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# On a Maximum Principle for Weak Solutions of the Stationary Stokes System

J. NAUMANN

## 1. - Introduction. Statement of the Result

Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain. We consider the homogeneous stationary Stokes system with unit viscosity:

$$(1.1) \quad -\Delta u + \nabla p = 0 \quad \text{in } \Omega,$$

$$(1.2) \quad \operatorname{div} u = 0 \quad \text{in } \Omega;$$

here  $u = \{u_1, u_2, u_3\}$  and  $p$  represent the velocity field of the flow, and the undetermined pressure, respectively ( $\nabla p = \{p_{x_1}, p_{x_2}, p_{x_3}\}$ <sup>1)</sup>).

By  $\partial\Omega$  we denote the boundary of  $\Omega$ . Without any further reference, throughout the whole paper we suppose that  $\partial\Omega \in C^2$  (cf. e.g. [8] for the definition). System (1.1), (1.2) will be completed by the boundary condition

$$(1.3) \quad u = f \quad \text{on } \partial\Omega$$

where  $f$  is a given vector field on  $\partial\Omega$ .

We introduce some notations used in what follows. Let  $D \subset \mathbb{R}^3$  be any bounded domain with Lipschitz boundary  $\partial D$  (cf. e.g. [8]). Then  $H^k(D) \equiv W_2^k(D)$  ( $k = 1, 2, \dots$ ) denotes the usual Sobolev space of all functions in  $L^2(D)$  having their generalized derivatives up to order  $k$  (including) in  $L^2(D)$ . Further, let

$$H_0^1(D) = \{v \in H^1(D) : v = 0 \text{ a.e. on } \partial D\},$$

$$H^1(D; \mathbb{R}^3) = [H^1(D)]^3$$

$$H_0^1(D; \mathbb{R}) = [H_0^1(D)]^3$$

and

$$V(D) = \{v \in H_0^1(D; \mathbb{R}^3) : \operatorname{div} v = 0 \text{ a.e. in } D\}.$$

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<sup>1)</sup>  $\varphi_{x_i} = \frac{\partial \varphi}{\partial x_i}$  (with respect to a Cartesian frame;  $i=1,2,3$ ).

In order to define the notion of weak solution of (1.1)-(1.3) let  $f \in H^{1/2}(\partial\Omega; \mathbb{R}^3) \equiv [W_2^{1/2}(\partial\Omega)]^3$  <sup>2)</sup> be given such that

$$\int_{\partial\Omega} f_i n_i dS = 0$$

( $n = \{n_1, n_2, n_3\}$  = outward unit normal along  $\partial\Omega$ ).

The function  $u \in H^1(\Omega; \mathbb{R}^3)$  is called a *weak solution* of (1.1)-(1.3) if

$$(1.4) \quad \int_{\Omega} \nabla u_i \cdot \nabla \varphi_i dx = 0 \quad \forall \varphi \in V(\Omega),$$

$$(1.5) \quad \operatorname{div} u = 0 \quad \text{a.e. in } \Omega,$$

$$(1.6) \quad u = f \quad \text{a.e. on } \partial\Omega.$$

It is well-known that the above conditions on  $f$  guarantee the existence and uniqueness of a weak solution of (1.1)-(1.3) (cf. e.g. [6]). Furthermore, (1.4) implies the existence of an element  $\hat{p} \in L^2(\Omega)/\mathbb{R}$  such that

$$(1.4') \quad \int_{\Omega} \nabla u_i \cdot \nabla \chi_i dx = \int_{\Omega} p \operatorname{div} \chi dx \quad \forall \chi \in H_0^1(\Omega; \mathbb{R}^3), \quad \forall p \in \hat{p}$$

(cf. [5], [7], [11]). In addition, there holds  $u \in [C^\infty(\Omega)]^3$  and  $p \in C^\infty(\Omega)$  (for all  $p \in \hat{p}$ ) (cf. [6], [7]).

The aim of the present paper is to prove a global  $L^\infty$ -bound on the Euclidean norm of the weak solution of (1.1)-(1.3) in terms of  $f$ . We follow an idea of Cannarsa [4] and make essential use of results by Giaquinta, Modica [5] and Solonnikov, Ščadilov [11]. Moreover, our approach gives an additional information on  $p$  near the boundary  $\partial\Omega$  ( $p$  according to (1.4'); cf. (3.2) below).

For any  $\xi \in \mathbb{R}^3$ , let

$$B_r(\xi) = \{x \in \mathbb{R}^3 : |x - \xi| < r\}.$$

The main result of our paper is the following

**THEOREM.** *Let  $f \in H^1(\Omega; \mathbb{R}^3)$ . Let  $\operatorname{div} f = 0$  a.e. in  $\Omega$ , and let there exist an  $0 < R_0 \leq \operatorname{diam} \Omega$  such that*

$$(1.7) \quad \Lambda_1 := \operatorname{ess\,sup}_{\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < R_0\}} |f|^2 < +\infty,$$

<sup>2)</sup> Cf. e.g. [8] for a discussion of the spaces  $W_p^r(\partial\Omega)$  ( $1 \leq p < +\infty$ ,  $0 < r < +\infty$ ). In what follows, we do not make, however, any explicit use of these spaces. Throughout repeated Latin subscripts imply summation over 1,2,3.

$$(1.8) \quad \Lambda_2 := \sup \left\{ \frac{1}{r} \int_{B_r(\xi) \cap \Omega} |\nabla f|^2 dx \mid 0 < r \leq R_0, \xi \in \partial\Omega \right\} < +\infty.$$

Let  $u \in H^1(\Omega; \mathbb{R}^3)$  be the weak solution of (1.1)-(1.3). Then

$$(1.9) \quad \operatorname{ess\,sup}_{\Omega} |u|^2 \leq c \left( \Lambda_1 + \Lambda_2 + \int_{\Omega} (|f|^2 + |\nabla f|^2) dx \right)$$

where the constant  $c$  depends on geometric properties of  $\partial\Omega$  only.

The paper is organized as follows. Section 2 is devoted to the proof of an inequality on the weak solution of the Stokes system in a semi-ball. This inequality is of an independent interest; it relies essentially on the square integrability of the second order derivatives of the solution near the base of the semi-ball, which we are going to prove in the appendix. The proof of our main theorem is then given in the third and fourth section.

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## 2. - The Stokes System in a Semi-Ball

Let

$$B_r^+ = B_r^+(0) = \{x \in \mathbb{R}^3 : |x| < r, x_3 > 0\}.$$

Suppose we are given a function  $w \in H^1(B_r^+; \mathbb{R}^3)$  satisfying

$$(2.1) \quad \operatorname{div} w = 0 \text{ a.e. in } B_r^+, \quad w = 0 \text{ a.e. on } \partial B_r^+ \cap \{x_3 = 0\}.$$

By the Lax-Milgram lemma, there exists a uniquely determined function  $U \in H^1(B_r^+; \mathbb{R}^3)$  such that

$$(2.2) \quad \int_{B_r^+} \nabla U_i \cdot \nabla \varphi_i dx = 0 \quad \forall \varphi \in V(B_r^+),$$

$$(2.3) \quad \operatorname{div} U = 0 \quad \text{a.e. in } B_r^+,$$

$$(2.4) \quad U = w \quad \text{a.e. on } \partial B_r^+.$$

As above, (2.2) implies the existence of an element  $\hat{q} \in L^2(B_r^+)/\mathbb{R}$  such that, for any  $q \in \hat{q}$ ,

$$(2.2') \quad \int_{B_r^+} \nabla U_i \cdot \nabla \chi_i dx = \int_{B_r^+} (q - q_{B_r^+}) \operatorname{div} \chi dx \quad \forall \chi \in H_0^1(B_r^+; \mathbb{R}^3).$$

In addition, there holds

$$(2.5) \quad \int_{B_r^+} (q - q_{B_r^+})^2 dx \leq c_0 \int_{B_r^+} |\nabla U|^2 dx$$

where

$$q_{B_r^+} = \frac{3}{2\pi r^3} \int_{B_r^+} q dy$$

and  $c_0$  is an absolute constant. Indeed,  $B_r^+$  being star-shaped with respect to any interior point of it, there exists a  $\zeta \in H_0^1(B_r^+; \mathbb{R}^3)$  such that

$$\operatorname{div} \zeta = q - q_{B_r^+} \text{ a.e. in } B_r^+, \quad \int_{B_r^+} |\nabla \zeta|^2 dx \leq c_0 \int_{B_r^+} q^2 dx$$

(cf. [1]). By a homothetical argument, the constant  $c_0$  can be easily seen to be independent of  $r$ . Now, letting  $\chi = \zeta$  in (2.2') gives (2.5).

The proof of our main result is based on the estimate (2.6) below.

**PROPOSITION** (Campanato type estimate). *Let  $U \in H^1(B_r^+; \mathbb{R}^3)$  satisfy (2.2)-(2.4). Then*

$$(2.6) \quad \int_{B_\rho^+} |\nabla U|^2 dx \leq c \left(\frac{\rho}{r}\right)^2 \int_{B_r^+} |\nabla U|^2 dx \quad \forall 0 < \rho \leq r$$

with  $c = \text{const}$  independent of both  $\rho$  and  $r$ .

**REMARK.** Estimates of the type (2.6) [with  $\left(\frac{\rho}{r}\right)^3$  in place of  $\left(\frac{\rho}{r}\right)^2$ ; more general, with  $\left(\frac{\rho}{r}\right)^n$  when  $\mathbb{R}^n$  is the underlying space] have been firstly proved in [2] for weak solutions of homogeneous linear elliptic equations with constant coefficients (cf. [3] for a detailed discussion of estimates of this type).

We note that estimate (2.6) with  $\left(\frac{\rho}{r}\right)^3$  can be proved when the third order derivatives of  $U$  are in  $L^2$  near the boundary  $\partial B_r^+ \cap \{x_3 = 0\}$  and appropriate estimates on these derivatives are available (cf. (2.8) below). However, (2.6) is sufficient for our later purposes.

**PROOF OF THE PROPOSITION.** We begin by observing that

$$(2.7) \quad U_{i x_k x_l}, q_{x_k} \in L^2(B_{r/4}^+),$$

$$(2.8) \quad \int_{B_{r/4}^+} [(U_{ix_k x_l})^2 + (q_{x_k})^2] dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx$$

( $i, k, l = 1, 2, 3$ ;  $c = \text{const}$  independent of  $r$ ). The proof of (2.7) and (2.8) will be given in the appendix.

Estimate (2.6) is now easily deduced from (2.8). Indeed, let  $0 < \rho \leq \frac{r}{4}$ . By Hölder's inequality and Sobolev's imbedding theorem,

$$\begin{aligned} \int_{B_\rho^+} |\nabla U|^2 dx &\leq \left(\frac{2\pi}{3}\right)^{2/3} \rho^2 \left(\int_{B_\rho^+} |\nabla U|^6 dx\right)^{1/3} \\ &\leq c\rho^2 \left(\frac{1}{r^2} \int_{B_{r/4}^+} |\nabla U|^2 dx + \sum_{i,k,l=1}^3 \int_{B_{r/4}^+} (U_{ix_k x_l})^2 dx\right) \end{aligned}$$

where the constant  $c$  is independent of both  $\rho$  and  $r$ <sup>3)</sup>. This can be readily seen by a homothetical argument. Thus, by (2.8),

$$\int_{B_\rho^+} |\nabla U|^2 dx \leq c \left(\frac{\rho}{r}\right)^2 \int_{B_r^+} |\nabla U|^2 dx.$$

This inequality is trivial for  $\frac{r}{4} < \rho \leq r$ . Whence (2.6).

### 3. - Proof of the Theorem

We begin by proving the following statement which is of an independent interest:

*Let  $f \in H^1(\Omega; \mathbb{R}^3)$  with  $\text{div } f = 0$  a.e. in  $\Omega$ , and let  $u \in H^1(\Omega; \mathbb{R}^3)$  be the weak solution of (1.1)-(1.3). Suppose there exist constants  $0 < R_0 \leq \text{diam } \Omega$  and  $0 < \lambda < 2$  such that*

$$(*) \quad \Lambda := \sup \left\{ \frac{1}{r^\lambda} \int_{B_r(\xi) \cap \Omega} |\nabla f|^2 dx \mid 0 < r \leq R_0, \xi \in \partial\Omega \right\} < +\infty.$$

*Then there exists an  $0 < R_1 \leq R_0$  and a constant  $c > 0$  which both depend*

<sup>3)</sup> By  $c$  we denote different positive constants possibly changing their numerical value from line to line.

on  $\lambda$  and on geometric properties of  $\partial\Omega$  only, such that

$$(3.1) \quad \int_{B_r(\xi) \cap \Omega} |\nabla(u-f)|^2 dx \leq c \left( \Lambda + \int_{\Omega} |\nabla f|^2 dx \right) r^\lambda \quad \forall 0 < r \leq R_1, \quad \forall \xi \in \partial\Omega.$$

REMARK. Let  $x \in \Omega$  and  $r = \text{dist}(x, \partial\Omega) \leq \frac{R_1}{2}$ . Let  $p \in L^2(\Omega)$  satisfy (1.4'). Then

$$(3.2) \quad \int_{B_r(x)} (p - p_{B_r(x)})^2 dy \leq c \left( \Lambda_2 + \int_{\Omega} |\nabla f|^2 dx \right) r$$

where

$$p_{B_r(x)} = \frac{3}{4\pi r^3} \int_{B_r(x)} p dy$$

and  $c = \text{const}$  independent of  $x$  and  $r$ .

Indeed, there exists an  $\eta \in H_0^1(B_r(x); \mathbb{R}^3)$  such that

$$\begin{aligned} \text{div } \eta &= p - p_{B_r(x)} \text{ a.e. in } B_r(x), \\ \int_{B_r(x)} |\nabla \eta|^2 dy &\leq c_0 \int_{B_r(x)} (p - p_{B_r(x)})^2 dy \end{aligned}$$

with an absolute constant  $c_0$  (cf. [1], [10]). Let  $\chi = \eta$  a.e. in  $B_r(x)$  and  $\chi = 0$  a.e. in  $\Omega \setminus B_r(x)$ . Then  $\chi \in H_0^1(\Omega; \mathbb{R}^3)$ , and (1.4') implies

$$\int_{B_r(x)} (p - p_{B_r(x)})^2 dy \leq c \int_{B_r(x)} |\nabla u|^2 dy.$$

Let  $\xi \in \partial\Omega$  satisfy  $|\xi - x| = r$ . Clearly,  $B_r(x) \subset B_{2r}(\xi) \cap \Omega$ , and (3.2) follows by combining (3.1) (with  $\Lambda = \Lambda_2$  (from (1.8)) and  $\lambda = 1$ ) and the latter estimate.

We divide the proof of (3.1) into four steps.

1° Let  $\xi \in \partial\Omega$  be arbitrary. We introduce Cartesian coordinates  $y = \{y_1, y_2, y_3\}$  by

$$y = A(x - \xi)$$

where the direction of the negative  $y_3$ -axis coincides with the direction of the outward normal (with respect to  $\Omega$ ) at  $\xi$ , and  $A = \{a_{ij}\}$  is an orthogonal matrix (with  $a_{ij}$  depending on  $\xi$ ).

Our assumption  $\partial\Omega \in C^2$  guarantees the existence of a real  $\sigma = \sigma(\xi) > 0$  and a function  $F = F(\xi) \in C^2(\Delta_\sigma)$  ( $\Delta_\sigma = [-\sigma, \sigma] \times [-\sigma, \sigma]$ ) such that

$$\{y \in \mathbb{R}^3 : \{y_1, y_2\} \in \Delta_\sigma, y_3 = F(y_1, y_2)\} \subset \partial\Omega,$$

$$\{y \in \mathbb{R}^3 : \{y_1, y_2\} \in \Delta_\sigma, F(y_1, y_2) < y_3 \leq F(y_1, y_2) + \sigma\} \subset \Omega,$$

$$(3.3) \quad \begin{cases} F(0, 0) = 0, \quad \nabla F(0, 0) = 0, \\ |\nabla F(y_1, y_2)| + \sum_{\alpha, \beta=1}^2 |F_{y_\alpha y_\beta}(y_1, y_2)| \leq M = \text{const} \quad \forall \{y_1, y_2\} \in \Delta_\sigma. \end{cases}$$

Now, for all  $\xi \in \partial\Omega$ , the reals  $\sigma = \sigma(\xi)$  are uniformly bounded from below by a fixed positive constant, while the constants  $M$  (possibly depending on  $\xi$ ) are uniformly bounded from above by a fixed constant. This can be established by the aid of the compactness of  $\partial\Omega$ . Thus, in all that follows, both  $\sigma$  and  $M$  are assumed to be independent of  $\xi \in \partial\Omega$ .

Set  $\bar{u} = u - f$  a.e. in  $\Omega$ . Then from (1.4) we get

$$(3.4) \quad \int_{\Omega} |\nabla \bar{u}|^2 dx \leq \int_{\Omega} |\nabla f|^2 dx,$$

$$(3.5) \quad \int_{B_\sigma(\xi) \cap \Omega} \nabla \bar{u}_i \cdot \nabla \varphi_i dx = - \int_{B_\sigma(\xi) \cap \Omega} \nabla f_i \cdot \nabla \varphi_i dx \quad \forall \varphi \in V(B_\sigma(\xi) \cap \Omega).$$

Next, for any  $0 < r \leq \sigma$  let

$$C_r(0) = \{y \in \mathbb{R}^3 : |y| < r, y_3 > F(y_1, y_2)\}.$$

The orthogonality of  $A$  implies  $B_r(\xi) \cap \Omega = C_r(0)$ .

We introduce functions  $v$  and  $g$  on  $C_\sigma(0)$  by setting

$$v(y) = A\bar{u}(x), \quad g(y) = Af(x) \quad \text{for a.a. } y \in C_\sigma(0).$$

Then (3.5) takes the form

$$(3.6) \quad \int_{C_\sigma(0)} \nabla v_i \cdot \nabla \chi_i dy = - \int_{C_\sigma(0)} \nabla g_i \cdot \nabla \chi_i dy \quad \forall \chi \in V(C_\sigma(0)).$$

Further,

$$\text{div } v = 0 \text{ a.e. in } C_\sigma(0), \quad v = 0 \text{ a.e. on } \partial C_\sigma(0) \cap \{y_3 = F(y_1, y_2)\},$$

$$(3.7) \quad \int_{B_r(\xi) \cap \Omega} |\nabla \bar{u}|^2 dx = \int_{C_r(0)} |\nabla v|^2 dy,$$

$$(3.8) \quad \int_{B_r(\xi) \cap \Omega} |\nabla f|^2 dx = \int_{C_r(0)} |\nabla g|^2 dy$$



for all  $0 < r \leq \sigma$ .

2° We introduce new variables  $z = \{z_1, z_2, z_3\}$  by the transformation

$$z = T(y) = \{y_1, y_2, y_3 - F(y_1, y_2)\}, \quad y \in C_\sigma(0).$$

Clearly,  $T$  is a one-to-one mapping (with Jacobian  $\equiv 1$ ) from  $C_\sigma(0)$  onto  $D_\sigma = T(C_\sigma(0))$ .

Define

$$\delta := (1 + \max\{1, 2M^2\})^{1/2} (M \text{ according to (3.3)}), \quad r_1 := \frac{\sigma}{\delta},$$

$$B_{r_1}^+ := \{z \in \mathbb{R}^3 : |z| < r_1, z_3 > 0\}.$$

Then  $B_{r_1}^+ \subset D_\sigma$ . Indeed,  $z \in B_{r_1}^+$  implies  $\{z_1, z_2\} \in \Delta_\sigma$ . Letting denote  $y_1 = z_1, y_2 = z_2, y_3 = z_3 + F(z_1, z_2)$  we have  $y_3 > F(y_1, y_2)$  and

$$|y|^2 \leq |z|^2 + z_3^2 + 2(F(z_1, z_2))^2 \leq |z|^2 + \max\{1, 2M^2\}|z|^2,$$

i.e.  $y \in C_\sigma(0)$  and therefore  $z = T(y) \in D_\sigma$ . Furthermore, a simple calculation shows

$$(3.9) \quad \partial(T^{-1}(B_{r_1}^+)) = T^{-1}(\partial B_{r_1}^+).$$

Now we introduce new functions  $w$  and  $h$  by

$$w_\alpha(z) = v_\alpha(y) \quad (\alpha = 1, 2), \quad w_3(z) = v_3(y) - F_{y_\beta}(y_1, y_2)v_\beta(y) \quad ^4),$$

$$h(z) = g(y)$$

for a.a.  $y \in C_\sigma(0)$  ( $z = T(y)$ ) (cf. [11]). Then

$$w_{\alpha z_\beta} = v_{\alpha y_\beta} + F_{y_\beta} v_{\alpha y_3}, \quad w_{\alpha z_3} = v_{\alpha y_3},$$

$$w_{3z_\alpha} = v_{3y_\alpha} + F_{y_\alpha} v_{3y_3} - F_{y_\alpha y_\gamma} v_\gamma - F_{y_\gamma} (v_{\gamma y_\alpha} + F_{y_\alpha} v_{\gamma y_3}),$$

$$w_{3z_3} = v_{3y_3} - F_{y_\gamma} v_{\gamma y_3} \quad (\alpha, \beta = 1, 2).$$

Thus,  $w \in H^1(D_\sigma; \mathbb{R}^3)$ ,  $\operatorname{div} w = 0$  a.e. in  $D_\sigma$  and  $w = 0$  a.e. on  $\partial D_\sigma \cap \{z_3 = 0\}$ . Analogously,  $h \in H^1(D_\sigma; \mathbb{R}^3)$ .

Let  $\psi \in V(B_{r_1}^+)$  be arbitrary. Set

$$\chi_\alpha(y) = \psi_\alpha(z) \quad (\alpha = 1, 2), \quad \chi_3(y) = \psi_3(z) + F_{z_\beta}(z_1, z_2)\psi_\beta(z)$$

for a.a.  $z \in B_{r_1}^+$  ( $y = T^{-1}(z)$ ). As above,  $\chi \in H^1(T^{-1}(B_{r_1}^+); \mathbb{R}^3)$  and  $\operatorname{div} \chi = 0$  a.e. in  $T^{-1}(B_{r_1}^+)$ . By (3.9),  $\chi = 0$  a.e. on  $\partial(T^{-1}(B_{r_1}^+))$ . Hence  $\chi \in V(T^{-1}(B_{r_1}^+))$ .

<sup>4)</sup> Repeated Greek subscripts imply summation over 1 and 2.

We extend  $\chi$  by zero onto  $C_\sigma(0) \setminus T^{-1}(B_{r_1}^+)$  and obtain an admissible test function for (3.6). This gives

$$(3.10) \quad \int_{B_{r_1}^+} \nabla w_i \cdot \nabla \psi_i dz = \int_{B_{r_1}^+} A(w, \psi) dz + \int_{B_{r_1}^+} B(h, \psi) dz$$

where

$$\begin{aligned} A(w, \psi) &= A_{kl}^{ij} w_{iz}, \psi_{kz_l} + A_\alpha^{ij} (w_{iz}, \psi_\alpha + w_\alpha \psi_{iz}) + F_{z_\alpha z_\gamma} F_{z_\beta z_\gamma} w_\alpha \psi_\beta, \\ B(h, \psi) &= -\nabla h_i \cdot \nabla \psi_i + B_{kl}^{ij} h_{iz}, \psi_{kz_l} + F_{z_\alpha z_\beta} (h_{3z_\alpha} - F_{z_\alpha} h_{3z_\beta}) \psi_\beta. \end{aligned}$$

Here  $A_{kl}^{ij} = B_{kl}^{ij} \equiv 0$  if  $i + j + k + l \leq 5$ , while the coefficients  $A_{kl}^{ij}$  and  $B_{kl}^{ij}$  with  $6 \leq i + j + k + l \leq 12$  (at least one index = 3) are of the form  $\pm F_{z_\alpha}$ ,  $\pm F_{z_\alpha} F_{z_\beta}$ ,  $\pm F_{z_\alpha} F_{z_\beta} F_{z_\gamma}$  or  $F_{z_\alpha} F_{z_\beta} |\nabla F|^2$ , respectively (e.g.  $A_{\alpha 3}^{\alpha\beta} = -F_{z_\beta}$ ,  $A_{\beta 3}^{\alpha 3} = F_{z_\alpha} F_{z_\beta} |\nabla F|^2$  ( $\alpha, \beta = 1, 2$ )); the coefficients  $A_\alpha^{ij}$  are composed by the functions  $F_{z_\alpha z_\beta}$ ,  $\pm F_{z_\alpha} F_{z_\beta z_\gamma}$  or  $F_{z_\alpha} F_{z_\beta} F_{z_\beta z_\gamma}$ , respectively. Thus,  $A_{kl}^{ij}$ ,  $B_{kl}^{ij}$  and  $A_\alpha^{ij}$  are continuous functions on  $\Delta_\sigma$  and the following estimates hold:

$$(3.11) \quad |A_{kl}^{ij}|, |B_{kl}^{ij}| \leq c_0 (1 + |\nabla F| + |\nabla F|^2 + |\nabla F|^3) |\nabla F|,$$

$$(3.12) \quad |A_\alpha^{ij}| \leq c_0 (1 + |\nabla F| + |\nabla F|^2) \sum_{\beta, \gamma=1}^2 |F_{z_\beta z_\gamma}|$$

for all  $z_1^2 + z_2^2 \leq r_1^2$  ( $i, j, k, l = 1, 2, 3, \alpha = 1, 2$ ;  $c_0 = \text{const}$ ).

3° Let  $0 < r \leq r_1$  be arbitrary (recall that  $r_1 = \sigma(1 + \max\{1, 2M^2\})^{-1/2}$ ). Let  $U \in H^1(B_r^+; \mathbb{R}^3)$  denote the uniquely determined solution of (2.2)-(2.4) [ $w = v \circ T^{-1}$  in (2.4)]. Then

$$(3.13) \quad \int_{B_\rho^+} |\nabla w|^2 dz \leq 4c_0 \left(\frac{\rho}{r}\right)^2 \int_{B_r^+} |\nabla w|^2 dz + 2(1 + 2c_0) \int_{B_r^+} |\nabla(w - U)|^2 dz$$

for all  $0 < \rho \leq r$  where  $c_0$  is the constant occurring in (2.6).

The function

$$\psi = \begin{cases} w - U & \text{a.e. in } B_r^+, \\ 0 & \text{a.e. in } B_{r_1}^+ \setminus B_r^+ \end{cases}$$

is admissible in (3.10). Adding (3.10) and (2.2) with  $\varphi = w - U$  we find

$$(3.14) \quad \begin{aligned} \int_{B_r^+} |\nabla(w - U)|^2 dz &= \int_{B_r^+} A(w, w - U) dz + \int_{B_r^+} B(h, w - U) dz \\ &= I_1 + I_2. \end{aligned}$$

In order to estimate  $I_1$  we first note that  $|\nabla F| \leq c(M)(z_1^2 + z_2^2)^{1/2}$ <sup>5)</sup> for all  $z_1^2 + z_2^2 \leq r_1^2$  (cf. (3.3)). Thus, by (3.11),

$$\begin{aligned} & \left| \int_{B_r^+} A_{ki}^{ij} w_{iz} (w - U)_{kz} dz \right| \\ & \leq \frac{1}{8} \int_{B_r^+} |\nabla(w - U)|^2 dz + c(M)r^2 \int_{B_r^+} |\nabla w|^2 dz \end{aligned}$$

[for what follows it is decisive that the factor of  $\int_{B_r^+} |\nabla w|^2 dz$  can be made arbitrarily small to obtain (3.16) below; this explains the introduction of the coordinate system  $y = A(x - \xi)$  at each  $\xi \in \partial\Omega$ ]. The estimation of the remaining two integrals forming  $I_1$ , is readily seen when taking into account (3.3), (3.12) and  $w - U = 0$  a.e. on  $\partial B_r^+$ . Thus,

$$I_1 \leq \frac{1}{4} \int_{B_r^+} |\nabla(w - U)|^2 dz + c(M)r^2 \int_{B_r^+} |\nabla w|^2 dz.$$

Next, using (3.3) and (3.11) one easily obtains

$$I_2 \leq \frac{1}{4} \int_{B_r^+} |\nabla(w - U)|^2 dz + c(\sigma, M) \int_{B_r^+} |\nabla h|^2 dz$$

( $0 < r \leq r_1$ ). Inserting these estimates into (3.14) and combining this result with (3.13) we find

$$(3.15) \quad \int_{B_\rho^+} |\nabla w|^2 dz \leq c(M) \left[ \left( \frac{\rho}{r} \right)^2 + r^2 \right] \int_{B_r^+} |\nabla w|^2 dz + c(\sigma, M) \int_{B_r^+} |\nabla h|^2 dz$$

for all  $0 < \rho \leq r \leq r_1$ .

It remains to estimate the second integral on the right of (3.15). To this end, we note that  $T^{-1}(B_r^+) \subset C_{\delta r}(0)$  [for  $z \in B_r^+$  implies  $|T^{-1}(z)|^2 \leq |z|^2 + z_3^2 + 2(F(z_1, z_2))^2 \leq \delta^2 |z|^2$ ]. Therefore,

$$\int_{B_r^+} |\nabla h|^2 dz = \int_{T^{-1}(B_r^+)} |\nabla h(T(y))|^2 dy \leq \int_{C_{\delta r}(0)} |\nabla h(T(y))|^2 dy.$$

On the other hand, from  $h(z) = g(y)$  ( $z = T(y)$ ) we infer that  $|\nabla h(z)| \leq c_0(1 + \max_{\Delta_\sigma} |\nabla F|) |\nabla g(y)|$  for a.a.  $y \in C_\sigma(0)$  ( $c_0 = \text{const}$ ). Thus, by (\*), (3.3)

<sup>5)</sup> In what follows, we denote by  $c(M)$  (resp.  $c(\sigma, M)$ ) different positive constants which only depend on  $M$  (resp.  $\sigma$  and  $M$ ).

and (3.8),

$$\int_{B_r^+} |\nabla h|^2 dz \leq c(M) \int_{B_{\delta r}(\xi) \cap \Omega} |\nabla f|^2 dx \leq c(M) \Lambda (\delta r)^\lambda$$

for all  $0 < r \leq \min \left\{ \frac{R_0}{\delta}, r_1 \right\}$  <sup>6)</sup>.

Now, (3.15) gives

$$\int_{B_\rho^+} |\nabla w|^2 dz \leq c(M) \left[ \left( \frac{\rho}{r} \right)^2 + r^2 \right] \int_{B_r^+} |\nabla w|^2 dz + c(\sigma, \lambda, M) \Lambda r^\lambda$$

for all  $0 < \rho \leq r \leq \min \left\{ \frac{R_0}{\delta}, r_1 \right\}$ . Hence there exists an  $0 < r_2 \leq \min \left\{ \frac{R_0}{\delta}, r_1 \right\}$  such that

$$(3.16) \quad \int_{B_r^+} |\nabla w|^2 dz \leq c(\sigma, \lambda, M) \left( \frac{1}{r_2} \int_{B_{r_2}^+} |\nabla w|^2 dz + \Lambda \right) r^\lambda$$

for all  $0 < r \leq r_2$  (cf. e.g. [5; Lemma 0.6]). Here  $r_2$  only depends on  $M$  via  $c(M)$ .

4° In (3.16) we return from  $w$  to  $u$ . To begin with, we note that  $T(C_r(0)) \subset B_{\delta r}^+$  for any  $0 < r \leq \frac{r_1}{\delta}$  ( $= \sigma(1 + \max\{1, 2M^2\})^{-1}$ ). Observing that  $|\nabla v(y)| \leq c(M)(|w(z)| + |\nabla w(z)|)$  for a.a.  $z \in B_{r_1}^+$  ( $y = T^{-1}(z)$ ) we get by virtue of (3.7)

$$(3.17) \quad \begin{aligned} \int_{B_r(\xi) \cap \Omega} |\nabla \bar{u}|^2 dx &= \int_{T(C_r(0))} |\nabla v(T^{-1}(z))|^2 dz \\ &\leq c(M) \int_{B_{\delta r}^+} (|w|^2 + |\nabla w|^2) dz \\ &\leq c(\sigma, M) \int_{B_{\delta r}^+} |\nabla w|^2 dz \end{aligned}$$

for all  $0 < r \leq \frac{r_1}{\delta}$  [note that  $w = 0$  a.e. on  $\partial B_{\delta r}^+ \cap \{z_3 = 0\}$ ,  $\frac{r_1}{\delta} < \sigma$ ].

Next, we have  $T^{-1}(B_r^+) \subset C_{\delta r}(0)$  for all  $0 < r \leq r_1$ ,  $|\nabla w(z)| \leq c(M)(|v(y)| + |\nabla v(y)|)$  for a.a.  $y \in C_\sigma(0)$  ( $z = T(y)$ ) and  $v = 0$  a.e. on

<sup>6)</sup> We emphasize that the components  $a_i$ , of the matrix  $A$  occurring in  $y=A(x-\xi)$ , do not explicitly enter into (3.7) and (3.8). Therefore, all estimates in step 4° are independent of the  $a_i$ 's and thus on  $\xi \in \partial\Omega$ , too.

$C_\sigma(0) \cap \{y_3 = F(y_1, y_2)\}$ . Thus, (3.7) and (3.4) imply

$$\begin{aligned}
 \int_{B_{r_2}^+} |\nabla w|^2 dz &= \int_{T^{-1}(B_{r_2}^+)} |\nabla w(T(y))|^2 dy \\
 &\leq c(\sigma, M) \int_{C_{\delta r_2}(0)} |\nabla v|^2 dy \\
 &\leq c(\sigma, M) \int_{B_{\delta r_2}(\xi) \cap \Omega} |\nabla \bar{u}|^2 dx \\
 (3.18) \qquad &\leq c(\sigma, M) \int_{\Omega} |\nabla f|^2 dx
 \end{aligned}$$

[for  $r_2 \leq \min \left\{ \frac{R_\Omega}{\delta}, r_1 \right\}$ , i.e.  $\delta r_2 \leq \delta r_1 = \sigma$ ].

Combining (3.16) with (3.17) and (3.18) one finally obtains

$$\begin{aligned}
 \int_{B_r(\xi) \cap \Omega} |\nabla \bar{u}|^2 dx &\leq c(\sigma, M) \left( \frac{1}{r_2} \int_{B_{r_2}^+} |\nabla w|^2 dz + \Lambda \right) (\delta r)^\lambda \\
 &\leq c(\sigma, \lambda, M) \left( \int_{\Omega} |\nabla f|^2 dx + \Lambda \right) r^\lambda
 \end{aligned}$$

for all  $0 < r \leq \frac{r_2}{\delta}$  [ $r_2 = r_2(M)$  being fixed]. Thus, (3.1) is satisfied with  $R_1 := \frac{r_2}{\delta}$ .

#### 4. - Proof of the Theorem completed

Let  $x \in \Omega$  be arbitrary. Let  $d = \text{dist}(x, \partial\Omega)$ . Then there holds

$$(4.1) \qquad \int_{B_\rho(x)} |u|^2 dy \leq c_0 \left( \frac{\rho}{d} \right)^3 \int_{B_d(x)} |u|^2 dy \quad \forall 0 < \rho \leq d$$

with  $c_0$  an absolute constant (cf. [5; Prop. 1.9]).

We distinguish two cases.

(i)  $d \geq \frac{R_1}{2}$  ( $R_1$  according to (3.1) with  $\Lambda = \Lambda_2$  (from (1.8)) and  $\lambda = 1$ ).

Then (4.1) combined with (3.4) gives

$$\begin{aligned} \int_{B_\rho(x)} |u|^2 dy &\leq 16c_0 R_1^{-3} \rho^3 \int_{\Omega} (|f|^2 + |u - f|^2) dy \\ &\leq c\rho^3 \int_{\Omega} (|f|^2 + |\nabla f|^2) dy \end{aligned}$$

where the constant  $c$  depends on  $\Omega$  only.

(ii)  $d < \frac{R_1}{2}$ . There exists a  $\xi \in \partial\Omega$  such that  $|\xi - x| = d$ . Following [4] we combine (1.7) and (3.1) to obtain

$$\begin{aligned} \int_{B_d(x)} |u|^2 dy &\leq \frac{8}{3}\pi d^3 \operatorname{ess\,sup}_{B_d(x)} |f|^2 + 2 \int_{B_d(x)} |u - f|^2 dy \\ &\leq \frac{8}{3}\pi d^3 \Lambda_1 + c_0 d^2 \int_{B_{2d}(\xi) \cap \Omega} |\nabla(u - f)|^2 dy \\ &\leq c(\sigma, M) \left( \Lambda_1 + \Lambda_2 + \int_{\Omega} |\nabla f|^2 dy \right) d^3, \end{aligned}$$

$c_0$  being an absolute constant.

Thus, in both cases,

$$\int_{B_\rho(x)} |u|^2 dy \leq c \left( \Lambda_1 + \Lambda_2 + \int_{\Omega} (|f|^2 + |\nabla f|^2) dy \right) \rho^3$$

for all  $0 < \rho \leq d = \operatorname{dist}(x, \partial\Omega)$ . Since almost all points  $x \in \Omega$  are Lebesgue points of  $|u|^2$ , the latter inequality implies the assertion of the Theorem.

### Appendix: Proof of (2.7) and (2.8)

We apply an idea from Solonnikov, Ščadilov [11] (cf. step 3 below). In that paper, the authors prove the square integrability of the second order derivatives of any generalized solution to the inhomogeneous Stokes system near the boundary of a bounded domain with  $C^3$ -boundary (i.e. after introducing the new variables  $z = \{z_1, z_2, z_3\}$  (cf. above) the reasoning in [11] refers to an equation of type (3.10)). In contrast to that, we start immediately from (2.2). Therefore, our proof of (2.7) is technically simpler than the one in [11]. In addition, we establish the estimate (2.8) which is crucial for the proof of (2.6).

To begin with, we introduce the following notations. Let  $\zeta \in L^1(B_r^+)$ . We extend  $\zeta$  by zero onto  $\mathbb{R}_+^3 \setminus B_r^+$  <sup>7)</sup> and denote this function on  $\mathbb{R}_+^3$  again by  $\zeta$ .

<sup>7)</sup>  $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_3 > 0\}$ .

Then define

$$\zeta_\varepsilon(x) = \int_{\mathbb{R}^2} \omega_\varepsilon(x' - y') \zeta(y', x_3) dy'$$

for any  $\varepsilon > 0$  and almost all  $x \in \mathbb{R}_+^3$ , where  $x' = \{x_1, x_2\}$ ,  $y' = \{y_1, y_2\} \in \mathbb{R}^2$  and  $\omega_\varepsilon(x') = \frac{1}{\varepsilon^2} \omega\left(\frac{x'}{\varepsilon}\right)$ ,  $\omega$  being the standard mollifying kernel in  $\mathbb{R}^2$ . We have:

(i) Let  $\zeta \in L^2(B_r^+)$ . Then

$$\int_{B_r^+} \zeta_\varepsilon^2 dx \leq \int_{B_r^+} \zeta^2 dx \quad \forall \varepsilon > 0.$$

(ii) Let  $\zeta \in H^1(B_r^+)$ . Then  $\zeta_{\varepsilon x_i} = (\zeta_{x_i})_\varepsilon$  a.e. in  $B_{3r/4}^+$  for  $i = 1, 2, 3$  and  $0 < \varepsilon < \frac{r}{4}$ .

1. PROOF OF

$$(A.1) \quad U_{ix_j x_\alpha} \in L^2(B_{r/2}^+), \quad \int_{B_{r/2}^+} (U_{ix_j x_\alpha})^2 dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx$$

( $i, j = 1, 2, 3, \alpha = 1, 2, c = \text{const} > 0$  independent of  $r$ ).

Let  $\psi \in [C^\infty(B_r^+)]^3$ ,  $\text{supp}(\psi) \subset B_{3r/4}^+$ . We extend  $\psi$  by zero onto  $\mathbb{R}_+^3 \setminus B_r^+$ , denote this function on  $\mathbb{R}_+^3$  again by  $\psi$  and form

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^2} \omega_\varepsilon(x' - y') \psi(y', x_3) dy'$$

for a.a.  $x \in B_r^+$  and all  $0 < \varepsilon < \frac{r}{4}$ . Then  $\psi_{\varepsilon x_\alpha} = 0$  near  $\partial B_r^+$  ( $\alpha = 1, 2$ ). Using  $\psi_{\varepsilon x_\alpha}$  as test function in (2.2') (in place of  $\chi$ ), changing variables and observing (ii) gives

$$\int_{B_{3r/4}^+} \nabla U_{\varepsilon i} \cdot \nabla \psi_{i x_\alpha} dx = \int_{B_{3r/4}^+} (q - q_{B_r^+})_\varepsilon \text{div } \psi_{x_\alpha} dx.$$

Thus, by integration by parts,

$$(A.2) \quad \int_{B_{3r/4}^+} \nabla U_{\varepsilon i x_\alpha} \cdot \nabla \psi_i dx = \int_{B_{3r/4}^+} (q - q_{B_r^+})_{\varepsilon x_\alpha} \text{div } \psi dx.$$

By an approximation argument, (A.2) is in fact true for any  $\psi \in H_0^1(B_{3r/4}^+; \mathbb{R}^3)$  (cf. e.g. [8; Th. 4.10, p. 87]).

Let  $\eta \in C^\infty(\mathbb{R}^3)$  be a cut-off function for  $B_{3r/4} : \eta \equiv 1$  on  $B_{r/2}$ ,  $\eta \equiv 0$  in  $\mathbb{R}^3 \setminus B_{3r/4}$  and  $0 \leq \eta \leq 1$ ,  $|\nabla \eta| \leq \frac{c_0}{r}$  and  $|\eta_{x_i x_j}| \leq \frac{c_0}{r^2}$  in  $\mathbb{R}^3$  ( $i, j = 1, 2, 3$ ,  $c_0 = \text{const} > 0$  independent of  $r$ ). Then  $\psi = U_{\varepsilon x_\alpha} \eta^2 \in \dot{H}_0^1(B_{3r/4}^+; \mathbb{R}^3)$  ( $0 < \varepsilon < \frac{r}{4}$ ,  $\alpha = 1, 2$ ) [note that  $U_{\varepsilon x_\alpha} = 0$  a.e. on  $\partial B_{3r/4}^+ \cap \{x_3 = 0\}$  by virtue of (2.1) and (2.4)]. Observing that  $\text{div } U_{\varepsilon x_\alpha} = (\text{div } U)_{\varepsilon x_\alpha} = 0$  a.e. in  $B_{3r/4}^+$  (cf. (2.3) and (ii) above) we obtain from (A.2)

$$\begin{aligned}
 & \int_{B_{3r/4}^+} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx \\
 &= -2 \int_{B_{3r/4}^+} U_{\varepsilon i x_j x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_j} dx + 2 \int_{B_{3r/4}^+} (q - q_{B_r^+})_{\varepsilon x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_i} dx \\
 \text{(A.3)} \quad &= -2 \int_{B_{3r/4}^+} U_{\varepsilon i x_j x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_j} dx \\
 &\quad - 2 \int_{B_{3r/4}^+} (q - q_{B_r^+})_\varepsilon [U_{\varepsilon i x_\alpha x_\alpha} \eta \eta_{x_i} + U_{\varepsilon i x_\alpha} (\eta_{x_\alpha} \eta_{x_i} + \eta \eta_{x_\alpha x_i})] dx \\
 &= I_1 + I_2
 \end{aligned}$$

[no summation over  $\alpha$ ].

The estimation of  $I_1$  is standard:

$$\begin{aligned}
 I_1 &\leq \frac{1}{4} \int_{B_{3r/4}^+} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B_{3r/4}^+} |\nabla U_\varepsilon|^2 dx \\
 &\leq \frac{1}{4} \int_{B_{3r/4}^+} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B_{3r/4}^+} |\nabla U|^2 dx
 \end{aligned}$$

(cf. (i) and (ii) above). Next, to estimates  $I_2$  we make use of (2.5), (i) and (ii):

$$\begin{aligned}
 & -2 \int_{B_{3r/4}^+} (q - q_{B_r^+})_\varepsilon U_{\varepsilon i x_\alpha x_\alpha} \eta \eta_{x_i} dx \\
 & \leq \frac{1}{4} \int_{B_{3r/4}^+} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B_{3r/4}^+} [(q - q_{B_r^+})_\varepsilon]^2 dx \\
 & \leq \frac{1}{4} \int_{B_{3r/4}^+} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx,
 \end{aligned}$$



$$\begin{aligned}
& -2 \int_{B_{3r/4}^+} (q - q_{B_r^+})_\varepsilon U_{\varepsilon i x_\alpha} (\eta_{x_i} \eta_{x_\alpha} + \eta \eta_{x_i x_\alpha}) dx \\
& \leq \frac{c}{r^2} \int_{B_{3r/4}^+} |\nabla U|^2 dx.
\end{aligned}$$

Inserting these estimates into (A.3) we get

$$(A.4) \quad \int_{B_{r/2}^+} |\nabla U_{\varepsilon x_\alpha}|^2 dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx \quad \forall 0 < \varepsilon \leq \frac{r}{4}.$$

Letting  $\varepsilon \rightarrow 0$  implies (A.1).

## 2. PROOF OF

$$(A.5) \quad \begin{cases} U_{3x_3 x_3}, q_{x_3} \in L^2(B_{r/2}^+), \\ \int_{B_{r/2}^+} [(U_{3x_3 x_3})^2 + (q_{x_3})^2] dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx. \end{cases}$$

Firstly,  $\operatorname{div} U = 0$  a.e. in  $B_r^+$  and (A.1) imply

$$U_{3x_3 x_3} = -(U_{1x_1} + U_{2x_2})_{x_3} = -U_{1x_3 x_1} - U_{2x_3 x_2}$$

a.e. in  $B_{r/2}^+$ . Whence the statement on  $U_{3x_3 x_3}$  in (A.5).

Secondly, let  $h \in H_0^1(B_{r/2}^+)$ . We extend  $h$  by zero onto  $B_r^+ \setminus B_{r/2}^+$  and denote this function on  $B_r^+$  again by  $h$ . Then  $\chi = \{0, 0, h\}$  is admissible in (2.2.'):

$$-\int_{B_{r/2}^+} (\Delta U_3) h dx = \int_{B_{r/2}^+} q h_{x_3} dx.$$

The statement on  $q_{x_3}$  in (A.5) is now readily seen.

## 3. PROOF OF

$$(A.6) \quad q_{x_\alpha} \in L^2(B_{r/4}^+), \quad \int_{B_{r/4}^+} (q_{x_\alpha})^2 dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx \quad (\alpha = 1, 2).$$

In order to prove (A.6) we need the following result.

*Let  $f \in H^1(\mathbb{R}_+^3)$  have bounded support. Then there exists a function  $\phi \in H^2(\mathbb{R}_+^3; \mathbb{R}^3)$  such that*

$$(A.7) \quad \operatorname{div} \phi = f \text{ a.e. in } \mathbb{R}_+^3,$$

(A.8)  $\phi = 0$  a.e. on  $\{x_3 = 0\}$ ,

(A.9) 
$$\int_{\mathbb{R}_+^3} |\nabla\phi|^2 dx \leq c \int_{\mathbb{R}_+^3} f^2 dx,$$

(A.10) 
$$\int_{\mathbb{R}_+^3} |\nabla\phi_{x_\alpha}|^2 dx \leq c \int_{\mathbb{R}_+^3} (f_{x_\alpha})^2 dx \quad (\alpha = 1, 2)$$

( $c = \text{const} > 0$  independent of  $f$ ).

This result is stated without proof in [11]. A proof of (A.7)-(A.10) can be given by using the explicit representation of the solution of

$$-\Delta v + \nabla p = 0, \text{ div } v = f \text{ in } \mathbb{R}_+^3, v = 0 \text{ on } \{x_3 = 0\}$$

in terms of potentials the kernels of which involve only differences with respect to  $x_1$  and  $x_2$  ( $x = \{x_1, x_2, x_3\} \in \mathbb{R}_+^3$ ; cf. [9; pp. 163-165]) [private communication by V.A. Solonnikov].

An entirely different and more elementary solution of (A.7)-(A.10) can be given as follows [private communication by V.A. Solonnikov]. Define

$$\phi_i(x) = \int_{\mathbb{R}^3} K\left(\frac{x-y}{|x-y|}\right) \frac{x_i - y_i}{|x-y|^3} \tilde{f}(y) dy, \quad x \in \mathbb{R}_+^3 \quad (i = 1, 2, 3);$$

here  $K$  is any function in  $C^2(\mathbb{R}^3)$  with  $\text{supp}(K) \subset \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$  and  $\int_{\partial B_1(0)} K dS = 1$ , and

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for a.a. } x \in \mathbb{R}_+^3, \\ 0 & \text{for a.a. } x \in \mathbb{R}^3 \setminus \mathbb{R}_+^3. \end{cases}$$

Then (A.7) and (A.8) are easily verified. Further, the derivatives  $\phi_{ix_k}$  as well as  $\phi_{ix_\alpha x_k}$  ( $i, k = 1, 2, 3; \alpha = 1, 2$ ) give rise to a singular integral to which the well-known Calderon-Zygmund theorem applies. Whence (A.9) and (A.10) (cf. also [10; Lemma 2.1, p. 252]).

Now, let  $\eta \in C^\infty(\mathbb{R}^3)$  be a cut-off function for  $B_{r/2} : \eta \equiv 1$  on  $B_{r/4}$ ,  $\eta \equiv 0$  in  $\mathbb{R}^3 \setminus B_{r/2}$  and  $0 \leq \eta \leq 1$ ,  $|\nabla\eta| \leq \frac{c_0}{r}$  and  $|\eta_{x_j x_j}| \leq \frac{c_0}{r^2}$  in  $\mathbb{R}^3$  ( $i, j = 1, 2, 3; c_0 = \text{const} > 0$  independent of  $r$ ). We apply the result just stated with  $f = (q - q_{B_r^+})_\epsilon \eta$  a.e. in  $B_r^+$ ,  $f = 0$  a.e. in  $\mathbb{R}_+^3 \setminus B_r^+$  ( $0 < \epsilon < \frac{r}{4}$ ). Thus, there exists a function  $\phi^{(\epsilon)} \in H^2(\mathbb{R}_+^3; \mathbb{R}^3)$  such that

(A.7 $_\epsilon$ )  $\text{div } \phi^{(\epsilon)} = (q - q_{B_r^+})_\epsilon \eta$  a.e. in  $B_r^+$ ,

(A.8 $_\epsilon$ )  $\phi^{(\epsilon)} = 0$  a.e. on  $B_r^+ \cap \{x_3 = 0\}$ ,

$$(A.9_\epsilon) \quad \int_{\mathbb{R}_+^3} |\nabla \phi^{(\epsilon)}|^2 dx \leq c \int_{B_{r/2}^+} (q - q_{B_r^+})^2 dx,$$

$$(A.10_\epsilon) \quad \int_{\mathbb{R}_+^3} |\nabla \phi_{x_\alpha}^{(\epsilon)}|^2 dx \leq c \int_{B_{r/2}^+} \left[ (q_{\epsilon x_\alpha})^2 \eta^2 + \frac{1}{r^2} (q - q_{B_r^+})^2 \right] dx$$

( $\alpha = 1, 2$ ;  $c = \text{const} > 0$  independent of  $r$ ; note that  $(q - q_{B_r^+})_\epsilon = q_\epsilon - q_{B_r^+}$ ).

Clearly,  $\phi_{x_\alpha}^{(\epsilon)} \eta \in H_0^1(B_{3r/4}^+; \mathbb{R}^3)$  ( $\alpha = 1, 2$ ). Thus,  $\chi = \phi_{x_\alpha}^{(\epsilon)} \eta$  is admissible in (A.2). Taking into account (A.7 $_\epsilon$ ) and

$$\begin{aligned} & \int_{B_{3r/4}^+} (q - q_{B_r^+})_{\epsilon x_\alpha} \phi_{i x_\alpha}^{(\epsilon)} \eta_{x_i} dx \\ &= - \int_{B_{3r/4}^+} (q - q_{B_r^+})_\epsilon \left( \phi_{i x_\alpha x_\alpha}^{(\epsilon)} \eta_{x_i} + \phi_{i x_\alpha}^{(\epsilon)} \eta_{x_i x_\alpha} \right) dx \end{aligned}$$

we get

$$\begin{aligned} & \int_{B_{r/2}^+} (q_{\epsilon x_\alpha})^2 \eta^2 dx \\ (A.11) \quad &= \int_{B_{r/2}^+} \nabla U_{\epsilon i x_\alpha} \cdot \nabla \eta \phi_{i x_\alpha}^{(\epsilon)} dx + \int_{B_{r/2}^+} \nabla U_{\epsilon i x_\alpha} \cdot \nabla \phi_{i x_\alpha}^{(\epsilon)} \eta dx \\ &+ \int_{B_{r/2}^+} (q - q_{B_r^+})_\epsilon \left( \phi_{i x_\alpha x_\alpha}^{(\epsilon)} \eta_{x_i} + \phi_{i x_\alpha}^{(\epsilon)} \eta_{x_i x_\alpha} \right) dx \\ &- \int_{B_{r/2}^+} q_{\epsilon x_\alpha} (q - q_{B_r^+})_\epsilon \eta \eta_{x_\alpha} dx \\ &= J_1 + J_2 + J_3 + J_4 \end{aligned}$$

[no summation over  $\alpha$ ]. To estimate  $J_1$  and  $J_2$  we combine (2.5) and (A.4), (A.9 $_\epsilon$ ), (A.10 $_\epsilon$ ):

$$\begin{aligned} J_1 &\leq \frac{1}{2} \int_{B_{r/2}^+} |\nabla U_{\epsilon x_\alpha}|^2 dx + \frac{c_0^2}{2r^2} \int_{B_{r/2}^+} |\nabla \phi^{(\epsilon)}|^2 dx \\ &\leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx, \end{aligned}$$

$$\begin{aligned}
 J_2 &\leq \left( \int_{B_{r/2}^+} |\nabla U_{\varepsilon x_\alpha}|^2 dx \right)^{1/2} \left( \int_{B_{r/2}^+} |\nabla \phi_{x_\alpha}^{(\varepsilon)}|^2 dx \right)^{1/2} \\
 &\leq \frac{1}{4} \int_{B_{r/2}^+} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx.
 \end{aligned}$$

Analogously, by (2.5) and (A.9<sub>ε</sub>), (A.10<sub>ε</sub>),

$$\begin{aligned}
 J_3 &\leq \frac{c}{r} \left\{ \int_{B_r^+} |\nabla U|^2 dx \right\}^{1/2} \left\{ \int_{B_{r/2}^+} \left( \frac{1}{r^2} |\nabla \phi^{(\varepsilon)}|^2 + |\nabla \phi_{x_\alpha}^{(\varepsilon)}|^2 \right) dx \right\}^{1/2} \\
 &\leq \frac{1}{4} \int_{B_{r/2}^+} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx.
 \end{aligned}$$

Finally,

$$J_4 \leq \frac{1}{4} \int_{B_{r/2}^+} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx.$$

Inserting the estimates on  $J_1, \dots, J_4$  into (A.11) and letting  $\varepsilon \rightarrow 0$  we get (A.6).

#### 4. PROOF OF

$$U_{\alpha x_3 x_3} \in L^2(B_{r/4}^+), \quad \int_{B_{r/4}^+} (U_{\alpha x_3 x_3})^2 dx \leq \frac{c}{r^2} \int_{B_r^+} |\nabla U|^2 dx \quad (\alpha = 1, 2).$$

Let  $h \in H_0^1(B_{r/4}^+)$ . We extend  $h$  by zero onto  $B_r^+ \setminus B_{r/4}^+$  and denote this function on  $B_r^+$  again by  $h$ . Then we let  $\chi = \{h, 0, 0\}$  in (2.2') and find

$$\int_{B_{r/4}^+} U_{1x_3} h_{x_3} dx = - \int_{B_{r/4}^+} (U_{1x_1} h_{x_1} + U_{1x_2} h_{x_2}) dx + \int_{B_{r/4}^+} q h_{x_1} dx.$$

Hence, the claim follows for  $\alpha = 1$  when observing (A.4) and (A.6). To prove the claim for  $\alpha = 2$  we let  $\chi = \{0, h, 0\}$  in (2.2') and argue analogously.

The proof of (2.7) and (2.8) is complete.

