}) \in \overline{N}.$$

Since M normalizes N, this gives $k(k_1^{-1}n_1\overline{n}_1)M = k(\overline{n}')M$, and $\overline{n}'$ is the only element of $\overline{N}$ with this property. Notice that

$$|\overline{n}'|/|\overline{n}^{H_1}| = 1 + 0(r), \quad r \to 0,$$

and

$$|\overline{n}^{H_1}| = e^{-\alpha(H_1)}r.$$

We conclude that the symmetric difference

$$U\Delta(\overline{N}\backslash B(\epsilon e^{-\alpha(H_1)}))$$

is contained in the annulus

$$R_\epsilon = B((1 + C\epsilon)\epsilon e^{-\alpha(H_1)})\backslash B((1 - C\epsilon)\epsilon e^{-\alpha(H_1)}).$$

But if $\overline{n}_1$ is a Lebesgue point of $|f|$ with respect to the gauge, one easily gets

$$\int_{R_\epsilon} |f(\overline{n}_1\overline{n})||\overline{n}|^{-Q}d\overline{n} \to 0, \quad \epsilon \to 0.$$

Now (3.8) implies that for a.a. $\overline{n}_1$

$$(3.9) \qquad \int_{\overline{N}\backslash B(\epsilon)} f(k_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n}$$
$$= e^{<\lambda+\rho,H_1>} \int_{\overline{N}\backslash B(\epsilon e^{-\alpha(H_1)})} f(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n} + o(1), \quad \epsilon \to 0.$$

For almost all $\overline{n}_1$, we know that

$$\mathrm{cpv}\int f(\overline{n}_1\overline{n})|\overline{n}|^{-Q(1+i\gamma)}d\overline{n}$$

exists, which means that the value of the integral in the right-hand side of (3.9) describes an approximate circle in $\mathbb{C}$ as ϵ appproaches 0. The same must then be true of the left-hand integral. Since the centres of these circles are the central principal values of (3.7), the lemma follows.

4. - The result for trees

Let T be a homogeneous tree with branching degree $q+1 \geq 3$. We essentially follow the notation from [8], see also Figà-Talamanca and Picardello [2]. In particular, we fix a vertex $\circ \in T$ and identify any $x \in T$ with the shortest (geodesic) path from $\circ$ to x. The boundary Ω of T then consists of all infinite geodesic paths. If $x,\ x' \in T$, we denote by $N(x, x')$ the number of edges that x and x' have in common, and similarly for $N(x,\omega)$ and $N(\omega,\omega')$ with $\omega,\omega' \in \Omega$. One sets $|x| = N(x,x)$.

The sets

$$E_n(\omega) = \{\omega' \in \Omega : N(\omega,\omega') \geq n\}$$

define a basis of a topology in Ω. Similarly, one lets $E_n(x) = \{\omega \in \Omega : N(x,\omega) \geq n\}$. In particular, $E_n(x) = \emptyset$ for $n > |x|$. The disjoint union $T \cup \Omega$ also has a natural topology. If α is a nonnegative integer, an admissible approach region at $\omega \in \Omega$ is defined as

$$\Gamma_\alpha(\omega) = \{x \in T : N(x,\omega) \geq |x| - \alpha\}.$$

A complex-valued function f in Ω is said to be Hölder if it satisfies $|f(\omega) - f(\omega')| \leq \text{const } e^{-\epsilon N(\omega,\omega')}$ for some $\epsilon > 0$ and all $\omega,\ \omega' \in \Omega$.

The standard normalized measure ν in Ω satisfies $\nu(E_n(\omega)) = q^{1-n}/(q+1)$ for $n \geq 1$ and $\omega \in \Omega$.

The Poisson kernel of T is

$$K(x,\omega) = q^{2N(x,\omega)-|x|}, \quad x \in T, \quad \omega \in \Omega.$$

Let $z \in \mathbb{C}$. Any $f \in L^1(\nu)$ (and any martingale in Ω) has a z-Poisson integral

$$K_z f(x) = \int_\Omega K(x,\omega)^z f(\omega) d\nu(\omega), \quad x \in T.$$

The function $K_z f$ is an eigenfunction of the isotropic transition operator P in T, with eigenvalue $\gamma(z) = (q^z + q^{1-z})/(q+1)$.

Korànyi and Picardello [8] study the convergence of normalized z-Poisson integrals, defined for Re $z > 1/2$ and for $z = 1/2$ and $z = 1/2 + i\pi/\log\ q$ by $\mathcal{K}_z f = K_z f / K_z 1$. For these values of z, they prove that $\mathcal{K}_z f$ converges admissibly to f almost everywhere in Ω for $f \in L^1(\nu)$. When f is continuous in Ω, one can extend $\mathcal{K}_z f$ by f to a continuous functions in $T \cup \Omega$. Because of the properties of the expression for $\gamma(z)$, this takes care of all eigenvalues of P except those corresponding to Re $z = 1/2,\ 0 < |\text{Im } z| < \pi/\log\ q$.

Therefore, we shall have $z = (1+i\tau)/2$ in the sequel, with $0 < |\tau| < 2\pi/\log\ q$. The "spherical function" $K_z 1(x)$ then equals $\text{Re}(c_z q^{-z|x|})$, where

$$c_z = \frac{q}{q+1} \frac{1 - q^{i\tau - 1}}{1 - q^{i\tau}},$$

see [8] and [2, §3.2]. To avoid the zeroes of $K_z 1$, we define now $\mathcal{K}_z f(x) = K_z f(x)/(c_z q^{-z|x|})$. We shall obtain a formula for the asymptotic behaviour of $\mathcal{K}_z f(x)$ like (3.2). For this we need intertwining operators.

The mean values of a function $f \in L^1(\nu)$ are

$$E_n f(\omega) = \frac{1}{\nu(E_n(\omega))} \int_{E_n(\omega)} f \, d\nu, \quad \omega \in \Omega, \quad n = 0, 1, \cdots.$$

The differences of f are $\Delta_n f(\omega) = E_n f(\omega) - E_{n-1} f(\omega)$, $n \geq 0$, where $E_{-1} f \equiv 0$. Clearly,

$$f = \sum_0^\infty \Delta_n f \text{ a.e.}$$

An operator I_z is defined for $z \in \mathbb{C}$ by

$$I_z f = \sum_0^\infty c(n, z) \Delta_n f,$$

where $c(0, z) = 1$ and

$$c(n, z) = \frac{1 - q^{2(z-1)}}{1 - q^{-2z}} q^{(1-2z)n}, \quad n > 0.$$

For Re $z = 1/2$ we see that I_z is unitary in $L^2(\nu)$ and of weak type $(1, 1)$ for ν. When q is odd, T has a natural free group structure.

Then I_z intertwines representations π_z and π_{1-z} of T, see [2, §4.4]. These representations are unitary and belong to the principal series of T when Re $z = 1/2$.

THEOREM 4.1. *Let $z = (1+i\tau)/2$, $0 < |\tau| < 2\pi/\log q$, and take $f \in L^1(\nu)$ and $\alpha > 0$. Then for* a.a. $\omega_1 \in \Omega$

$$\mathcal{K}_z f(x) = (c_z^{-1} - 1) q^{i\tau|x|} f(\omega_1) + I_{\bar{z}} f(\omega_1) + o(1) \tag{4.1}$$

as $x \to \omega_1$, $x \in \Gamma_\alpha(\omega_1)$. If f is Hölder, (4.1) *holds as $x \in T$ approaches ω_1 in the topology of $T \cup \Omega$, uniformly for $\omega_1 \in \Omega$.*

PROOF. We have

$$\mathcal{K}_z f(x) = c_z^{-1} \int_\Omega q^{2N(x,\omega)z} f(\omega) d\nu(\omega)$$

$$= c_z^{-1} \sum_{n=0}^{|x|} q^{(1+i\tau)n} \int_{E_n(x) \setminus E_{n+1}(x)} f(\omega) d\nu(\omega).$$

If to begin with $x \in \Gamma_0(\omega_1)$, one gets

$$\int_{E_n(\omega)} f d\nu(\omega) = \frac{q}{q+1} q^{-n} E_n f(\omega_1)$$

for $n \leq |x|$, except that the factor $q/q+1$ must be deleted when $n = 0$.

It follows that

$$\mathcal{K}_z f(x) = c_z^{-1}\Big(\frac{q}{q+1}\Big(\sum_{n=1}^{|x|} q^{i\tau n} E_n f(\omega_1) - \sum_{n=0}^{|x|-1} q^{i\tau n-1} E_{n+1} f(\omega_1)\Big)$$

$$+ E_0 f(\omega_1)\Big) = c_z^{-1}\Big(\frac{q}{q+1}(1 - q^{-1-i\tau}) \sum_1^{|x|} q^{i\tau n} E_n f(\omega_1) + E_0 f(\omega_1)\Big).$$

Now we write $E_n f$ as $\sum_0^n \Delta_m f$ and change the order of summation:

$$\mathcal{K}_z f(x) = \sum_{m=1}^{|x|} c(m, \bar{z}) \Delta_m f(\omega_1) - \frac{1 - q^{-1-i\tau}}{1 - q^{-1+i\tau}} q^{i\tau} q^{i\tau|x|} \sum_{m=0}^{|x|} \Delta_m f(\omega_1)$$

$$+ \Big(\frac{1 - q^{-1-i\tau}}{1 - q^{-1+i\tau}} q^{i\tau} + c_z^{-1}\Big) \Delta_0 f(\omega_1).$$

The coefficients of $\Delta_0 f(\omega_1)$ here equals $1 = c(0, \bar{z})$, and we conclude that

$$\mathcal{K}_z f(x) = (c_z^{-1} - 1) q^{i\tau|x|} E_{|x|} f(\omega_1) + \sum_0^{|x|} c(m, \bar{z}) \Delta_m f(\omega_1) \tag{4.2}$$

for $x \in \Gamma_0(\omega_1)$.

By standard martingale theory, this implies (4.1) for a.a. ω_1 when $\alpha = 0$. If f is Hölder, the differences $\Delta_m f$ decrease exponentially in m, uniformly in Ω. Then (4.1) holds as $x \to \omega_1$ staying in $\Gamma_0(\omega_1)$, uniformly in ω_1. Since f and $I_{\bar{z}} f$ are now continuous in Ω, the last statement of Theorem 4.1 follows.

It remains to consider $f \in L^1$ and $\alpha > 0$. Approximating with Hölder or locally constant functions, we see that it is enough to show that the maximal operator

$$M_\alpha f(\omega_1) = \sup_{x \in \Gamma_\alpha(\omega_1)} |\mathcal{K}_z f(x)|$$

is of weak type $(1, 1)$ in Ω. Letting $x \in \Gamma_\alpha(\omega_1)$, we choose $\omega_2 \in \Omega$ with $x \in \Gamma_0(\omega_2)$. Then we apply (4.2) with ω_2 instead of ω_1 and estimate the two terms. Since $E_{|x|}(\omega_2) \subset E_{|x|-\alpha}(\omega_1)$, one has

$$|E_{|x|} f(\omega_2)| \leq C \ E_{|x|-\alpha} |f(\omega_1)|.$$

All constants C may depend on α. Further, $\Delta_m f(\omega_2)$ equals $\Delta_m f(\omega_1)$ for $m \leq |x| - \alpha$ and is dominated by $C \ E_{|x|-\alpha}|f(\omega_1)|$ for $|x| - \alpha < m \leq |x|$. As a result,

$$|\mathcal{K}_z f(x)| \leq | \sum_0^{|x|-\alpha} c(m, \bar{z}) \Delta_m f(\omega_1)| + C \ E_{|x|-\alpha} |f(\omega_1)|.$$

Thus $M_\alpha f$ is dominated by the maximal function of the martingale $\sum c(m, \bar{z}) \Delta_m f$ and the standard maximal function of f. This gives the weak type $(1, 1)$ estimate which ends the proof of Theorem 4.1.

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